DYNAMICAL NUMBER OF BASE-POINTS OF NON BASE-WANDERING JONQUIÈRES TWISTS

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ABSTRACT. We give some properties of the dynamical number of base-points of birational self-maps of the complex projective plane.

In particular we give a formula to determine the dynamical number of base-points of non base-wandering Jonquières twists.

1. INTRODUCTION

The *plane Cremona group* $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$ is the group of birational maps of the complex projective plane $\mathbb{P}^2_\mathbb{C}$. It is isomorphic to the group of $\mathbb{C}$-algebra automorphisms of $\mathbb{C}(X,Y)$, the function field of $\mathbb{P}^2_\mathbb{C}$. Using a system of homogeneous coordinates $(x:y:z)$ a birational map $f\in\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$ can be written as

$$(x:y:z) \dashrightarrow (P_0(x,y,z):P_1(x,y,z):P_2(x,y,z))$$

where P_0, P_1 and P_2 are homogeneous polynomials of the same degree without common factor. This degree does not depend on the system of homogeneous coordinates. We call it the *degree* of f and denote it by $\deg(f)$. Geometrically it is the degree of the pull-back by f of a general projective line. Birational maps of degree 1 are homographies and form the group $\mathrm{Aut}(\mathbb{P}^2_\mathbb{C})=\mathrm{PGL}(3,\mathbb{C})$ of automorphisms of the projective plane.

$\diamond$ **Four types of elements**.

The elements $f\in\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$ can be classified into exactly one of the four following types according to the growth of the sequence $(\deg(f^k))_{k\in\mathbb{N}}$ (*see* [DF01, BD15]):

(1) The sequence $(\deg(f^k))_{k\in\mathbb{N}}$ is bounded, f is either of finite order or conjugate to an automorphism of $\mathbb{P}^2_\mathbb{C}$; we say that f is an *elliptic element*.
(2) The sequence $(\deg(f^k))_{k\in\mathbb{N}}$ grows linearly, f preserves a unique pencil of rational curves and f is not conjugate to an automorphism of any rational projective surface; we call f a *Jonquières twist*.
(3) The sequence $(\deg(f^k))_{k\in\mathbb{N}}$ grows quadratically, f is conjugate to an automorphism of a rational projective surface preserving a unique elliptic fibration; we call f a *Halphen twist*.
(4) The sequence $(\deg(f^k))_{k\in\mathbb{N}}$ grows exponentially and we say that f is *hyperbolic*.

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◇ **The Jonquières group**.

Let us fix an affine chart of $\mathbb{P}^2_{\mathbb{C}}$ with coordinates (x,y). The *Jonquières group* J is the subgroup of the Cremona group of all maps of the form

$$(x,y) \dashrightarrow \left(\frac{A(y)x+B(y)}{C(y)x+D(y)}, \frac{ay+b}{cy+d}\right) \tag{1.1}$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}(2,\mathbb{C})$ and $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{PGL}(2,\mathbb{C}(y))$. The group J is the group of all birational maps of $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ permuting the fibers of the projection onto the second factor; it is isomorphic to the semi-direct product $\mathrm{PGL}(2,\mathbb{C}(y)) \rtimes \mathrm{PGL}(2,\mathbb{C})$.

We can check with (1.1) that if f belongs to J, then $(\deg(f^k))_{k\in\mathbb{N}}$ grows at most linearly; elements of J are either elliptic or Jonquières twists. Let us denote by $\mathcal{J}$ the set of Jonquières twist:

$$\mathcal{J} = \left\{ f \in \mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}}) \mid f \text{ Jonquières twist } \right\}.$$

A Jonquières twist is called a *base-wandering Jonquières twist* if its action on the basis of the rational fibration has infinite order. Let us denote by J_0 the normal subgroup of J that preserves fiberwise the rational fibration, that is the subgroup of those maps of the form

$$(x,y) \dashrightarrow \left(\frac{A(y)x+B(y)}{C(y)x+D(y)}, y\right).$$

The group J_0 is isomorphic to $\mathrm{PGL}(2,\mathbb{C}(y))$. The group J_0 has three maximal (for the inclusion) uncountable abelian subgroups

$$\mathrm{J}_a = \left\{ (x+a(y),y) \,|\, a \in \mathbb{C}(y) \right\}, \qquad \mathrm{J}_m = \left\{ (b(y)x,y) \,|\, b \in \mathbb{C}(y)^* \right\}$$

and

$$\mathrm{J}_F = \left\{ (x,y), \left(\frac{c(y)x+F(y)}{x+c(y)}, y\right) \,|\, c \in \mathbb{C}[y] \right\}$$

where F denotes an element of $\mathbb{C}[y]$ that is not a square ([D0́6]).

Let us associate to $f = \left(\frac{A(y)x+B(y)}{C(y)x+D(y)}, y\right) \in \mathrm{J}_0$ the matrix $M_f = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. The *Baum Bott index* of f is $\mathrm{BB}(f) = \frac{\left(\mathrm{Tr}(M_f)\right)^2}{\det(M_f)}$ (by analogy with the Baum Bott index of a foliation) which is well defined in PGL and is invariant by conjugation. This invariant BB indicates the degree growth:

Proposition 1.1 ([CD12])**.** *Let f be a Jonquières twist that preserves fiberwise the rational fibration. The rational function* $\mathrm{BB}(f)$ *is constant if and only if f is an elliptic element.*

A direct consequence is the following:

Corollary 1.2. *Let f be a non-base wandering Jonquières twist; the rational function* $\mathrm{BB}(f)$ *is constant if and only if f is an elliptic element.*

Every $f \in \mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ admits a resolution

$$\begin{array}{ccc} & S & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ \mathbb{P}^2_{\mathbb{C}} & \overset{}{\underset{f}{\dashrightarrow}} & \mathbb{P}^2_{\mathbb{C}} \end{array}$$

where π_1, π_2 are sequences of blow-ups. The resolution is *minimal* if and only if no (-1)-curve of S are contracted by both π_1 and π_2. Assume that the resolution is minimal; the *base-points* of f are the points blown-up by π_1, which can be points of S or infinitely near points. If f belongs to J, then f has one base-point p_0 of multiplicity $d-1$ and $2d-2$ base-points $p_1, p_2, \ldots, p_{2d-2}$ of multiplicity 1. Similarly the map f^{-1} has one base-point q_0 of multiplicity $d-1$ and $2d-2$ base-points $q_1, q_2, \ldots, q_{2d-2}$ of multiplicity 1. Let us denote by $f_\sharp$ the action of f on the Picard-Manin space of $\mathbb{P}^2_{\mathbb{C}}$, by $\mathbf{e}_m \in \mathrm{NS}(S)$ the Néron-Severi class of the total transform of m under π_j (for $1 \leq j \leq 2$), and by ℓ the class of a line in $\mathbb{P}^2_{\mathbb{C}}$. The action of f on ℓ and the classes $(\mathbf{e}_{p_j})_{0\leq j\leq 2d-2}$ is given by:

$$\begin{cases} f_\sharp(\ell) = d\ell - (d-1)\mathbf{e}_{q_0} - \displaystyle\sum_{i=1}^{2d-2} \mathbf{e}_{q_i} \\ f_\sharp(\mathbf{e}_{p_0}) = (d-1)\ell - (d-2)\mathbf{e}_{q_0} - \displaystyle\sum_{i=1}^{2d-2} \mathbf{e}_{q_i} \\ f_\sharp(\mathbf{e}_{p_i}) = \ell - \mathbf{e}_{q_0} - \mathbf{e}_{q_i} \qquad \forall\, 1 \leq i \leq 2d-2 \end{cases}$$

◇ **Dynamical degree**.

Given a birational self-map $f\colon S \dashrightarrow S$ of a complex projective surface, its dynamical degree $\lambda(f)$ is a positive real number that measures the complexity of the dynamics of f. Indeed $\log(\lambda(f))$ provides an upper bound for the topological entropy of f and is equal to it under natural assumptions (*see* [BD05, DS05]). The dynamical degree is invariant under conjugacy; as shown in [BC16] precise knowledge on $\lambda(f)$ provides useful information on the conjugacy class of f. By definition a *Pisot number* is an algebraic integer $\lambda \in]1,+\infty[$ whose other Galois conjugates lie in the open unit disk; Pisot numbers include integers $d \geq 2$ as well as reciprocal quadratic integers $\lambda > 1$. A *Salem number* is an algebraic integer $\lambda \in]1,+\infty[$ whose other Galois conjugates are in the closed unit disk, with at least one on the boundary. Diller and Favre proved the following statement:

Theorem 1.3 ([DF01])**.** *Let f be a birational self-map of a complex projective surface. If $\lambda(f)$ is different from 1, then $\lambda(f)$ is a Pisot number or a Salem number.*

One of the goal of [BC16] is the study of the structure of the set of all dynamical degrees $\lambda(f)$ where f runs over the group of birational maps $\mathrm{Bir}(S)$ and S over the collection of all projective surfaces. In particular they get:

Theorem 1.4 ([BC16]). *Let Λ be the set of all dynamical degrees of birational maps of complex projective surfaces. Then*

- ◇ *Λ is a well ordered subset of $\mathbb{R}_+$;*
- ◇ *if λ is an element of Λ, there is a real number $\varepsilon > 0$ such that $]\lambda, \lambda + \varepsilon]$ does not intersect Λ;*
- ◇ *there is a non-empty interval $]\lambda_G, \lambda_G + \varepsilon]$, with $\varepsilon > 0$, on the right of the golden mean that contains infinitely many Pisot and Salem numbers, but does not contain any dynamical degree.*

◇ **Dynamical number of base-points** ([BD15]).

If S is a projective smooth surface, every $f \in \mathrm{Bir}(S)$ admits a resolution

$$\begin{array}{ccc} & Z & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ S & \underset{f}{\dashrightarrow} & S \end{array}$$

where π_1, π_2 are sequences of blow-ups. The resolution is *minimal* if and only if no (-1)-curve of Z are contracted by both π_1 and π_2. Assume that the resolution is minimal; the *base-points* of f are the points blown-up by π_1, which can be points of S or infinitely near points. We denote by $\mathfrak{b}(f)$ the number of such points, which is also equal to the difference of the ranks of $\mathrm{Pic}(Z)$ and $\mathrm{Pic}(S)$, and thus equal to $\mathfrak{b}(f^{-1})$.

Let us define the *dynamical number of base-points* of f by

$$\mu(f) = \lim_{k \to +\infty} \frac{\mathfrak{b}(f^k)}{k}.$$

Since $\mathfrak{b}(f \circ \varphi) \leq \mathfrak{b}(f) + \mathfrak{b}(\varphi)$ for any f, $\varphi \in \mathrm{Bir}(S)$ we see that $\mu(f)$ is a non-negative real number. Moreover, $\mathfrak{b}(f^{-1})$ and $\mathfrak{b}(f)$ being equal we get $\mu(f^k) = |k\mu(f)|$ for any $k \in \mathbb{Z}$. Furthermore, the dynamical number of base-points is an invariant of conjugation: if $\psi \colon S \dashrightarrow Z$ is a birational map between smooth projective surfaces and if f belongs to $\mathrm{Bir}(S)$, then $\mu(f) = \mu(\psi \circ f \circ \psi^{-1})$. In particular if f is conjugate to an automorphism of a smooth projective surface, then $\mu(f) = 0$. The converse holds, *i.e.* $f \in \mathrm{Bir}(S)$ is conjugate to an automorphism of a smooth projective surface if and only if $\mu(f) = 0$ ([BD15, Proposition 3.5]). This follows from the geometric interpretation of μ we will recall now. If $f \in \mathrm{Bir}(S)$ is a birational map, a (possibly infinitely near) base-point p of f is a *persistent base-point* of f if there exists an integer N such that p is a base-point of f^k for any $k \geq N$ but is not a base-point of f^{-k} for any $k \geq N$. We put an equivalence relation on the set of points that belongs to S or are infinitely near: take a minimal resolution of f

$$\begin{array}{ccc} & Z & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ S & \underset{f}{\dashrightarrow} & S \end{array}$$

where π_1, π_2 are sequences of blow-ups; the point p is *equivalent* to q if there exists an integer k such that $(\pi_2 \circ \pi_1^{-1})^k(p) = q$. Denote by ν the number of equivalence classes of persistent base-points of f; then the set

$$\left\{\mathfrak{b}(f^k) - \nu k \,|\, k \geq 0\right\} \subset \mathbb{Z}$$

is bounded. In particular, $\mu(f)$ is an integer, equal to ν (*see* [BD15, Proposition 3.4]). This gives a bound for $\mu(f)$; indeed, if $f \in \mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ is a map whose base-points have multiplicities $m_1 \geq m_2 \geq \ldots \geq m_r$ then (*see for instance* [ACn02, §2.5] and [ACn02, Corollary 2.6.7])

$$\begin{cases} \displaystyle\sum_{i=1}^{r} m_i = 3(\deg(f) - 1) \\ \displaystyle\sum_{i=1}^{r} m_i^2 = \deg(f)^2 - 1 \\ m_1 + m_2 + m_3 \geq \deg(f) + 1 \end{cases}$$

in particular, $r \leq 2\deg(f) - 1$ so $\nu \leq 2\deg(f) - 1$ and $\mu(f) \leq 2\deg(f) - 1$.

If $f \in \mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ is a Jonquières twist, then there exists an integer $a \in \mathbb{N}$ such that

$$\lim_{k \to +\infty} \frac{\deg(f^k)}{k} = a^2 \frac{\mu(f)}{2};$$

moreover, a is the degree of the curves of the unique pencil of rational curves invariant by f (*see* [BD15, Proposition 4.5]). In particular, $a = 1$ if and only if f preserves a pencil of lines. On the one hand $\left\{\mu(f) \,|\, f \in \mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})\right\} \subseteq \mathbb{N}$ and on the other hand if f belongs to $\mathcal{J}$, then $\mu(f) > 0$; as a result

$$\left\{\mu(f) \,|\, f \in \mathcal{J}\right\} \subseteq \mathbb{N} \smallsetminus \{0\}.$$

Let us recall that if $f_{\alpha,\beta} = \left(\frac{\alpha x + y}{x+1}, \beta y\right)$ then $\mu(f_{\alpha,\beta}) = 1$. Indeed, by induction one can prove that $f_{\alpha,\beta}^{2n} = \left(\frac{P_n(x,y)}{Q_n(x,y)}, \beta^{2n} y\right)$ with

$$P_n(x,y) = \sum_{0 \leq i+j \leq n+1} a_{ij} x^i y^j \qquad\qquad Q_n(x,y) = \sum_{0 \leq i+j \leq n} b_{ij} x^i y^j$$

and $a_{ij} \geq 0$, $b_{ij} \geq 0$ for any $n \geq 0$, so that $\deg f_{\alpha,\beta}^{2n} = n + 1$ for any $n \geq 0$; we conclude using the fact that $\mu(f) = 2 \displaystyle\lim_{k \to +\infty} \frac{\deg(f^k)}{k}$. Furthermore, $\mu(f_{\alpha,\beta}^k) = |k\mu(f_{\alpha,\beta})| = |k|$ for any $k \in \mathbb{Z}$. Hence

$$\left\{\mu(f) \,|\, f \in \mathcal{J}\right\} = \mathbb{N} \smallsetminus \{0\}$$

and

$$\big\{\mu(f)\,|\,f\in\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})\big\}=\mathbb{N}.$$

As we have seen if f belongs to $\mathcal{J}$, then $\mu(f)=2\lim\limits_{k\to+\infty}\dfrac{\deg(f^k)}{k}$. Can we express $\mu(f)$ in a simpliest way ? We will see that if f is a non base-wandering Jonquières twist, the answer is yes.

◇ **Results**.

The dynamical number of base-points of birational self maps of the complex projective plane satisfies the following properties:

Theorem A. 1. *If f is a birational self-map from $\mathbb{P}^2_{\mathbb{C}}$ into itself, then its dynamical number of base-points is bounded: if $f\in\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$, then $\mu(f)\leq 2\deg(f)-1$.*

2. *We can precise the set of all dynamical numbers of base-points of birational maps of $\mathbb{P}^2_{\mathbb{C}}$ (resp. of Jonquières maps of $\mathbb{P}^2_{\mathbb{C}}$) :*

$$\big\{\mu(f)\,|\,f\in\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})\big\}=\mathbb{N}\qquad\textit{and}\qquad\big\{\mu(f)\,|\,f\in\mathcal{J}\big\}=\mathbb{N}\smallsetminus\{0\}.$$

3. *There exist sequences $(f_n)_n$ of birational self-maps of $\mathbb{P}^2_{\mathbb{C}}$ such that*
   - ◇ $\mu(f_n)>0$ *for any* $n\in\mathbb{N}$*;*
   - ◇ $\mu\big(\lim\limits_{n\to+\infty}f_n\big)=0.$

4. *There exist sequences $(f_n)_n$ of birational self-maps of $\mathbb{P}^2_{\mathbb{C}}$ such that*
   - ◇ $\mu(f_n)=0$ *for any* $n\in\mathbb{N}$*;*
   - ◇ $\mu\big(\lim\limits_{n\to+\infty}f_n\big)>0.$

Let us now give a formula to determine the dynamical number of base-points of Jonquières twists that preserves fiberwise the fibration.

Theorem B. *Let $f=\left(\frac{A(y)x+B(y)}{C(y)x+D(y)},y\right)$ be a Jonquières twist that preserves fiberwise the fibration, and let M_f be its associated matrix. Denote by $\mathrm{Tr}(M_f)$ the trace M_f, by χ_f the characteristic polynomial of M_f, and by Δ_f the discriminant of χ_f. Then exactly one of the following holds:*

1. *If χ_f has two distinct roots in $\mathbb{C}[y]$, then f is conjugate to $g=\left(\frac{\mathrm{Tr}(M_f)+\delta_f}{\mathrm{Tr}(M_f)-\delta_f}x,y\right)$, where $\delta_f^2=\Delta_f$, and*

$$\mu(f)=\mu(g)=2(\deg(g)-1).$$

2. *If χ_f has no root in $\mathbb{C}[y]$, set*

$$\Omega_f=\gcd\left(\frac{\mathrm{Tr}(M_f)}{2},\left(\frac{\mathrm{Tr}(M_f)}{2}\right)^2-\det(M_f)\right)$$

and let us define P_f and S_f as

$$\frac{\mathrm{Tr}(M_f)}{2}=P_f\Omega_f, \qquad\qquad \left(\frac{\mathrm{Tr}(M_f)}{2}\right)^2-\det(M_f)=S_f\Omega_f.$$

2.a. *If* $\gcd(\Omega_f,S_f)=1$*, then*
- ◇ *if* $\deg(S_f)\leq\deg(\Omega_f)+2\deg(P_f)$*, then* $\mu(f)=\deg(\Omega_f)+2\deg(P_f)$*;*
- ◇ *otherwise* $\mu(f)=\deg(S_f)$*.*

2.b. *If* $S_f=\Omega_f^pT_f$ *with* $p\geq 1$ *and* $\gcd(T_f,\Omega_f)=1$*, then*
- ◇ *if* $\deg(S_f)\leq\deg(\Omega_f)+2\deg(P_f)$*, then* $\mu(f)=2\deg(P_f)$*;*
- ◇ *otherwise* $\mu(f)=\deg(S_f)-\deg(\Omega_f)$*.*

2.c. *If* $\Omega_f=S_f^pT_f$ *with* $p\geq 1$ *and* $\gcd(T_f,S_f)=1$*, then* $\mu(f)=2\deg(P_f)+\deg(\Omega_f)-\deg(S_f)$*.*

As a consequence we are able to determine the dynamical number of base-points of non base-wandering Jonquières twists:

Corollary C. *Let $f=(f_1,f_2)$ be a non base-wandering Jonquières twist.*

If ℓ is the order of f_2, then $\mu(f)=\frac{\mu(f^\ell)}{\ell}$ where $\mu(f^\ell)$ is given by Theorem B.

Combining the inequalities obtained in Theorem A and Theorem B we get the following statement (we use the notations introduced in Theorem B):

Corollary D. *Let f be a Jonquières twist that preserves fiberwise the fibration. Assume that χ_f has two distinct roots in $\mathbb{C}[y]$.*

Then there exists a conjugate g of f such that g belongs to J_m *and* $\deg(g)\leq\deg(f)$*. For instance* $g=\left(\frac{\mathrm{Tr}(M_f)+\delta_f}{\mathrm{Tr}(M_f)-\delta_f}x,y\right)$ *suits.*

2. Dynamical number of base-points of Jonquières twists

In this section we will prove Theorem B.

Let f be an element of J_0; write f as $\left(\frac{A(y)x+B(y)}{C(y)x+D(y)},y\right)$ with A, B, C, $D\in\mathbb{C}[y]$. The characteristic polynomial of $M_f=\begin{pmatrix}A & B\\ C & D\end{pmatrix}$ is $\chi_f(X)=X^2-\mathrm{Tr}(M_f)X+\det(M_f)$. There are three possibilities:

(1) χ_f has one root of multiplicity 2 in $\mathbb{C}[y]$;
(2) χ_f has two distinct roots in $\mathbb{C}[y]$;
(3) χ_f has no root in $\mathbb{C}[y]$.

Let us consider these three possibilities.

(1) If χ_f has one root of multiplicity 2 in $\mathbb{C}[y]$, then f is conjugate to the elliptic birational map $(x+a(y),y)$ of J_a. In particular f does not belong to $\mathcal{J}$.

(2) Assume that χ_f has two distinct roots. The discriminant of χ_f is

$$\Delta_f = \big(\mathrm{Tr}(M_f)\big)^2 - 4\det(M_f) = \delta_f^2$$

and the roots of χ_f are

$$\frac{\mathrm{Tr}(M_f)+\delta_f}{2} \qquad \text{and} \qquad \frac{\mathrm{Tr}(M_f)-\delta_f}{2}.$$

Furthermore, M_f is conjugate to $\begin{pmatrix} \frac{\mathrm{Tr}(M_f)+\delta_f}{2} & 0 \\ 0 & \frac{\mathrm{Tr}(M_f)-\delta_f}{2} \end{pmatrix}$, *i.e.* f is conjugate to $g=(a(y)x,y)\in \mathrm{J}_m$ with $a(y)=\frac{\mathrm{Tr}(M_f)+\delta_f}{\mathrm{Tr}(M_f)-\delta_f}$. Let us first express $\mu(g)$ thanks to $\deg(g)$. Remark that $g^k = (a(y)^kx,y)$. Write $a(y)^j$ as $\frac{P_j(y)}{Q_j(y)}$ where P_j, $Q_j\in\mathbb{C}[y]$, $\gcd(P_j,Q_j)=1$, then $\deg(g^j) = \max(\deg(P_j),\deg(Q_j))+1$. But $\deg(P_j)=j\deg(P)$ and $\deg(Q_j)=j\deg(Q)$ so

$$\deg(g^k) = \max(k\deg(P_f),k\deg(Q_1))+1 = k\underbrace{\max(\deg(P_f),\deg(Q_1))}_{\deg(g)-1}+1.$$

As a consequence $\deg(g^k)=k\deg(g)-k+1$. According to $\mu(g)=2\lim_{k\to+\infty}\frac{\deg(g^k)}{k}$ we get

$$\mu(g) = 2\lim_{k\to+\infty}\Big(\deg(g)-1+\frac{1}{k}\Big) = 2(\deg(g)-1).$$

Let us now express $\mu(f)$ thanks to f. Since f and g are conjugate $\mu(f)=\mu(g)$ hence $\mu(f)=2(\deg(g)-1)$. But $g=\Big(\frac{\mathrm{Tr}(M_f)+\delta_f}{\mathrm{Tr}(M_f)-\delta_f}x,y\Big)$; in particular

$$\deg(g)\leq 1+\max\big(\deg(\mathrm{Tr}(M_f)+\delta_f),\deg(\mathrm{Tr}(M_f)-\delta_f)\big)$$

and $\mu(f)\leq 2\max\big(\deg(\mathrm{Tr}(M_f)+\delta_f),\deg(\mathrm{Tr}(M_f)-\delta_f)\big)$.

(3) Suppose that χ_f has no root in $\mathbb{C}[y]$. This means that $\big(\mathrm{Tr}(M_f)\big)^2-4\det(M_f)$ is not a square in $\mathbb{C}[y]$ (hence $BC\neq 0$). Note that

$$\begin{pmatrix} C & \frac{D-A}{2} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} C & \frac{D-A}{2} \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{\mathrm{Tr}(M_f)}{2} & \Big(\frac{\mathrm{Tr}(M_f)}{2}\Big)^2-\det(M_f) \\ 1 & \frac{\mathrm{Tr}(M_f)}{2} \end{pmatrix}.$$

In other words f is conjugate to

$$g=\left(\frac{\frac{\mathrm{Tr}(M_f)}{2}x+\Big(\frac{\mathrm{Tr}(M_f)}{2}\Big)^2-\det(M_f)}{x+\frac{\mathrm{Tr}(M_f)}{2}},y\right)\in \mathrm{J}_{\frac{\mathrm{Tr}(M_f)}{2}}.$$

Set $P(y)=\frac{\mathrm{Tr}(M_f)}{2}\in\mathbb{C}[y]$ and $F(y)=\left(\frac{\mathrm{Tr}(M_f)}{2}\right)^2-\det(M_f)\in\mathbb{C}[y]$, *i.e.* f is conjugate to $g=\left(\frac{P(y)x+F(y)}{x+P(y)},y\right)$ with P, $F\in\mathbb{C}[y]$. Denote by d_P (resp. d_F) the degree of P (resp. F). Remark that $\deg(g)=\max(d_P+1,d_F,2)$.

Let us now express $\deg(g^k)$. Consider $M_g=\begin{pmatrix} P & F\\ 1 & P\end{pmatrix}$ and set

$$Q=\begin{pmatrix}\sqrt{F} & -\sqrt{F}\\ 1 & 1\end{pmatrix}\qquad\text{and}\qquad D=\begin{pmatrix}P+\sqrt{F} & 0\\ 0 & P-\sqrt{F}\end{pmatrix}$$

Then $M_g^k=QD^kQ^{-1}$ hence

$$M_g^k=\begin{pmatrix}\sqrt{F}\frac{(P+\sqrt{F})^k+(P-\sqrt{F})^k}{(P+\sqrt{F})^k-(P-\sqrt{F})^k} & F\\ 1 & \sqrt{F}\frac{(P+\sqrt{F})^k+(P-\sqrt{F})^k}{(P+\sqrt{F})^k-(P-\sqrt{F})^k}\end{pmatrix}$$

Let us set

$$\Upsilon_k=\sqrt{F}\frac{(P+\sqrt{F})^k+(P-\sqrt{F})^k}{(P+\sqrt{F})^k-(P-\sqrt{F})^k}$$

and let us denote by D_k (resp. N_k) the denominator (resp. numerator) of Υ_k.

Lemma 2.1. *Let $\Omega_f=\gcd(P,F)$ and write P (resp. F) as Ω_fP_f (resp. Ω_fS_f). Assume* $\gcd(S_f,\Omega_f)=1$. *Then*

- ◇ *if $d_{S_f}\leq d_{\Omega_f}+2d_{P_f}$, then $\mu(g)=d_{\Omega_f}+2d_{P_f}$;*
- ◇ *otherwise $\mu(g)=d_{S_f}$.*

Proof. (a) Assume k even, write k as 2ℓ. A straightforward computation yields to

$$\Upsilon_{2\ell}=\frac{\displaystyle\sum_{j=0}^{\ell}\binom{2\ell}{2j}\Omega_f^{\ell-j}P_f^{2(\ell-j)}S_f^j}{\displaystyle P_f\sum_{j=0}^{\ell-1}\binom{2\ell}{2j+1}\Omega_f^{\ell-1-j}P_f^{2(\ell-1-j)}S_f^j}$$

Recall that $\gcd(\Omega_f,S_f)=1$ by assumption and $\gcd(\Omega_f,P_f)=1$ by construction. On the one hand

$$\deg(N_{2\ell})=\begin{cases}\ell(d_{\Omega_f}+2d_{P_f}) & \text{if } d_{S_f}\leq d_{\Omega_f}+2d_{P_f}\\ \ell d_{S_f} & \text{otherwise}\end{cases}$$

On the other hand

$$\deg(D_{2\ell})=\begin{cases}d_{P_f}+(\ell-1)\,(d_{\Omega_f}+2d_{P_f}) & \text{if } d_{S_f}\leq d_{\Omega_f}+2d_{P_f}\\ d_{P_f}+(\ell-1)\,d_{S_f} & \text{otherwise}\end{cases}$$

Finally

$$\deg(g^{2\ell})=\begin{cases}\max\Big(\ell(d_{\Omega_f}+2d_{P_f})+1, d_{S_f}+\ell d_{\Omega_f}+(2\ell-1)d_{P_f}, (\ell-1)\,d_{\Omega_f}+(2\ell-1)d_{P_f}+2\Big) & \text{if } d_{S_f}\leq d_{\Omega_f}+2d_{P_f}\\ \max\Big(\ell d_{S_f}+1, d_{\Omega_f}+d_{P_f}+\ell d_{S_f}, d_{P_f}+(\ell-1)\,d_{S_f}+2\Big) & \text{otherwise}\end{cases}$$

(b) Suppose k odd, write k as $2\ell+1$. A straightforward computation yields to

$$\Upsilon_{2\ell+1}=P_f\Omega_f\,\frac{\displaystyle\sum_{j=0}^{\ell}\binom{2\ell+1}{2j}\Omega_f^{\ell-j}P_f^{2(\ell-j)}S_f^j}{\displaystyle\sum_{j=0}^{\ell}\binom{2\ell+1}{2j+1}\Omega_f^{\ell-j}P_f^{2(\ell-j)}S_f^j}$$

Let us recall that $\gcd(\Omega_f,S_f)=1$ by assumption and $\gcd(\Omega_f,P_f)=1$ by construction. On the one hand

$$\deg(N_{2\ell+1})=\begin{cases}\ell(d_{\Omega_f}+2d_{P_f})+d_{\Omega_f}+d_{P_f} & \text{if } d_{S_f}\leq d_{\Omega_f}+2d_{P_f}\\ \ell d_{S_f}+d_{P_f}+d_{\Omega_f} & \text{otherwise}\end{cases}$$

On the other hand

$$\deg(D_{2\ell+1})=\begin{cases}\ell(d_{\Omega_f}+2d_{P_f}) & \text{if } d_{S_f}\leq d_{\Omega_f}+2d_{P_f}\\ \ell d_{S_f} & \text{otherwise}\end{cases}$$

Finally

$$\deg(g^{2\ell+1})=\begin{cases}\max\Big((\ell+1)d_{\Omega_f}+(2\ell+1)d_{P_f}+1, (\ell+1)d_{\Omega_f}+2\ell d_{P_f}+d_{S_f}, \ell(d_{\Omega_f}+2d_{P_f})+2\Big) & \text{if } d_{S_f}\leq d_{\Omega_f}+2d_{P_f}\\ \max\Big(\ell d_{S_f}+d_{P_f}+d_{\Omega_f}+1, (\ell+1)d_{S_f}+d_{\Omega_f}, \ell d_{S_f}+2\Big) & \text{otherwise}\end{cases}$$

We conclude with the equality $\mu(g)=2\lim_{k\to+\infty}\dfrac{\deg(g^k)}{k}$. □

Lemma 2.2. *Let $\Omega_f=\gcd(P,F)$ and write P (resp. F) as Ω_fP_f (resp. Ω_fS_f). Suppose that $S_f=\Omega_f^pT_f$ with $p\geq 1$ and $\gcd(T_f,\Omega_f)=1$. Then*

- ◇ *if $d_{S_f}\leq d_{\Omega_f}+2d_{P_f}$, then $\mu(g)=2d_{P_f}$;*
- ◇ *otherwise $\mu(g)=d_{S_f}-d_{\Omega_f}$.*

Proof. (a) Assume k even, write k as 2ℓ. We get

$$\Upsilon_{2\ell}=\frac{\displaystyle\sum_{j=0}^{\ell}\binom{2\ell}{2j}\Omega_f^{\ell-j}P_f^{2(\ell-j)}S_f^j}{\displaystyle P_f\sum_{j=0}^{\ell-1}\binom{2\ell}{2j+1}\Omega_f^{\ell-1-j}P_f^{2(\ell-1-j)}S_f^j}=\frac{\displaystyle\Omega_f\sum_{j=0}^{\ell}\binom{2\ell}{2j}\Omega_f^{(p-1)j}P_f^{2(\ell-j)}T_f^j}{\displaystyle P_f\sum_{j=0}^{\ell-1}\binom{2\ell}{2j+1}\Omega_f^{(p-1)j}P_f^{2(\ell-1-j)}T_f^j}$$

Recall that $\gcd(\Omega_f,T_f)=1$ and that $d_{S_f}=pd_{\Omega_f}+d_{T_f}$, *i.e.* $d_{T_f}=d_{S_f}-pd_{\Omega_f}$. On the one hand

$$\deg(N_{2\ell})=\begin{cases}2\ell d_{P_f}+d_{\Omega_f} & \text{if } d_{S_f}\leq d_{\Omega_f}+2d_{P_f}\\ \ell d_{S_f}+(1-\ell)d_{\Omega_f} & \text{otherwise}\end{cases}$$

On the other hand

$$\deg(D_{2\ell}) = \begin{cases} (2\ell-1)d_{P_f} & \text{if } d_{S_f} \leq d_{\Omega_f} + 2d_{P_f} \\ (\ell-1)(d_{S_f} - d_{\Omega_f}) + d_{P_f} & \text{otherwise} \end{cases}$$

Finally

$$\deg(g^{2\ell}) = \begin{cases} \max\Big(2\ell d_{P_f} + d_{\Omega_f} + 1, (2\ell-1)d_{P_f} + d_{\Omega_f} + d_{S_f}, (2\ell-1)d_{P_f} + 1\Big) & \text{if } d_{S_f} \leq d_{\Omega_f} + 2d_{P_f} \\ \max\Big(\ell d_{S_f} - (\ell-1)d_{\Omega_f} + 1, \ell d_{S_f} + (2-\ell)d_{\Omega_f} + d_{P_f}, (\ell-1)(d_{S_f} - d_{\Omega_f}) + d_{P_f} + 1\Big) & \text{otherwise} \end{cases}$$

(b) Suppose k odd, write k as $2\ell+1$. We get

$$\Upsilon_{2\ell+1} = \frac{P_f\Omega_f \displaystyle\sum_{j=0}^{\ell} \binom{2\ell+1}{2j} \Omega_f^{\ell-j} P_f^{2(\ell-j)} S_f^j}{\displaystyle\sum_{j=0}^{\ell} \binom{2\ell+1}{2j+1} \Omega_f^{\ell-j} P_f^{2(\ell-j)} S_f^j} = \frac{P_f\Omega_f \displaystyle\sum_{j=0}^{\ell} \binom{2\ell+1}{2j} \Omega_f^{(p-1)j} P_f^{2(\ell-j)} T_f^j}{\displaystyle\sum_{j=0}^{\ell} \binom{2\ell+1}{2j+1} \Omega_f^{(p-1)j} P_f^{2(\ell-j)} T_f^j}$$

On the one hand

$$\deg(N_{2\ell+1}) = \begin{cases} (2\ell+1)d_{P_f} + d_{\Omega_f} & \text{if } d_{S_f} \leq d_{\Omega_f} + 2d_{P_f} \\ \ell d_{S_f} - (\ell-1)d_{\Omega_f} + d_{P_f} & \text{otherwise} \end{cases}$$

On the other hand

$$\deg(D_{2\ell+1}) = \begin{cases} 2\ell d_{P_f} & \text{if } d_{S_f} \leq d_{\Omega_f} + 2d_{P_f} \\ \ell d_{S_f} - \ell d_{\Omega_f} & \text{otherwise} \end{cases}$$

Finally

$$\deg(g^{2\ell+1}) = \begin{cases} \max\Big((2\ell+1)d_{P_f} + d_{\Omega_f} + 1, 2\ell d_{P_f} + d_{\Omega_f} + d_{S_f}, 2\ell d_{P_f} + 1\Big) & \text{if } d_{S_f} \leq d_{\Omega_f} + 2d_{P_f} \\ \max\Big(\ell d_{S_f} - (\ell-1)d_{\Omega_f} + d_{P_f} + 1, (\ell+1)d_{S_f} - (\ell-1)d_{\Omega_f}, \ell d_{S_f} - \ell d_{\Omega_f} + 1\Big) & \text{otherwise} \end{cases}$$

We conclude with the equality $\mu(g) = 2\lim_{k\to+\infty} \dfrac{\deg(g^k)}{k}$. □

Lemma 2.3. *Let* $\Omega_f = \gcd(P,F)$ *and write* P *(resp.* F*) as* $\Omega_f P_f$ *(resp.* $\Omega_f S_f$*). Suppose that* $\Omega_f = S_f^p T_f$ *with* $p \geq 1$ *and* $\gcd(T_f, S_f) = 1$*. Then*

$$\mu(g) = 2d_{P_f} + d_{\Omega_f} - d_{S_f}.$$

Proof. (a) Assume k even, write k as 2ℓ. We obtain

$$\Upsilon_{2\ell} = \frac{\displaystyle\sum_{j=0}^{\ell} \binom{2\ell}{2j} \Omega_f^{\ell-j} P_f^{2(\ell-j)} S_f^j}{P_f \displaystyle\sum_{j=0}^{\ell-1} \binom{2\ell}{2j+1} \Omega_f^{\ell-1-j} P_f^{2(\ell-1-j)} S_f^j} = \frac{S_f \displaystyle\sum_{j=0}^{\ell} \binom{2\ell}{2j} S_f^{j(p-1)} P_f^{2j} T_f^j}{P_f \displaystyle\sum_{j=0}^{\ell-1} \binom{2\ell}{2j+1} S_f^{j(p-1)} P_f^{2j} T_f^j}$$

Recall that $\gcd(S_f, T_f) = 1$; one has

$$\deg(N_{2\ell}) = (p\ell - \ell + 1)d_{S_f} + 2\ell d_{P_f} + \ell d_{T_f}$$

and

$$\deg(D_{2\ell}) = (\ell-1)(p-1)d_{S_f} + (2\ell-1)d_{P_f} + (\ell-1)d_{T_f}$$

Finally

$$\begin{aligned}\deg(g^{2\ell}) = \max\Big((p\ell-\ell+1)d_{S_f} + 2\ell d_{P_f} + \ell d_{T_f} + 1,\\ (\ell(p-1)+2)d_{S_f} + (2\ell-1)d_{P_f} + \ell d_{T_f},\\ (\ell-1)(p-1)d_{S_f} + (2\ell-1)d_{P_f} + (\ell-1)d_{T_f} + 2\Big)\end{aligned}$$

(b) Suppose k odd, write k as $2\ell+1$. We get

$$\Upsilon_{2\ell+1} = \frac{\sum_{j=0}^{\ell}\binom{2\ell+1}{2j}P^{2\ell+1-2j}F^j}{\sum_{j=0}^{\ell}\binom{2\ell+1}{2j+1}P^{2\ell-2j}F^j} = \frac{S_f^pT_fP_f\sum_{j=0}^{\ell}\binom{2\ell+1}{2(\ell-j)}S_f^{j(p-1)}T_f^jP_f^{2j}}{\sum_{j=0}^{\ell}\binom{2\ell+1}{2j}S_f^{j(p-1)}T_f^jP_f^{2j}}.$$

On the one hand

$$\deg(N_{2\ell+1}) = (p+\ell(p-1))d_{S_f} + (\ell+1)d_{T_f} + (2\ell+1)d_{P_f},$$

and on the other hand

$$\deg(D_{2\ell+1}) = 2\ell d_{P_f} + \ell d_{T_f} + \ell(p-1)d_{S_f}.$$

Finally

$$\begin{aligned}\deg(g^{2\ell+1}) = \max\Big((p+\ell(p-1))d_{S_f} + (\ell+1)d_{T_f} + (2\ell+1)d_{P_f} + 1,\\ (p+1+\ell(p-1))d_{S_f} + 2\ell d_{P_f} + (\ell+1)d_{T_f},\\ 2\ell d_{P_f} + \ell d_{T_f} + \ell(p-1)d_{S_f} + 2\Big)\end{aligned}$$

We conclude with the equality $\mu(g) = 2\lim_{k\to+\infty}\frac{\deg(g^k)}{k}$. □

3. Examples

In this section we will give examples that illustrate Theorem B; more precisely §3.1 (resp. §3.2) illustrates Theorem B.1. (resp. Theorem B.2.)

3.1. **Example that illustrates Theorem B.1.**

3.1.1. *First example.* Consider the birational map of J given in the affine chart $x=1$ by $f=\big(y,(1-y)yz\big)$. The matrix associated to f is

$$M_f=\begin{pmatrix}(1-y)y & 0\\ 0 & 1\end{pmatrix},$$

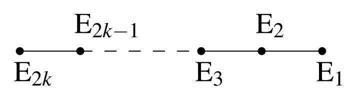
and the Baum Bott index $\mathrm{BB}(f)$ of f is $\frac{\big((1-y)y+1\big)^2}{(1-y)y}$; in particular f belongs to $\mathcal{J}$ (Proposition 1.1). The characteristic polynomial of M_f is

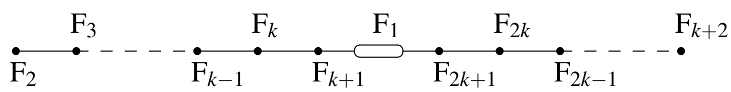
$$\chi_f(X)=\big(X-(1-y)y\big)(X-1).$$

According to Theorem B.1. one has $\mu(f)=4\leq 2\max\big(\deg(2),\deg(2(1-y)y\big)=4$.

We can see it another way: [CD12] asserts that $\deg(f^k)=k\deg(f)-k+1=3k-k+1=2k+1$. Consequently $\mu(f)=2\lim\limits_{k\to+\infty}\dfrac{\deg(f^k)}{k}=2\lim\limits_{k\to+\infty}\dfrac{2k+1}{k}=4$.

A third way to see this is to look at the configuration of the exceptional divisors. For any $k\geq 1$ one has $f^k=\big(x^{2k+1}:x^{2k}y:(x-y)^ky^kz\big)$. The configuration of the exceptional divisors of f^k is





where

- ⋄ two curves are related by an edge if their intersection is positive;
- ⋄ the self-intersections correspond to the shape of the vertices;
- ⋄ the point means self-intersection -1, the rectangle means self-intersection $-2k$.

In particular the number of base-points of f^k is $2k+2k+1=4k+1$ and

$$\mu(f)=\lim_{k\to+\infty}\frac{\#\mathfrak{b}(f^k)}{k}=4.$$

3.1.2. *Second example.* Consider the birational map of J given in the affine chart $z=1$ by $f=\big(x,xy+x(x-1)\big)$. The matrix associated to f is

$$M_f=\begin{pmatrix}x & x(x-1)\\ 0 & 1\end{pmatrix};$$

according to Proposition 1.1 the map f is a Jonquières twist (indeed $\mathrm{BB}(f)=\frac{(1+x)^2}{x}\in\mathbb{C}(x)\smallsetminus\mathbb{C}$). The characteristic polynomial of M_f is $\chi_f(X)=\big(X-x\big)(X-1)$. and f is conjugate to $g=(x,xy)$. According to Theorem B.1. one has

$$\mu(f)=\mu(g)=2(\deg(g)-1)=2\leq 2\max\big(\deg(2),\deg(2(1-y)y\big)=2.$$

We can see it another way: for any $k\geq 1$ one has $f^k=\big(x,x^ky+x^{k+1}-x)\big)$ and thus $\deg(f^k)=k+1$. As a result $\mu(f)=2\lim_{k\to+\infty}\dfrac{\deg(f^k)}{k}=2\times 1=2$.

3.2. **Examples that illustrate Theorem B.2.**

3.2.1. *First example.* Consider the map of J given in the affine chart $y=1$ by

$$f=\left(x,\frac{x(1-xz)}{z}\right).$$

The matrix associated to f is

$$M_f=\begin{pmatrix}-x^2 & x\\ 1 & 0\end{pmatrix},$$

the Baum Bott index $\mathrm{BB}(f)$ of f is $-x^3$ and f belongs to $\mathcal{J}$ (Proposition 1.1).

Theorem B.2.a. asserts that $\mu(f)=3$. We can see it another way: a computation gives $\deg(f^{2k})=3k+1$ and $\deg(f^{2k+1})=3(k+1)$ for any $k\geq 0$. Since $\mu(f)=2\lim_{k\to+\infty}\dfrac{\deg(f^k)}{k}$ one gets $\mu(f)=3$.

3.2.2. *Second example.* Consider the map f of J associated to the matrix

$$M_f=\begin{pmatrix}y & 2y^8\\ y & 1\end{pmatrix}.$$

The Baum Bott index $\mathrm{BB}(f)$ of f is $\frac{(y+1)^2}{y(1-2y^8)}$ and f belongs to $\mathcal{J}$ (Proposition 1.1). Theorem B.2.a. asserts that $\mu(f)=9$. We can see it another way: a computation gives $\deg(f^{2k})=9k+1$ and $\deg(f^{2k+1})=9k+8$ for any $k\geq 0$. Since $2\lim_{k\to+\infty}\dfrac{\deg(f^k)}{k}=\mu(f)$ one gets $\mu(f)=9$.

3.2.3. *Third example.* Let us consider the Jonquières map of $\mathbb{P}^2_{\mathbb{C}}$ given in the affine chart $z=1$ by

$$f=\left(\frac{y(y+2)x+y^5}{x+y(y+2)},y\right).$$

The matrix associated to f is

$$M_f=\begin{pmatrix}y(y+2) & y^5\\ 1 & y(y+2)\end{pmatrix}$$

and the Baum Bott index $\mathrm{BB}(f)$ of f is $\frac{4(y+2)^2}{(y+2)^2-y^5}$. In particular f is a Jonquières twist (Proposition 1.1).

According to Theorem B.2.b. one has $\mu(f)=3$. An other way to see that is to compute $\deg f^k$ for any k: for any $\ell\geq 1$ one has

$$\deg(f^{2\ell})=3(\ell+1), \qquad \deg(f^{2\ell+1})=3\ell+5.$$

Then we find again $\mu(f)=2\lim_{k\to+\infty}\dfrac{\deg(f^k)}{k}=3$.

3.2.4. *Fourth example.* Consider the map f of J associated to the matrix

$$M_f=\begin{pmatrix} y(y+2)^8 & y^5\\ 1 & y(y+2)^8\end{pmatrix}.$$

The Baum Bott index $\mathrm{BB}(f)$ of f is $\frac{4(y+2)^{16}}{(y+2)^{16}-y^3}$ and f belongs to $\mathcal{J}$ (Proposition 1.1). According to Theorem B.2.b. one has $\mu(f)=16$. An other way to see that is to compute $\deg f^k$ for any k: for any $k\geq 1$ one has $\deg f^k=8k+2$. Then we find again $\mu(f)=2\lim_{k\to+\infty}\dfrac{\deg(f^k)}{k}=2\times 8=16$.

3.2.5. *Fifth example.* Let us consider the Jonquières map of $\mathbb{P}^2_{\mathbb{C}}$ given in the affine chart $z=1$ by

$$f=\left(\frac{y(y+1)(y+2)x+y^2}{(y+2)x+y(y+1)(y+2)},y\right).$$

The matrix associated to f is

$$M_f=\begin{pmatrix} y(y+1)(y+2) & y^2\\ y+2 & y(y+1)(y+2)\end{pmatrix}$$

and the Baum-Bott index $\mathrm{BB}(f)$ of f is $\frac{4(y+1)^2(y+2)}{(y+1)^2(y+2)-1}$; in particular f is a Jonquières twist (Proposition 1.1).

Theorem B.2.c. asserts that $\mu(f)=3$. An other way to see that is to compute $\deg f^k$ for any k: for any $k\geq 1$

$$\deg(f^{2k})=3k+2 \qquad \deg(f^{2k+1})=3k+4$$

so $2\lim_{k\to+\infty}\dfrac{\deg(f^k)}{k}=3$ and we find again $\mu(f)=3$.

3.3. **Families.**

3.3.1. *First family.* Let us consider the family $(f_t)_t$ of elements of J given by $f_t=\left(x+t,y\frac{x}{x+1}\right)$. A straightforward computation yields to

$$f_t^n=\left(x+nt,y\frac{x}{x+1}\frac{x+t}{x+t+1}\cdots\frac{x+(n-1)t}{x+(n-1)t+1}\right)$$

The birational map f_t belongs to $\mathcal{J}$ if some multiple of t is equal to 1, and to $\mathrm{J}\smallsetminus\mathcal{J}$ otherwise. Furthermore,

◇ if no multiple of t is equal to 1, then $\mu(f_t)=2\left(\text{because} \lim_{k\to+\infty}\dfrac{\deg f_t^k}{k}=1\right)$;

◇ otherwise $\mu(f_t)=0$.

3.3.2. *Second family, illustration of Theorem A.3.* Let us recall a result of [CDX21]: let f be any element of $\mathrm{PGL}_3(\mathbb{C})$, or any elliptic element of $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ of infinite order; then f is a limit of pairwise conjugate loxodromic elements (resp. Jonquières twists) in the Cremona group. Hence there exist families $(f_n)_n$ of birational self-maps of the complex projective plane such that

◇ $\mu(f_n)>0$ for any $n\in\mathbb{N}$;

◇ $\mu\big(\lim_{n\to+\infty} f_n\big)=0$.

3.3.3. *Third family, illustration of Theorem A.4.* Let us recall a construction given in [CDX21]. Consider a pencil of cubic curves with nine distinct base points p_i in $\mathbb{P}^2_{\mathbb{C}}$. Given a point m in $\mathbb{P}^2_{\mathbb{C}}$, draw the line (p_1m) and denote by m' the third intersection point of this line with the cubic of our pencil that contains m: the map $m\mapsto\sigma_1(m)=m'$ is a birational involution. Replacing p_1 by p_2, we get a second involution and, for a very general pencil, $\sigma_1\circ\sigma_2$ is a Halphen twist that preserves our cubic pencil. At the opposite range, consider the degenerate cubic pencil, the members of which are the union of a line through the origin and the circle $C=\{x^2+y^2=z^2\}$. Choose $p_1=(1:0:1)$ and $p_2=(0:1:1)$ as our distinguished base points. Then, $\sigma_1\circ\sigma_2$ is a Jonquières twist preserving the pencil of lines through the origin; if the plane is parameterized by $(s,t)\mapsto(st,t)$, this Jonquières twist is conjugate to $(s,t)\mapsto\left(s,\frac{(s-1)t+1}{(s^2+1)t+s-1}\right)$. Now, if we consider a family of general cubic pencils converging towards this degenerate pencil, we obtain a sequence of Halphen twists converging to a Jonquières twist. So there exists a sequence $(f_n)_n$ of birational self-maps of $\mathbb{P}^2_{\mathbb{C}}$ whose limit is also a birational self-map of $\mathbb{P}^2_{\mathbb{C}}$ and such that

◇ $\mu(f_n)=0$ for any $n\in\mathbb{N}$;

◇ $\mu\big(\lim_{n\to+\infty} f_n\big)>0$.

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