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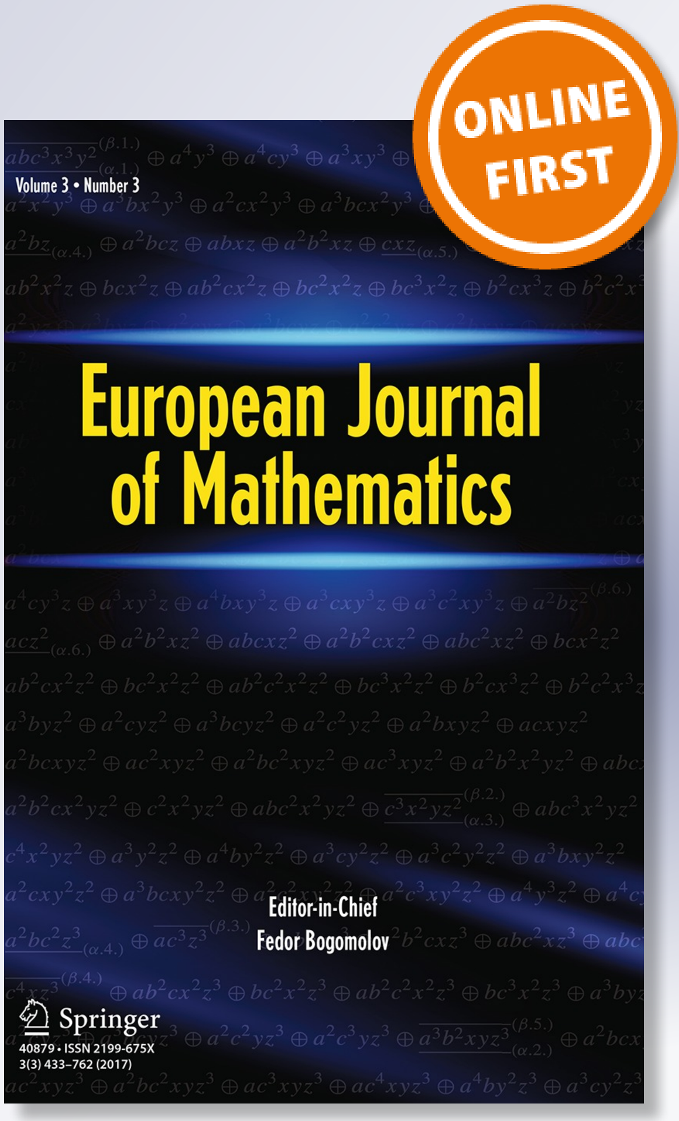

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RESEARCH ARTICLE

Degree growth of polynomial automorphisms and birational maps: some examples

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Abstract We give examples of new degree growths for polynomial automorphisms of $\mathbb{C}^k$ and birational maps of $\mathbb{P}^k_{\mathbb{C}}$. Namely, if k is an integer $\geqslant 3$, then for any $\ell \leqslant [(k-1)/2]$ we provide polynomial automorphisms f of $\mathbb{C}^k$ such that $\deg f^n \sim n^\ell$, and for all $0 \leqslant \ell \leqslant k$ we provide birational maps ϕ of $\mathbb{P}^k_{\mathbb{C}}$ such that $\deg \phi^n \sim n^\ell$.

Keywords Iteration problems · Polynomial automorphisms · Cremona maps

Mathematics Subject Classification 32H50 · 37F10 · 14E07

1 Introduction

Let f be a polynomial automorphism of $\mathbb{C}^2$, then either $(\deg f^n)_{n\in\mathbb{N}}$ is bounded or $(\deg f^n)_{n\in\mathbb{N}}$ grows exponentially. In higher dimensions there are intermediate growths:

Theorem A *Let k be an integer $\geqslant 3$. For any $\ell \leqslant [(k-1)/2]$ there exist polynomial automorphisms f of $\mathbb{C}^k$ such that*

$$\deg f^n \sim n^\ell.$$

The group of polynomial automorphisms of $\mathbb{C}^2$ has a structure of amalgamated product [6]. Using this rigidity a lot of properties of polynomial automorphisms of $\mathbb{C}^2$ have

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been established. All these properties show a dichotomy. Up to conjugacy there are two types of polynomial automorphisms of $\mathbb{C}^2$: the Jonquières ones and the Hénon ones. If f and g are two polynomial automorphisms of $\mathbb{C}^2$ and H_∞ denotes the line at infinity (we view $\mathbb{C}^2$ in $\mathbb{P}^2_\mathbb{C}$), then

($\mathcal{P}_1$) if $(\deg f^n)_n$ is not bounded, then f does not preserve a rational fibration;
($\mathcal{P}_2$) $\deg f^2 = (\deg f)^2$ if and only if $\deg f^n = (\deg f)^n$ for any $n \in \mathbb{N}$ (see [5, Proposition 3]).

We give counter-examples to properties ($\mathcal{P}_1$) and ($\mathcal{P}_2$) in dimension $\geqslant 3$. Nevertheless one can prove a result similar to property ($\mathcal{P}_2$):

Proposition B *Let f be a polynomial automorphism of $\mathbb{C}^k$. Then* $\deg f^i = (\deg f)^i$ *for* $1 \leqslant i \leqslant k$ *if and only if* $\deg f^n = (\deg f)^n$ *for any* $n \geqslant 1$.

If ϕ is a birational self-map of $\mathbb{P}^2_\mathbb{C}$, then $(\deg \phi^n)_{n\in\mathbb{N}}$ is either bounded, or grows linearly, or grows quadratically, or grows exponentially [3]. In this context there exist other types of growth in higher dimension. Lin has studied the degree growth of monomial maps of $\mathbb{P}^k_\mathbb{C}$ (see [8]). He proved in particular that if A is a $k\times k$ integer matrix with nonzero determinant, then there exist two constants $\alpha \geqslant \beta > 0$ and a unique integer $0 \leqslant \ell \leqslant k-1$ such that for any $n \in \mathbb{N}$

$$\beta\rho(A)^\ell n^\ell \leqslant \deg \phi_A^n \leqslant \alpha\rho(A)^\ell n^\ell$$

where $\rho(A)$ denotes the spectral radius of A and ϕ_A the monomial map associated to A. This leads to the question: Do there exist in dimension k birational maps of $\mathbb{P}^k_\mathbb{C}$ with growth n^ℓ, $\ell > k-1$? We will show that:

Theorem C *Assume $k \geqslant 3$. For all $0 \leqslant \ell \leqslant k$ there exist a birational self-map ϕ of $\mathbb{P}^k_\mathbb{C}$ and two constants $\alpha \geqslant \beta > 0$ such that for any $n \geqslant 0$*

$$\beta n^\ell \leqslant \deg \phi^n \leqslant \alpha n^\ell.$$

Note that k is not an upper bound, at least in dimension 3: there exists a birational self-map f of $\mathbb{P}^3_\mathbb{C}$ such that $\deg f^n \sim n^4$ (see [11]).

The Newman–Shanks–Williams primes were first described by Newman, Shanks and Williams in 1981 during the study of finite simple groups with square order [9]. They often appear in the literature. We will give an example of a birational self-map of $\mathbb{P}^3_\mathbb{C}$ whose degrees of iterates are Newman–Shanks–Williams primes.

2 Recalls, definitions, notations

2.1 The group of polynomial automorphisms of $\mathbb{C}^k$

A *polynomial automorphism* f of $\mathbb{C}^k$ is a polynomial map of the type

$$\begin{aligned} f\colon \mathbb{C}^k &\to \mathbb{C}^k \\ (z_0, z_1, \ldots, z_{k-1}) &\mapsto \big(f_0(z_0, z_1, \ldots, z_{k-1}), f_1(z_0, z_1, \ldots, z_{k-1}), \ldots, \\ &\qquad f_{k-1}(z_0, z_1, \ldots, z_{k-1})\big) \end{aligned}$$

that is bijective. The set of polynomial automorphisms of $\mathbb{C}^k$ forms a group denoted by $\mathrm{Aut}(\mathbb{C}^k)$.

The automorphisms of $\mathbb{C}^k$ of the form $(f_0, f_1, \ldots, f_{k-1})$ where f_i depends only on $z_i, z_{i+1}, \ldots,_{k-1}$ form a subgroup E_k of $\mathrm{Aut}(\mathbb{C}^k)$. Moreover we have the inclusions

$$\mathrm{GL}(\mathbb{C}^k) \subset \mathrm{Aff}_k \subset \mathrm{Aut}(\mathbb{C}^k)$$

where Aff_k denotes the *group of affine maps*

$$\begin{aligned} f\colon (z_0, z_1, \ldots, z_{k-1}) \mapsto \big(f_0(z_0, z_1, \ldots, z_{k-1}), f_1(z_0, z_1, \ldots, z_{k-1}), \ldots, \\ f_{k-1}(z_0, z_1, \ldots, z_{k-1})\big) \end{aligned}$$

with f_i affine; Aff_k is the semi-direct product of $\mathrm{GL}(\mathbb{C}^k)$ with the commutative subgroups of translations. The subgroup $\mathrm{Tame}_k \subset \mathrm{Aut}(\mathbb{C}^k)$ generated by E_k and Aff_k is called the *group of tame automorphisms*. If $k = 2$ one has:

Theorem 2.1 ([6]) *In dimension* 2 *the group of tame automorphisms coincides with the whole group of polynomial automorphism, more precisely*

$$\mathrm{Aut}(\mathbb{C}^2) = \mathrm{Aff}_2 *_{\mathrm{Aff}_2 \cap \mathrm{E}_2} \mathrm{E}_2.$$

But this is not the case in higher dimension: $\mathrm{Tame}_3 \subsetneq \mathrm{Aut}(\mathbb{C}^3)$ (see [10]).

Another important result in dimension 2 is the following:

Theorem 2.2 ([4]) *Let f be an element of* $\mathrm{Aut}(\mathbb{C}^2)$. *Then, up to conjugacy,*

- *either f belongs to* E_2*,*
- *or f can be written as*

$$\varphi_\ell \circ \varphi_{\ell-1} \circ \cdots \circ \varphi_1$$

where $\varphi_i\colon (z_0, z_1) \mapsto (z_1, P_i(z_1) - \delta_i z_0)$, $\delta_i \in \mathbb{C}^*$, $P_i \in \mathbb{C}[z_1]$, $\deg P_i \geqslant 2$.

We denote by $\mathcal{H}$ the set of polynomial automorphisms of $\mathbb{C}^2$ that can be written up to conjugacy as $\varphi_\ell \circ \varphi_{\ell-1} \circ \cdots \circ \varphi_1$, where $\varphi_i\colon (z_0, z_1) \mapsto (z_1, P_i(z_1) - \delta_i z_0)$, $\delta_i \in \mathbb{C}^*$, $P_i \in \mathbb{C}[z_1]$, $\deg P_i \geqslant 2$. The elements of $\mathcal{H}$ are of *Hénon type*.

From now on we will write $f = (f_0, f_1, \ldots, f_{k-1})$ instead of

$$f\colon (z_0, z_1, \ldots, z_{k-1}) \mapsto \big(f_0(z_0, z_1, \ldots, z_{k-1}), f_1(z_0, z_1, \ldots, z_{k-1}), \ldots, f_{k-1}(z_0, z_1, \ldots, z_{k-1})\big).$$

The *algebraic degree* $\deg f$ of $f = (f_0, f_1, \ldots, f_{k-1}) \in \mathrm{Aut}(\mathbb{C}^k)$ is

$$\max(\deg f_0, \deg f_1, \ldots, \deg f_{k-1}).$$

2.2 The Cremona group

A *rational self-map* f of $\mathbb{P}^k_{\mathbb{C}}$ can be written

$$(z_0{:}z_1{:}\ldots{:}z_k) \dashrightarrow \big(f_0(z_0, z_1, \ldots, z_k){:}f_1(z_0, z_1, \ldots, z_k){:}\ldots{:}f_k(z_0, z_1, \ldots, z_k)\big)$$

where f_i's are homogeneous polynomials of the same degree $\geqslant 1$ and without common factor of positive degree. The *degree* of f is the degree of f_i. If there exists a rational self-map g of $\mathbb{P}^k_{\mathbb{C}}$ such that $f \circ g = g \circ f = \mathrm{id}$ we say that the rational self-map f of $\mathbb{P}^k_{\mathbb{C}}$ is *birational*.

The set of birational self-maps of $\mathbb{P}^k_{\mathbb{C}}$ forms a group denoted by $\mathrm{Bir}(\mathbb{P}^k_{\mathbb{C}})$ and is called the *Cremona group*. Of course, $\mathrm{Aut}(\mathbb{C}^k)$ is a subgroup of $\mathrm{Bir}(\mathbb{P}^k_{\mathbb{C}})$. Another natural subgroup of $\mathrm{Bir}(\mathbb{P}^k_{\mathbb{C}})$ is the group $\mathrm{Aut}(\mathbb{P}^k_{\mathbb{C}}) \simeq \mathrm{PGL}(k+1;\mathbb{C})$ of automorphisms of $\mathbb{P}^k_{\mathbb{C}}$.

The *indeterminacy set* $\mathrm{Ind}(f)$ of f is the set of the common zeros of f_i's. The *exceptional set* $\mathrm{Exc}(f)$ of f is the (finite) union of subvarieties M_i of $\mathbb{P}^k_{\mathbb{C}}$ such that f is not injective on any open subset of M_i.

2.3 A little bit of dynamics

Let f be a polynomial automorphism of $\mathbb{C}^k$. One can see f as a birational self-map of $\mathbb{P}^k_{\mathbb{C}}$ and we still denote it by f. We will say that f is *algebraically stable* if for any $n > 0$

$$f^n\big(\{z_k = 0\} \setminus \mathrm{Ind}(f^n)\big)$$

is not contained in $\mathrm{Ind}(f)$. This is equivalent to the fact that $(\deg f)^n = \deg f^n$ for any $n > 0$. For instance, elements of $\mathcal{H}$ are algebraically stable.

Remark 2.3 Note that in dimension 2 the map f is algebraically stable if and only if for any $n > 0$

$$f^n\big(\{z_2 = 0\} \setminus \mathrm{Ind}(f^n)\big) \cap \mathrm{Ind}(f) = \varnothing.$$

Be careful, this is not the case in higher dimension. Consider for instance

$$f = \big(5z_0^2 + z_2^2 + 6z_0z_2 + z_1, z_2^2 + z_0, z_2\big),$$

then

- on the one hand, $(-1{:}0{:}1{:}0)$ belongs to $\{z_3 = 0\} \setminus \mathrm{Ind}(f)$ and

$$f(-1{:}0{:}1{:}0) = (0{:}1{:}0{:}0) \in \mathrm{Ind}(f) = \{(0{:}1{:}0{:}0)\},$$

  i.e. $f(\{z_2 = 0\} \setminus \mathrm{Ind}(f)) \cap \mathrm{Ind}(f) \neq \varnothing$,
- on the other hand, for any $n \geqslant 1$, $\deg f^n = (\deg f)^n$.

The algebraic degree of a birational map f of $\mathbb{P}^k_{\mathbb{C}}$ (resp. a polynomial automorphism of $\mathbb{C}^k$) is not a dynamical invariant so we introduce the *dynamical degree*

$$\lambda(f) = \lim_{n \to +\infty} (\deg f^n)^{1/n}$$

which is a dynamical invariant, that is for any $g \in \mathrm{Bir}(\mathbb{P}^k_{\mathbb{C}})$ (resp. $g \in \mathrm{Aut}(\mathbb{C}^k)$) one has $\lambda(f) = \lambda(gfg^{-1})$.

For any element f in $\mathcal{H}$ the algebraic and dynamical degrees coincide, more precisely if

$$f = \varphi_\ell \circ \varphi_{\ell-1} \circ \cdots \circ \varphi_1$$

where $\varphi_i = (z_1, P_i(z_1) - \delta_i z_0)$, $\delta_i \in \mathbb{C}^*$, $P_i \in \mathbb{C}[z_1]$, $\deg P_i \geqslant 2$, one has [4]

$$\lambda(f) = \prod_{i=1}^{\ell} \deg \varphi_i \geqslant 2.$$

A polynomial automorphism f is conjugate to an element of E_2 if and only if $\lambda(f) = 1$. A polynomial automorphism f of $\mathbb{C}^2$ belongs to $\mathcal{H}$ if and only if $\lambda(f) > 1$. There is another characterization of the automorphisms of Hénon type:

Theorem 2.4 ([7]) *The centralizer of f in $\mathrm{Aut}(\mathbb{C}^2)$, that is $\{g \in \mathrm{Aut}(\mathbb{C}^2) : fg = gf\}$, is countable if and only if f belongs to $\mathcal{H}$.*

3 Birational maps and automorphisms with polynomial growths

3.1 The growths of a birational self-map and its inverse

If f is a birational self-map of $\mathbb{P}^k_{\mathbb{C}}$, then $(\deg f, \deg f^{-1})$ is called the *bidegree* of f. There is the following relationship between $\deg f$ and $\deg f^{-1}$ (see [1]):

$$\begin{cases} \deg f^{-1} \leqslant (\deg f)^{k-1}, \\ \deg f \leqslant (\deg f^{-1})^{k-1}. \end{cases}$$

As a result, if f is a birational self-map of $\mathbb{P}^k_{\mathbb{C}}$, the degree growths of f and f^{-1} are linked:

Proposition 3.1 *Let f be a birational self-map of $\mathbb{P}^k_{\mathbb{C}}$.*

- *The sequence $(\deg f^n)_{n\in\mathbb{N}}$ is bounded if and only if the sequence $(\deg f^{-n})_{n\in\mathbb{N}}$ is bounded.*
- *The sequence $(\deg f^n)_{n\in\mathbb{N}}$ grows exponentially if and only if $(\deg f^{-n})_{n\in\mathbb{N}}$ grows exponentially.*
- *If $\deg f^n \simeq n^p$ and $\deg f^{-n} \simeq n^q$ for some integers $p, q \geqslant 1$, then*

$$(p, q) \in \left\{\left(\left[\frac{q+1}{k}\right], q\right), \dots, (kq, q)\right\}.$$

Remark 3.2 When we write "the sequence $(\deg f^n)_{n\in\mathbb{N}}$ grows exponentially if and only if $(\deg f^{-n})_{n\in\mathbb{N}}$ grows exponentially", it does not mean that $(\deg f^n)_{n\in\mathbb{N}}$ and $(\deg f^{-n})_{n\in\mathbb{N}}$ have exactly the same behavior, e.g. the polynomial automorphism of $\mathbb{C}^3$ given by

$$f = \left(z_0^2 + z_1 + z_2, z_0^2 + z_1, z_0\right)$$

satisfies for all $n \geqslant 1$

$$\begin{cases} \deg f^n = 2^n, \\ \deg f^{-n} = 2^{[(n+1)/2]}. \end{cases}$$

3.2 Examples of polynomial automorphisms with new polynomial growths

Let us now give examples of polynomial automorphisms with polynomial growths. Consider the polynomial automorphism of $\mathbb{C}^3$ given by

$$f = \left(z_1 + z_0 z_2^d, z_0, z_2\right)$$

where $d \geqslant 1$. Assume $n \geqslant 1$. Set $f^n = (f_{0,n}, f_{1,n}, z_2)$ and $\delta_n = \deg f^n$. Note that $\delta_1 = d + 1$ and since

$$f^n = f f^{n-1} = \left(f_{1,n-1} + f_{0,n-1} z_2^d, f_{0,n-1}, z_2\right),$$

one has

$$\delta_n = \max\left(\deg f_{1,n-1}, \deg f_{0,n-1} + d, \deg f_{0,n-1}, 1\right).$$

But $f_{1,n-1} = f_{0,n-2}$ and $d \geqslant 1$ so $\delta_n = \deg f_{0,n-1} + d = \delta_{n-1} + d$. In other words,

$$\deg f^n = dn + 1.$$

One can also prove that $\deg f^n = \deg f^{-n}$ for any $n \geqslant 1$. From f one can construct an example of a polynomial automorphism g of $\mathbb{C}^5$ such that $\deg g^n \sim n^2$. For instance, if

$$g = \left(z_1 + z_0 z_2^d, z_0, z_2, z_4 + z_0^p z_3, z_3\right)$$

where $p \geqslant d \geqslant 1$, then for any $n \geqslant 1$

$$\deg g^n = \frac{pd}{2} n^2 + \frac{p(2-d)}{2} n + 1$$

and $\deg g^n = \deg g^{-n}$ for any $n \geqslant 1$. From g one can construct a polynomial automorphism h of $\mathbb{C}^7$ such that $\deg h^n \sim n^3$. Indeed, let us consider

$$h = \left(z_1 + z_0 z_2^d, z_0, z_2, z_4 + z_0^p z_3, z_3, z_6 + z_3^\ell z_5, z_5\right)$$

where $\ell \geqslant p \geqslant d \geqslant 1$. For any $n \geqslant 1$

$$\deg h^n = 1 + \ell\left(1 - \frac{p}{2} + \frac{pd}{3}\right) n + \frac{\ell p(1-d)}{2} n^2 + \frac{\ell pd}{6} n^3$$

and $\deg h^n = \deg h^{-n}$. By repeating this process one gets the following statement:

Proposition 3.3 *There exist polynomial automorphisms f of $\mathbb{C}^{2k+1}$, $k \geqslant 2$, such that $\deg f^n \sim n^k$.*

Theorem A follows from Proposition 3.3.

3.3 Birational maps with new polynomial growths

Let us first recall the following example [3]. If φ is the birational map of $\mathbb{P}^2_{\mathbb{C}}$, given in the affine chart $z_2 = 1$ by

$$\varphi(z_0, z_1) = \left(z_1 + \frac{2}{3}, z_0 \frac{z_1 - 1/3}{z_1 + 1}\right),$$

then there exist two constants $\beta \geqslant \alpha > 0$ such that for any $n \geqslant 0$

$$\alpha n^2 \leqslant \deg \varphi^n \leqslant \beta n^2.$$

Using this example we will construct birational self-maps ϕ of $\mathbb{P}^k_{\mathbb{C}}$ satisfying $\deg \phi^n \sim n^k$.

One can write φ^n as $(P_n/Q_n, R_n/S_n)$ where P_n, Q_n, R_n and S_n denote some elements of $\mathbb{C}[z_0, z_1]$ without common factor. Set $p_n = \deg P_n$, $q_n = \deg Q_n$, $r_n = \deg R_n$ and $s_n = \deg S_n$. The following equalities hold (by iteration):

$$\begin{cases} p_n = s_{n-1} + 1, \\ q_n = s_{n-1}, \\ r_n = s_n + 1, \\ \deg \varphi^n = s_{n-1} + s_n + 1. \end{cases}$$

Let us now consider the birational self-map of $\mathbb{P}^3_{\mathbb{C}}$ given in the affine chart $z_3 = 1$ by

$$\Psi_3(z_0, z_1, z_2) = \left(z_1 + \frac{2}{3}, z_0 \frac{z_1 - 1/3}{z_1 + 1}, z_0 z_2 \right).$$

One can check that

$$\Psi_3^n = \left(\frac{P_n}{Q_n}, \frac{R_n}{S_n}, \frac{U_n}{V_n} \right)$$

where $U_n = z_0 z_2 P_1 P_2 \ldots P_{n-1}$ and $V_n = Q_1 Q_2 \ldots Q_{n-1}$ have no common factor. Since $s_i \sim i^2$ there exist two constants $\beta \geqslant \alpha > 0$ such that for any $n \geqslant 0$ the following inequalities hold:

$$\alpha n^3 \leqslant \deg \Psi_3^n \leqslant \beta n^3.$$

Let us now consider the birational self-map of $\mathbb{P}^4_{\mathbb{C}}$ defined in the affine chart $z_4 = 1$ by

$$\Psi_4(z_0, z_1, z_2, z_3) = \left(z_1 + \frac{2}{3}, z_0 \frac{z_1 - 1/3}{z_1 + 1}, z_0 z_2, z_2 z_3 \right).$$

One has $\Psi_4^n = (P_n/Q_n, R_n/S_n, U_n/V_n, W_n/X_n)$ where

$$W_n = W_1 U_1 U_2 \ldots U_{n-1} \quad \text{and} \quad X_n = X_1 V_1 V_2 \ldots V_{n-1}$$

have no common factor. Since $\deg U_n \sim n^3$ and $\deg V_n \sim n^3$, there exist two constants $\beta \geqslant \alpha > 0$ such that for any $n \geqslant 0$

$$\alpha n^4 \leqslant \deg \Psi_4^n \leqslant \beta n^4.$$

By repeating this process one gets:

Theorem 3.4 *Let k be an integer $\geqslant 3$. There exist birational self-maps ϕ of $\mathbb{P}^k_{\mathbb{C}}$ with the following property: there exist two constants $\beta \geqslant \alpha > 0$ such that for any $n \geqslant 0$*

$$\alpha n^k \leqslant \deg \phi^n \leqslant \beta n^k.$$

This statement and Lin's result mentioned just before Theorem C imply Theorem C.

4 Properties $(\mathcal{P}_i)$

Note that Theorem 2.4 can be also stated as follows: the centralizer of a polyonomial automorphism f of $\mathbb{C}^2$ is countable if and only if $(\deg f^n)_{n\in\mathbb{N}}$ grows exponentially. This property is not true in higher dimension: there exist polynomial automorphisms of $\mathbb{C}^3$ with uncountable centralizer and exponential degree growth [2]. Let us now show that this is also the case for other properties, and in particular for $(\mathcal{P}_1)$, $(\mathcal{P}_2)$.

4.1 Property $(\mathcal{P}_1)$

Property $(\mathcal{P}_1)$ does not hold in higher dimension:

Proposition 4.1 *The polynomial automorphism f of $\mathbb{C}^3$, given by*

$$\big(z_0^2+z_1,\, z_0,\, z_2+1\big),$$

preserves the fibration $z_2=\text{const}$ *and for all $n\geqslant 1$ the equality* $\deg f^n=2^n$ *holds.*

The polynomial automorphism g of $\mathbb{C}^3$, given by

$$\big(z_1^2+z_0z_1+z_2,\, z_1+1,\, z_0\big),$$

preserves the fibration $z_1=\text{const}$ *and for all $n\geqslant 1$ the equality* $\deg g^n=n+1$ *holds.*

4.2 Property $(\mathcal{P}_2)$

In [5], Furter proved that if f is a polynomial automorphism of $\mathbb{C}^2$ then $\deg f^2=(\deg f)^2$ if and only if $\deg f^n=(\deg f)^n$ for all $n\in\mathbb{N}$. This property does not hold in higher dimension. Consider for instance the polynomial automorphism f given by

$$f=\big(z_1^2+z_5,\, z_5^2+z_4,\, z_2,\, z_1,\, z_0,\, z_4^2+z_3\big).$$

One can check that $\deg f=2$, $\deg f^2=4$, $\deg f^3=8$ but $\deg f^4=8$.

Let f be a polynomial automorphism of $\mathbb{C}^k$. For any integer $n\geqslant 0$ set

$$\Omega_n=f^n\big((z_{k-1}=0)\setminus\mathrm{Ind}(f^n)\big).$$

Note that $\Omega_n\subset(z_{k-1}=0)$ for any n. We say that f *is not algebraically stable after ℓ steps* if ℓ is the smallest integer such that $\Omega_\ell\subset\mathrm{Ind}(f)$.

Let us first remark that if $\Omega_1\cap\mathrm{Ind}(f)=\varnothing$, then $\Omega_2=f(\Omega_1)\subseteq\Omega_1$ so $\Omega_2\cap\mathrm{Ind}(f)=\varnothing$. By induction one gets for any $n\geqslant 1$ that $\Omega_n\cap\mathrm{Ind}(f)=\varnothing$ and $\deg f^n=(\deg f)^n$, i.e. f is algebraically stable.

Let us now assume that $\Omega_1\cap\mathrm{Ind}(f)\neq\varnothing$. Then:

- Either $\Omega_1\subset\mathrm{Ind}(f)$, that is f is not algebraically stable after one step.
- Or $\Omega_1\not\subset\mathrm{Ind}(f)$ hence $\Omega_2=f(\Omega_1\setminus\mathrm{Ind}(f))\subseteq\Omega_1$.

- Either $\dim \Omega_2 = \dim \Omega_1$, so $\Omega_2 = \Omega_1$ and then $\Omega_n = \Omega_1$ for any n. In particular, $\Omega_n \not\subset \mathrm{Ind}(f)$ for any n and f is algebraically stable.
- Or $\dim \Omega_2 < \dim \Omega_1$, then either $\Omega_2 \subset \mathrm{Ind}(f)$ and f is not algebraically stable after two steps or $\Omega_2 \not\subset \mathrm{Ind}(f)$ and we come back to the previous alternative, that is either $\dim \Omega_3 = \dim \Omega_2$ or $\dim \Omega_3 < \dim \Omega_2$. Since for any n one has $0 \leqslant \dim \Omega_n \leqslant k - 1$, one gets that either f is algebraically stable, or f is not algebraically stable after at most $k - 1$ steps.

Hence one can state:

Proposition 4.2 *Let f be a polynomial automorphism of $\mathbb{C}^k$. Either f is algebraically stable, or f is not algebraically stable after ℓ steps, with $\ell \leqslant k - 1$. In other words,* $\deg f^i = (\deg f)^i$ *for* $1 \leqslant i \leqslant k$ *if and only if* $\deg f^n = (\deg f)^n$ *for any* $n \geqslant 1$.

5 Newman–Shanks–Williams primes

We mentioned at the beginning of Sect. 4 that there exist polynomial automorphisms of $\mathbb{C}^3$ with exponential growth and uncountable centralizer; we will see in this section that it also holds for birational maps of $\mathbb{P}^3_{\mathbb{C}}$ that are not conjugate to polynomial automorphisms of $\mathbb{C}^3$.

A *Newman–Shanks–Williams* prime is a prime number p which can be written as

$$s_{2m+1} = \frac{(1+\sqrt{2})^{2m+1} + (1-\sqrt{2})^{2m+1}}{2}.$$

They were first described by Newman, Shanks and Williams in 1981 during the study of finite simple groups with square order [9]. Let us briefly explain their result. Call a finite simple group G a special group if G has square order, and call the numbers s_{2m+1} special numbers. They proved that

- if a special number s_{2m+1} is a prime, then the symplectic group $\mathrm{Sp}(4, s_{2m+1})$ of dimension 4 over $\mathbb{F}_{s_{2m+1}}$ is a special group of order

$$\left(s_{2m+1}^2 \left(s_{2m+1}^2 - 1 \right) \frac{\left(1+\sqrt{2}\right)^{2m+1} - \left(1-\sqrt{2}\right)^{2m+1}}{2\sqrt{2}} \right)^2;$$

- conversely, if a symplectic group is special, then it is $\mathrm{Sp}(4, p)$ with $p = s_{2m+1}$ a prime, for some m.

The sequence of Newman–Shanks–Williams numbers can be described by the following recurrence relation:

$$\text{for all } \ n \geqslant 2, \quad s_n = 2s_{n-1} + s_{n-2}$$

with $s_0 = s_1 = 1$.

Let us study the behavior of the birational self-map ϕ of $\mathbb{P}^3_{\mathbb{C}}$ given by $(z_0 z_1^2, z_0 z_1, z_2)$, or in homogeneous coordinates by $(z_0 z_1^2 : z_0 z_1 z_3 : z_2 z_3^2 : z_3^3)$. We look at the degree growth of ϕ, and the Newman–Shanks–Williams primes reappear:

Proposition 5.1 *For any $n \geqslant 1$ one has*

$$\phi^n = z_3^{3^n-(a_n+2b_n)}\big(z_0^{a_n}z_1^{2b_n}:z_0^{b_n}z_1^{a_n}z_3^{b_n}:z_2z_3^{a_n+2b_n-1}:z_3^{a_n+2b_n}\big)$$

where

$$\mu_1 = 1+\sqrt{2},\quad \mu_2 = 1-\sqrt{2},\quad a_n = \frac{1}{2}\left(\mu_1^n+\mu_2^n\right),\quad b_n = \frac{\sqrt{2}}{4}\left(\mu_1^n-\mu_2^n\right).$$

For all $n \geqslant 1$ one has $\deg\phi^{\pm n} = s_{n+1}$, *and so* $\lambda(\phi) = \mu_1$.

Proof By iteration one gets that

$$\phi^n = z_3^{3^n-(a_n+c_n)}\big(z_0^{a_n}z_1^{c_n}:z_0^{b_n}z_1^{d_n}z_3^{a_n+c_n-(b_n+d_n)}:z_2z_3^{a_n+c_n-1}:z_3^{a_n+c_n}\big)$$

with

$$\begin{bmatrix} a_{n+1}\\ b_{n+1}\\ c_{n+1}\\ d_{n+1}\end{bmatrix} = \begin{bmatrix} 1&0&2&0\\ 0&1&0&2\\ 1&0&1&0\\ 0&1&0&1\end{bmatrix}\begin{bmatrix} a_n\\ b_n\\ c_n\\ d_n\end{bmatrix}.$$

But on the one hand,

$$\begin{bmatrix} 1&0&2&0\\ 0&1&0&2\\ 1&0&1&0\\ 0&1&0&1\end{bmatrix} = \begin{bmatrix} 2&0&2&0\\ 0&2&0&2\\ \sqrt{2}&0&-\sqrt{2}&0\\ 0&\sqrt{2}&0&-\sqrt{2}\end{bmatrix}\begin{bmatrix} \lambda_1&0&0&0\\ 0&\lambda_1&0&0\\ 0&0&\lambda_2&0\\ 0&0&0&\lambda_2\end{bmatrix}\begin{bmatrix} 2&0&2&0\\ 0&2&0&2\\ \sqrt{2}&0&-\sqrt{2}&0\\ 0&\sqrt{2}&0&-\sqrt{2}\end{bmatrix}^{-1}$$

with $\mu_1 = 1+\sqrt{2}$, $\mu_2 = 1-\sqrt{2}$ and, on the other hand, $a_1 = c_1 = d_1 = 1$, $b_1 = 2$ hence

$$a_n = d_n = \frac{\mu_1^n+\mu_2^n}{2},\qquad b_n = \frac{\sqrt{2}\,(\mu_1^n-\mu_2^n)}{4},\qquad c_n = 2b_n.$$

It follows that for any $n \geqslant 1$ one has

$$\deg\phi^{\pm n} = a_n + 2b_n = s_{n+1}$$

and $\lim_{n\to+\infty}(s_n)^{1/n} = \mu_1$. □

In the context of birational maps of $\mathbb{P}^k_{\mathbb{C}}$, $k \geqslant 3$, we also get that uncountable centralizers and exponential degree growth can cohabit:

Proposition 5.2 *The centralizer of ϕ is uncountable. Indeed it contains*

$$\{(z_0^p z_1^{2q}, z_0^q z_1^p, z_2) : p, q \in \mathbb{Z}\} \cup \{(z_0, z_1, z_2+\alpha) : \alpha \in \mathbb{C}\}$$
$$\cup\, \{(z_0, z_1, \alpha z_2) : \alpha \in \mathbb{C}^*\} \simeq \mathbb{Z}^2 \cup \mathbb{C} \cup \mathbb{C}^*.$$

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