ACTION OF THE CREMONA GROUP ON FOLIATIONS ON $\mathbb{P}^2_{\mathbb{C}}$: SOME CURIOUS FACTS

by

Dominique CERVEAU & Julie DÉSERTI

Abstract. — The Cremona group of birational transformations of $\mathbb{P}^2_{\mathbb{C}}$ acts on the space $\mathbb{F}(2)$ of holomorphic foliations on the complex projective plane. Since this action is not compatible with the natural graduation of $\mathbb{F}(2)$ by the degree, its description is complicated. The fixed points of the action are essentially described by Cantat-Favre in [**3**]. In that paper we are interested in problems of "aberration of the degree" that is pairs $(\phi,\mathcal{F})\in\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})\times\mathbb{F}(2)$ for which $\deg\phi^*\mathcal{F}<(\deg\mathcal{F}+1)\deg\phi+\deg\phi-2$, the generic degree of such pull-back. We introduce the notion of numerical invariance ($\deg\phi^*\mathcal{F}=\deg\mathcal{F}$) and relate it in small degrees to the existence of transversal structure for the considered foliations.
2010 Mathematics Subject Classification. — 14E07, 37F75

1. Introduction

Let us consider on the complex projective plane $\mathbb{P}^2_{\mathbb{C}}$ a foliation $\mathcal{F}$ of degree d and a birational map ϕ of degree k. If the pair $(\mathcal{F},\phi)$ is generic then

$$\deg\phi^*\mathcal{F}=(d+1)k+k-2.$$

For example if $\mathcal{F}$ and ϕ are both of degree 2, then $\phi^*\mathcal{F}$ is of degree 6. Nevertheless one has the following statement which says that "aberration of the degree" is not exceptional:

Theorem A. — *For any foliation $\mathcal{F}$ of degree 2 on $\mathbb{P}^2_{\mathbb{C}}$ there exists a quadratic birational map ψ of $\mathbb{P}^2_{\mathbb{C}}$ such that* $\deg\psi^*\mathcal{F}\leq 3$.

Holomorphic singular foliations on compact complex projective surfaces have been classified up to birational equivalence by Brunella, McQuillan and Mendes ([**1**]). Let $\mathcal{F}$ be a holomorphic singular foliation on a compact complex projective surface S. Let $\mathrm{Bir}(\mathcal{F})$ (resp. $\mathrm{Aut}(\mathcal{F})$) denote the group of birational (resp. biholomorphic) maps of S that send leaf to leaf. If $\mathcal{F}$ is of general type, then $\mathrm{Bir}(\mathcal{F})=\mathrm{Aut}(\mathcal{F})$ is a finite group. In [**3**] Cantat and Favre classify the pairs $(S,\mathcal{F})$ for which $\mathrm{Bir}(\mathcal{F})$ (resp. $\mathrm{Aut}(\mathcal{F})$) is infinite; in the case of $\mathbb{P}^2_{\mathbb{C}}$ such foliations are given by closed rational 1-forms.

In this article we introduce a weaker notion: the numerical invariance. We consider on $\mathbb{P}^2_{\mathbb{C}}$ a pair $(\mathcal{F},\phi)$ of a foliation $\mathcal{F}$ of degree d and a birational map ϕ of degree $k\geq 2$. The foliation $\mathcal{F}$ is ***numerically invariant***

Second author supported by the Swiss National Science Foundation grant no PP00P2_128422 /1 and by ANR Grant "BirPol" ANR-11-JS01-004-01.

under the action of ϕ if $\deg\phi^*\mathcal{F}=\deg\mathcal{F}$. We characterize such pairs $(\mathcal{F},\phi)$ with $\deg\mathcal{F}=\deg\phi=2$ which is the first degree with deep (algebraic and dynamical) phenomena, both for foliations and birational maps. We prove that a numerically invariant foliation under the action of a generic quadratic map is special:

Theorem B. — *Let $\mathcal{F}$ be a foliation of degree 2 on $\mathbb{P}^2_{\mathbb{C}}$ numerically invariant under the action of a generic quadratic birational map of $\mathbb{P}^2_{\mathbb{C}}$. Then $\mathcal{F}$ is transversely projective.*

In that statement generic means outside an hypersurface in the space $\mathring{\mathrm{B}}\mathrm{ir}_2$ of quadratic birational maps of $\mathbb{P}^2_{\mathbb{C}}$.

For any quadratic birational map ϕ of $\mathbb{P}^2_{\mathbb{C}}$ there exists at least one foliation of degree 2 on $\mathbb{P}^2_{\mathbb{C}}$ numerically invariant under the action of ϕ and we give "normal forms" for such foliations. We don't know if the foliations numerically invariant under the action of a non-generic quadratic birational map have a special transversal structure. Problem: for any birational map ϕ of degree $d\geq 3$, does there exist a foliation numerically invariant under the action of ϕ ?

A foliation $\mathcal{F}$ on $\mathbb{P}^2_{\mathbb{C}}$ is ***primitive*** if $\deg\mathcal{F}\leq\deg\phi^*\mathcal{F}$ for any birational map ϕ. Foliations of degree 0 and 1 are defined by a rational closed 1-form (it is a well-known fact, see for example [**2**]). Hence a non-primitive foliation of degree 2 is also defined by a closed 1-form that is a very special case of transversely projective foliations. Generically a foliation of degree 2 is primitive. Remark that there are foliations that are pull-back by a rational map of degree greater than 1, and that are nevertheless primitive. This is the case of the foliation given by $Q_1\,\mathrm{d}Q_2-Q_2\,\mathrm{d}Q_1$ where Q_1 and Q_2 denote two generic polynomials of degree 3, in other words a generic pencil of elliptic curves. The following problem seems relevant: classify in any degree the primitive foliations numerically invariant under the action of birational maps of degree $\geq$ 2; are such foliations transversely projective or is this situation specific to the degree 2 ? In this vein we get the following statement.

Theorem C. — *A foliation $\mathcal{F}$ of degree 2 on $\mathbb{P}^2_{\mathbb{C}}$ numerically invariant under the action of a generic cubic birational map of $\mathbb{P}^2_{\mathbb{C}}$ satisfies the following properties:*

- *$\mathcal{F}$ is given by a closed rational 1-form (Liouvillian integrability);*
- *$\mathcal{F}$ is non-primitive.*

Is it a general fact, *i.e.* if $\mathcal{F}$ is numerically invariant under the action of ϕ and $\deg\phi\gg\deg\mathcal{F}$ is $\mathcal{F}$ Liouvillian integrable ?

The text is organized as follows: we first give some definitions, notations and properties of birational maps of $\mathbb{P}^2_{\mathbb{C}}$ and foliations on $\mathbb{P}^2_{\mathbb{C}}$. In §3 we give a proof of Theorem A; we focus on foliations of degree 2 on $\mathbb{P}^2_{\mathbb{C}}$ that have at least two singular points, and then on foliations of degree 2 on $\mathbb{P}^2_{\mathbb{C}}$ with exactly one singular point. The section 4 is devoted to the description of foliations of degree 2 on $\mathbb{P}^2_{\mathbb{C}}$ numerically invariant under the action of any quadratic birational map. This allows us to prove Theorem B. At the end of the paper, §5, we describe the foliations of degree 2 numerically invariant under some cubic birational maps of $\mathbb{P}^2_{\mathbb{C}}$, and we finally establish Theorem C.

Acknowledgment. — We thank Alcides Lins Neto for helpful discussions, and the anonymous referee for remarks and suggestions.

2. Some definitions, notations and properties

2.1. About birational maps of $\mathbb{P}^2_{\mathbb{C}}$. — A ***rational map*** ϕ of $\mathbb{P}^2_{\mathbb{C}}$ is a "map" of the type

$$\phi\colon \mathbb{P}^2_{\mathbb{C}} \dashrightarrow \mathbb{P}^2_{\mathbb{C}}, \qquad (x:y:z) \dashrightarrow (\phi_0(x,y,z):\phi_1(x,y,z):\phi_2(x,y,z))$$

where the ϕ_i's are homogeneous polynomials of the same degree and without common factor. The ***degree*** of ϕ is by definition the degree of the ϕ_i's. A ***birational map*** ϕ of $\mathbb{P}^2_{\mathbb{C}}$ is a rational map having a rational "inverse" ψ, *i.e.* $\phi\circ\psi=\psi\circ\phi=\mathrm{id}$. The first examples are the birational maps of degree 1 which generate the group $\mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}})=\mathrm{PGL}(3;\mathbb{C})$. Let us give some examples of quadratic birational maps:

$$\sigma\colon (x:y:z)\dashrightarrow(yz:xz:xy), \qquad \rho\colon (x:y:z)\dashrightarrow(xy:z^2:yz),$$
$$\tau\colon (x:y:z)\dashrightarrow(x^2:xy:y^2-xz).$$

These three maps, which are involutions, play an important role in the description of the set of quadratic birational maps of $\mathbb{P}^2_{\mathbb{C}}$.

The birational maps of $\mathbb{P}^2_{\mathbb{C}}$ form a group denoted $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ and called ***Cremona group***. If ϕ is an element of $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ then $\mathscr{O}(\phi)$ is the orbit of ϕ under the action of $\mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}})\times\mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}})$:

$$\mathscr{O}(\phi)=\big\{\ell\phi\ell'\,|\,\ell,\,\ell'\in\mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}})\big\}.$$

A very old theorem, often called Nœther Theorem, says that any element of $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ can be written, up to the action of an automorphism of $\mathbb{P}^2_{\mathbb{C}}$, as a composition of quadratic birational maps ([**4**]). In [**5**, Chapters 1 & 6] the structure of the set Bir_d (resp. $\mathring{\mathrm{B}}\mathrm{ir}_d$) of birational maps of $\mathbb{P}^2_{\mathbb{C}}$ of degree $\leq d$ (resp. of degree d) has been studied when $d=2$ and $d=3$.

Theorem 2.1 **(Corollary 1.10, Theorem 1.31, [5]).** — *One has the following decomposition*

$$\mathring{\mathrm{B}}\mathrm{ir}_2=\mathscr{O}(\sigma)\cup\mathscr{O}(\rho)\cup\mathscr{O}(\tau).$$

Furthermore

$$\mathrm{Bir}_2=\overline{\mathscr{O}(\sigma)}$$

where $\overline{\mathscr{O}(\sigma)}$ denotes the ordinary closure of $\mathscr{O}(\sigma)$, and

$$\dim\mathscr{O}(\tau)=12, \qquad \dim\mathscr{O}(\rho)=13, \qquad \dim\mathscr{O}(\sigma)=14.$$

Note that there is a more precise description of Bir_2 in [**5**, Chapter 1].

We will further do some computations with birational maps of degree 3. Let us consider the following family of cubic birational maps:

$$\Phi_{a,b}\colon (x:y:z)\dashrightarrow\big(x(x^2+y^2+axy+bxz+yz):y(x^2+y^2+axy+bxz+yz):xyz\big)$$

with a, $b\in\mathbb{C}$, $a^2\neq4$ and $2b\not\in\{a\pm\sqrt{a^2-4}\}$. The structure of Bir_3 is much more complicated than the structure of Bir_2 (*see* [**5**, Chapter 6]), nevertheless one has the following result.

Theorem 2.2 **(Proposition 6.35, Theorem 6.38, [5]).** — *The closure of*

$$\mathscr{X}=\big\{\mathscr{O}(\Phi_{a,b})\,|\,a,b\in\mathbb{C},\,a^2\neq4,\,2b\not\in\{a\pm\sqrt{a^2-4}\}\big\}$$

in the set of rational maps of degree 3 *is an irreducible algebraic variety of dimension* 18.

Furthermore the closure of $\mathscr{X}$ in $\mathring{\mathrm{B}}\mathrm{ir}_3$ is $\mathring{\mathrm{B}}\mathrm{ir}_3$.

In the sequel we will say that a $\Phi_{a,b}$ is a generic element of Bir_3.

The "most degenerate model" [1] is up to automorphisms of $\mathbb{P}^2_{\mathbb{C}}$

$$\Psi\colon (x:y:z) \dashrightarrow \big(xz^2+y^3:yz^2:z^3\big).$$

2.2. About foliations. —

Definition 2.3. — Let $\mathcal{F}$ be a foliation (maybe singular) on a complex manifold M; the foliation $\mathcal{F}$ is a ***singular transversely projective*** one if there exists

a) $\pi\colon P\to M$ a $\mathbb{P}^1$-bundle over M,

b) $\mathcal{G}$ a codimension one singular holomorphic foliation on P transversal to the generic fibers of π,

c) $\varsigma\colon M\to P$ a meromorphic section generically transverse to $\mathcal{G}$,

such that $\mathcal{F}=\varsigma^*\mathcal{G}$.

Let us give an other characterization of singular transversely projective foliations. Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2_{\mathbb{C}}$; assume that there exist three rational 1-forms θ_0, θ_1 and θ_2 on $\mathbb{P}^2_{\mathbb{C}}$ such that

i) $\mathcal{F}$ is described by θ_0, *i.e.* $\mathcal{F}=\mathcal{F}_{\theta_0}$,

ii) the θ_i's form a $\mathfrak{sl}(2;\mathbb{C})$-triplet, that is

$$\mathrm{d}\theta_0=\theta_0\wedge\theta_1,\qquad \mathrm{d}\theta_1=\theta_0\wedge\theta_2,\qquad \mathrm{d}\theta_2=\theta_1\wedge\theta_2.$$

Then $\mathcal{F}$ is a singular transversely projective foliation. To see it one considers the manifolds $M=\mathbb{P}^2_{\mathbb{C}}$, $P=\mathbb{P}^2_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}$, the canonical projection $\pi\colon P\to M$, and the foliation $\mathcal{G}$ given by

$$\theta=\mathrm{d}z+\theta_0+z\theta_1+\frac{z^2}{2}\theta_2$$

where z is an affine coordinate of $\mathbb{P}^1_{\mathbb{C}}$; in that case the transverse section is $z=0$. When one can choose the θ_i's such that $\theta_1=\theta_2=0$ (resp. $\theta_2=0$) the foliation $\mathcal{F}$ is ***defined by a closed*** 1***-form*** (resp. is ***transversely affine***).

Classical examples of singular transversely projective foliations are given by Riccati foliations.

Definition 2.4. — A ***Riccati equation*** is a differential equation of the following type

$$\mathcal{E}_R\colon\ y'=a(x)y^2+b(x)y+c(x)$$

where a, b and c are meromorphic functions on an open subset $\mathcal{U}$ of $\mathbb{C}$. To the equation $\mathcal{E}_R$ one associates the meromorphic differential form

$$\omega_{\mathcal{E}_R}=\mathrm{d}y-\big(a(x)y^2+b(x)y+c(x)\big)\,\mathrm{d}x$$

defined on $\mathcal{U}\times\mathbb{C}$. In fact $\omega_{\mathcal{E}_R}$ induces a foliation $\mathcal{F}_{\omega_{\mathcal{E}_R}}$ on $\mathcal{U}\times\mathbb{P}^1_{\mathbb{C}}$ that is transverse to the generic fiber of the projection $\mathcal{U}\times\mathbb{P}^1_{\mathbb{C}}\to\mathcal{U}$. One can check that

$$\theta_0=\omega_{\mathcal{E}_R},\qquad \theta_1=-(2a(x)y+b(x))\,\mathrm{d}x,\qquad \theta_2=-2a(x)\,\mathrm{d}x$$

is a $\mathfrak{sl}(2;\mathbb{C})$-triplet associated to the foliation $\mathcal{F}_{\omega_{\mathcal{E}_R}}$.

We say that $\omega_{\mathcal{E}_R}$ is a ***Riccati*** 1***-form*** and $\mathcal{F}_{\omega_{\mathcal{E}_R}}$ is a ***Riccati foliation***.

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1. In the following sense: the exceptional locus of any element of Bir_3 is a union of degree 6 of conics and lines; the exceptional locus of Ψ is reduced to a single line of multiplicity 6.

Let S be a ruled surface, that is a surface S endowed with $f\colon S \to \mathcal{C}$, where $\mathcal{C}$ denotes a curve and $f^{-1}(c) \simeq \mathbb{P}^1_{\mathbb{C}}$. Let us consider a singular foliation $\mathcal{F}$ on S transverse to the generic fibers of f. The foliation $\mathcal{F}$ is transversely projective.

Recall that a foliation $\mathcal{F}$ on a surface S is ***radial*** at a point m of S if in local coordinates (x,y) around m the foliation $\mathcal{F}$ is given by a holomorphic 1-form of the following type

$$\omega = x\,\mathrm{d}y - y\,\mathrm{d}x + \text{ h.o.t.}$$

Let us denote by $\mathbb{F}(n;d)$ the set of foliations of degree d on $\mathbb{P}^n_{\mathbb{C}}$ (*see* [**2**]). The following statement gives a criterion which asserts that an element of $\mathbb{F}(2;2)$ is transversely projective.

Proposition 2.5. — *Let $\mathcal{F} \in \mathbb{F}(2;2)$ be a foliation of degree 2 on $\mathbb{P}^2_{\mathbb{C}}$. If a singular point of $\mathcal{F}$ is radial, then $\mathcal{F}$ is transversely projective.*

Proof. — Assume that the singular point is the origin 0 in the affine chart $z = 1$, the foliation $\mathcal{F}$ is thus defined by a 1-form of the following type

$$\omega = x\,\mathrm{d}y - y\,\mathrm{d}x + q_1\,\mathrm{d}x + q_2\,\mathrm{d}y + q_3(x\,\mathrm{d}y - y\,\mathrm{d}x)$$

where the q_i's denote quadratic forms. Let us consider the complex projective plane $\mathbb{P}^2_{\mathbb{C}}$ blown up at the origin; this space is denoted by $\mathrm{Bl}(\mathbb{P}^2_{\mathbb{C}},0)$. Let

$$\pi\colon \mathrm{Bl}(\mathbb{P}^2_{\mathbb{C}},0) \to \mathbb{P}^2_{\mathbb{C}}$$

be the canonical projection. Then $\pi^*\mathcal{F}$ is transverse to the generic fibers of π, and in fact transverse to all the fibers excepted the strict transforms of the lines $xq_1 + yq_2 = 0$. Hence the foliation $\pi^*\mathcal{F}$ is transversely projective; since this notion is invariant under the action of a birational map, $\mathcal{F}$ is transversely projective. □

Remark 2.6. — The same argument can be involved for foliations of degree 2 on $\mathbb{P}^2_{\mathbb{C}}$ having a singular point with zero 1-jet.

Remark 2.7. — The closure of the set Δ_R of foliations in $\mathbb{F}(2;2)$ having a radial singular point is irreducible, of codimension 2 in $\mathbb{F}(2;2)$. Indeed Δ_R is the $\mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}})$-orbit of the set

$$\big\{x\,\mathrm{d}y - y\,\mathrm{d}x + q_1\,\mathrm{d}x + q_2\,\mathrm{d}y + q_3(x\,\mathrm{d}y - y\,\mathrm{d}x)\,\big|\, q_i \text{ quadratic form}\big\};$$

in fact it is easy to see that $\overline{\Delta_R}$ is the $\mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}})$-orbit of

$$\big\{\lambda(x\,\mathrm{d}y - y\,\mathrm{d}x) + q_1\,\mathrm{d}x + q_2\,\mathrm{d}y + q_3(x\,\mathrm{d}y - y\,\mathrm{d}x)\,\big|\, \lambda \in \mathbb{C},\, q_i \text{ quadratic form}\big\}.$$

In particular $\overline{\Delta_R}$ is an unirational set in $\mathbb{F}(2;2)$.

3. Proof of Theorem A

We establish Theorem A in two steps: we first look at foliations that have at least two singular points and then at foliations with exactly one singular point.

3.1. Foliations of degree 2 on $\mathbb{P}^2_{\mathbb{C}}$ with at least two singularities. — Any $\mathcal{F}\in\mathbb{F}(2;2)$ is described in homogeneous coordinates by a 1-form ω that can be written

$$\omega=q_1yz\left(\frac{\mathrm{d}y}{y}-\frac{\mathrm{d}z}{z}\right)+q_2xz\left(\frac{\mathrm{d}z}{z}-\frac{\mathrm{d}x}{x}\right)+q_3xy\left(\frac{\mathrm{d}x}{x}-\frac{\mathrm{d}y}{y}\right) \tag{3.1}$$

where

$$q_1=a_0x^2+a_1y^2+a_2z^2+a_3xy+a_4xz+a_5yz,$$
$$q_2=b_0x^2+b_1y^2+b_2z^2+b_3xy+b_4xz+b_5yz,$$
$$q_3=c_0x^2+c_1y^2+c_2z^2+c_3xy+c_4xz+c_5yz.$$

Proposition 3.1. — *For any $\mathcal{F}\in\mathbb{F}(2;2)$ with at least two distinct singularities there exists a quadratic birational map $\psi\in\mathscr{O}(\rho)$ such that $\deg\psi^*\mathcal{F}\leq 3$.*

Proof. — In homogeneous coordinates $\mathcal{F}$ is described by a 1-form ω as in (3.1).

Up to an automorphism of $\mathbb{P}^2_{\mathbb{C}}$ one can suppose that $(1:0:0)$ and $(0:1:0)$ are singular points of $\mathcal{F}$, that is $a_1=b_0=c_0=c_1=0$. If $c_3\neq 0$, resp. $c_3=0$ and $b_4\neq 0$, resp. $c_3=b_4=0$, then let us consider the quadratic birational map ψ of $\mathscr{O}(\rho)$ defined as follows

$$\psi\colon (x:y:z)\dashrightarrow\left(xy:z^2+\frac{b_3-c_4+\sqrt{(b_3-c_4)^2+4b_4c_3}}{2c_3}yz:yz\right),$$

resp.

$$\psi\colon (x:y:z)\dashrightarrow\left(xy:z^2+yz:-\frac{b_3-c_4}{b_4}yz\right),$$

resp. $\psi=\rho$. A direct computation shows that $\psi^*\omega=yz^2\omega'$ where ω' denotes a homogeneous 1-form of degree 4. The foliation $\mathcal{F}'$ defined by ω' has degree at most 3. □

3.2. Foliations of degree 2 on $\mathbb{P}^2_{\mathbb{C}}$ with exactly one singularity. — Such foliations have been classified:

Theorem 3.2 **([6])**. — *Up to automorphisms of $\mathbb{P}^2_{\mathbb{C}}$ there are four foliations of degree 2 on $\mathbb{P}^2_{\mathbb{C}}$ having exactly one singularity. They are described in affine chart by the following 1-forms:*

- $\Omega_1=x^2\,\mathrm{d}x+y^2(x\,\mathrm{d}y-y\,\mathrm{d}x)$,
- $\Omega_2=x^2\mathrm{d}x+(x+y^2)(x\,\mathrm{d}y-y\,\mathrm{d}x)$,
- $\Omega_3=xy\,\mathrm{d}x+(x^2+y^2)(x\,\mathrm{d}y-y\,\mathrm{d}x)$,
- $\Omega_4=(x+y^2-x^2y)\,\mathrm{d}y+x(x+y^2)\,\mathrm{d}x$.

Proposition 3.3. — *There exists a quadratic birational map $\psi_1\in\mathscr{O}(\rho)$ such that $\deg\psi_1^*\mathcal{F}_{\Omega_1}=2$; furthermore $\mathcal{F}_{\Omega_1}$ has a rational first integral and is non-primitive.*

For $k=2$, 3, there is no birational map φ_k such that $\deg\varphi_k^*\mathcal{F}_{\Omega_k}=0$ but there is ψ_k in $\mathscr{O}(\tau)$ such that $\deg\psi_k^*\mathcal{F}_{\Omega_k}=1$. In particular $\mathcal{F}_{\Omega_2}$ and $\mathcal{F}_{\Omega_3}$ are non-primitive.

There is a quadratic birational map $\psi_4\in\mathscr{O}(\tau)$ such that $\deg\psi_4^*\mathcal{F}_{\Omega_4}=3$, and $\mathcal{F}_{\Omega_4}$ is primitive.

Remark 3.4. — If $\phi=(x^2:xy:xz+y^2)$, then $\deg\phi^*\mathcal{F}_{\Omega_2}=\deg\phi^*\mathcal{F}_{\Omega_3}=2$. A contrario we will see later there is no quadratic birational map ϕ such that $\deg\phi^*\mathcal{F}_{\Omega_4}=2$ (*see* Corollary 4.15).

Corollary 3.5. — *For any element $\mathcal{F}$ of $\mathbb{F}(2;2)$ with exactly one singularity there exists a quadratic birational map ψ such that $\deg\psi^*\mathcal{F}\leq 3$.*

Proof of Proposition 3.3. — The foliation $\mathcal{F}_{\Omega_1}$ is given in homogeneous coordinates by

$$\Omega_1' = (x^2z - y^3)\,\mathrm{d}x + xy^2\,\mathrm{d}y - x^3\,\mathrm{d}z;$$

if $\psi_1\colon (x:y:z) \dashrightarrow (x^2:xy:yz)$ then

$$\psi_1^*\Omega_1' \wedge \big(y(2xz-y^2)\,\mathrm{d}x + x(y^2-xz)\,\mathrm{d}y - x^2y\,\mathrm{d}z\big) = 0.$$

The foliation $\mathcal{F}_{\Omega_1}$ has a rational first integral and is non-primitive, it is the image of a foliation of degree 0 by a cubic birational map:

$$(x^3 : x^2y : x^2z + y^3/3)^*\Omega_1' \wedge \big(z\,\mathrm{d}x - x\,\mathrm{d}z\big) = 0.$$

The foliation $\mathcal{F}_{\Omega_2}$ is described in homogeneous coordinates by

$$\Omega_2' = (x^2z - xyz - y^3)\,\mathrm{d}x + x(xz+y^2)\,\mathrm{d}y - x^3\,\mathrm{d}z;$$

let us consider the birational map $\psi_2\colon (x:y:z) \dashrightarrow (x^2 : xy : xz - 2x^2 - 2xy - y^2)$ then

$$\psi_2^*\Omega_2' \wedge \big((xz-yz)\,\mathrm{d}x + xz\,\mathrm{d}y - x^2\,\mathrm{d}z\big) = 0.$$

One can verify that

$$\left(2 + \frac{1}{x} + 2\frac{y}{x} + \frac{y^2}{x^2}\right)\exp\left(-\frac{y}{x}\right)$$

is a first integral of $\mathcal{F}_{\Omega_2}$; it is easy to see that $\mathcal{F}_{\Omega_2}$ has no rational first integral so there is no birational map φ_2 such that $\deg\varphi_2^*\mathcal{F}_{\Omega_2} = 0$.

The foliation $\mathcal{F}_{\Omega_3}$ is given in homogeneous coordinates by the 1-form

$$\Omega_3' = y(xz - x^2 - y^2)\,\mathrm{d}x + x(x^2+y^2)\,\mathrm{d}y - x^2y\,\mathrm{d}z;$$

if $\psi_3\colon (x:y:z) \dashrightarrow (x^2 : xy : xz + y^2/2)$ then

$$\psi_3^*\Omega_3' \wedge \big(y(z-x)\,\mathrm{d}x + x^2\,\mathrm{d}y - xy\,\mathrm{d}z\big) = 0.$$

The function

$$\left(\frac{y}{x}\right)\exp\left(\frac{1}{2}\frac{y^2}{x^2} - \frac{1}{x}\right)$$

is a first integral of $\mathcal{F}_{\Omega_3}$, and $\mathcal{F}_{\Omega_3}$ has no rational first integral so there is no birational map φ_3 such that $\deg\varphi_3^*\mathcal{F}_{\Omega_3} = 0$.

Let us consider the birational map of $\mathbb{P}^2_{\mathbb{C}}$ given by

$$\psi_4\colon (x:y:z) \dashrightarrow (-x^2 : xy : y^2 - xz)$$

In homogeneous coordinates

$$\Omega_4' = x(xz+y^2)\,\mathrm{d}x + (xz^2 + y^2z - x^2y)\,\mathrm{d}y + (xyz - y^3 - x^3)\,\mathrm{d}z;$$

a direct computation shows that $\psi_4^*\Omega_4' \wedge \eta = 0$ where

$$\eta = (3y^3z - x^2y^2 + x^3z - 2xyz^2)\,\mathrm{d}x + (x^3y - 4y^4 - x^2z^2 + 3xy^2z)\,\mathrm{d}y$$
$$+ x(2y^3 - x^3 - xyz)\,\mathrm{d}z.$$

The foliation $\mathcal{F}_{\Omega_4}$ has no invariant algebraic curve so $\mathcal{F}_{\Omega_4}$ is not transversely projective ([**6**, Proposition 1.3]). In fact a foliation of degree 2 without invariant algebraic curve is primitive (*see* Introduction); as a consequence $\mathcal{F}_{\Omega_4}$ is a primitive foliation. □

4. Numerical invariance

In the sequel num. inv. means numerically invariant.

In this section we determine the foliations $\mathcal{F}$ of $\mathbb{F}(2;2)$ num. inv. under the action of σ (resp. ρ, resp. τ). Note that if ϕ is a birational map of $\mathbb{P}^2_{\mathbb{C}}$ and ℓ an element of $\mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}})$ then $\deg(\phi\ell)^*\mathcal{F} = \deg\phi^*\mathcal{F}$; hence following Theorem 2.1 we get the description of foliations num. inv. under the action of a quadratic birational map of $\mathbb{P}^2_{\mathbb{C}}$ by giving normal forms.

Recall that σ is given in a fixed system of homogeneous coordinates $(x:y:z)$ by

$$\sigma\colon (x:y:z) \dashrightarrow (yz:xz:xy),$$

and remark that σ is invariant under conjugacy by elements of the group $\mathfrak{S}_3$ of standard permutations of coordinates.

Lemma 4.1. — *An element $\mathcal{F}$ of $\mathbb{F}(2;2)$ is num. inv. under the action of σ if and only if it is given up to permutations of coordinates and standard affine charts by 1-forms of the following type*

- *either* $\omega_1 = y\big(\kappa+\varepsilon y\big)\,\mathrm{d}x + \big(\beta x+\delta y+\alpha x^2+\gamma xy\big)\,\mathrm{d}y,$
- *or* $\omega_2 = \big(\delta+\beta y+\kappa y^2\big)\,\mathrm{d}x + \big(\alpha+\varepsilon x+\gamma x^2\big)\,\mathrm{d}y,$

where α, β, γ, δ, ε, κ (*resp.* α, β, γ, δ, ε, κ) *are complex numbers such that* $\deg\mathcal{F}_{\omega_1} = 2$ (*resp.* $\deg\mathcal{F}_{\omega_2} = 2$).

Proof. — The foliation $\mathcal{F}$ is defined by a homogeneous 1-form ω of degree 3. The map σ is an automorphism of $\mathbb{P}^2_{\mathbb{C}} \smallsetminus \{xyz=0\}$; hence if $\sigma^*\omega = P\omega'$, with ω' a 1-form of degree 3 and P a homogeneous polynomial, then $P = x^iy^jz^k$ for some integers i, j, k such that $i+j+k=4$. Up to permutation of coordinates it is sufficient to look at the four following cases: $P=x^4$, $P=x^3y$, $P=x^2y^2$ and $P=x^2yz$. Let us write ω as in (3.1). Computations show that x^4 (resp. x^3y) cannot divide $\sigma^*\omega$. If $P=x^2yz$, then $\sigma^*\omega = P\omega'$ if and only if

$$c_0=b_0=a_2=b_2=a_1=c_1=b_4=c_3=0, \quad b_3=c_4$$

that gives ω_1. Finally one has $\sigma^*\omega = x^2y^2\omega'$ if and only if

$$c_1=c_0=b_0=a_1=b_4=c_3=a_5=0, \quad b_3=c_4, \quad c_5=a_3;$$

in that case we obtain ω_2. □

Proposition 4.2. — *A foliation $\mathcal{F}\in\mathbb{F}(2;2)$ num. inv. under the action of an element of $\mathscr{O}(\sigma)$ is $\mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}})$-conjugate either to a foliation of type $\mathcal{F}_{\omega_1}$, or to a foliation of type $\mathcal{F}_{\omega_2}$. In particular it is transversely projective.*

Proof. — Let ϕ be an element of $\mathscr{O}(\sigma)$ such that $\deg\phi^*\mathcal{F}=2$; the map ϕ can be written $\ell_1\sigma\ell_2$ where ℓ_1 and ℓ_2 denote automorphisms of $\mathbb{P}^2_{\mathbb{C}}$. By assumption the degree of $(\ell_1\sigma\ell_2)^*\mathcal{F} = \ell_2^*(\sigma^*(\ell_1^*\mathcal{F}))$ is 2. Hence $\deg\sigma^*(\ell_1^*\mathcal{F})=2$ and the foliation $\ell_1^*\mathcal{F}$ is num. inv. under the action of σ. Since $\ell_1^*\mathcal{F}$ and $\mathcal{F}$ are conjugate and since the notion of transversal projectivity is invariant by conjugacy it is sufficient to establish the statement for $\phi=\sigma$. The proposition thus follows from the fact that 1-forms of Lemma 4.1 are Riccati ones (up to multiplication). □

Remark 4.3. — For generic values of parameters α, β, γ, δ, ε, κ a foliation of type $\mathcal{F}_{\omega_1}$ given by the corresponding form ω_1 is not given by a closed meromorphic 1-form. This can be seen by studying the holonomy group of $\mathcal{F}_{\omega_1}$ that can be identified with a subgroup of $\mathrm{PGL}(2;\mathbb{C})$ generated by two elements f and g. For generic values of the parameters f and g are also generic, in particular the group $\langle f,g\rangle$ is free. When $\mathcal{F}_{\omega_1}$ is given by a closed 1-form, then the holonomy group is an abelian one.

Remark that a contrario the foliations given by 1-forms of type ω_2 are given by a closed meromorphic 1-form.

Remark 4.4. — Let Δ_i denote the closure of the set of elements of $\mathbb{F}(2;2)$ conjugate to a foliation of type $\mathcal{F}_{\omega_i}$.

In the affine chart $x=1$ an element of type $\mathcal{F}_{\omega_1}$ is radial at $(0,0)$ as soon as $\alpha\neq 0$. Since Δ_R is closed, the inclusion $\Delta_1\subset\Delta_R$ holds.

If the components of ω_2 are not constant, then an element of type $\mathcal{F}_{\omega_2}$ has a singular point in $\mathbb{C}^2$, and up to an ad-hoc translation $\mathcal{F}_{\omega_2}$ is an element of type $\mathcal{F}_{\omega_1}$. As Δ_1 is closed, one has $\Delta_2\subset\Delta_1$.

Remark 4.5. — The notion of num. inv. is not related to the dynamic of the map (*see* [**3**] for example): the foliations num. inv. by the involution σ ("without dynamic") are conjugate to the foliations num. inv. by $A\sigma$, $A\in\mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}})$, which has a rich dynamic for generic A.

The foliations of $\mathbb{F}(2;2)$ invariant by σ are particular cases of num. inv. foliations:

Proposition 4.6. — *An element of $\mathbb{F}(2;2)$ invariant by σ is given up to permutations of coordinates and affine charts*

- *either by* $y(1+y)\,\mathrm{d}x+(\beta x+\alpha y+\alpha x^2+\beta xy)\,\mathrm{d}y$,
- *or by* $y(1-y)\,\mathrm{d}x+(\beta x-\alpha y+\alpha x^2-\beta xy)\,\mathrm{d}y$,
- *or by* $y\,\mathrm{d}x+(\alpha+\varepsilon x+\alpha x^2)\,\mathrm{d}y$,

where the parameters are complex numbers such that the degree of the associated foliations is 2.

Proof. — With the notations of Lemma 4.1 one has

$$\sigma^*\omega_1=-y(\varepsilon+\kappa y)\,\mathrm{d}x-(\gamma x+\alpha y+\delta x^2+\beta xy)\,\mathrm{d}y;$$

thus $\sigma^*\omega_1\wedge\omega_1=0$ if and only if either $\gamma=\beta$, $\delta=\alpha$, $\varepsilon=\kappa$, or $\gamma=-\beta$, $\delta=-\alpha$, $\varepsilon=-\kappa$.

One has $\sigma^*\omega_2=-(\kappa+\beta y+\delta y^2)\,\mathrm{d}x-(\gamma+\varepsilon x+\alpha x^2)\,\mathrm{d}y$, and $\omega_2\wedge\sigma^*\omega_2=0$ if and only if $\gamma=\alpha$, $\delta=0$ and $\kappa=0$. □

Remark 4.7. — The foliations associated to the two first 1-forms with parameters α, β of Proposition 4.6 are conjugate by the automorphism $(x,y)\mapsto(x,-y)$.

Lemma 4.8. — *A foliation $\mathcal{F}\in\mathbb{F}(2;2)$ is num. inv. under the action of ρ if and only if $\mathcal{F}$ is given in affine chart*

- *either by* $\omega_3=y\big(\kappa+\varepsilon y+\lambda y^2\big)\,\mathrm{d}x+\big(\beta+\kappa x+\delta y+\gamma xy+\alpha y^2-\lambda xy^2\big)\,\mathrm{d}y$,
- *or by* $\omega_4=y\big(\mu+\delta x+\gamma y+\varepsilon xy\big)\,\mathrm{d}x+\big(\alpha+\beta x+\lambda y+\delta x^2+\kappa xy-\varepsilon x^2y\big)\,\mathrm{d}y$,
- *or by* $\omega_5=\big(\lambda+\gamma y+\kappa xy+\varepsilon y^2\big)\,\mathrm{d}x+\big(\beta+\delta x+\alpha x^2\big)\,\mathrm{d}y$,

where the parameters of ω_i are such that $\deg\mathcal{F}_{\omega_i}=2$.

Proof. — Let us take the notations of the proof of Lemma 4.1. The map ρ is an automorphism of $\mathbb{P}^2_{\mathbb{C}}\smallsetminus\{yz=0\}$; therefore if $\rho^*\omega=P\omega'$ with ω' a 1-form of degree 3 and P a homogeneous polynomial then $P=y^jz^k$ for some integers j, k such that $j+k=4$. We have to look at the five following cases: $P=z^4$, $P=yz^3$, $P=y^2z^2$, $P=y^3z$ and $P=y^4$. Computations show that y^4 (resp. y^3z) cannot divide $\rho^*\omega$. If $P=z^4$ then $\rho^*\omega=P\omega'$ if and only if

$$c_0=b_0=c_3=b_4=b_2=0,\quad a_0=c_4,\quad b_3=c_4,\quad a_4=2c_2-b_5;$$

this gives the first case ω_3. The equality $\rho^*\omega=yz^3\omega'$ holds if and only if

$$b_0=c_0=b_4=c_1=a_1=b_2=0,\quad a_0=2c_4-b_3$$

and we obtain ω_4. Finally one has $\rho^*\omega=y^2z^2\omega'$ if and only if

$$c_1=b_0=c_3=a_5=a_1=c_0=b_4=0,\quad c_5=a_3$$

which corresponds to ω_5. □

Proposition 4.9. — *The foliations of type $\mathcal{F}_{\omega_3}$ and $\mathcal{F}_{\omega_5}$ are transversely projective. In fact the $\mathcal{F}_{\omega_3}$ are transversely affine, and the $\mathcal{F}_{\omega_5}$ are Riccati ones.*

Proof. — A foliation of type $\mathcal{F}_{\omega_3}$ is described by the 1-form

$$\theta_0 = \mathrm{d}x - \frac{\big(\beta+\delta y+\alpha y^2\big)+\big(\kappa+\gamma y-\lambda y^2\big)x}{y\big(\kappa+\varepsilon y+\lambda y^2\big)}\,\mathrm{d}y$$

and it is transversely affine; to see it consider the $\mathfrak{sl}(2;\mathbb{C})$-triplet

$$\theta_0, \qquad \theta_1 = \frac{\kappa+\gamma y-\lambda y^2}{y(\kappa+\varepsilon y+\lambda y^2)}\,\mathrm{d}y, \qquad \theta_2 = 0.$$

A foliation of type $\mathcal{F}_{\omega_5}$ is given by

$$\mathrm{d}y + \frac{\lambda+(\gamma+\kappa)y+\varepsilon y^2}{\beta+\delta x+\alpha x^2}\,\mathrm{d}x$$

and thus is a Riccati foliation. In fact the fibration $x/z=$ constant is transverse to $\mathcal{F}_{\omega_5}$ that generically has three invariant lines. □

We don't know if the $\mathcal{F}_{\omega_4}$ are transversely projective. For generic values of the parameters a foliation of type $\mathcal{F}_{\omega_4}$ hasn't meromorphic uniform first integral in the affine chart $y=1$. Thus if $\mathcal{F}_{\omega_4}$ is transversely projective then it must have an invariant algebraic curve different from $y=0$ (*see* [**7**]). We don't know if it is the case. A foliation of degree 2 is conjugate to a generic $\mathcal{F}_{\omega_4}$ (by an automorphism of $\mathbb{P}^2_{\mathbb{C}}$) if and only if it has an invariant line (say $y=0$) with a singular point (say 0) and local model $2x\,\mathrm{d}y-y\,\mathrm{d}x$. The closure of the set of such foliations has codimension 2. Note that the three families $\mathcal{F}_{\omega_3}$, $\mathcal{F}_{\omega_4}$ and $\mathcal{F}_{\omega_5}$ have non trivial intersection. The set $\overline{\{\mathcal{F}_{\omega_4}\}}$ contains many interesting elements such that the famous Euler foliation given by $y^2\,\mathrm{d}x+(y-x)\,\mathrm{d}y$; this foliation is transversely affine but is not given by a closed rational 1-form.

Proposition 4.10. — *A foliation $\mathcal{F}\in\mathbb{F}(2;2)$ num. inv. under the action of an element of $\mathscr{O}(\rho)$ is conjugate to a foliation either of type $\mathcal{F}_{\omega_3}$, or of type $\mathcal{F}_{\omega_4}$, or of type $\mathcal{F}_{\omega_5}$.*

Let us look at special num. inv. foliations, those invariant by ρ.

Proposition 4.11. — *An element of $\mathbb{F}(2;2)$ invariant by ρ is given by a 1-form of one of the following type*

- $y(1-y)\,\mathrm{d}x+(\beta+x)\mathrm{d}y$,
- $y^2\,\mathrm{d}x+(-1+y)\mathrm{d}y$,
- $y(1-y)(\gamma+\delta x)\,\mathrm{d}x+(1+y)(\alpha+\beta x+\delta x^2)\,\mathrm{d}y$,
- $y(1+y)(\gamma+\delta x)\,\mathrm{d}x+(1-y)(\alpha+\beta x+\delta x^2)\,\mathrm{d}y$,
- $(1-y^2)\,\mathrm{d}x+(\beta+\delta x+\alpha x^2)\,\mathrm{d}y$,

where the parameters are complex numbers such that the degree of the associated foliations is 2.

Corollary 4.12. — *An element of $\mathbb{F}(2;2)$ invariant by ρ is defined by a closed 1-form.*

Remark 4.13. — The third and fourth cases with parameters α, β, γ, δ are conjugate by the automorphism $(x,y)\mapsto(x,-y)$.

From Lemmas 4.1 and 4.8 one gets the following statement.

Proposition 4.14. — *A foliation num. inv. by an element of $\mathscr{O}(\phi)$, with $\phi = \sigma$, ρ, preserves an algebraic curve.*

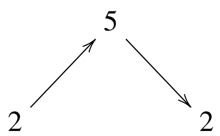
Corollary 4.15. — *There is no quadratic birational map ϕ of the complex projective plane such that* $\deg\phi^*\mathcal{F}_{\Omega_4} = 2$.

Proof. — The foliation $\mathcal{F}_{\Omega_4}$ has no invariant algebraic curve ([**6**, Proposition 1.3]); according to Proposition 4.14 it is thus sufficient to show that there is no birational map $\phi \in \mathscr{O}(\tau)$ such that $\deg\phi^*\mathcal{F}_{\Omega_4} = 2$ that can be established with a direct and tedious computation. □

Remark 4.16. — The map ρ can be written $\ell_1\sigma\ell_2\sigma\ell_3$ with

$$\ell_1 = (z-y : y-x : y), \qquad \ell_2 = (y+z : z : x), \qquad \ell_3 = (x+z : y-z : z).$$

We are interested by the "intermediate" degrees of a numerically invariant foliation $\mathcal{F}$, that is the sequence $\deg\mathcal{F}$, $\deg(\ell_1\sigma)^*\mathcal{F}$, $\deg(\ell_1\sigma\ell_2\sigma\ell_3)^*\mathcal{F} = \deg\mathcal{F}$. A tedious computation shows that for generic values of the parameters the sequence is 2, 5, 2. We schematize this fact by the diagram



A similar argument to Lemma 4.1 yields to the following result.

Lemma 4.17. — *An element $\mathcal{F}$ of $\mathbb{F}(2;2)$ is num. inv. under the action of τ if and only if $\mathcal{F}$ is given in affine chart by a 1-form of type*

$$\begin{aligned}\omega_6 \;=\; & \big(-\delta x+\alpha y-\varepsilon x^2+\theta xy+\beta y^2+\kappa x^2y+\mu xy^2+\lambda y^3\big)\,\mathrm{d}x \\ & +\big(-3\alpha x+\xi x^2+2(\delta-\beta)xy+\alpha y^2-\kappa x^3-\mu x^2y-\lambda xy^2\big)\,\mathrm{d}y\end{aligned}$$

where the parameters are such that $\deg\mathcal{F}_{\omega_6} = 2$.

We don't know the qualitative description of foliations of type $\mathcal{F}_{\omega_6}$. For example we don't know if the $\mathcal{F}_{\omega_6}$ are transversely projective. If it is the case, this implies the existence of invariant algebraic curves, and that fact is unknown.

Proposition 4.18. — *A foliation $\mathcal{F} \in \mathbb{F}(2;2)$ num. inv. under the action of an element of $\mathscr{O}(\tau)$ is conjugate to $\mathcal{F}_{\omega_6}$ for suitable values of the parameters.*

Let us describe some special num. inv. foliations under the action of τ, those invariant by τ.

Proposition 4.19. — *An element of $\mathbb{F}(2;2)$ invariant by τ is given*

- *either by*

$$\big(-\varepsilon x^2+\theta xy+\beta y^2+\varepsilon xy^2-(\frac{\xi}{2}+\theta)y^3\big)\,\mathrm{d}x+x\big(\xi x-2\beta y-\varepsilon xy+(\frac{\xi}{2}+\theta)y^2\big)\,\mathrm{d}y,$$

- *or by*

$$\big(-\delta x+\alpha y+\frac{3}{2}\delta y^2+\kappa x^2y+\mu xy^2+\lambda y^3\big)\,\mathrm{d}x-\big(3\alpha x+\delta xy-\alpha y^2+\kappa x^3+\mu x^2y+\lambda xy^2\big)\,\mathrm{d}y,$$

where the parameters are complex numbers such that the degree of the associated foliations is 2.

The foliations associated to the first 1-form are transversely affine.

Proof. — The 1-jet at the origin of the 1-form

$$\omega=\big(-\varepsilon x^2+\theta xy+\beta y^2+\varepsilon xy^2-(\frac{\xi}{2}+\theta)y^3\big)\,\mathrm{d}x+x\big(\xi x-2\beta y-\varepsilon xy+(\frac{\xi}{2}+\theta)y^2\big)\,\mathrm{d}y$$

is zero so after one blow-up $\mathcal{F}_\omega$ is transverse to the generic fiber of the Hopf fibration; furthermore as the exceptional divisor is invariant, $\mathcal{F}_\omega$ is transversely affine. □

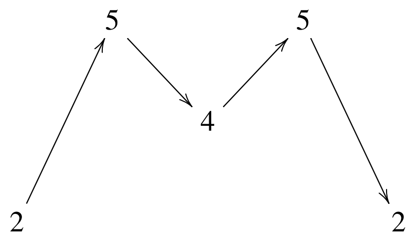
Remark 4.20. — The map τ can be written $\ell_1\sigma\ell_2\sigma\ell_3\sigma\ell_2\sigma\ell_4$ with

$$\ell_1=(x-y:x-2y:-x+y-z),\qquad \ell_2=(x+z:x:y),$$
$$\ell_3=(-y:x-3y+z:x),\qquad \ell_4=(y-x:z-2x:2x-y).$$

Let us consider a foliation $\mathcal{F}$ num. inv. under the action of τ; set $\mathcal{F}'=\ell_1^*\mathcal{F}$. We compute the intermediate degrees:

$$\deg\sigma^*\mathcal{F}'=5,\qquad \deg(\sigma\ell_2\sigma)^*\mathcal{F}'=4,\qquad \deg(\sigma\ell_3\sigma\ell_2\sigma)^*\mathcal{F}'=5.$$

To summarize:



5. Higher degree

We will now focus on similar questions but with cubic birational maps of $\mathbb{P}^2_{\mathbb{C}}$ and elements of $\mathbb{F}(2;2)$. The generic model of such birational maps is:

$$\Phi_{a,b}\colon(x:y:z)\dashrightarrow\big(x(x^2+y^2+axy+bxz+yz):y(x^2+y^2+axy+bxz+yz):xyz\big)$$

with a, $b\in\mathbb{C}$, $a^2\neq 4$ and $2b\notin\{a\pm\sqrt{a^2-4}\}$.

Lemma 5.1. — *An element $\mathcal{F}$ of $\mathbb{F}(2;2)$ is num. inv. under the action of $\Phi_{a,b}$ if and only if $\mathcal{F}$ is given in affine chart*

- *either by $\omega_7=y(\alpha+\gamma y)\,\mathrm{d}x-x(\alpha+\kappa x)\,\mathrm{d}y$,*
- *or by*

$$\omega_8=b\big(b^2-ab+1+(a-2b)y+y^2\big)\,\mathrm{d}x+\big((b^2-ab+1)+(ab-2)x+x^2\big)\,\mathrm{d}y,$$

where the parameters are such that $\deg\mathcal{F}_{\omega_7}=\deg\mathcal{F}_{\omega_8}=2$.

Remark 5.2. — Remark that the foliations $\mathcal{F}_{\omega_7}$ do not depend on the parameters of $\Phi_{a,b}$, that is, the $\mathcal{F}_{\omega_7}$ are num. inv. by all $\Phi_{a,b}$, whereas the $\mathcal{F}_{\omega_8}$ only depend on a and b.

Furthermore $\mathcal{F}_{\omega_7}$ is num. inv. by σ and ρ.

Proposition 5.3. — *Any $\mathcal{F}\in\mathbb{F}(2;2)$ num. inv. under the action of $\Phi_{a,b}$, and more generally any $\mathcal{F}\in\mathbb{F}(2;2)$ num. inv. under the action of a generic cubic birational map of $\mathbb{P}^2_{\mathbb{C}}$, satisfies the following properties:*

- *$\mathcal{F}$ is given by a rational closed 1-form;*
- *$\mathcal{F}$ is non-primitive.*

Proof. — Let us establish the properties for $\mathcal{F}_{\omega_7}$; remark that $\mathcal{F}_{\omega_7}$ is given by

$$\frac{\mathrm{d}x}{x(\alpha+\kappa x)}-\frac{\mathrm{d}y}{y(\alpha+\gamma y)}$$

which is a closed rational 1-form (remark that $(|\alpha|+|\kappa|)(|\alpha|+|\gamma|)\neq 0$ since $\deg\mathcal{F}_{\omega_7}=2$). The foliation $\mathcal{F}_{\omega_7}$ is non-primitive: indeed one has

$$\sigma^*\omega_7=\frac{1}{x^2y^2}\Big((\alpha x+\kappa)\,\mathrm{d}x-(\alpha y+\gamma)\,\mathrm{d}y\Big)$$

that defines a foliation of degree 0.

The idea and result are the same for the foliations $\mathcal{F}_{\omega_8}$ (except that it gives a birational map ϕ such that $\deg\phi^*\mathcal{F}_{\omega_8}=1$). □

Let us consider an element $\mathcal{F}$ of $\mathbb{F}(2;2)$ num. inv. under the action of a birational map of degree ≥ 3; is $\mathcal{F}$ defined by a closed 1-form ?

Remark 5.4. — The foliations $\mathcal{F}_{\omega_7}$ are contained in the orbit of the foliation $\mathcal{F}_{\eta'}$.

Remark 5.5. — Any map $\Phi_{a,b}$ can be written $\ell_1\sigma\ell_2\sigma\ell_3$ with

$$\ell_2=(a_0y+a_1z:a_2y+a_3z:a_4x+a_5y+a_6z)$$

(*see* [**5**, proof of Proposition 6.36]). Let us consider the birational map $\xi=\sigma\ell_2\sigma$ with

$$\ell_2=(a_0y+a_1z:a_2y+a_3z:a_4x+a_5y+a_6z)\in\mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}}).$$

As in Lemma 5.1 there are two families of foliations $\mathscr{F}_1$, $\mathscr{F}_2$ of degree 2, one that does not depend on the parameters of ξ and the other one depending only on the parameters of ξ, such that $\xi^*\mathscr{F}_1$ and $\xi^*\mathscr{F}_2$ are of degree 2. One question is the following: what is the intermediate degree ? A computation shows that for generic parameters $\deg\sigma^*\mathscr{F}_1=4$ and that $\deg\sigma^*\mathscr{F}_2=2$. This implies in particular that $\mathcal{F}_{\omega_8}$ is num. inv. under the action of σ. For $\mathscr{F}_1$ and $\mathcal{F}_{\omega_7}$ one has

$$\begin{array}{ccc} & 4 & \\ \nearrow & & \searrow \\ 2 & & 2 \end{array}$$

and for $\mathscr{F}_2$ and $\mathcal{F}_{\omega_8}$

$$2\longrightarrow 2\longrightarrow 2$$

Let us now consider the "most degenerate" cubic birational map

$$\Psi\colon(x:y:z)\dashrightarrow\big(xz^2+y^3:yz^2:z^3\big).$$

Lemma 5.6. — *An element $\mathcal{F}$ of $\mathbb{F}(2;2)$ is num. inv. under the action of Ψ if and only if $\mathcal{F}$ is given in affine chart by*

$$\omega_9=\big(-\alpha+\beta y+\gamma y^2\big)\,\mathrm{d}x+\big(\varepsilon-3\beta x+\kappa y-3\gamma xy+\lambda y^2\big)\mathrm{d}y$$

where the parameters are such that $\deg\mathcal{F}_{\omega_9}=2$. *In particular $\mathcal{F}$ is transversely affine.*

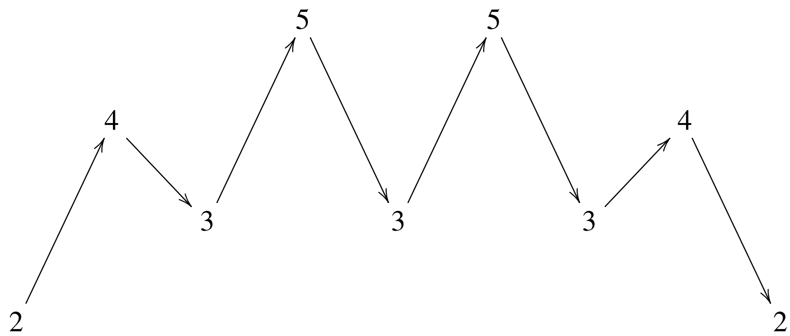
Remark 5.7. — The map ψ can be written $\ell_1\sigma\ell_2\sigma\ell_3\sigma\ell_4\sigma\ell_5\sigma\ell_4\sigma\ell_6\sigma\ell_2\sigma\ell_7$ with

$$\ell_1=(z-y:y:y-x),\qquad \ell_2=(y+z:z:x),\qquad \ell_3=(-z:-y:x-y),$$
$$\ell_4=(x+z:x:y),\qquad \ell_5=(-y:x-3y+z:x),\qquad \ell_6=(-x:-y-z:x+y),$$
$$\ell_7=(x+y:z-y:y).$$

As previously we consider the problem of the intermediate degrees; if $\mathcal{F}'=\ell_1^*\mathcal{F}$, a computation shows that for generic parameters

$$\deg\sigma^*\mathscr{F}'=4,\quad \deg(\sigma\ell_2\sigma)^*\mathscr{F}'=3,\quad \deg(\sigma\ell_2\sigma\ell_3\sigma)^*\mathscr{F}'=5,$$
$$\deg(\sigma\ell_2\sigma\ell_3\sigma\ell_4\sigma)^*\mathscr{F}'=3,\quad \deg(\sigma\ell_2\sigma\ell_3\sigma\ell_4\sigma\ell_5\sigma)^*\mathscr{F}'=5,$$
$$\deg(\sigma\ell_2\sigma\ell_3\sigma\ell_4\sigma\ell_5\sigma\ell_4\sigma)^*\mathscr{F}'=3,\quad \deg(\sigma\ell_2\sigma\ell_3\sigma\ell_4\sigma\ell_5\sigma\ell_4\sigma\ell_6\sigma)^*\mathscr{F}'=4,$$

that is



We have not studied the quadratic foliations numerically invariant by (any) cubic birational transformation. It is reasonable to think that such foliations are transversely projective.

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DOMINIQUE CERVEAU, Membre de l'Institut Universitaire de France. IRMAR, UMR 6625 du CNRS, Université de Rennes 1, 35042 Rennes, France. • *E-mail :* dominique.cerveau@univ-rennes1.fr

JULIE DÉSERTI, IMJ-PRG, UMR 7586, Université Paris 7, Bâtiment Sophie Germain, Case 7012, 75205 Paris Cedex 13, France. *E-mail :* julie.deserti@imj-prg.fr