ACTION OF THE CREMONA GROUP ON FOLIATIONS ON $\mathbb{P}^2_{\mathbb{C}}$: SOME CURIOUS FACTS

by

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Abstract. — The Cremona group of birational transformations of $\mathbb{P}^2_{\mathbb{C}}$ acts on the space $\mathbb{F}(2)$ of holomorphic foliations on the complex projective plane. Since this action is not compatible with the natural graduation of $\mathbb{F}(2)$ by the degree, its description is complicated. The fixed points of the action are essentially described by Cantat-Favre in [3]. In that paper we are interested in problems of "aberration of the degree" that is pairs $(\phi, \mathcal{F}) \in \text{Bir}(\mathbb{P}^2_{\mathbb{C}}) \times \mathbb{F}(2)$ for which $\deg \phi^* \mathcal{F} < (\deg \mathcal{F} + 1)\deg \phi + \deg \phi - 2$, the generic degree of such pull-back. We introduce the notion of numerical invariance ($\deg \phi^* \mathcal{F} = \deg \mathcal{F}$) and relate it in small degrees to the existence of transversal structure for the considered foliations.

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1. Introduction

Let us consider on the complex projective plane $\mathbb{P}^2_{\mathbb{C}}$ a foliation $\mathcal{F}$ of degree $d$ and a birational map $\phi$ of degree $k$. If the pair $(\mathcal{F}, \phi)$ is generic then

$$\deg \phi^* \mathcal{F} = (d + 1)k + k - 2.$$ 

For example if $\mathcal{F}$ and $\phi$ are both of degree 2, then $\phi^* \mathcal{F}$ is of degree 6. Nevertheless one has the following statement which says that "aberration of the degree" is not exceptional:

Theorem A. — For any foliation $\mathcal{F}$ of degree 2 on $\mathbb{P}^2_{\mathbb{C}}$, there exists a quadratic birational map $\psi$ of $\mathbb{P}^2_{\mathbb{C}}$ such that $\deg \psi^* \mathcal{F} \leq 3$.

Holomorphic singular foliations on compact complex projective surfaces have been classified up to birational equivalence by Brunella, McQuillan and Mendes ([11]). Let $\mathcal{F}$ be a holomorphic singular foliation on a compact complex projective surface $S$. Let Bir($\mathcal{F}$) (resp. Aut($\mathcal{F}$)) denote the group of birational (resp. biholomorphic) maps of $S$ that send leaf to leaf. If $\mathcal{F}$ is of general type, then Bir($\mathcal{F}$) = Aut($\mathcal{F}$) is a finite group. In [3] Cantat and Favre classify the pairs $(S, \mathcal{F})$ for which Bir($\mathcal{F}$) (resp. Aut($\mathcal{F}$)) is infinite; in the case of $\mathbb{P}^2_{\mathbb{C}}$ such foliations are given by closed rational 1-forms.

In this article we introduce a weaker notion: the numerical invariance. We consider on $\mathbb{P}^2_{\mathbb{C}}$ a pair $(\mathcal{F}, \phi)$ of a foliation $\mathcal{F}$ of degree $d$ and a birational map $\phi$ of degree $k \geq 2$. The foliation $\mathcal{F}$ is **numerically invariant**
under the action of $\phi$ if $\deg \phi^* F = \deg F$. We characterize such pairs $(F, \phi)$ with $\deg F = \deg \phi = 2$ which is the first degree with deep (algebraic and dynamical) phenomena, both for foliations and birational maps. We prove that a numerically invariant foliation under the action of a generic quadratic map is special:

**Theorem B.** — Let $F$ be a foliation of degree 2 on $\mathbb{P}_\mathbb{C}^2$ numerically invariant under the action of a generic quadratic birational map of $\mathbb{P}_\mathbb{C}^2$. Then $F$ is transversely projective.

In that statement generic means outside an hypersurface in the space $\text{Bir}_2$ of quadratic birational maps of $\mathbb{P}_\mathbb{C}^2$.

For any quadratic birational map $\phi$ of $\mathbb{P}_\mathbb{C}^2$ there exists at least one foliation of degree 2 on $\mathbb{P}_\mathbb{C}^2$ numerically invariant under the action of $\phi$ and we give "normal forms" for such foliations. We don’t know if the foliations numerically invariant under the action of a non-generic quadratic birational map have a special transversal structure. Problem: for any birational map $\phi$ of degree $d \geq 3$, does there exist a foliation numerically invariant under the action of $\phi$?

A foliation $F$ on $\mathbb{P}_\mathbb{C}^2$ is primitive if $\deg F \leq \deg \phi^* F$ for any birational map $\phi$. Foliations of degree 0 and 1 are defined by a rational closed 1-form (it is a well-known fact, see for example [2]). Hence a non-primitive foliation of degree 2 is also defined by a closed 1-form that is a very special case of transversely projective foliations. Generically a foliation of degree 2 is primitive. Remark that there are foliations that are pull-back by a rational map of degree greater than 1, and that are nevertheless primitive. This is the case of the foliation given by $Q_1 dQ_2 - Q_2 dQ_1$ where $Q_1$ and $Q_2$ denote two generic polynomials of degree 3, in other words a generic pencil of elliptic curves. The following problem seems relevant: classify in any degree the primitive foliations numerically invariant under the action of birational maps of degree $\geq 2$; are such foliations transversely projective or is this situation specific to the degree 2 ? In this vein we get the following statement.

**Theorem C.** — A foliation $F$ of degree 2 on $\mathbb{P}_\mathbb{C}^2$ numerically invariant under the action of a generic cubic birational map of $\mathbb{P}_\mathbb{C}^2$ satisfies the following properties:

- $F$ is given by a closed rational 1-form (Liouvillean integrability);
- $F$ is non-primitive.

Is it a general fact, i.e. if $F$ is numerically invariant under the action of $\phi$ and $\deg \phi \gg \deg F$ is $F$ Liouvillean integrable?

The text is organized as follows: we first give some definitions, notations and properties of birational maps of $\mathbb{P}_\mathbb{C}^2$ and foliations on $\mathbb{P}_\mathbb{C}^2$. In §3 we give a proof of Theorem A. we focus on foliations of degree 2 on $\mathbb{P}_\mathbb{C}^2$ that have at least two singular points, and then on foliations of degree 2 on $\mathbb{P}_\mathbb{C}^2$ with exactly one singular point. The section 4 is devoted to the description of foliations of degree 2 on $\mathbb{P}_\mathbb{C}^2$ numerically invariant under the action of any quadratic birational map. This allows us to prove Theorem B. At the end of the paper, §5 we describe the foliations of degree 2 numerically invariant under some cubic birational maps of $\mathbb{P}_\mathbb{C}^2$, and we finally establish Theorem C.

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2. Some definitions, notations and properties

2.1. About birational maps of $\mathbb{P}^2 \mathbb{C}$. — A rational map $\phi$ of $\mathbb{P}^2 \mathbb{C}$ is a "map" of the type

$$\phi : \mathbb{P}^2 \mathbb{C} \dashrightarrow \mathbb{P}^2 \mathbb{C}, \quad (x : y : z) \mapsto (\phi_0(x, y, z) : \phi_1(x, y, z) : \phi_2(x, y, z))$$

where the $\phi_i$'s are homogeneous polynomials of the same degree and without common factor. The degree of $\phi$ is by definition the degree of the $\phi_i$'s. A birational map $\phi$ of $\mathbb{P}^2 \mathbb{C}$ is a rational map having a rational "inverse" $\psi$, i.e. $\phi \circ \psi = \psi \circ \phi = \text{id}$. The first examples are the birational maps of degree 1 which generate the group $\text{Aut}(\mathbb{P}^2 \mathbb{C}) = \text{PGL}(3, \mathbb{C})$. Let us give some examples of quadratic birational maps:

$$\sigma : (x : y : z) \mapsto (yz : xz : xy), \quad \rho : (x : y : z) \mapsto (xy : z^2 : yz),$$

$$\tau : (x : y : z) \mapsto (x^2 : xy : y^2 - xz).$$

These three maps, which are involutions, play an important role in the description of the set of quadratic birational maps of $\mathbb{P}^2 \mathbb{C}$.

The birational maps of $\mathbb{P}^2 \mathbb{C}$ form a group denoted $\text{Bir}(\mathbb{P}^2 \mathbb{C})$ and called Cremona group. If $\phi$ is an element of $\text{Bir}(\mathbb{P}^2 \mathbb{C})$ then $\mathcal{O}(\phi)$ is the orbit of $\phi$ under the action of $\text{Aut}(\mathbb{P}^2 \mathbb{C}) \times \text{Aut}(\mathbb{P}^2 \mathbb{C})$:

$$\mathcal{O}(\phi) = \{ \ell \phi \ell' | \ell, \ell' \in \text{Aut}(\mathbb{P}^2 \mathbb{C}) \}.$$ A very old theorem, often called Noether Theorem, says that any element of $\text{Bir}(\mathbb{P}^2 \mathbb{C})$ can be written, up to the action of an automorphism of $\mathbb{P}^2 \mathbb{C}$, as a composition of quadratic birational maps ([4]). In [5] Chapters 1 & 6 the structure of the set $\text{Bir}_d$ (resp. $\text{Bir}_d$) of birational maps of $\mathbb{P}^2 \mathbb{C}$ of degree $\leq d$ (resp. of degree $d$) has been studied when $d = 2$ and $d = 3$.

**Theorem 2.1 (Corollary 1.10, Theorem 1.31, [5]).** — One has the following decomposition

$$\text{Bir}_2 = \mathcal{O}(\sigma) \cup \mathcal{O}(\rho) \cup \mathcal{O}(\tau).$$

Furthermore

$$\text{Bir}_2 = \overline{\mathcal{O}(\sigma)}$$

where $\overline{\mathcal{O}(\sigma)}$ denotes the ordinary closure of $\mathcal{O}(\sigma)$, and

$$\dim \mathcal{O}(\tau) = 12, \quad \dim \mathcal{O}(\rho) = 13, \quad \dim \mathcal{O}(\sigma) = 14.$$

Note that there is a more precise description of $\text{Bir}_2$ in [5], Chapter 1.

We will further do some computations with birational maps of degree 3. Let us consider the following family of cubic birational maps:

$$\Phi_{a,b} : (x : y : z) \mapsto (x(x^2 + y^2 + axy + bxz + yz) : y(x^2 + y^2 + axy + bxz + yz) : xyz)$$

with $a, b \in \mathbb{C}$, $a^2 \neq 4$ and $2b \notin \{a \pm \sqrt{a^2 - 4}\}$. The structure of $\text{Bir}_3$ is much more complicated than the structure of $\text{Bir}_2$ (see [5] Chapter 6), nevertheless one has the following result.

**Theorem 2.2 (Proposition 6.35, Theorem 6.38, [5]).** — The closure of

$$\mathcal{X} = \{ \mathcal{O}(\Phi_{a,b}) | a, b \in \mathbb{C}, a^2 \neq 4, 2b \notin \{a \pm \sqrt{a^2 - 4}\} \}$$

in the set of rational maps of degree 3 is an irreducible algebraic variety of dimension 18. Furthermore the closure of $\mathcal{X}$ in $\text{Bir}_3$ is $\text{Bir}_3$. 

In the sequel we will say that a $\Phi_{a,b}$ is a generic element of Bir$_3$. The "most degenerate model" is up to automorphisms of $\mathbb{P}^2_\mathbb{C}$

$$\Psi: (x : y : z) \rightarrow (xz^2 + y^3 : yz^2 : z^3).$$

2.2. About foliations. —

**Definition 2.3.** Let $\mathcal{F}$ be a foliation (maybe singular) on a complex manifold $M$; the foliation $\mathcal{F}$ is a **singular transversely projective** one if there exists

a) $\pi: P \rightarrow M$ a $\mathbb{P}^1$-bundle over $M$,

b) $\mathcal{G}$ a codimension one singular holomorphic foliation on $P$ transversal to the generic fibers of $\pi$,

c) $\varsigma: M \rightarrow P$ a meromorphic section generically transverse to $\mathcal{G}$, such that $\mathcal{F} = \varsigma^* \mathcal{G}$.

Let us give another characterization of singular transversely projective foliations. Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2_\mathbb{C}$; assume that there exist three rational 1-forms $\theta_0, \theta_1$ and $\theta_2$ on $\mathbb{P}^2_\mathbb{C}$ such that

i) $\mathcal{F}$ is described by $\theta_0$, i.e. $\mathcal{F} = \mathcal{F}_{\theta_0}$,

ii) the $\theta_i$'s form a $\text{sl}(2; \mathbb{C})$-triplet, that is

$$d\theta_0 = \theta_0 \wedge \theta_1, \quad d\theta_1 = \theta_0 \wedge \theta_2, \quad d\theta_2 = \theta_1 \wedge \theta_2.$$ 

Then $\mathcal{F}$ is a singular transversely projective foliation. To see it one considers the manifolds $M = \mathbb{P}^2_\mathbb{C}$, $P = \mathbb{P}^2_\mathbb{C} \times \mathbb{P}^1_\mathbb{C}$, the canonical projection $\pi: P \rightarrow M$, and the foliation $\mathcal{G}$ given by

$$\theta = dz + \theta_0 + z\theta_1 + \frac{z^2}{2} \theta_2$$

where $z$ is an affine coordinate of $\mathbb{P}^1_\mathbb{C}$; in that case the transverse section is $z = 0$. When one can choose the $\theta_i$'s such that $\theta_1 = \theta_2 = 0$ (resp. $\theta_2 = 0$) the foliation $\mathcal{F}$ is **defined by a closed 1-form** (resp. is **transversely affine**).

Classical examples of singular transversely projective foliations are given by Riccati foliations.

**Definition 2.4.** A **Riccati equation** is a differential equation of the following type

$$\mathcal{E}_R: \ y' = a(x)y^2 + b(x)y + c(x)$$

where $a$, $b$ and $c$ are meromorphic functions on an open subset $\mathcal{U}$ of $\mathbb{C}$. To the equation $\mathcal{E}_R$ one associates the meromorphic differential form

$$\omega_{\mathcal{E}_R} = dy - (a(x)y^2 + b(x)y + c(x)) \, dx$$

defined on $\mathcal{U} \times \mathbb{C}$. In fact $\omega_{\mathcal{E}_R}$ induces a foliation $\mathcal{F}_{\omega_{\mathcal{E}_R}}$ on $\mathcal{U} \times \mathbb{P}^1_\mathbb{C}$ that is transverse to the generic fiber of the projection $\mathcal{U} \times \mathbb{P}^1_\mathbb{C} \rightarrow \mathcal{U}$. One can check that

$$\theta_0 = \omega_{\mathcal{E}_R}, \quad \theta_1 = -(2a(x)y + b(x)) \, dx, \quad \theta_2 = -2a(x) \, dx$$

is a $\text{sl}(2; \mathbb{C})$-triplet associated to the foliation $\mathcal{F}_{\omega_{\mathcal{E}_R}}$.

We say that $\omega_{\mathcal{E}_R}$ is a **Riccati 1-form** and $\mathcal{F}_{\omega_{\mathcal{E}_R}}$ is a **Riccati foliation**.

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1. In the following sense: the exceptional locus of any element of Bir$_3$ is a union of degree 6 of conics and lines; the exceptional locus of $\Psi$ is reduced to a single line of multiplicity 6.
Let $S$ be a ruled surface, that is a surface $S$ endowed with $f : S \to C$, where $C$ denotes a curve and $f^{-1}(c) \simeq \mathbb{P}^1_C$. Let us consider a singular foliation $\mathcal{F}$ on $S$ transverse to the generic fibers of $f$. The foliation $\mathcal{F}$ is transversely projective.

Recall that a foliation $\mathcal{F}$ on a surface $S$ is radial at a point $m$ of $S$ if in local coordinates $(x, y)$ around $m$ the foliation $\mathcal{F}$ is given by a holomorphic 1-form of the following type

$$\omega = x \, dy - y \, dx + \text{h.o.t.}$$

Let us denote by $\mathcal{F}(n; d)$ the set of foliations of degree $d$ on $\mathbb{P}^n_C$ (see [2]). The following statement gives a criterion which asserts that an element of $\mathcal{F}(2; 2)$ is transversely projective.

**Proposition 2.5.** — Let $\mathcal{F} \in \mathcal{F}(2; 2)$ be a foliation of degree 2 on $\mathbb{P}^2_C$. If a singular point of $\mathcal{F}$ is radial, then $\mathcal{F}$ is transversely projective.

**Proof.** — Assume that the singular point is the origin 0 in the affine chart $z = 1$, the foliation $\mathcal{F}$ is thus defined by a 1-form of the following type

$$\omega = x \, dy - y \, dx + q_1 \, dx + q_2 \, dy + q_3 (x \, dy - y \, dx)$$

where the $q_i$’s denote quadratic forms. Let us consider the complex projective plane $\mathbb{P}^2_C$ blown up at the origin; this space is denoted by $\text{Bl}(\mathbb{P}^2_C, 0)$. Let

$$\pi : \text{Bl}(\mathbb{P}^2_C, 0) \to \mathbb{P}^2_C$$

be the canonical projection. Then $\pi^* \mathcal{F}$ is transverse to the generic fibers of $\pi$, and in fact transverse to all the fibers excepted the strict transforms of the lines $xq_1 + yq_2 = 0$. Hence the foliation $\pi^* \mathcal{F}$ is transversely projective; since this notion is invariant under the action of a birational map, $\mathcal{F}$ is transversely projective. \( \square \)

**Remark 2.6.** — The same argument can be involved for foliations of degree 2 on $\mathbb{P}^2_C$ having a singular point with zero 1-jet.

**Remark 2.7.** — The closure of the set $\Delta_R$ of foliations in $\mathcal{F}(2; 2)$ having a radial singular point is irreducible, of codimension 2 in $\mathcal{F}(2; 2)$. Indeed $\Delta_R$ is the $\text{Aut}(\mathbb{P}^2_C)$-orbit of the set

$$\left\{ x \, dy - y \, dx + q_1 \, dx + q_2 \, dy + q_3 (x \, dy - y \, dx) \mid q_i \text{ quadratic form} \right\};$$

in fact it is easy to see that $\overline{\Delta_R}$ is the $\text{Aut}(\mathbb{P}^2_C)$-orbit of

$$\left\{ \lambda (x \, dy - y \, dx) + q_1 \, dx + q_2 \, dy + q_3 (x \, dy - y \, dx) \mid \lambda \in \mathbb{C}, q_i \text{ quadratic form} \right\}.$$

In particular $\overline{\Delta_R}$ is an unirational set in $\mathcal{F}(2; 2)$.

### 3. Proof of Theorem

We establish Theorem $\square$ in two steps: we first look at foliations that have at least two singular points and then at foliations with exactly one singular point.
3.1. Foliations of degree 2 on $\mathbb{P}^2_C$ with at least two singularities. — Any $\mathcal{F} \in \mathbb{F}(2; 2)$ is described in homogeneous coordinates by a 1-form $\omega$ that can be written
\[
\omega = q_1yz \left( \frac{dy}{y} - \frac{dz}{z} \right) + q_2xz \left( \frac{dz}{z} - \frac{dx}{x} \right) + q_3xy \left( \frac{dx}{x} - \frac{dy}{y} \right)
\] (3.1)
where
\[
q_1 = a_2x^2 + a_1y^2 + a_2z^2 + a_3xy + a_4xz + a_5yz,
q_2 = b_2x^2 + b_1y^2 + b_2z^2 + b_3xy + b_4xz + b_5yz,
q_3 = c_0x^2 + c_1y^2 + c_2z^2 + c_3xy + c_4xz + c_5yz.
\]

**Proposition 3.1.** — For any $\mathcal{F} \in \mathbb{F}(2; 2)$ with at least two distinct singularities there exists a quadratic birational map $\psi \in \mathcal{O}(\rho)$ such that $\deg \psi^* \mathcal{F} \leq 3$.

**Proof.** — In homogeneous coordinates $\mathcal{F}$ is described by a 1-form $\omega$ as in (3.1).

Up to an automorphism of $\mathbb{P}^2_C$, one can suppose that $(1 : 0 : 0)$ and $(0 : 1 : 0)$ are singular points of $\mathcal{F}$, that is $a_1 = b_2 = c_0 = c_1 = 0$. If $c_3 \neq 0$, resp. $c_3 = 0$ and $b_4 \neq 0$, resp. $c_3 = b_4 = 0$, then let us consider the quadratic birational map $\psi \in \mathcal{O}(\rho)$ defined as follows
\[
\psi: (x : y : z) \rightarrow \left( xy : z^2 + \frac{3b - c_4 + \sqrt{(b_3 - c_4)^2 + 4b_4c_3}}{2c_3} yz : yz \right),
\]
resp.
\[
\psi: (x : y : z) \rightarrow \left( xy : z^2 + \frac{b_3 - c_4}{b_4} yz \right),
\]
resp. $\psi = \rho$. A direct computation shows that $\psi^* \omega = yz^2 \omega'$ where $\omega'$ denotes a homogeneous 1-form of degree 4. The foliation $\mathcal{F}'$ defined by $\omega'$ has degree at most 3. \qed

3.2. Foliations of degree 2 on $\mathbb{P}^2_C$ with exactly one singularity. — Such foliations have been classified:

**Theorem 3.2 (§6).** — Up to automorphisms of $\mathbb{P}^2_C$ there are four foliations of degree 2 on $\mathbb{P}^2_C$ having exactly one singularity. They are described in affine chart by the following 1-forms:
- $\Omega_1 = x^2dx + y^2(xdy - ydx)$,
- $\Omega_2 = x^2dx + (x + y^2)(xdy - ydx)$,
- $\Omega_3 = xydx + (x^2 + y^2)(xdy - ydx)$,
- $\Omega_4 = (x + y^2 - x^2y)dy + x(x + y^2)dx$.

**Proposition 3.3.** — There exists a quadratic birational map $\psi_1 \in \mathcal{O}(\rho)$ such that $\deg \psi_1^* \Omega_1 = 2$; furthermore $\mathcal{F}_{\Omega_1}$ has a rational first integral and is non-primitive.

For $k = 2, 3$, there is no birational map $\varphi_k$ such that $\deg \varphi_k^* \mathcal{F}_{\Omega_1} = 0$ but there is $\psi_k$ in $\mathcal{O}(\tau)$ such that $\deg \psi_k^* \mathcal{F}_{\Omega_1} = 1$. In particular $\mathcal{F}_{\Omega_2}$ and $\mathcal{F}_{\Omega_3}$ are non-primitive.

There exists a birational map $\psi_4 \in \mathcal{O}(\tau)$ such that $\deg \psi_4^* \mathcal{F}_{\Omega_4} = 3$, and $\mathcal{F}_{\Omega_4}$ is primitive.

**Remark 3.4.** — If $\phi = (x^2 : xy : xz + y^2)$, then $\deg \phi^* \mathcal{F}_{\Omega_2} = \deg \phi^* \mathcal{F}_{\Omega_3} = 2$. A contrario we will see later there is no quadratic birational map $\phi$ such that $\deg \phi^* \mathcal{F}_{\Omega_4} = 2$ (see Corollary 4.15).

**Corollary 3.5.** — For any element $\mathcal{F}$ of $\mathbb{F}(2; 2)$ with exactly one singularity there exists a quadratic birational map $\psi$ such that $\deg \psi^* \mathcal{F} \leq 3$. 
Proof of Proposition 3.3 — The foliation $F_{\Omega_1}$ is given in homogeneous coordinates by

$$\Omega_1' = (x^2z - y^3) \, dx + xy^2 \, dy - y^3 \, dz;$$

if $\psi_1: (x : y : z) \to (x^2 : xy : yz)$ then

$$\psi_1^* \Omega_1' \wedge (y(2xz - y^2) \, dx + x(y^2 - xz) \, dy - x^2y \, dz) = 0.$$

The foliation $F_{\Omega_1}$ has a rational first integral and is non-primitive, it is the image of a foliation of degree 0 by a cubic birational map:

$$(x^3 : x^2y : x^2z + y^3/3) \Omega_1' \wedge (z \, dx - x \, dz) = 0.$$

The foliation $F_{\Omega_2}$ is described in homogeneous coordinates by

$$\Omega_2' = (x^2z - x^2y - y^3) \, dx + x(xz + y^2) \, dy - x^3 \, dz;$$

let us consider the birational map $\psi_2: (x : y : z) \to (x^2 : xy : xz - 2x^2 - 2xy - y^2)$ then

$$\psi_2^* \Omega_2' \wedge ((xz - yz) \, dx + xz \, dy - x^2 \, dz) = 0.$$

One can verify that

$$\left(2 + \frac{1}{x} + 2 \frac{y}{x^2} \right) \exp \left(-\frac{y}{x} \right)$$

is a first integral of $F_{\Omega_2}$; it is easy to see that $F_{\Omega_2}$ has no rational first integral so there is no birational map $\phi_2$ such that $\deg \phi_2^* F_{\Omega_2} = 0$.

The foliation $F_{\Omega_3}$ is given in homogeneous coordinates by the 1-form

$$\Omega_3' = y(xz - x^2 - y^2) \, dx + x(x^2 + y^2) \, dy - x^2y \, dz;$$

if $\psi_3: (x : y : z) \to (x^2 : xy : xz + y^2/2)$ then

$$\psi_3^* \Omega_3' \wedge (y(z - x) \, dx + x^2 \, dy - xy \, dz) = 0.$$

The function

$$\left(\frac{y}{x} \right) \exp \left(\frac{1}{2} \frac{y^2}{x^2} - \frac{1}{x} \right)$$

is a first integral of $F_{\Omega_3}$, and $F_{\Omega_3}$ has no rational first integral so there is no birational map $\phi_3$ such that $\deg \phi_3^* F_{\Omega_3} = 0$.

Let us consider the birational map of $\mathbb{P}^2_{\mathbb{C}}$ given by

$$\psi_4: (x : y : z) \to (-x^2 : xy : y^2 - xz)$$

In homogeneous coordinates

$$\Omega_4' = x(xz + y^2) \, dx + (xz^2 + x^2y - x^2y) \, dy + (xyz - y^3 - x^3) \, dz;$$

a direct computation shows that $\psi_4^* \Omega_4' \wedge \eta = 0$ where

$$\eta = (3y^3z - x^3y^2 + x^3z - 2xyz^2) \, dx + (x^3y - 4y^4 - x^2z^2 + 3xy^2z) \, dy + x(2y^3 - x^3 - xyz) \, dz.$$
4. Numerical invariance

In the sequel num. inv. means numerically invariant.

In this section we determine the foliations \( F \) of \( \mathbb{P}(2; 2) \) num. inv. under the action of \( \sigma \) (resp. \( \rho \), resp. \( \tau \)). Note that if \( \phi \) is a birational map of \( \mathbb{P}^2_\mathbb{C} \) and \( \ell \) an element of \( \text{Aut}(\mathbb{P}^2_\mathbb{C}) \) then \( \text{deg}(\phi \ell)^*F = \text{deg}\phi^*F \); hence following Theorem 2.1 we get the description of foliations num. inv. under the action of a quadratic birational map of \( \mathbb{P}^2_\mathbb{C} \) by giving normal forms.

Recall that \( \sigma \) is given in a fixed system of homogeneous coordinates \((x : y : z)\) by

\[
\sigma: (x : y : z) \rightarrow (yz : xz : xy),
\]

and remark that \( \sigma \) is invariant under conjugacy by elements of the group \( S_3 \) of standard permutations of coordinates.

**Lemma 4.1.** — An element \( \sigma \) of \( \mathbb{P}(2; 2) \) is num. inv. under the action of \( \sigma \) if and only if it is given up to permutations of coordinates and standard affine charts by 1-forms of the following type

- either \( \omega_1 = y(\kappa + \varepsilon y) \, dx + (\beta x + \delta y + \alpha x^2 + \gamma xy) \, dy \),
- or \( \omega_2 = (\delta + \beta y + \kappa y^2) \, dx + (\alpha + \varepsilon x + \gamma x^2) \, dy \),

where \( \alpha, \beta, \gamma, \delta, \varepsilon, \kappa \) (resp. \( \alpha, \beta, \gamma, \delta, \varepsilon, \kappa \)) are complex numbers such that \( \text{deg} \, \omega_1 = 2 \) (resp. \( \text{deg} \, \omega_2 = 2 \)).

**Proof.** — The foliation \( \sigma \) is defined by a homogeneous 1-form \( \omega \) of degree 3. The map \( \sigma \) is an automorphism of \( \mathbb{P}^2_\mathbb{C} \setminus \{xyz = 0\} \); hence if \( \sigma^* \omega = P \omega' \), with \( \omega' \) a 1-form of degree 3 and \( P \) a homogeneous polynomial, then \( P = x^iy^jz^k \) for some integers \( i, j, k \) such that \( i + j + k = 4 \). Up to permutation of coordinates it is sufficient to look at the four following cases: \( P = x^3i, P = x^3y, P = x^2y^2 \) and \( P = x^2yz \). Let us write \( \omega \) as in (3.1).

Computations show that \( x^3 \) (resp. \( x^3y \)) cannot divide \( \sigma^* \omega \). If \( P = x^2yz \), then \( \sigma^* \omega = P \omega' \) if and only if

\[
c_0 = b_0 = a_2 = b_2 = a_1 = c_1 = b_4 = c_3 = 0, \quad b_3 = c_4
\]

that gives \( \omega_1 \). Finally one has \( \sigma^* \omega = x^2y^2 \omega' \) if and only if

\[
c_1 = c_0 = b_0 = a_1 = b_4 = c_3 = a_5 = 0, \quad b_3 = c_4, \quad c_5 = a_3;
\]

in that case we obtain \( \omega_2 \). \( \square \)

**Proposition 4.2.** — A foliation \( F \in \mathbb{P}(2; 2) \) num. inv. under the action of an element of \( \mathcal{O}(\sigma) \) is \( \text{Aut}(\mathbb{P}^2_\mathbb{C}) \)-conjugate either to a foliation of type \( F_{\omega_1} \), or to a foliation of type \( F_{\omega_2} \). In particular it is transversely projective.

**Proof.** — Let \( \phi \) be an element of \( \mathcal{O}(\sigma) \) such that \( \text{deg} \phi^* F = 2 \); the map \( \phi \) can be written \( \ell_1 \sigma \ell_2 \) where \( \ell_1 \) and \( \ell_2 \) denote automorphisms of \( \mathbb{P}^2_\mathbb{C} \). By assumption the degree of \( (\ell_1 \sigma \ell_2)^* F = \ell_1^2 (\sigma^*(\ell_1^* F)) \) is 2. Hence \( \text{deg} \sigma^*(\ell_1^* F) = 2 \) and the foliation \( \ell_1^* F \) is num. inv. under the action of \( \sigma \). Since \( \ell_1 F \) and \( F \) are conjugate and since the notion of transversal projectivity is invariant by conjugacy it is sufficient to establish the statement for \( \phi = \sigma \). The proposition thus follows from the fact that 1-forms of Lemma 4.1 are Riccati ones (up to multiplication). \( \square \)

**Remark 4.3.** — For generic values of parameters \( \alpha, \beta, \gamma, \delta, \varepsilon, \kappa \) a foliation of type \( F_{\omega_1} \) given by the corresponding form \( \omega_1 \) is not given by a closed meromorphic 1-form. This can be seen by studying the holonomy group of \( F_{\omega_1} \) that can be identified with a subgroup of \( \text{PGL}(2; \mathbb{C}) \) generated by two elements \( f \) and \( g \). For generic values of the parameters \( f \) and \( g \) are also generic, in particular the group \( \langle f, g \rangle \) is free. When \( F_{\omega_1} \) is given by a closed 1-form, then the holonomy group is an abelian one.

Remark that a contrario the foliations given by 1-forms of type \( \omega_2 \) are given by a closed meromorphic 1-form.
Remark 4.4. — Let \( \Delta_1 \) denote the closure of the set of elements of \( \mathbb{F}(2;2) \) conjugate to a foliation of type \( F_{\omega_0} \).

In the affine chart \( x = 1 \) an element of type \( F_{\omega_0} \) is radial at \( (0,0) \) as soon as \( \alpha \neq 0 \). Since \( \Delta_R \) is closed, the inclusion \( \Delta_1 \subset \Delta_R \) holds.

If the components of \( \omega_2 \) are not constant, then an element of type \( F_{\omega_0} \) has a singular point in \( \mathbb{C}^2 \), and up to an ad-hoc translation \( F_{\omega_2} \) is an element of type \( F_{\omega_0} \). As \( \Delta_1 \) is closed, one has \( \Delta_2 \subset \Delta_1 \).

Remark 4.5. — The notion of num. inv. is not related to the dynamic of the map (see \([3]\) for example): the foliations num. inv. by the involution \( \sigma \) ("without dynamic") are conjugate to the foliations num. inv. by \( A\sigma \), \( A \in \text{Aut}(\mathbb{P}^2) \), which has a rich dynamic for generic \( A \).

The foliations of \( \mathbb{F}(2;2) \) invariant by \( \sigma \) are particular cases of num. inv. foliations:

Proposition 4.6. — An element of \( \mathbb{F}(2;2) \) invariant by \( \sigma \) is given up to permutations of coordinates and affine charts

- either by \( y(1+y)dx + (\beta x + \alpha y + \alpha^2 + \beta xy)dy \),
- or by \( y(1-y)dx + (\beta x - \alpha y + \alpha^2 - \beta xy)dy \),
- or by \( ydx + (\alpha + \epsilon x + \alpha^2)dy \),

where the parameters are complex numbers such that the degree of the associated foliations is 2.

Proof. — With the notations of Lemma 4.1 one has

\[
\sigma^* \omega_1 = -y(\epsilon + \kappa y)dx - (\gamma x + \alpha y + \delta x^2 + \beta xy)dy;
\]

thus \( \sigma^* \omega_1 \wedge \omega_1 = 0 \) if and only if either \( \gamma = \beta, \delta = \alpha, \epsilon = \kappa \) or \( \gamma = -\beta, \delta = -\alpha, \epsilon = -\kappa \).

One has \( \sigma^* \omega_2 = -(\kappa + \beta y + \delta y^2)dx - (\gamma + \epsilon x + \alpha^2)dy \), and \( \omega_2 \wedge \sigma^* \omega_2 = 0 \) if and only if \( \gamma = \alpha, \delta = 0 \) and \( \kappa = 0 \).

Remark 4.7. — The foliations associated to the two first 1-forms with parameters \( \alpha, \beta \) of Proposition 4.6 are conjugate by the automorphism \((x, y) \mapsto (x, -y)\).

Lemma 4.8. — A foliation \( F \in \mathbb{F}(2;2) \) is num. inv. under the action of \( \rho \) if and only if \( F \) is given in affine chart

- either by \( \omega_3 = y(\kappa + \epsilon y + \lambda y^2)dx + (\beta + \kappa x + \delta y + \gamma xy + \alpha^2 - \lambda xy^2)dy \),
- or by \( \omega_4 = y(\mu + \delta x + \epsilon y + \epsilon xy)dx + (\alpha + \beta x + \lambda y + \delta x^2 + \kappa xy - \epsilon x^2 y)dy \),
- or by \( \omega_5 = (\lambda + \gamma + \kappa xy + \epsilon y^2)dx + (\beta + \delta x + \alpha^2)dy \),

where the parameters of \( \omega_i \) are such that \( \deg F_{\omega_i} = 2 \).

Proof. — Let us take the notations of the proof of Lemma 4.1. The map \( \rho \) is an automorphism of \( \mathbb{P}^2 \setminus \{yz = 0\} \); therefore if \( \rho^* \omega = P\omega' \) with \( \omega' \) a 1-form of degree 3 and \( P \) a homogeneous polynomial then \( P = y^j z^k \) for some integers \( j, k \) such that \( j + k = 4 \). We have to look at the five following cases: \( P = z^4 \), \( P = yz^3 \), \( P = y^2 z^2 \), \( P = y^3 z \) and \( P = y^4 \). Computations show that \( y^4 \) (resp. \( y^3 z \)) cannot divide \( \rho^* \omega \). If \( P = z^4 \) then \( \rho^* \omega = P\omega' \) if and only if

\[
c_0 = b_0 = c_3 = b_4 = b_2 = 0, \quad a_0 = c_4, \quad b_3 = c_4, \quad a_4 = 2c_2 - b_5;
\]

this gives the first case \( \omega_3 \). The equality \( \rho^* \omega = yz^3 \omega' \) holds if and only if

\[
b_0 = c_0 = b_4 = c_1 = a_1 = b_2 = 0, \quad a_0 = 2c_4 - b_3
\]

and we obtain \( \omega_4 \). Finally one has \( \rho^* \omega = y^2 z^2 \omega' \) if and only if

\[
c_1 = b_0 = c_3 = a_5 = a_1 = c_0 = b_4 = 0, \quad c_5 = a_3
\]
which corresponds to $\omega_5$. 

Proposition 4.9. — The foliations of type $\mathcal{F}_{\omega_3}$ and $\mathcal{F}_{\omega_5}$ are transversely projective. In fact the $\mathcal{F}_{\omega_3}$ are transversely affine, and the $\mathcal{F}_{\omega_5}$ are Riccati ones.

Proof. — A foliation of type $\mathcal{F}_{\omega_3}$ is described by the 1-form

$$\theta_0 = dx - \frac{(\beta + \delta y + \alpha y^2) + (\kappa + \gamma y - \lambda y^2)x}{y(\kappa + \epsilon y + \lambda y^2)} dy$$

and it is transversely affine; to see it consider the $sl(2;\mathbb{C})$-triplet

$$\theta_0, \quad \theta_1 = \frac{\kappa + \gamma y - \lambda y^2}{y(\kappa + \epsilon y + \lambda y^2)} dy, \quad \theta_2 = 0.$$

A foliation of type $\mathcal{F}_{\omega_5}$ is given by

$$dy + \frac{\lambda + (\gamma + \kappa)y + \epsilon y^2}{\beta + \delta x + \alpha x^2} dx$$

and thus is a Riccati foliation. In fact the fibration $x/z = \text{constant}$ is transverse to $\mathcal{F}_{\omega_5}$ that generically has three invariant lines.

We don’t know if the $\mathcal{F}_{\omega_i}$ are transversely projective. For generic values of the parameters a foliation of type $\mathcal{F}_{\omega_i}$ hasn’t meromorphic uniform first integral in the affine chart $y = 1$. Thus if $\mathcal{F}_{\omega_i}$ is transversely projective then it must have an invariant algebraic curve different from $y = 0$ (see [7]). We don’t know if it is the case. A foliation of degree 2 is conjugate to a generic $\mathcal{F}_{\omega_i}$ (by an automorphism of $\mathbb{P}^2_\mathbb{C}$) if and only if it has an invariant line (say $y = 0$) with a singular point (say 0) and local model $2x dy - y dx$. The closure of the set of such foliations has codimension 2. Note that the three families $\mathcal{F}_{\omega_0}, \mathcal{F}_{\omega_3}$ and $\mathcal{F}_{\omega_5}$ have non trivial intersection. The set $\{\mathcal{F}_{\omega_i}\}$ contains many interesting elements such that the famous Euler foliation given by $y^2 dx + (y - x) dy$; this foliation is transversely affine but is not given by a closed rational 1-form.

Proposition 4.10. — A foliation $\mathcal{F} \in \mathbb{P}(2;2)$ num. inv. under the action of an element of $\Theta(\rho)$ is conjugate to a foliation either of type $\mathcal{F}_{\omega_0}$, or of type $\mathcal{F}_{\omega_1}$, or of type $\mathcal{F}_{\omega_5}$.

Let us look at special num. inv. foliations, those invariant by $\rho$.

Proposition 4.11. — An element of $\mathbb{P}(2;2)$ invariant by $\rho$ is given by a 1-form of one of the following type

- $y(1-y) dx + (\beta + x) dy$,
- $y^2 dx + (\alpha x) dy$,
- $y(1-y)(y + \delta x) dx + (1+y)(\alpha + \beta x + \delta x^2) dy$,
- $y(1+y)(y + \delta x) dx + (1-y)(\alpha + \beta x + \delta x^2) dy$,
- $(1-y^2) dx + (\beta + \delta x + \alpha x^2) dy$,

where the parameters are complex numbers such that the degree of the associated foliations is 2.

Corollary 4.12. — An element of $\mathbb{P}(2;2)$ invariant by $\rho$ is defined by a closed 1-form.

Remark 4.13. — The third and fourth cases with parameters $\alpha, \beta, \gamma, \delta$ are conjugate by the automorphism $(x,y) \mapsto (x,-y)$.

From Lemmas [4.1] and [4.8] one gets the following statement.
Proposition 4.14. — A foliation num. inv. by an element of $\mathcal{O}(\phi)$, with $\phi = \sigma, \rho$, preserves an algebraic curve.

Corollary 4.15. — There is no quadratic birational map $\phi$ of the complex projective plane such that $\deg \phi^* \mathcal{F}_{\Omega_4} = 2$.

Proof. — The foliation $\mathcal{F}_{\Omega_4}$ has no invariant algebraic curve ([6 Proposition 1.3]); according to Proposition 4.14 it is thus sufficient to show that there is no birational map $\phi \in \mathcal{O}(\tau)$ such that $\deg \phi^* \mathcal{F}_{\Omega_4} = 2$ that can be established with a direct and tedious computation.

Remark 4.16. — The map $\rho$ can be written $\ell_1 \sigma \ell_2 \sigma \ell_3$ with

$$\ell_1 = (z - y : y - x : y), \quad \ell_2 = (y + z : z : x), \quad \ell_3 = (x + z : y - z : z).$$

We are interested by the "intermediate" degrees of a numerically invariant foliation $\mathcal{F}$, that is the sequence $\deg \mathcal{F}, \deg (\ell_1 \sigma)^* \mathcal{F}, \deg (\ell_1 \sigma \ell_2 \sigma \ell_3)^* \mathcal{F} = \deg \mathcal{F}$. A tedious computation shows that for generic values of the parameters the sequence is 2, 5, 2. We schematize this fact by the diagram

```
  5
 / \ /
/   \/
\   /\2
  \  /
    2
```

A similar argument to Lemma 4.1 yields to the following result.

Lemma 4.17. — An element $\mathcal{F}$ of $\mathbb{P}(2; 2)$ is num. inv. under the action of $\tau$ if and only if $\mathcal{F}$ is given in affine chart by a 1-form of type

$$\omega_0 = \left( - \delta x + \alpha y - \varepsilon x^2 + \theta xy + \beta y^2 + \kappa x y + \mu x y^2 + \lambda y^3 \right) dx + \left( - 3 \alpha x + \xi x^2 + 2(\delta - \beta)xy + \alpha y^2 - \kappa x^3 - \mu x^2 y - \lambda xy^2 \right) dy$$

where the parameters are such that $\deg \mathcal{F}_{\omega_0} = 2$.

We don’t know the qualitative description of foliations of type $\mathcal{F}_{\omega_0}$. For example we don’t know if the $\mathcal{F}_{\omega_0}$ are transversely projective. If it is the case, this implies the existence of invariant algebraic curves, and that fact is unknown.

Proposition 4.18. — A foliation $\mathcal{F} \in \mathbb{P}(2; 2)$ num. inv. under the action of an element of $\mathcal{O}(\tau)$ is conjugate to $\mathcal{F}_{\omega_0}$ for suitable values of the parameters.

Let us describe some special num. inv. foliations under the action of $\tau$, those invariant by $\tau$.

Proposition 4.19. — An element of $\mathbb{P}(2; 2)$ invariant by $\tau$ is given

- either by

$$\left( - \varepsilon x^2 + \theta xy + \beta y^2 + \varepsilon xy^2 - \left( \frac{\xi}{2} + \theta \right) y^3 \right) dx + x \left( \xi x - 2 \beta y - \varepsilon xy + \left( \frac{\xi}{2} + \theta \right) y^2 \right) dy,$$

- or by

$$\left( - \delta x + \alpha y + \frac{3}{2} \delta y^2 + \kappa x^2 y + \mu x y^2 + \lambda y^3 \right) dx - \left( 3 \alpha x + \delta xy - \alpha y^2 + \kappa x^3 + \mu x^2 y + \lambda xy^2 \right) dy,$$

where the parameters are complex numbers such that the degree of the associated foliations is 2.

The foliations associated to the first 1-form are transversely affine.
Proof. — The 1-jet at the origin of the 1-form
\[ \omega = (-\varepsilon x^2 + \Theta xy + \beta y^2 + \varepsilon xy^2 - (\frac{\xi}{2} + \Theta) y^3) \, dx + x(\xi x - 2\beta y - \varepsilon xy + (\frac{\xi}{2} + \Theta) y^2) \, dy \]
is zero so after one blow-up \( F_\omega \) is transverse to the generic fiber of the Hopf fibration; furthermore as the exceptional divisor is invariant, \( F_\omega \) is transversely affine. \( \square \)

Remark 5.20. — The map \( \tau \) can be written \( \ell_1 \sigma \ell_2 \sigma \ell_3 \sigma \ell_4 \sigma \ell_4 \) with
\[
\begin{align*}
\ell_1 &= (x - y : x - 2y : -x + y - z), \\
\ell_2 &= (x + z : x : y), \\
\ell_3 &= (-y : x - 3y + z : x), \\
\ell_4 &= (y - x : z - 2x : 2x - y).
\end{align*}
\]
Let us consider a foliation \( F \) num. inv. under the action of \( \tau \); set \( F' = \ell_1^* F \). We compute the intermediate degrees:
\[
\deg \sigma^* F' = 5, \quad \deg (\sigma \ell_2 \sigma)^* F' = 4, \quad \deg (\sigma \ell_3 \sigma \ell_2 \sigma)^* F' = 5.
\]
To summarize:

\[
\begin{array}{cccc}
5 & & 5 \\
& 4 & & \\
2 & & & 2
\end{array}
\]

5. Higher degree

We will now focus on similar questions but with cubic birational maps of \( \mathbb{P}^2_\mathbb{C} \) and elements of \( \mathfrak{F}(2; 2) \). The generic model of such birational maps is:
\[
\Phi_{a,b}: (x : y : z) \mapsto (x(x^2 + y^2 + axy + bxz + yz) : y(x^2 + y^2 + axy + bxz + yz) : xyz)
\]
with \( a, b \in \mathbb{C}, a^2 \neq 4 \) and \( 2b \notin \{a \pm \sqrt{a^2 - 4}\} \).

Lemma 5.1. — An element \( F \) of \( \mathfrak{F}(2; 2) \) is num. inv. under the action of \( \Phi_{a,b} \) if and only if \( F \) is given in affine chart
- either by \( \omega_7 = y(\alpha + \gamma y) \, dx - x(\alpha + \kappa x) \, dy \),
- or by
\[
\omega_8 = b(b^2 - ab + 1 + (a - 2b)y + y^2) \, dx + ((b^2 - ab + 1) + (ab - 2)x + x^2) \, dy,
\]
where the parameters are such that \( \deg F_{\omega_7} = \deg F_{\omega_8} = 2 \).

Remark 5.2. — Remark that the foliations \( F_{\omega_7} \), do not depend on the parameters of \( \Phi_{a,b} \), that is, the \( F_{\omega_7} \), are num. inv. by all \( \Phi_{a,b} \), whereas the \( F_{\omega_8} \) only depend on \( a \) and \( b \).

Furthermore \( F_{\omega_7} \) is num. inv. by \( \sigma \) and \( \rho \).

Proposition 5.3. — Any \( F \in \mathfrak{F}(2; 2) \) num. inv. under the action of \( \Phi_{a,b} \), and more generally any \( F \in \mathfrak{F}(2; 2) \) num. inv. under the action of a generic cubic birational map of \( \mathbb{P}^2_\mathbb{C} \), satisfies the following properties:
- \( F \) is given by a rational closed 1-form;
- \( F \) is non-primitive.
Proof. — Let us establish the properties for $F_{\omega_7}$; remark that $F_{\omega_7}$ is given by
\[
\frac{dx}{x(\alpha + \kappa y)} - \frac{dy}{y(\alpha + \gamma y)}
\]
which is a closed rational 1-form (remark that $(|\alpha| + |\kappa|)(|\alpha| + |\gamma|) \neq 0$ since $\deg F_{\omega_7} = 2$). The foliation $F_{\omega_7}$ is non-primitive: indeed one has
\[
\sigma^* \omega_7 = \frac{1}{x^r y^s} \left( (\alpha x + \kappa) dx - (\alpha y + \gamma) dy \right)
\]
that defines a foliation of degree 0.

The idea and result are the same for the foliations $F_{\omega_8}$ (except that it gives a birational map $\phi$ such that $\deg \phi^* F_{\omega_8} = 1$).

Let us consider an element $F$ of $F_{(2;2)}$ num. inv. under the action of a birational map of degree $\geq 3$; is $F$ defined by a closed 1-form ?

Remark 5.4. — The foliations $F_{\omega_7}$ are contained in the orbit of the foliation $F_{\eta'}$.

Remark 5.5. — Any map $\Phi_{a,b}$ can be written $\ell_1 \sigma \ell_2 \sigma_3$ with
\[
\ell_2 = (a_0 y + a_1 z : a_2 y + a_3 z : a_4 x + a_5 y + a_6 z)
\]
(see [5] proof of Proposition 6.36). Let us consider the birational map $\xi = \sigma \ell_2 \sigma$ with
\[
\ell_2 = (a_0 y + a_1 z : a_2 y + a_3 z : a_4 x + a_5 y + a_6 z) \in \text{Aut}(\mathbb{P}^2_C).
\]
As in Lemma 5.1 there are two families of foliations $F_1, F_2$ of degree 2, one that does not depend on the parameters of $\xi$ and the other one depending only on the parameters of $\xi$, such that $\xi^* F_1$ and $\xi^* F_2$ are of degree 2. One question is the following: what is the intermediate degree? A computation shows that for generic parameters $\deg \sigma^* F_1 = 4$ and that $\deg \sigma^* F_2 = 2$. This implies in particular that $F_{\omega_8}$ is num. inv. under the action of $\sigma$. For $F_1$ and $F_{\omega_7}$ one has

\[
\begin{array}{c}
4 \\
2 \\
\end{array}
\begin{array}{c}
2 \\
\end{array}
\]

and for $F_2$ and $F_{\omega_8}$

\[
\begin{array}{c}
2 \\
2 \\
2 \\
\end{array}
\]

Let us now consider the ”most degenerate” cubic birational map
\[
\Psi: (x : y : z) \rightarrow (xz^2 + y^3 : yz^2 : z^3).
\]

Lemma 5.6. — An element $F$ of $F(2;2)$ is num. inv. under the action of $\Psi$ if and only if $F$ is given in affine chart by
\[
\omega_0 = (-\alpha + \beta y + \gamma z^2) dx + (\epsilon - 3\beta x + \kappa y - 3\gamma xy + \lambda y^2) dy
\]
where the parameters are such that $\deg F_{\omega_0} = 2$. In particular $F$ is transversely affine.
Remark 5.7. — The map $\psi$ can be written $\ell_1 \sigma \ell_2 \sigma \ell_3 \sigma \ell_4 \sigma \ell_5 \sigma \ell_6 \sigma \ell_7$ with

\[
\begin{align*}
\ell_1 &= (z-y:y:y-x), & \ell_2 &= (y+z:z:x), & \ell_3 &= (-z:-y:x-y), \\
\ell_4 &= (x+z:x:y), & \ell_5 &= (-y:x-3y+z:x), & \ell_6 &= (-x:-y-z:x+y), \\
\ell_7 &= (x+y:z-y:y).
\end{align*}
\]

As previously we consider the problem of the intermediate degrees; if $\mathcal{F}' = \ell_1^* \mathcal{F}$, a computation shows that for generic parameters

\[
\begin{align*}
\deg \sigma^* \mathcal{F}' &= 4, & \deg (\sigma \ell_2 \sigma)^* \mathcal{F}' &= 3, & \deg (\sigma \ell_2 \sigma \ell_3 \sigma)^* \mathcal{F}' &= 5, \\
\deg (\sigma \ell_2 \sigma \ell_3 \sigma \ell_4 \sigma)^* \mathcal{F}' &= 3, & \deg (\sigma \ell_2 \sigma \ell_3 \sigma \ell_4 \sigma \ell_5 \sigma)^* \mathcal{F}' &= 5, \\
\deg (\sigma \ell_2 \sigma \ell_3 \sigma \ell_4 \sigma \ell_5 \sigma \ell_6 \sigma)^* \mathcal{F}' &= 3, & \deg (\sigma \ell_2 \sigma \ell_3 \sigma \ell_4 \sigma \ell_5 \sigma \ell_6 \sigma)^* \mathcal{F}' &= 4,
\end{align*}
\]

that is

\[
\begin{array}{c}
\begin{array}{c}
4 \\
3 \\
4 \\
3 \\
3 \\
5 \\
5 \\
3 \\
2
\end{array}
\end{array}
\]

We have not studied the quadratic foliations numerically invariant by (any) cubic birational transformation. It is reasonable to think that such foliations are transversely projective.

References


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