THE EMBEDDINGS OF THE HEISENBERG GROUP INTO THE CREMONA GROUP

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ABSTRACT. In this note we describe the embeddings of the Heisenberg group into the Cremona group.

INTRODUCTION

The Heisenberg group is the non-abelian nilpotent group given by

$$\mathcal{H} = \langle \mathrm{f},\, \mathrm{g} \,|\, [\mathrm{f},\mathrm{g}] = \mathrm{h},\, [\mathrm{f},\mathrm{h}] = [\mathrm{g},\mathrm{h}] = \mathrm{id}\rangle.$$

It has two generators, f and g, and h is the generator of the center of $\mathcal{H}$.

The Cremona group is the group $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ of birational maps of the projective plane $\mathbb{P}^2_{\mathbb{C}}$ into itself. Such maps can be written in the form

$$(x:y:z) \dashrightarrow \big(P_0(x,y,z) : P_1(x,y,z) : P_2(x,y,z)\big)$$

where P_0, P_1, $P_2 \in \mathbb{C}[x,y,z]$ are homogeneous polynomials of the same degree, and this degree is the degree of the map, if the polynomials have no common factor (of positive degree). Recall that if ϕ is a birational self map of the complex projective plane, then one of the following holds ([Giz80, Can01, DF01, BD15]):

- ⋄ the sequence $(\deg(\phi^n))_{n\in\mathbb{N}}$ is bounded, and ϕ is said to be *elliptic*;
- ⋄ the sequence $(\deg(\phi^n))_{n\in\mathbb{N}}$ grows linearly with n, and ϕ is said to be a *Jonquières twist*;
- ⋄ the sequence $(\deg(\phi^n))_{n\in\mathbb{N}}$ grows quadratically with n, and ϕ is said to be a *Halphen twist*;
- ⋄ $(\deg(\phi^n))_{n\in\mathbb{N}}$ grows exponentially fast with n, and ϕ is said to be *hyperbolic*.

Proposition A. *Let ρ be an embedding of $\mathcal{H}$ into the Cremona group. Then $\rho(\mathcal{H})$ does not contain hyperbolic birational maps.*

More precisely $\rho(\mathrm{f})$ and $\rho(\mathrm{g})$ are either elliptic birational maps, or Jonquières twists.

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We describe the embeddings of $\mathcal{H}$ into the Cremona group. In [D0́6] we already looked at such embeddings but with the following assumption: the images of f and g are elliptic birational self maps. There are other embeddings of $\mathcal{H}$ into the Cremona group:

Theorem B. *Let* ρ *be an embedding of* $\mathcal{H}$ *into the Cremona group. Then up to birational conjugacy*

- *either* $\rho(\mathcal{H})$ *is a subgroup of* $\mathrm{PGL}(3,\mathbb{C})$ *and*
$$\rho(\mathrm{f}) = (x+\alpha y, y+\beta) \qquad\qquad \rho(\mathrm{g}) = (x+\gamma y, y+\delta)$$
with α, β, γ, δ *in* $\mathbb{C}$ *such that* $\alpha\delta-\beta\gamma = 1$*;*
- *or* $\rho(\mathcal{H})$ *is a subgroup of the group of polynomial automorphisms of* $\mathbb{C}^2$ *and* $(\rho(\mathrm{f}),\rho(\mathrm{g}))$ *is one of the following pairs*
$$\big((ax+Q(y),y+c),\,(\alpha x+P(y),y+\gamma)\big),$$
$$\left(\left(ax+Q(y),by+\frac{\gamma(b-1)}{\beta-1}\right),\big(\alpha x+P(y),\beta y+\gamma\big)\right)$$
with $\beta\in\mathbb{C}^*\smallsetminus\{1\}$*,* a*,* α*,* b *in* $\mathbb{C}^*$*,* c*,* γ *in* $\mathbb{C}$ *and* P*,* Q *in* $\mathbb{C}[y]$*;*
- *or* $\rho(\mathrm{f})$ *is a Jonquières twist and* $(\rho(\mathrm{f}),\rho(\mathrm{g}))$ *is one of the following pairs*
$$\Big(\big(x,\delta x^{\pm1}y\big),\,\big(\gamma x,ya(x)\big)\Big), \qquad\qquad \Big(\big(x,\delta x^{\pm2}y\big),\,\big(\gamma x,ya(x)\big)\Big),$$
$$\big((-x,\delta x^{\pm1}y),\,(\gamma x,yb(x))\big), \qquad\qquad \big((\lambda x,yc(x)),(\delta x,yd(x))\big)$$
with δ*,* $\gamma\in\mathbb{C}^*$*,* $\lambda\in\mathbb{C}^*\smallsetminus\{1,-1\}$ *and* a*,* b*,* c*,* $d\in\mathbb{C}(x)^*$ *such that*
$$\frac{b(x)}{b(-x)}\in\mathbb{C}^*, \qquad\qquad \frac{c(\delta x)d(x)}{c(x)d(\lambda x)}\in\mathbb{C}^*.$$

Remark C. *Note that the two last families are not empty. For instance*
$$\big((-x,\alpha x^{\pm1}y),\,(\beta x,\gamma x^2y)\big), \qquad\qquad \big((\lambda x,\alpha x^p y),(\gamma x,\beta x^q y)\big)$$
with α*,* β*,* $\gamma\in\mathbb{C}^*$*,* $\lambda\in\mathbb{C}^*\smallsetminus\{1,-1\}$*,* p*,* $q\in\mathbb{N}$ *are such pairs.*

1. Some recalls

1.1. **About birational maps of the complex projective plane.** Let ϕ be a birational self map of the complex projective plane. Then one of the following holds ([Giz80, Can01, DF01, BD15]):

- ϕ is *elliptic* if and only if the sequence $(\deg(\phi^n))_{n\in\mathbb{N}}$ is bounded. In this case there exist a birational map $\psi\colon S \dashrightarrow \mathbb{P}^2_{\mathbb{C}}$ and an integer $k\geq 1$ such that $\psi^{-1}\circ\phi^k\circ\psi$ belongs to the connected component of the identity of the group

$\mathrm{Aut}(S)$. Either ϕ is of finite order, or ϕ is conjugate to an automorphism of $\mathbb{P}^2_{\mathbb{C}}$, which restricts to one of the following automorphisms on some open subset isomorphic to $\mathbb{C}^2$:

- $(x,y) \mapsto (\alpha x, \beta y)$ where the kernel of the group morphism
$$\mathbb{Z}^2 \to \mathbb{C}^2 \qquad\qquad (i,j) \mapsto \alpha^i\beta^j$$
is generated by $(k,0)$ for some $k \in \mathbb{Z}$;
- $(x,y) \mapsto (\alpha x, y+1)$ where $\alpha \in \mathbb{C}^*$.

◇ ϕ is *parabolic* if and only if the sequence $(\deg(\phi^n))_{n\in\mathbb{N}}$ grows linearly or quadratically with n. If ϕ is parabolic, there exist a birational map $\psi\colon S \dashrightarrow \mathbb{P}^2_{\mathbb{C}}$ and a fibration $\pi\colon S \to B$ onto a curve B such that $\psi^{-1}\circ\phi\circ\psi$ permutes the fibers of π. If $(\deg(\phi^n))_{n\in\mathbb{N}}$ grows linearly, then the fibration π is rational and ϕ is said to be a *Jonquières twist*. If $(\deg(\phi^n))_{n\in\mathbb{N}}$ grows quadratically, then the fibration π is elliptic and ϕ is said to be a *Halphen twist*.

◇ ϕ is *hyperbolic* if and only if $(\deg(\phi^n))_{n\in\mathbb{N}}$ grows exponentially fast with n: there is a constant $c(\phi)$ such that $\deg(\phi^n) = c(\phi)\lambda(\phi)^n + O(1)$.

1.2. **About distorted elements.** If G is a group generated by a finite subset $F \subset \mathrm{G}$ the F-length $|g|_F$ of an element g of G is defined as the least non-negative integer ℓ such that g admits an expression of the form $g = f_1f_2\dots f_\ell$ where each f_i belongs to $F \cup F^{-1}$. We say that g is *distorted* if $\lim\limits_{k\to+\infty}\dfrac{|g^k|_F}{k} = 0$ (note that the limit $\lim\limits_{k\to+\infty}\dfrac{|g^k|_F}{k}$ always exists and is a real number since the sequence $k \mapsto |g^k|_F$ is subadditive). This notion actually does not depend on the chosen F, but only on the pair (g,G).

If G is any group, an element $g \in \mathrm{G}$ is *distorted* if it is distorted in some finitely generated subgroup of G.

The element h of

$$\mathcal{H} = \langle \mathrm{f}, \mathrm{g} \,|\, [\mathrm{f},\mathrm{g}] = \mathrm{h},\ [\mathrm{f},\mathrm{h}] = [\mathrm{g},\mathrm{h}] = \mathrm{id}\rangle$$

satisfies the following property:

$$\forall k \in \mathbb{Z} \qquad \mathrm{h}^{k^2} = [\mathrm{f}^k, \mathrm{g}^k] = \mathrm{f}^k\mathrm{g}^k\mathrm{f}^{-k}\mathrm{g}^{-k}$$

so $||\mathrm{h}^{k^2}|| \leq 4k$ and $\lim\limits_{k\to+\infty}\dfrac{||\mathrm{h}^k||}{k^2} = 0$. Hence h is distorted.

An element $\phi \in \mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ is said to be *algebraic* if it is contained in an algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$. By [BF13, §2.6] the map $\phi \in \mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ is algebraic if and only if the sequence $(\deg(\phi^n))_{n\in\mathbb{N}}$ is bounded. In other words elliptic elements and algebraic elements coincide. By [BD15, Proposition 2.3] this is also equivalent to say that ϕ is of finite order or conjugate to an element of $\mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}})$. A straightforward computation shows that every element of $\mathrm{Aut}(\mathbb{P}^2_\mathbb{C})$ is distorted in $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$ (*see* [BF19, Lemma 4.40]). As a consequence every algebraic element of $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$ is distorted. The converse statement also holds:

Theorem 1.1 ([BF19, CC19]). *Any distorted element of* $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$ *is elliptic.*

Corollary 1.2. *Let* ρ *be an embedding of* $\mathcal{H}$ *into* $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$. *Then* $\rho(\mathrm{h})$ *is elliptic.*

We will use this corollary and the following description of the centralizer of hyperbolic birational maps to prove Proposition A:

Proposition 1.3 ([BC16]). *Let* ϕ *be a birational map of the complex projective plane. If* ϕ *is hyperbolic, then the infinite cyclic group generated by* ϕ *is a finite index subgroup of the centralizer of* ϕ *in* $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$.

Proof of the first part of Proposition A. Assume that $\rho(\mathcal{H})$ contains an hyperbolic element ϕ. Since $\rho(\mathrm{h})$ is the generator of the center of $\rho(\mathcal{H})$, $\rho(\mathrm{h})$ commutes with ϕ. Proposition 1.3 implies that either $\rho(\mathrm{h})$ is hyperbolic, or $\rho(\mathrm{h})$ is of finite order. But $\rho(\mathrm{h})$ is not hyperbolic Corollary (1.2) and by definition $\rho(\mathrm{h})$ is of infinite order. As a result $\rho(\mathcal{H})$ does not contain hyperbolic element. □

1.3. **About centralizers of elliptic birational maps.** Let us recall the description of the centralizers of the elliptic birational self maps of infinite order of the complex projective plane obtained in [BD15].

Consider ϕ an elliptic element of $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$. Assume that ϕ is of infinite order. As recalled in §1.1 the map ϕ is conjugate to an automorphism of $\mathbb{P}^2_\mathbb{C}$ which restricts to one of the following automorphisms on some open subset isomorphic to $\mathbb{C}^2$:

(1) $(\alpha x, \beta y)$ where α, β belong to $\mathbb{C}^*$;
(2) $(\alpha x, y+1)$ where $\alpha \in \mathbb{C}^*$.

In case (1) the centralizer of ϕ in $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$ is

$$\big\{(\eta(x), ya(x^k))\,|\, a \in \mathbb{C}(x),\, \eta \in \mathrm{PGL}(2,\mathbb{C}),\, \eta(\alpha x) = \alpha\eta(x)\big\};$$

in particular the elements of the centralizer of ϕ are elliptic birational maps or Jonquières twists. In case (2) the centralizer of ϕ in $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$ is

$$\big\{(\eta(x), y+a(x))\,|\, \eta \in \mathrm{PGL}(2,\mathbb{C}),\, \eta(\alpha x) = \alpha\eta(x),\, a \in \mathbb{C}(x),\, a(\alpha x) = a(x)\big\};$$

in particular the elements of the centralizer of ϕ are elliptic birational maps.

Corollary 1.2 and the previous description imply:

Lemma 1.4. *Let* ρ *be an embedding of* $\mathcal{H}$ *into* $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$.

Then $\rho(\mathrm{h})$ *is elliptic and up to birational conjugacy*

- *either* $\rho(\mathrm{h}) = (\alpha x, \beta y)$*, where the kernel of the group morphism*

$$\mathbb{Z}^2 \to \mathbb{C}^2 \qquad\qquad (i,j) \mapsto \alpha^i\beta^j$$

is generated by $(k,0)$ *for some* $k \in \mathbb{Z}$ *and both* $\rho(\mathrm{f})$, $\rho(\mathrm{g})$ *belong to*

$$\big\{(\eta(x), ya(x^k)) \,|\, a \in \mathbb{C}(x),\, \eta \in \mathrm{PGL}(2,\mathbb{C}),\, \eta(\alpha x) = \alpha\eta(x)\big\};$$

- *or* $\rho(\mathrm{h}) = (\alpha x, y+1)$*, and both* $\rho(\mathrm{f})$, $\rho(\mathrm{g})$ *belong to*

$$\big\{(\eta(x), y + a(x)) \,|\, \eta \in \mathrm{PGL}(2,\mathbb{C}),\, \eta(\alpha x) = \alpha\eta(x),\, a \in \mathbb{C}(x),\, a(\alpha x) = a(x)\big\}.$$

In particular $\rho(\mathrm{f})$ *and* $\rho(\mathrm{g})$ *are elliptic birational maps or Jonquières twists.*

It ends the proof of Proposition A.

2. Proof of Theorem B

2.1. **Assume that all the generators of $\rho(\mathcal{H})$ are elliptic.** The group $\mathrm{Aut}(\mathbb{C}^2)$ of polynomial automorphisms of $\mathbb{C}^2$ is a subgroup of $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$. It is generated by the group

$$\mathrm{A} = \big\{(a_0x + a_1y + a_2, b_0x + b_1y + b_2) \,|\, a_i,\, b_i \in \mathbb{C},\, a_0b_1 - a_1b_0 \neq 0\big\}$$

and

$$\mathrm{E} = \big\{(\alpha x + P(y), \beta y + \gamma) \,|\, \alpha,\, \beta \in \mathbb{C}^*,\, \gamma \in \mathbb{C},\, P \in \mathbb{C}[y]\big\}.$$

Let us recall the following result obtained when we study the embeddings of $\mathrm{SL}(n,\mathbb{Z})$ into the Cremona group:

Lemma 2.1 ([D0́6])**.** *Let* ρ *be an embedding of* $\mathcal{H}$ *into* $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$.

If $\rho(\mathrm{f})$, $\rho(\mathrm{g})$ *and* $\rho(\mathrm{h})$ *are elliptic, then up to birational conjugacy*

- *either* $\rho(\mathcal{H})$ *is a subgroup of* $\mathrm{PGL}(3,\mathbb{C})$*, and more precisely*

$$\rho(\mathrm{f}) = (x + \alpha y, y + \beta) \qquad\qquad \rho(\mathrm{g}) = (x + \gamma y, y + \delta)$$

with α, β, γ, $\delta \in \mathbb{C}$ *such that* $\alpha\delta - \beta\gamma = 1$*;*

- *or* $\rho(\mathcal{H})$ *is a subgroup of* E *and* $\rho(h^2) = (x + P(y), y)$ *for some* $P \in \mathbb{C}[y]$.

This statement implies the following one:

Proposition 2.2. *Let* ρ *be an embedding from* $\mathcal{H}$ *into* $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$*. Assume that* $\rho(\mathrm{f})$, $\rho(\mathrm{g})$ *and* $\rho(\mathrm{h})$ *are elliptic.*

Then up to birational conjugacy

- *either* $\rho(\mathcal{H})$ *is a subgroup of* $\mathrm{PGL}(3,\mathbb{C})$*, more precisely*

$$\rho(\mathrm{f}) = (x + \alpha y, y + \beta) \qquad\qquad \rho(\mathrm{g}) = (x + \gamma y, y + \delta)$$

with α, β, γ, $\delta \in \mathbb{C}$ *such that* $\alpha\delta - \beta\gamma = 1$*;*

◇ *or* $\rho(\mathcal{H})$ *is a subgroup of* E *and* $(\rho(\mathrm{f}),\rho(\mathrm{g}))$ *is one of the following pairs*

$$\big((ax+Q(y),y+c),\,(\alpha x+P(y),y+\gamma)\big)$$
$$\left(\left(ax+Q(y),by+\frac{\gamma(b-1)}{\beta-1}\right),\big(\alpha x+P(y),\beta y+\gamma\big)\right)$$

with a, α, b *in* $\mathbb{C}^*$, c, γ *in* $\mathbb{C}$, $\beta\in\mathbb{C}^*\smallsetminus\{1\}$ *and* P, Q *in* $\mathbb{C}[y]$.

Proof. The first assertion follows from Lemma 2.1. Let us focus on the second one.

If $\rho(\mathrm{h})$ belongs to E and $\rho(\mathrm{h}^2)=(x+P(y),y)$, then $\rho(\mathrm{h})=(\varepsilon x+Q(y),\eta(y))$ with $\varepsilon^2=1$, $Q\in\mathbb{C}[y]$ and $\eta(y)\in\big\{-y+\gamma,\,y\big\}$. But $\rho(\mathrm{f})$ and $\rho(\mathrm{g})$ belong to E so $[\rho(\mathrm{f}),\rho(\mathrm{g})]=\rho(\mathrm{h})$ implies that $\varepsilon=1$ and $\eta(y)=y$, *i.e.* $\rho(\mathrm{h})=(x+Q(y),y)$. Set

$$\rho(\mathrm{f})=(ax+R(y),by+c),\qquad\qquad \rho(\mathrm{g})=(\alpha x+P(y),\beta y+\gamma).$$

The second component of $\rho(\mathrm{f})\rho(\mathrm{g})$ has to be equal to the second component of $\rho(\mathrm{h})\rho(\mathrm{g})\rho(\mathrm{f})$, that is

$$\beta by+b\gamma+c=\beta by+\beta c+\gamma;$$

in other words either $\beta=b=1$, or $c=\frac{\gamma(b-1)}{\beta-1}$. □

2.2. **Assume that $\rho(\mathrm{f})$ is a Jonquières twist with trivial action on the basis of the fibration.** Since $\rho(\mathrm{h})$ is elliptic, then up to birational conjugacy either $\rho(\mathrm{h})=(\alpha x,\beta y)$, or $\rho(\mathrm{h})=(\alpha x,y+1)$ (*see* §1.1). But $\rho(\mathrm{f})$ belongs to the centralizer of $\rho(\mathrm{h})$ and is a Jonquières twist; therefore according to §1.3 one has: $\rho(\mathrm{h})=(\alpha x,\beta y)$, $\rho(\mathrm{f})$ can be written as $(x,ya(x))$ and $\rho(\mathrm{g})$ as $(\mu(x),yb(x))$ with $\mu\in\mathrm{PGL}(2,\mathbb{C})$ and a, $b\in\mathbb{C}(x)^*$.

Let us remark that if $\mu=\mathrm{id}$, then $[\rho(\mathrm{f}),\rho(\mathrm{g})]=\rho(\mathrm{h})$ implies $\alpha=\beta=1$ so $\mu\neq\mathrm{id}$.

The relation $[\rho(\mathrm{f}),\rho(\mathrm{g})]=\rho(\mathrm{h})$ implies that $\alpha=1$, and $a(\mu(x))=\beta a(x)$. Let us first look at polynomials P such that $P(\mu(x))=\beta P(x)$:

Claim 2.3. *If P is a non-zero polynomial such that $P(\mu(x))=\lambda^2P(x)$, $\lambda^2\neq 1$, then one of the following holds:*

◇ $P(x)=\delta\left(\frac{\gamma}{\lambda^2-1}+x\right)$, $\mu(x)=\gamma+\lambda^2x$ *with* a, $\delta\in\mathbb{C}$*;*

◇ $P(x)=\delta\left(\frac{\gamma}{\lambda+1}-x\right)^2$, $\mu(x)=\gamma-\lambda x$ *with* $\gamma\in\mathbb{C}$, *and* $\delta\in\mathbb{C}^*$*;*

◇ $P(x)=\delta\left(\frac{\gamma}{\lambda-1}+x\right)^2$, $\mu(x)=\gamma+\lambda x$ *with* $\gamma\in\mathbb{C}$, *and* $\delta\in\mathbb{C}^*$.

Proof. Let us consider the set $Z_P=\big\{z\,|\,P(z)=0\big\}$ of roots of P. It is a finite set invariant by μ. As a result $\mu^n_{|Z_P}=\mathrm{id}$ for some integer n.

If $\#Z_P\geq 3$, then $\mu^n_{|Z_P}=\mathrm{id}$ implies $\mu^n=\mathrm{id}$. Recall that

$$\rho(\mathrm{f})=(x,ya(x)),\qquad \rho(\mathrm{g})=(\mu(x),yb(x)),\qquad \rho(\mathrm{h})=(\alpha x,\beta y)$$

so

$$\rho(\mathrm{f})^n = (x, yA(x)), \quad \rho(\mathrm{g})^n = (\mu^n(x), yB(x)) = (x, yB(x)), \quad \rho(\mathrm{h})^{n^2} = (\alpha^{n^2}x, \beta^{n^2}y).$$

Then $[\rho(\mathrm{f})^n, \rho(\mathrm{g})^n] = \rho(\mathrm{h})^{n^2}$ implies $\alpha^{n^2} = \beta^{n^2} = 1$, that is $\rho(\mathrm{h})$ is of finite order: contradiction.

Hence $\#Z_P \leq 2$ so $\deg P \leq 2$. A straightforward computation implies the statement. □

Let us come back to $a(\mu(x)) = \beta a(x)$. As β is of infinite order and a belongs to $\mathbb{C}(x)^*$ we can rewrite this equality as follows: $\frac{P(\mu(x))}{Q(\mu(x))} = \frac{\lambda_1^2 P(x)}{\lambda_2^2 Q(x)}$ where

- ◇ λ_1, λ_2 are two elements of $\mathbb{C} \smallsetminus \{\pm 1\}$ such that $\beta = \frac{\lambda_1^2}{\lambda_2^2}$;
- ◇ P, Q are two polynomials without common factor.

As a result up to birational conjugacy $(\rho(\mathrm{f}), \rho(\mathrm{g}))$ is one of the following pairs

$$\Big((x, \delta xy), (\lambda x, yb(x))\Big) \qquad \Big((x, \delta x^2 y), (\lambda x, yb(x))\Big)$$
$$\Big((x, \frac{y}{\delta x}), (\lambda x, yb(x))\Big) \qquad \Big((x, \frac{y}{\delta x^2}), (\lambda x, yb(x))\Big)$$

with $\delta \in \mathbb{C}^*$, $\lambda \in \mathbb{C}^*$ of infinite order and $b \in \mathbb{C}(x)$.

We can thus state

Proposition 2.4. *Let ρ be an embedding of $\mathcal{H}$ into* $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$.

If $\rho(\mathrm{f})$ is a Jonquières twist with trivial action on the basis of the fibration, then up to birational conjugacy $(\rho(\mathrm{f}), \rho(\mathrm{g}))$ is one of the following pairs

$$\Big((x, \delta xy), (\lambda x, yb(x))\Big) \qquad \Big((x, \delta x^2 y), (\lambda x, yb(x))\Big)$$
$$\Big(\Big(x, \frac{y}{\delta x}\Big), (\lambda x, yb(x))\Big) \qquad \Big(\Big(x, \frac{y}{\delta x^2}\Big), (\lambda x, yb(x))\Big)$$

with $\delta \in \mathbb{C}^*$, $\lambda \in \mathbb{C}^*$ of infinite order and $b \in \mathbb{C}(x)$.

2.3. **Assume that $\rho(\mathrm{f})$ is a Jonquières twist with non-trivial action on the basis of the fibration.** Since $\rho(\mathrm{h})$ is elliptic and of infinite order, then up to birational conjugacy either $\rho(\mathrm{h}) = (\alpha x, \beta y)$, or $\rho(\mathrm{h}) = (\alpha x, y+1)$ (*see* §1.1). But $\rho(\mathrm{f})$ belongs to the centralizer of $\rho(\mathrm{h})$ and is a Jonquières twist; therefore according to §1.3 one has: $\rho(\mathrm{h}) = (\alpha x, \beta y)$, $\rho(\mathrm{f})$ can be written as $(\eta(x), ya(x))$ and $\rho(\mathrm{g})$ as $(\mu(x), yb(x))$ with η, μ in $\mathrm{PGL}(2,\mathbb{C})$ and a, b in $\mathbb{C}(x)$.

Up to conjugacy by an element of $\Big\{ \big(\frac{ax+b}{cx+d}, y\big) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}(2,\mathbb{C}) \Big\}$ one can assume that either $\eta(x) = x+1$, or $\eta(x) = \lambda x$ (remark that this conjugacy doesn't preserve the first component of $\rho(\mathrm{h})$).

Note that a direct computation implies

$$\big\{\nu\in\mathrm{PGL}(2,\mathbb{C})\,|\,\nu(\alpha x)=\alpha\nu(x)\big\}=\begin{cases}\mathrm{PGL}(2,\mathbb{C})\text{ if }\alpha=1\\ \big\{\beta x^{\pm1}\,|\,\beta\in\mathbb{C}^*\big\}\text{ if }\alpha=-1\\ \big\{\beta x\,|\,\beta\in\mathbb{C}^*\big\}\text{ if }\alpha^2\neq1\end{cases}\qquad(2.1)$$

so when η is an homothety we will have to distinguish the cases $\lambda=-1$ and $\lambda\neq-1$.

2.3.1. *Assume that* $\eta(x)=x+1$. Since $\rho(\mathrm{f})$ and $\rho(\mathrm{h})$ commute, $\rho(\mathrm{h})$ can be written as $(x+\gamma,\beta y)$.

If $\gamma\neq0$, then $[\rho(\mathrm{f}),\rho(\mathrm{h})]=\mathrm{id}$ leads to $a(x+\gamma)=a(x)$, that is $a(x)=a\in\mathbb{C}$: contradiction with the fact that $\rho(\mathrm{f})$ is a Jonquières twist.

If $\gamma=0$, then $\rho(\mathrm{h})=(x,\beta y)$ and $[\rho(\mathrm{f}),\rho(\mathrm{g})]=\rho(\mathrm{h})$ leads to $\rho(\mathrm{g})=(x+\mu,yb(x))$ and $b(x)a(x+\mu)=\beta a(x)b(x+1)$. Let us write a as $\frac{P}{Q}$ and b as $\frac{R}{S}$ with $P,Q,R,S\in\mathbb{C}[y]$ then $b(x)a(x+\mu)=\beta a(x)b(x+1)$ can be rewritten

$$P(x+\mu)Q(x)R(x)S(x+1)=\beta P(x)Q(x+\mu)R(x+1)S(x)\qquad(2.2)$$

Denote by p_i (resp. q_ℓ, resp. r_j, resp. s_k) the coefficient of the highest term of P (resp. Q, resp. R, resp. S). The coefficient of the highest term of the left-hand side of (2.2) has to be equal to the coefficient of the highest term of the right-hand side of (2.2), that is $p_iq_\ell r_js_k=\beta p_iq_\ell r_js_k$. So $\beta=1$, *i.e.* $\rho(\mathrm{h})=(x,y)$: contradiction.

2.3.2. *Suppose that* $\eta(x)=-x$, *i.e.* $\rho(\mathrm{f})=(-x,ya(x))$.

Remark 2.5. The map $\rho(\mathrm{f})^2=(x,ya(x)a(-x))$ is a Jonquières twist that preserves fiberwise the rational fibration $x=\mathrm{cst}$; consequently Proposition 2.4 says that $\rho(\mathrm{f})^2$ is one of the following:

$$\big(x,\delta xy\big),\qquad\big(x,\delta x^2y\big),\qquad\Big(x,\delta\frac{y}{x}\Big),\qquad\Big(x,\delta\frac{y}{x^2}\Big)$$

with $\delta\in\mathbb{C}^*$. Let us try to determine $\rho(\mathrm{f})$. If $\rho(\mathrm{f})^2=(x,\delta xy)$, then we have to consider the equation $a(x)a(-x)=\delta x$. The right-hand side of this equation is invariant by $x\mapsto-x$ whereas the left-hand side not, so there is no solution. The same holds if $\rho(\mathrm{f})^2=\big(x,\delta\frac{y}{x}\big)$. Consequently $\rho(\mathrm{f})^2$ is one of the following:

$$\big(x,\delta x^2y\big),\qquad\Big(x,\delta\frac{y}{x^2}\Big)$$

with $\delta\in\mathbb{C}^*$ and $\rho(\mathrm{f})$ is thus one of the following:

$$\big(-x,\zeta xy\big),\qquad\Big(-x,\zeta\frac{y}{x}\Big)$$

with $\zeta\in\mathbb{C}^*$.

Since f and h commute, then (2.1) implies that either $\rho(\mathrm{h})=\big(\frac{\alpha}{x},\beta y\big)$, or $\rho(\mathrm{h})=(\alpha x,\beta y)$. Let us consider these two cases.

- ⋄ Assume first that $\rho(\mathrm{h}) = \big(\frac{\alpha}{x}, \beta y\big)$. Note that $\big(\frac{\alpha}{x}, \beta y\big)$ does not commute neither to $\big(-x, \zeta xy\big)$, nor to $\big(-x, \zeta\frac{y}{x}\big)$: contradiction with $[\rho(\mathrm{f}), \rho(\mathrm{h})] = \mathrm{id}$.
- ⋄ Suppose now that $\rho(\mathrm{h}) = (\alpha x, \beta y)$.
  - If $\alpha^2 \neq 1$, then $[\rho(\mathrm{g}), \rho(\mathrm{h})] = \mathrm{id}$ and (2.1) imply that $\rho(\mathrm{g}) = (\gamma x, yb(x))$. Then $[\rho(\mathrm{f}), \rho(\mathrm{g})] = \rho(\mathrm{h})$ leads to $\alpha = 1$: contradiction with $\alpha^2 \neq 1$.
  - If $\alpha = -1$, that is $\rho(\mathrm{h}) = (-x, \beta y)$, then according to $[\rho(\mathrm{g}), \rho(\mathrm{h})] = \mathrm{id}$ and (2.1) we get that either $\rho(\mathrm{g}) = (\gamma x, yb(x))$, or $\rho(\mathrm{g}) = \big(\frac{\gamma}{x}, yb(x)\big)$. In both cases the relation $[\rho(\mathrm{f}), \rho(\mathrm{g})] = \rho(\mathrm{h})$ leads to a contradiction.
  - If $\alpha = 1$, *i.e.* $\rho(\mathrm{h}) = (x, \beta y)$, then $[\rho(\mathrm{f}), \rho(\mathrm{g})] = \rho(\mathrm{h})$ implies that either $\rho(\mathrm{g}) = (\gamma x, yb(x))$ or $\rho(\mathrm{g}) = \big(\frac{\gamma}{x}, yb(x)\big)$.
    First let us assume that $\rho(\mathrm{g}) = (\gamma x, yb(x))$. If $\rho(\mathrm{f}) = (-x, \zeta xy)$, then $[\rho(\mathrm{f}), \rho(\mathrm{g})] = \rho(\mathrm{h})$ leads to $\gamma b(x) = \beta b(-x)$, that is $\frac{b(x)}{b(-x)}$ belongs to $\mathbb{C}^*$. If $\rho(\mathrm{f}) = \big(-x, \zeta\frac{y}{x}\big)$, then $[\rho(\mathrm{f}), \rho(\mathrm{g})] = \rho(\mathrm{h})$ implies $b(x) = \beta\gamma b(-x)$, that is $\frac{b(x)}{b(-x)}$ belongs to $\mathbb{C}^*$.
    Finally suppose that $\rho(\mathrm{g}) = \big(\frac{\gamma}{x}, yb(x)\big)$. If $\rho(\mathrm{f}) = (-x, \zeta xy)$, then $[\rho(\mathrm{f}), \rho(\mathrm{g})] = \rho(\mathrm{h})$ leads to $\gamma b(x) = \beta x^2 b(-x)$. Write b as $\frac{P}{Q}$ with P, Q in $\mathbb{C}[x]$; then $\gamma b(x) = \beta x^2 b(-x)$ is equivalent to
    $$\gamma P(x)Q(-x) = \beta x^2 P(-x)Q(x)$$
    and the degree of the left-hand side is $\deg P + \deg Q$ whereas the degree of the right-hand side is $\deg P + \deg Q + 2$: contradiction. If $\rho(\mathrm{f}) = \big(-x, \zeta\frac{y}{x}\big)$, then a straightforward computation implies similarly a contradiction.

Proposition 2.6. *Let* ρ *be an embedding of* $\mathcal{H}$ *into* $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$.

If $\rho(\mathrm{f})$ *is a Jonquières twist with a order* 2 *action on the basis of the fibration, then up to birational conjugacy* $(\rho(\mathrm{f}), \rho(\mathrm{g}))$ *is one of the following pairs*

$$\big((-x, \alpha xy), (\beta x, ya(x))\big), \qquad \left(\left(-x, \alpha\frac{y}{x}\right), (\beta x, ya(x))\right)$$

with α, $\beta \in \mathbb{C}^*$ *and* $a \in \mathbb{C}(x)^*$ *such that* $\frac{a(x)}{a(-x)} \in \mathbb{C}^*$.

2.3.3. *Assume that* $\eta(x) = \lambda x$, $\lambda^2 \neq 1$. Recall that

$$\rho(\mathrm{f}) = (\lambda x, ya(x)), \qquad \rho(\mathrm{g}) = (\mu(x), yb(x)), \qquad \rho(\mathrm{h}) = (\upsilon(x), \beta y)$$

with λ in $\mathbb{C}^* \smallsetminus \{1, -1\}$, β in $\mathbb{C}^*$, μ, υ in $\mathrm{PGL}(2, \mathbb{C})$, and a, b in $\mathbb{C}(x)^*$.

First note that since $\rho(\mathrm{f})$ and $\rho(\mathrm{h})$ commute, $\upsilon(\lambda x) = \lambda\upsilon(x)$. According to (2.1), the homography υ is an homothety (recall that $\lambda^2 \neq 1$): $\upsilon(x) = \gamma x$ with $\gamma \in \mathbb{C}^*$.

The relations $[\rho(\mathrm{f}),\rho(\mathrm{g})]=\rho(\mathrm{h})$, $[\rho(\mathrm{f}),\rho(\mathrm{h})]=[\rho(\mathrm{g}),\rho(\mathrm{h})]=\mathrm{id}$ imply the following ones

$$a(x)=a(\gamma x) \tag{2.3}$$

$$b(x)=b(\gamma x) \tag{2.4}$$

$$\mu(\gamma x)=\gamma\mu(x) \tag{2.5}$$

$$\lambda\mu(x)=\gamma\mu(\lambda x) \tag{2.6}$$

$$b(x)a(\mu(x))=\beta a(x)b(\lambda x) \tag{2.7}$$

We will distinguish the cases $\gamma=1$, $\gamma=-1$, $\gamma^2\neq 1$.

- ◇ Assume that $\gamma^2\neq 1$. Then (2.1) and (2.5) lead to $\mu(x)=\mu x$ with $\mu\in\mathbb{C}^*$. Equation (2.6) can be rewritten $\lambda\mu x=\gamma\mu\lambda x$, that is $\gamma=1$: contradiction with the assumption $\gamma^2\neq 1$.
- ◇ Suppose that $\gamma=1$. Then $\lambda^2\neq 1$, (2.1) and (2.6) lead to $\mu(x)=\mu x$ with $\mu\in\mathbb{C}^*$. In other words
$$\rho(\mathrm{f})=(\lambda x, ya(x)), \qquad \rho(\mathrm{g})=(\mu x, yb(x))$$
with λ in $\mathbb{C}^*\smallsetminus\{1,-1\}$, μ in $\mathbb{C}^*$, a, b in $\mathbb{C}(x)$ such that $\frac{a(\mu x)b(x)}{a(x)b(\lambda x)}$ belongs to $\mathbb{C}^*$.
- ◇ Assume that $\gamma=-1$. Then (2.1) and (2.5) imply that $\mu(x)=\mu x^{\pm 1}$ with $\mu\in\mathbb{C}^*$. If $\mu(x)=\mu x$, then (2.6) can be rewritten $\lambda\mu x=-\lambda\mu x$: contradiction. If $\mu(x)=\frac{\mu}{x}$, then (2.6) can be rewritten $\frac{\lambda\mu}{x}=-\frac{\mu}{\lambda x}$; hence $\lambda^2=-1$.
  - If $\lambda=\mathbf{i}$, then $\rho(\mathrm{f})=(\mathbf{i}x, ya(x))$ and $\rho(\mathrm{f})^4=\big(x, ya(x)a(\mathbf{i}x)a(-x)a(-\mathbf{i}x)\big)$ preserves fiberwise the fibration $x=\mathrm{cst}$. According to Proposition 2.4 $\rho(\mathrm{f})^4$ can be written as $(x,\delta xy)$, or $(x,\delta x^2y)$, or $\big(x,\frac{y}{\delta x}\big)$, or $\big(x,\frac{y}{\delta x^2}\big)$. If $\rho(\mathrm{f})^4=(x,\delta xy)$, then $\delta x=a(x)a(\mathbf{i}x)a(-x)a(-\mathbf{i}x)$; but the right-hand side of this equality is invariant by $x\mapsto\mathbf{i}x$ whereas the left-hand side is not. As a consequence $\rho(\mathrm{f})^4$ can not be written $(x,\delta xy)$. Similarly one sees that $\rho(\mathrm{f})^4$ can not be written $(x,\delta x^2y)$, $\big(x,\frac{y}{\delta x}\big)$, and $\big(x,\frac{y}{\delta x^2}\big)$. Thus $\lambda\neq\mathbf{i}$.
  - Similarly one gets that the case $\lambda=-\mathbf{i}$ does not happen.

Proposition 2.7. *Let ρ be an embedding of $\mathcal{H}$ into* $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$.

If $\rho(\mathrm{f})$ is a Jonquières twist with an action on the basis of the fibration that is neither trivial, nor of order 2, then up to birational conjugacy $(\rho(\mathrm{f}),\rho(\mathrm{g}))$ is one of the following pairs

$$((\lambda x, ya(x)),(\mu x, yb(x)))$$

with $\lambda\in\mathbb{C}^*\smallsetminus\{1,-1\}$, $\mu\in\mathbb{C}^*$, a, $b\in\mathbb{C}(x)^*$ such that $\frac{a(\mu x)b(x)}{a(x)b(\lambda x)}\in\mathbb{C}^*$.

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