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# REPRESENTATIONS OF SOME LATTICES INTO THE GROUP OF ANALYTIC DIFFEOMORPHISMS OF THE SPHERE $\mathbb{S}^2$

by

Julie Déserti

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**Abstract.** — In [10] it is proved that any morphism from a subgroup of finite index of  $\mathrm{SL}(n, \mathbb{Z})$  to the group of analytic diffeomorphisms of  $\mathbb{S}^2$  has a finite image as soon as  $n \geq 5$ . The case  $n = 4$  is also claimed to follow along the same arguments; in fact this is not straightforward and this case indeed needs a modification of the argument. In this paper we recall the strategy for  $n \geq 5$  and then focus on the case  $n = 4$ .

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## 1. Introduction

After the works of Margulis ([14, 19]) on the linear representations of lattices of simple, real Lie groups with  $\mathbb{R}$ -rank larger than 1, some authors, like Zimmer, suggest to study the actions of lattices on compact manifolds ([20, 21, 22, 23]). One of the main conjectures of this program is the following: let us consider a connected, simple, real Lie group  $G$ , and let  $\Gamma$  be a lattice of  $G$  of  $\mathbb{R}$ -rank larger than 1. If there exists a morphism of infinite image from  $\Gamma$  to the group of diffeomorphisms of a compact manifold  $M$ , then the  $\mathbb{R}$ -rank of  $G$  is bounded by the dimension of  $M$ . There are a lot of contributions in that direction ([3, 4, 5, 7, 8, 9, 10, 11, 16, 17]). In this article we will focus on the embeddings of subgroups of finite index of  $\mathrm{SL}(n, \mathbb{Z})$  into the group  $\mathrm{Diff}^0(\mathbb{S}^2)$  of real analytic diffeomorphisms of  $\mathbb{S}^2$  (see [10]).

The article is organized as follows. First of all we will recall the strategy of [10]: the study of the nilpotent subgroups of  $\mathrm{Diff}^0(\mathbb{S}^2)$  implies that such subgroups are metabelian. But subgroups of finite index of  $\mathrm{SL}(n, \mathbb{Z})$ , for  $n \geq 5$ , contain nilpotent subgroups of length  $n - 1$  of finite index which are not metabelian; as a consequence Ghys gets the following statement.

**Theorem A** ([10]). — *Let  $\Gamma$  be a subgroup of finite index of  $\mathrm{SL}(n, \mathbb{Z})$ . As soon as  $n \geq 5$  there is no embedding of  $\Gamma$  into  $\mathrm{Diff}^0(\mathbb{S}^2)$ .*

To study nilpotent subgroups of  $\mathrm{Diff}^0(\mathbb{S}^2)$  one has to study nilpotent subgroups of  $\mathrm{Diff}_+^0(\mathbb{S}^1)$  (see §2), and then nilpotent subgroups of the group of formal diffeomorphisms of  $\mathbb{C}^2$  (see §3). The last section is devoted to establish the following result.

**Theorem B.** — Let  $\Gamma$  be a subgroup of finite index of  $\mathrm{SL}(n, \mathbb{Z})$ . As soon as  $n \geq 4$  there is no embedding of  $\Gamma$  into  $\mathrm{Diff}^{\omega}(\mathbb{S}^2)$ .

The proof relies on the characterization, up to isomorphism, of nilpotent subalgebras of length 3 of the algebra of formal vector fields of  $\mathbb{C}^2$  that vanish at the origin.

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## 2. Nilpotent subgroups of the group of analytic diffeomorphisms of $\mathbb{S}^1$

Let  $G$  be a group; let us set

$$G^{(0)} = G \quad \& \quad G^{(i)} = [G, G^{(i-1)}] \quad \forall i \geq 1.$$

The group  $G$  is *nilpotent* if there exists an integer  $n$  such that  $G^{(n)} = \{\mathrm{id}\}$ ; the *length of nilpotence* of  $G$  is the smallest integer  $k$  such that  $G^{(k)} = \{\mathrm{id}\}$ .

Set

$$G_{(0)} = G \quad \& \quad G_{(i)} = [G_{(i-1)}, G_{(i-1)}] \quad \forall i \geq 1.$$

The group  $G$  is *solvable* if  $G_{(n)} = \{\mathrm{id}\}$  for some integer  $n$ ; the *length of solvability* of  $G$  is the smallest integer  $k$  such that  $G_{(k)} = \{\mathrm{id}\}$ .

We say that the group  $G$  (resp. algebra  $\mathfrak{g}$ ) is *metabelian* if  $[G, G]$  (resp.  $[\mathfrak{g}, \mathfrak{g}]$ ) is abelian.

**Proposition 2.1** ([10]). — Any nilpotent subgroup of  $\mathrm{Diff}_+^{\omega}(\mathbb{S}^1)$  is abelian.

*Proof.* — Let  $G$  be a nilpotent subgroup of  $\mathrm{Diff}_+^{\omega}(\mathbb{S}^1)$ . Assume that  $G$  is not abelian; it thus contains a Heisenberg group

$$\langle f, g, h \mid [f, g] = h, [f, h] = [g, h] = \mathrm{id} \rangle.$$

The application “rotation number“

$$\mathrm{Diff}_+^{\omega}(\mathbb{S}^1) \rightarrow \mathbb{R}/\mathbb{Z}, \quad \psi \mapsto \lim_{n \rightarrow +\infty} \frac{\psi^n(x) - x}{n}$$

is not a morphism but its restriction to a solvable subgroup is a morphism ([1]). Hence the rotation number of  $h$  is zero, and the set  $\mathrm{Fix}(h)$  of fixed points of  $h$  is non-empty, and finite. Considering some iterates of  $f$  and  $g$  instead of  $f$  and  $g$  one can assume that  $f$  and  $g$  fix any point of  $\mathrm{Fix}(h)$ . The set of fixed points of a non trivial element of  $\langle f, g \rangle$  is finite and invariant by  $h$  so the action of  $\langle f, g \rangle$  is free <sup>(1)</sup> on each component of  $\mathbb{S}^1 \setminus \mathrm{Fix}(h)$ . But the action of a free group on  $\mathbb{R}$  is abelian: contradiction.  $\square$

1. The stabilizer of every point is trivial, *i.e.* the action of a non trivial element of  $\langle f, g \rangle$  has no fixed point.

### 3. Nilpotent subgroups of the group of formal diffeomorphisms of $\mathbb{C}^2$

Let us denote  $\widehat{\text{Diff}}(\mathbb{C}^2, 0)$  the group of formal diffeomorphisms of  $\mathbb{C}^2$ , *i.e.* the formal completion of the group of germs of holomorphic diffeomorphisms at 0. Let  $\text{Diff}_i$  be the quotient of  $\widehat{\text{Diff}}(\mathbb{C}^2, 0)$  by the normal subgroups of formal diffeomorphisms tangent to the identity with multiplicity  $i$ ; it can be viewed as the set of jets of diffeomorphisms at order  $i$  with the law of composition with truncation at order  $i$ . Note that  $\text{Diff}_i$  is a complex linear algebraic group. One can see  $\widehat{\text{Diff}}(\mathbb{C}^2, 0)$  as the projective limit of the  $\text{Diff}_i$ 's:  $\widehat{\text{Diff}}(\mathbb{C}^2, 0) = \varprojlim \text{Diff}_i$ . Let us denote by  $\widehat{\chi}(\mathbb{C}^2, 0)$  the algebra of formal vector fields in  $\mathbb{C}^2$  vanishing at 0. One can define the set  $\chi_i$  of the  $i$ -th jets of vector fields; one has  $\varprojlim \chi_i = \widehat{\chi}(\mathbb{C}^2, 0)$ .

Let  $\widehat{O}(\mathbb{C}^2)$  be the ring of formal series in two variables, and let  $\widehat{K}(\mathbb{C}^2)$  be its fraction field;  $O_i$  is the set of elements of  $\widehat{O}(\mathbb{C}^2)$  truncated at order  $i$ .

The family  $(\exp_i: \chi_i \rightarrow \text{Diff}_i)$  is filtered, *i.e.* compatible with the truncation. We then define the exponential application as follows:  $\exp = \varprojlim \exp_i: \widehat{\chi}(\mathbb{C}^2, 0) \rightarrow \widehat{\text{Diff}}(\mathbb{C}^2, 0)$ .

As in the classical case, if  $X$  belongs to  $\widehat{\chi}(\mathbb{C}^2, 0)$ , then  $\exp(X)$  can be seen as the “flow at time  $t = 1$ ” of  $X$ . Indeed an element  $X_i$  of  $\chi_i$  can be seen as a derivation of  $O_i$ ; so it can be written  $S_i + N_i$  where  $S_i$  and  $N_i$  are two semi-simple (resp. nilpotent) derivations that commute. Passing to the limit, one gets  $X = S + N$  where  $S$  is a semi-simple vector field,  $N$  a nilpotent one, and  $[S, N] = \text{id}$  (*see* [15]). A semi-simple vector field is a formal vector field conjugate to a diagonal linear vector field that is complete. A vector field is nilpotent if and only if its linear part is; let us remark that the usual flow  $\varphi_t$  of a nilpotent vector field is polynomial in  $t$

$$\varphi_t(x) = \sum_I P_I(t)x^I, \quad P_I \in (\mathbb{C}[t])^2$$

so  $\varphi_1(x)$  is well defined. As a consequence  $\exp(tX) = \exp(tS)\exp(tN)$  is well defined for  $t = 1$ . Note that the Jordan decomposition is purely formal: if  $X$  is holomorphic, then  $S$  and  $N$  are not necessary holomorphic.

**Proposition 3.1** ([10]). — *Any nilpotent subalgebra of  $\widehat{\chi}(\mathbb{C}^2, 0)$  is metabelian.*

*Proof.* — Let  $\mathfrak{l}$  be a nilpotent subalgebra of  $\widehat{\chi}(\mathbb{C}^2, 0)$ , and let  $Z(\mathfrak{l})$  be its center. Since  $\widehat{\chi}(\mathbb{C}^2, 0) \otimes \widehat{K}(\mathbb{C}^2)$  is a vector space of dimension 2 over  $\widehat{K}(\mathbb{C}^2)$  one has the following alternative:

- the dimension of the subspace generated by  $Z(\mathfrak{l})$  in  $\widehat{\chi}(\mathbb{C}^2, 0) \otimes \widehat{K}(\mathbb{C}^2)$  is 1;
- the dimension of the subspace generated by  $Z(\mathfrak{l})$  in  $\widehat{\chi}(\mathbb{C}^2, 0) \otimes \widehat{K}(\mathbb{C}^2)$  is 2.

Let us study these different cases.

Under the first assumption there exists an element  $X$  of  $Z(\mathfrak{l})$  having the following property: any vector field of  $Z(\mathfrak{l})$  can be written  $uX$  with  $u$  in  $\widehat{K}(\mathbb{C}^2)$ . Let us consider the subalgebra  $\mathfrak{g}$  of  $\mathfrak{l}$  given by

$$\mathfrak{g} = \{\tilde{X} \in \mathfrak{l} \mid \exists u \in \widehat{K}(\mathbb{C}^2), \tilde{X} = uX\}.$$

Since  $X$  belongs to  $Z(\mathfrak{l})$ , the algebra  $\mathfrak{g}$  is abelian; it is also an ideal of  $\mathfrak{l}$ . Let us assume that  $\mathfrak{l}$  is not abelian: let  $Y$  be an element of  $\mathfrak{l}$  whose projection on  $\mathfrak{l}/\mathfrak{g}$  is non trivial, and central. Any vector field of  $\mathfrak{l}$  can be written as  $uX + vY$  with  $u, v$  in  $\widehat{K}(\mathbb{C}^2)$ . As  $X$  belongs to  $Z(\mathfrak{l})$ , and  $Y$  is central modulo  $\mathfrak{g}$  one has

$$X(u) = X(v) = Y(v) = 0.$$

The vector fields  $\frac{\partial}{\partial x}$  and  $\frac{\partial}{\partial y}$  being some linear combinations of  $X$  and  $Y$  with coefficients in  $\widehat{K}(\mathbb{C}^2, 0)$ , the partial derivatives of  $v$  are zero so  $v$  is a constant. Therefore  $[\mathfrak{l}, \mathfrak{l}] \subset \mathfrak{g}$ ; but  $\mathfrak{g}$  is abelian thus  $\mathfrak{l}$  is metabelian.

In the second case  $Z(\mathfrak{l})$  contains two elements  $X$  and  $Y$  which are linearly independent on  $\widehat{K}(\mathbb{C}^2)$ . Any vector field of  $\mathfrak{l}$  can be written as  $uX + vY$  with  $u$  and  $v$  in  $\widehat{K}(\mathbb{C}^2)$ . Since  $X$  and  $Y$  belong to  $Z(\mathfrak{l})$  one has

$$X(u) = X(v) = Y(u) = Y(v) = 0.$$

As a consequence  $u$  and  $v$  are constant, *i.e.*  $\mathfrak{l} \subset \{uX + vY \mid u, v \in \mathbb{C}\}$ ; in particular  $\mathfrak{l}$  is abelian.  $\square$

**Proposition 3.2** ([10]). — *Any nilpotent subgroup of  $\widehat{\text{Diff}}(\mathbb{C}^2, 0)$  is metabelian.*

*Proof.* — Let  $G$  be a nilpotent subgroup of  $\widehat{\text{Diff}}(\mathbb{C}^2, 0)$  of length  $k$ . Let us denote by  $G_i$  the projection of  $G$  on  $\text{Diff}_i$ . The Zariski closure  $\overline{G_i}$  of  $G_i$  in  $\text{Diff}_i$  is an algebraic nilpotent subgroup of length  $k$ . It is sufficient to prove that  $\overline{G_i}$  is metabelian.

Since  $\overline{G_i}$  is a complex algebraic subgroup it is the direct product of the subgroup  $\overline{G_{i,u}}$  of its unipotent elements and the subgroup  $\overline{G_{i,s}}$  of its semi-simple elements (*see for example* [2]).

An element of  $\text{Diff}_i$  is unipotent if and only if its linear part, which belongs to  $\text{GL}(2, \mathbb{C})$ , is; so  $\overline{G_{i,s}}$  projects injectively onto a nilpotent subgroup of  $\text{GL}(2, \mathbb{C})$ . Therefore  $\overline{G_{i,s}}$  is abelian.

The group  $\overline{G_{i,u}}$  coincides with  $\exp \mathfrak{l}_i$  where  $\mathfrak{l}_i$  is a nilpotent Lie algebra of  $\chi_i$  of length  $k$ . Passing to the limit one thus obtains the existence of a nilpotent subalgebra  $\mathfrak{l}$  of  $\widehat{\chi}(\mathbb{C}^2, 0)$  of length  $k$  such that  $\exp(\mathfrak{l})$  projects onto  $\overline{G_{i,u}}$  for any  $i$ . According to Proposition 3.1 the subalgebra  $\mathfrak{l}$ , and thus  $\overline{G_{i,u}}$  are metabelian.  $\square$

#### 4. Nilpotent subgroups of the group of analytic diffeomorphisms of $\mathbb{S}^2$

**Proposition 4.1** ([10]). — *Any nilpotent subgroup of  $\text{Diff}^{\text{an}}(\mathbb{S}^2)$  has a finite orbit.*

*Proof.* — Let  $G$  be a nilpotent subgroup of  $\text{Diff}^{\text{an}}(\mathbb{S}^2)$ ; up to finite index one can assume that the elements of  $G$  preserve the orientation. Let  $\phi$  be a non trivial element of  $G$  that commutes with  $G$ . Let  $\text{Fix}(\phi)$  be the set of fixed points of  $\phi$ ; it is a non empty analytic subspace of  $\mathbb{S}^2$  invariant by  $G$ . If  $p$  is an isolated fixed point of  $\phi$ , then the orbit of  $p$  under the action of  $G$  is finite. So it is sufficient to study the case where  $\text{Fix}(\phi)$  only contains curves; there are thus two possibilities:

- $\text{Fix}(\phi)$  is a singular analytic curve whose set of singular points is a finite orbit for  $G$ ;
- $\text{Fix}(\phi)$  is a smooth analytic curve, not necessary connected. One of the connected component of  $\mathbb{S}^2 \setminus \text{Fix}(\phi)$  is a disk denoted by  $\mathbb{D}$ . Any subgroup  $\Gamma$  of finite index of  $G$  which contains  $\phi$  fixes  $\mathbb{D}$ . Let us consider an element  $\gamma$  of  $\Gamma$ , and a fixed point  $m$  of  $\gamma$  that belongs to  $\mathbb{D}$ . By construction  $\phi$  has no fixed point in  $\mathbb{D}$  so according to the Brouwer Theorem  $(\phi^k(m))_k$  has a limit point on the boundary  $\partial\mathbb{D}$  of  $\mathbb{D}$ . Therefore  $\gamma$  has at least one fixed point on  $\partial\mathbb{D}$ . The group  $\Gamma$  thus acts on  $\partial\mathbb{D}$ , and any of its elements has a fixed point on  $\mathbb{D}$ . Then  $\Gamma$  has a fixed point on  $\partial\mathbb{D}$  (Proposition 2.1).  $\square$

**Theorem 4.2** ([10]). — *Any nilpotent subgroup of  $\text{Diff}^{\text{an}}(\mathbb{S}^2)$  is metabelian.*

*Proof.* — Let  $G$  be a nilpotent subgroup of  $\text{Diff}^{\text{an}}(\mathbb{S}^2)$ , and let  $\Gamma$  be a subgroup of finite index of  $G$  having a fixed point  $m$  (such a subgroup exists according to Proposition 4.1). One can embed  $\Gamma$  into  $\widehat{\text{Diff}}(\mathbb{R}^2, 0)$ , and so into  $\widehat{\text{Diff}}(\mathbb{C}^2, 0)$ , by considering the jets of infinite order of elements of  $\Gamma$  in  $m$ . According to Proposition 3.2 the group  $\Gamma$  is metabelian.

One can suppose that  $G$  is a finitely generated group.

Let us first assume that  $G$  has no element of finite order. Then  $G$  is a cocompact lattice of the nilpotent, simply connected Lie group  $G \otimes \mathbb{R}$  (*see* [18]). The group  $G$  is metabelian if and only if  $G \otimes \mathbb{R}$  is; but  $\Gamma$  is metabelian so  $G \otimes \mathbb{R}$  also.

Finally let us consider the case where  $G$  contains at least one element of finite order. The set of such elements is a normal subgroup of  $G$  that thus intersects non trivially the center  $Z(G)$  of  $G$ . Let us consider a non trivial element  $\phi$  of  $Z(G)$  which has finite order. Let us recall that a finite group of diffeomorphisms of the sphere is conjugate to a group of isometries. Denote by  $G^+$  the subgroup of elements of  $G$  which preserve the orientation. It is thus sufficient to prove that  $G^+$  is metabelian; indeed if  $\phi$  does not preserve the orientation, then  $\phi$  has order 2, and  $G = \mathbb{Z}/2\mathbb{Z} \times G^+$ . So let us assume that  $\phi$  preserves the orientation;  $\phi$  is conjugate to a direct isometry of  $\mathbb{S}^2$ , and has exactly two fixed points on the sphere. The group  $G$  has thus an invariant set of two elements. By considering germs in the neighborhood of these two points, one gets that  $G$  can be embedded into  $2 \cdot \text{Diff}(\mathbb{R}^2, 0)$  <sup>(2)</sup> and thus into  $2 \cdot \text{Diff}(\mathbb{C}^2, 0)$ :

$$1 \longrightarrow \text{Diff}(\mathbb{C}^2, 0) \longrightarrow 2 \cdot \text{Diff}(\mathbb{C}^2, 0) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$$

Remark that  $2 \cdot \text{Diff}(\mathbb{C}^2, 0)$  is the projective limit of the algebraic groups  $2 \cdot \text{Diff}_i$ . One can conclude as in the proof of Proposition 3.2 except that the subgroup of the semi-simple elements of  $2 \cdot \text{Diff}_i$  embeds now in  $2 \cdot \text{GL}(2, \mathbb{C})$ ; it is metabelian because it contains an abelian subgroup of index 2.  $\square$

Let  $\Gamma$  be a subgroup of finite index of  $\text{SL}(n, \mathbb{Z})$  for  $n \geq 5$ . Since  $\Gamma$  contains nilpotent subgroups of finite index of length  $n - 1$  (for example the group of upper triangular unipotent matrices) which are not metabelian one gets the following statement.

**Corollary 4.3** ([10]). — *Let  $\Gamma$  be a subgroup of finite index of  $\text{SL}(n, \mathbb{Z})$ ; as soon as  $n \geq 5$  there is no embedding of  $\Gamma$  into  $\text{Diff}^{\text{an}}(\mathbb{S}^2)$ .*

### 5. Nilpotent subgroups of length 3 of the group of analytic diffeomorphisms of $\mathbb{S}^2$

Let us precise Proposition 3.1 for nilpotent subalgebras of length 3 of  $\widehat{\chi}(\mathbb{C}^2, 0)$ . Let  $\mathfrak{l}$  be such an algebra. The dimension of the subspace generated by  $Z(\mathfrak{l})$  in  $\widehat{\chi}(\mathbb{C}^2, 0) \otimes \widehat{K}(\mathbb{C}^2)$  has dimension at most 1, for else  $\mathfrak{l}$  would be abelian (Proposition 3.1) and this is impossible under our assumptions. So let us assume that the dimension of the subspace generated by  $Z(\mathfrak{l})$  in  $\widehat{\chi}(\mathbb{C}^2, 0) \otimes \widehat{K}(\mathbb{C}^2)$  is 1. There exists an element  $X$  in  $Z(\mathfrak{l})$  with the following property: any element of  $Z(\mathfrak{l})$  can be written  $uX$  with  $u$  in  $\widehat{K}(\mathbb{C}^2)$ . Let  $\mathfrak{g}$  denote the abelian ideal of  $\mathfrak{l}$  defined by

$$\mathfrak{g} = \{ \widetilde{X} \in \mathfrak{l} \mid \exists u \in \widehat{K}(\mathbb{C}^2), \widetilde{X} = uX \}.$$

By hypothesis  $\mathfrak{l}$  is not abelian. Let  $Y$  be in  $\mathfrak{l}$ ; assume that its projection onto  $\mathfrak{l}/\mathfrak{g}$  is a non trivial element of  $Z(\mathfrak{l}/\mathfrak{g})$ . Any vector field of  $\mathfrak{l}$  can be written

$$uX + vY, \quad u, v \in \widehat{K}(\mathbb{C}^2).$$

Since  $X$ , resp.  $Y$  belongs to  $Z(\mathfrak{l})$  (resp.  $Z(\mathfrak{l}/\mathfrak{g})$ ) and since the length of  $\mathfrak{l}$  is 3, one has

$$X(u) = Y^3(u) = X(v) = Y(v) = 0. \quad (5.1)$$

If  $X$  and  $Y$  are non singular, one can choose formal coordinates  $x$  and  $y$  such that  $X = \frac{\partial}{\partial x}$  and  $Y = \frac{\partial}{\partial y}$ . The previous conditions can be thus translated as follows:  $v$  is a constant and  $u$  is a polynomial in  $y$  of degree 2. We will see that we have a similar property without assumption on  $X$  and  $Y$ .

2. Let  $G$  be a group and let  $q$  be a positive integer;  $q \cdot G$  denotes the semi-direct product of  $G^q$  by  $\mathbb{Z}/q\mathbb{Z}$  under the action of the cyclic permutation of the factors.

**Lemma 5.1.** — *Let  $X$  and  $Y$  be two vector fields of  $\widehat{\mathcal{X}}(\mathbb{C}^2, 0)$  that commute and are not colinear. One can assume that  $(X, Y) = \left(\frac{\partial}{\partial \tilde{x}}, \frac{\partial}{\partial \tilde{y}}\right)$  where  $\tilde{x}$  and  $\tilde{y}$  are two independent variables in a Liouvillian extension of  $\widehat{K}(\mathbb{C}^2, 0)$ .*

*Proof.* — Since  $X$  and  $Y$  are non colinear, there exist two 1-forms  $\alpha, \beta$  with coefficients in  $\widehat{K}(\mathbb{C}^2)$  such that

$$\alpha(X) = 1, \quad \alpha(Y) = 0, \quad \beta(X) = 0, \quad \beta(Y) = 1.$$

The vector fields  $X$  and  $Y$  commute if and only if  $\alpha$  and  $\beta$  are closed (this statement of linear algebra is true for convergent meromorphic vector fields and is also true in the completion). The 1-form  $\alpha$  is closed so according to [6] one has

$$\alpha = \sum_{i=1}^r \lambda_i \frac{d\widehat{\phi}_i}{\widehat{\phi}_i} + d\left(\frac{\widehat{\Psi}_1}{\widehat{\Psi}_2}\right) = d\left(\sum_{i=1}^r \lambda_i \log \widehat{\phi}_i + \frac{\widehat{\Psi}_1}{\widehat{\Psi}_2}\right)$$

where  $\widehat{\Psi}_1, \widehat{\Psi}_2$ , and the  $\widehat{\phi}_i$  denote some formal series and the  $\lambda_i$  some complex numbers. One has a similar expression for  $\beta$ . So there exists a Liouvillian extension  $\kappa$  of  $\widehat{K}(\mathbb{C}^2)$  having two elements  $\tilde{x}$  and  $\tilde{y}$  with  $\alpha = d\tilde{x}$  and  $\beta = d\tilde{y}$ . One thus has

$$X(\tilde{x}) = 1, \quad X(\tilde{y}) = 0, \quad Y(\tilde{x}) = 0, \quad Y(\tilde{y}) = 1.$$

□

From (5.1) one gets:  $v$  is a constant, and  $u$  is a polynomial in  $\tilde{y}$  of degree 2; so one proves the following statement.

**Proposition 5.2.** — *Let  $\mathfrak{l}$  be a nilpotent subalgebra of  $\widehat{\mathcal{X}}(\mathbb{C}^2, 0)$  of length 3. Then  $\mathfrak{l}$  is isomorphic to a subalgebra of*

$$\mathfrak{n} = \left\{ P(\tilde{y}) \frac{\partial}{\partial \tilde{x}} + \alpha \frac{\partial}{\partial \tilde{y}} \mid \alpha \in \mathbb{C}, P \in \mathbb{C}[\tilde{y}], \deg P = 2 \right\}.$$

**Remark 5.3.** — We use a real version of this statement whose proof is an adaptation of the previous one: a nilpotent subalgebra  $\mathfrak{l}$  of length 3 of  $\widehat{\mathcal{X}}(\mathbb{R}^2, 0)$  is isomorphic to a subalgebra of

$$\mathfrak{n} = \left\{ P(\tilde{y}) \frac{\partial}{\partial \tilde{x}} + \alpha \frac{\partial}{\partial \tilde{y}} \mid \alpha \in \mathbb{R}, P \in \mathbb{R}[\tilde{y}], \deg P = 2 \right\}.$$

**Theorem 5.4.** — *Let  $\Gamma$  be a subgroup of finite index of  $\mathrm{SL}(n, \mathbb{Z})$ ; as soon as  $n \geq 4$  there is no embedding of  $\Gamma$  into  $\mathrm{Diff}^0(\mathbb{S}^2)$ .*

*Proof.* — Let  $U(4, \mathbb{Z})$  (resp.  $U(4, \mathbb{R})$ ) be the subgroup of unipotent upper triangular matrices of  $\mathrm{SL}(4, \mathbb{Z})$  (resp.  $\mathrm{SL}(4, \mathbb{R})$ ); it is a nilpotent subgroup of length 3. Assume that there exists an embedding from a subgroup  $\Gamma$  of finite index of  $\mathrm{SL}(4, \mathbb{Z})$  into  $\mathrm{Diff}^0(\mathbb{S}^2)$ . Up to finite index  $\Gamma$  contains  $U(4, \mathbb{Z})$ . Let us set  $H = \rho(U(4, \mathbb{Z}))$ . Up to finite index  $H$  has a fixed point (Proposition 4.1). One can thus see  $H$  as a subgroup of  $\mathrm{Diff}(\mathbb{R}^2, 0) \subset \widehat{\mathrm{Diff}}(\mathbb{R}^2, 0)$  up to finite index.

Let us denote by  $j^1$  the morphism from  $\widehat{\mathrm{Diff}}(\mathbb{R}^2, 0)$  to  $\mathrm{Diff}_j$ . Up to conjugation  $j^1(\rho(U(4, \mathbb{Z})))$  is a subgroup of

$$\left\{ \begin{bmatrix} \lambda & t \\ 0 & \lambda \end{bmatrix} \mid \lambda \in \mathbb{R}^*, t \in \mathbb{R} \right\}.$$

Up to index 2 one can thus assume that  $j^1 \circ \rho$  takes values in the connected, simply connected group  $T$  defined by

$$T = \left\{ \begin{bmatrix} \lambda & t \\ 0 & \lambda \end{bmatrix} \mid \lambda, t \in \mathbb{R}, \lambda > 0 \right\}.$$

Let us set

$$\text{Diff}_i(T) = \{f \in \text{Diff}_i \mid j^1(f) \in T\};$$

the group  $\text{Diff}_i(T)$  is a connected, simply connected, nilpotent and algebraic group. The morphism

$$\rho_i: U(4, \mathbb{Z}) \rightarrow \text{Diff}_i$$

can be extended to a unique continuous morphism  $\tilde{\rho}_i: U(4, \mathbb{R}) \rightarrow \text{Diff}_i(T)$  (see [13, 12]) so to an algebraic morphism<sup>(3)</sup>. Let us note that  $\tilde{\rho}_i(U(4, \mathbb{Z}))$  is an algebraic subgroup of  $\text{Diff}_i(T)$  that contains  $\rho_i(U(4, \mathbb{Z}))$ ; in particular  $\overline{H}_i = \overline{\rho_i(U(4, \mathbb{Z}))} \subset \tilde{\rho}_i(U(4, \mathbb{R}))$ . By construction the family  $(H_i)_i$  is filtered; since the extension is unique, the family  $(\tilde{\rho}_i)_i$  is also filtered. Therefore  $K = \varprojlim \overline{H}_i$  is well defined. Since  $\rho$  is injective,  $H$  is a nilpotent subgroup of length 3; as  $H \subset K$  and as any  $\overline{H}_i$  is nilpotent of length at most 3 the group  $K$  is nilpotent of length at most 3. For  $i$  sufficiently large  $\tilde{\rho}_i(U(4, \mathbb{R}))$  is nilpotent of length 3; this group is connected so its Lie algebra is also nilpotent of length 3. Therefore the image of

$$D\tilde{\rho} := \varprojlim D\tilde{\rho}_i: \mathfrak{u}(4, \mathbb{R}) \rightarrow \hat{\chi}(\mathbb{R}^2, 0)$$

is isomorphic to  $\mathfrak{n}$  (Proposition 5.2). So there exists a surjective map  $\psi$  from  $\mathfrak{u}(4, \mathbb{R})$  onto  $\mathfrak{n}$ . The kernel of  $\psi$  is an ideal of  $\mathfrak{u}(4, \mathbb{R})$  of dimension 2; hence  $\ker \psi = \langle \delta_{14}, a\delta_{13} + b\delta_{24} \rangle$  where the  $\delta_{ij}$  denote the Kronecker matrices. One concludes by noting that  $\dim Z(\mathfrak{u}(4, \mathbb{R})/\ker \psi) = 2$  whereas  $\dim Z(\mathfrak{n}) = 1$ .  $\square$

**Corollary 5.5.** — *The image of a morphism from a subgroup of  $SL(n, \mathbb{Z})$  of finite index to  $\text{Diff}^{\text{an}}(\mathbb{S}^2)$  is finite as soon as  $n \geq 4$ .*

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3. Let  $N_1$  and  $N_2$  be two connected, simply connected, nilpotent and algebraic subgroups of  $\mathbb{R}$ ; any continuous morphism from  $N_1$  to  $N_2$  is algebraic.

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, Institut de Mathématiques de Jussieu, UMR 7586, Université Paris 7, Bâtiment Sophie Germain, Case 7012, 75205 Paris Cedex 13, France.

JULIE DÉSERTI • *E-mail* : deserti@math.jussieu.fr