

## Quarto-quartic birational maps of $\mathbb{P}_3(\mathbb{C})$

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We construct a determinantal family of quarto-quartic transformations of a complex projective space of dimension 3 from trigonal curves of degree 8 and genus 5. Moreover, we show that the variety of  $(4, 4)$ -birational maps of  $\mathbb{P}_3(\mathbb{C})$  has at least four irreducible components and describe three of them.

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### 1. Introduction

Let  $\mathbb{P}_3$  and  $\mathbb{P}'_3$  be two complex projective spaces of dimension 3. Denote by  $\text{Bir}_d(\mathbb{P}_3, \mathbb{P}'_3)$  the set of birational maps from  $\mathbb{P}_3$  to  $\mathbb{P}'_3$  defined by a linear system of degree  $d$  with base locus of codimension at least 2. A rational map  $\phi : \mathbb{P}_3 \dashrightarrow \mathbb{P}'_3$  is said to have bidegree  $(d_1, d_2)$  if it is an element of  $\text{Bir}_{d_1}(\mathbb{P}_3, \mathbb{P}'_3)$  such that its inverse is in  $\text{Bir}_{d_2}(\mathbb{P}'_3, \mathbb{P}_3)$ . Let  $\text{Bir}_{d_1, d_2}(\mathbb{P}_3, \mathbb{P}'_3)$  be the quasi projective variety [13] of birational maps from  $\mathbb{P}_3$  to  $\mathbb{P}'_3$  of bidegree  $(d_1, d_2)$ . It is obvious that a general surface in a linear system of degree  $d$  defining an element of  $\text{Bir}_d(\mathbb{P}_3, \mathbb{P}'_3)$  must be rational, but it should have a strong impact for  $d > 3$ . Our first motivations to study  $\text{Bir}_{4,4}(\mathbb{P}_3, \mathbb{P}'_3)$  were to understand some typical differences between  $d \leq 3$  and  $d > 3$ . For  $d \leq 3$ , a large amount of examples was already known a long time ago [11] and recent works such as [15] or [4] were more involved with classification problems. But for  $d > 3$ , only few examples are known. Basically, for  $d > 3$ , the classical examples are built by lifting a birational transformation of a projective space of lower

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dimension [5, § 7.2.3]. In particular, for  $d \geq 2$ , monoidal linear systems give the de Jonquières family  $\mathcal{J}_{d,d}$  (see [13, 14]). Another classical family (denoted by  $\mathcal{R}_{d,d}$ ) can be constructed with surfaces of degree  $d$  with a line of multiplicity  $d - 1$ . As expected from the components of  $\text{Bir}_{3,3}(\mathbb{P}_3, \mathbb{P}'_3)$ , we show that the closure of  $\mathcal{R}_{4,4}$  is an irreducible component of  $\text{Bir}_{4,4}(\mathbb{P}_3, \mathbb{P}'_3)$  (Proposition 3.13). A first surprise arises from normality properties. Indeed, in  $\text{Bir}_3(\mathbb{P}_3, \mathbb{P}'_3)$ , linear systems without a normal surface were exceptional and similar to elements of  $\mathcal{R}_{3,3}$ , but it turns out that any element of  $\text{Bir}_{4,4}(\mathbb{P}_3, \mathbb{P}'_3)$  with a linear system containing a normal surface must be in  $\mathcal{J}_{4,4}$  (Lemma 3.5). So, the closure of  $\mathcal{J}_{4,4}$  is an irreducible component of  $\text{Bir}_{4,4}(\mathbb{P}_3, \mathbb{P}'_3)$  (Corollary 3.6) and most of the work occurs with nonnormal surfaces. On the other hand,  $\mathcal{J}_{3,3}$  is not an irreducible component of  $\text{Bir}_{3,3}(\mathbb{P}_3, \mathbb{P}'_3)$  because it is in the boundary of the classical cubo-cubic determinantal family [4, § 4.2.2]. The classical cubo-cubic transformations [5, 7.2.2] are built from Arithmetically Cohen Macaulay curves of degree 6 and genus 3. This family is so particular [12] that it was unexpected to find in  $\text{Bir}_{4,4}(\mathbb{P}_3, \mathbb{P}'_3)$  another ACM family. Hence, most of the present work is about the construction of the ACM family  $\mathcal{D}_{4,4}$  and its geometric properties. We first define elements of  $\mathcal{D}_{4,4}$  via a direct construction of the resolution of their base ideal. This resolution is nongeneric among the maps of the following type:

$$\mathcal{O}_{\mathbb{P}_3}^2(-H) \oplus \mathcal{O}_{\mathbb{P}_3}(-2H) \longrightarrow \mathcal{O}_{\mathbb{P}_3}^4$$

and we provide an explicit construction in Sec. 4.3. Then, from Corollary 4.14 and Proposition 4.20, we get the following geometric description of  $\mathcal{D}_{4,4}$ .

**Theorem A.** *Let  $\Gamma$  be a trigonal curve of genus 5, and let  $\bar{\Gamma}$  be its embedding in  $\mathbb{P}_3$  by a general linear system  $|\mathcal{O}_{\Gamma}(H)|$  of degree 8. Then,*

- *the quartic surfaces containing  $\bar{\Gamma}$  and singular along its unique 5-secant line give an element  $\phi_{\bar{\Gamma}}$  of  $\mathcal{D}_{4,4}$ ;*
- *the inverse of  $\phi_{\bar{\Gamma}}$  is obtained by the same construction with the same trigonal curve  $\Gamma$ , but embedded in  $\mathbb{P}'_3$  with  $|\omega_{\Gamma}^{\otimes 2}(-H)|$ .*

We also geometrically describe the  $P$ -locus of  $\phi_{\bar{\Gamma}}$  (see Sec. 4.4). Finally, we exhibit another family  $\mathcal{C}_{4,4}$  constructed with linear systems of quartic surfaces with a double conic. While studying components of  $\text{Bir}_{4,4}(\mathbb{P}_3, \mathbb{P}'_3)$ , it turns out that the following invariant allows one to distinguish all the components of  $\text{Bir}_{4,4}(\mathbb{P}_3, \mathbb{P}'_3)$  we find. Again, it is different in  $\text{Bir}_3(\mathbb{P}_3, \mathbb{P}'_3)$  where this invariant is not meaningful because in most examples it is 1 [4, Proposition 2.6].

**Definition 1.1** ([11, Chap. IX]). Let  $\phi$  be an element of  $\text{Bir}_d(\mathbb{P}_3, \mathbb{P}'_3)$ , and let  $H$  (respectively  $H'$ ) be a general hyperplane of  $\mathbb{P}_3$  (respectively  $\mathbb{P}'_3$ ). The genus of  $\phi$  is the geometric genus of the curve  $H \cap \phi^{-1}(H')$ .

**Theorem B.** *The quasi projective variety  $\text{Bir}_{4,4}(\mathbb{P}_3, \mathbb{P}'_3)$  has at least four irreducible components. Three of them are the closure of the families  $\mathcal{D}_{4,4}$ ,  $\mathcal{J}_{4,4}$  and  $\mathcal{R}_{4,4}$ . The*

family  $\mathcal{C}_{4,4}$  is in another irreducible component. Furthermore, we have

$$\dim(\mathcal{R}_{4,4}) = 37, \quad \dim(\mathcal{C}_{4,4}) = 37, \quad \dim(\mathcal{D}_{4,4}) = 46, \quad \dim(\mathcal{J}_{4,4}) = 54,$$

$$g(\phi_{\mathcal{R}_{4,4}}) = 0, \quad g(\phi_{\mathcal{C}_{4,4}}) = 1, \quad g(\phi_{\mathcal{D}_{4,4}}) = 2, \quad g(\phi_{\mathcal{J}_{4,4}}) = 3,$$

where  $g(\phi_X)$  denotes the genus of an element  $\phi_X$  general in the family  $X$ .

## 2. Definitions and First Properties

Let  $\phi : \mathbb{P}_3 \dashrightarrow \mathbb{P}'_3$  be a rational map given, for some choice of coordinates, by

$$(z_0 : z_1 : z_2 : z_3)$$

$$\dashrightarrow (\phi_0(z_0, z_1, z_2, z_3) : \phi_1(z_0, z_1, z_2, z_3) : \phi_2(z_0, z_1, z_2, z_3) : \phi_3(z_0, z_1, z_2, z_3)),$$

where the  $\phi_i$ 's are homogeneous polynomials of the same degree  $d$  without common factors. The *indeterminacy set* of  $\phi$  is the set of the common zeros of the  $\phi_i$ 's. Denote by  $J_\phi$  the ideal generated by the  $\phi_i$ 's. Let  $F_\phi$  be the scheme defined by  $J_\phi$ ; it is called *base locus* or *base scheme* of  $\phi$ . If  $\dim F_\phi = 1$ , then define  $F_\phi^1$  as the maximal subscheme of  $F_\phi$  of dimension 1 without isolated point and without embedded point. The *P-locus* of  $\phi$  is the union of irreducible hypersurfaces that are blown down to subvarieties of codimension at least 2 (see [5, § 7.1.4]).

**Lemma 2.1.** *Let  $\phi : \mathbb{P}_3 \dashrightarrow \mathbb{P}'_3$  be a  $(4, d)$ -birational map. Let  $H'$  be a general hyperplane of  $\mathbb{P}'_3$ . Let  $\gamma$  be the reduced scheme with support the singular locus of  $\phi^{-1}(H')$  and consider the secant variety*

$$\text{Sec}(\gamma) = \overline{\bigcup_{\substack{p, q \in \gamma \\ p \neq q}} \delta_{p,q}} \subset \mathbb{P}_3,$$

where  $\delta_{p,q}$  denotes the line through  $p$  and  $q$ . Then, the codimension of  $\text{Sec}(\gamma)$  in  $\mathbb{P}_3$  is strictly positive.

**Proof.** Let  $p, q$  be two distinct points of  $\gamma$ . The line  $\delta_{p,q}$  passing through  $p$  and  $q$  intersects  $F_\phi^1$  with length at least 4 since we have the inclusion of ideals  $J_{F_\phi^1} \subset J_\gamma^2$ ; the line  $\delta_{p,q}$  is thus blown down by  $|J_{F_\phi^1}(4H)|$ .

But  $\phi$  is dominant, so a general point of  $\mathbb{P}_3$  is not in the  $P$ -locus of  $\phi$ . Hence,  $\text{codim}_{\mathbb{P}_3}(\text{Sec}(\gamma)) > 0$ . □

**Corollary 2.2.** *Let  $\phi : \mathbb{P}_3 \dashrightarrow \mathbb{P}'_3$  be a  $(4, d)$ -birational map. Let  $H'$  be a general hyperplane of  $\mathbb{P}'_3$ . Let  $\gamma$  be the reduced scheme defined by the singular locus of  $\phi^{-1}(H')$ . If  $\dim \gamma = 1$ , then the one-dimensional part of  $\gamma$  is either a line, or a conic (possibly degenerated), or 3 lines through a fixed point.*

**Proof.** A general hyperplane section of the quartic surface  $\phi^{-1}(H')$  is an irreducible reduced plane quartic curve. So, it has at most 3 singular points hence

$\deg(\gamma) \leq 3$  when  $\dim(\gamma) = 1$ . Moreover,  $\gamma$  cannot contain a plane curve of degree 3 because  $\phi^{-1}(H')$  is an irreducible quartic surface. From Lemma 2.1, neither a space cubic curve, nor two disjoint lines, nor a line and a smooth conic can be in  $\gamma$ . Hence only a line, a conic or 3 lines through a fixed point are possible.  $\square$

Unfortunately, we have not studied<sup>a</sup> families with 3 lines through a point because we first missed such examples.

**Example 2.3 (Loria, 1890, see for instance [11, Chap. XIV §8]).** Let  $l_0, l_1, l_2$  be lines in  $\mathbb{P}_3$  of ideal  $(z_1, z_2), (z_2, z_0), (z_0, z_1)$  and  $O_1, O_2, O_3$  be three general points in the plane  $z_3 = 0$ . For  $1 \leq j \leq 3$ , let  $q_j$  be the quadric cone containing the lines  $l_0, l_1, l_2$  and the two points of  $O_1, O_2, O_3$  distinct from  $O_j$ . Then, the following quartics

$$(q_1q_2, q_0q_2, q_0q_1, z_0z_1z_2z_3)$$

gives a quarto-quartic birational transformation; the general element of this linear system is a Steiner quartic.

### 2.1. Families in $\text{Bir}_{p,q}(\mathbb{P}_3, \mathbb{P}'_3)$ and semicontinuity

**Definition 2.4.** Let  $p, q$  be integers such that  $\text{Bir}_{p,q}(\mathbb{P}_3, \mathbb{P}'_3)$  is not empty, and let  $S$  be a reduced subscheme of  $\text{Bir}_{p,q}(\mathbb{P}_3, \mathbb{P}'_3)$ . From the natural embedding of

$$\text{Bir}_{p,q}(\mathbb{P}_3, \mathbb{P}'_3) \subset \mathbb{P}(\text{Hom}(H^0(\mathcal{O}_{\mathbb{P}'_3}(1)), H^0(\mathcal{O}_{\mathbb{P}_3}(p))))$$

we obtain by pull back from  $S$  to  $S \times \mathbb{P}_3$  the tautological sequence

$$H^0(\mathcal{O}_{\mathbb{P}'_3}(1)) \otimes \mathcal{O}_{S \times \mathbb{P}_3}(-1, 0) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}_3}(p)) \otimes \mathcal{O}_{S \times \mathbb{P}_3}.$$

It gives the following two maps and we can globalize over  $S$  the previous definitions of the base scheme and its singular locus.

— Let  $F_S$  be the subscheme of  $S \times \mathbb{P}_3$  defined by the vanishing of the tautological map

$$H^0(\mathcal{O}_{\mathbb{P}'_3}(1)) \otimes \mathcal{O}_{S \times \mathbb{P}_3}(-1, 0) \longrightarrow \mathcal{O}_{S \times \mathbb{P}_3}(0, p).$$

— Let  $F_S^\sigma$  be the subscheme of  $S \times \mathbb{P}_3$  defined by the vanishing of the Jacobian map

$$H^0(\mathcal{O}_{\mathbb{P}'_3}(1)) \otimes (H^0(\mathcal{O}_{\mathbb{P}_3}(1)))^\vee \otimes \mathcal{O}_{S \times \mathbb{P}_3}(-1, 0) \longrightarrow \mathcal{O}_{S \times \mathbb{P}_3}(0, p-1).$$

— For an hyperplane  $H$  of  $\mathbb{P}_3$ , we define the restrictions to  $H$  as the following intersections in  $S \times \mathbb{P}_3$ :

$$F_S \cdot H = F_S \cap (S \times H), \quad F_S^\sigma \cdot H = F_S^\sigma \cap (S \times H).$$

<sup>a</sup>We don't expect that this example is related to one of the four components described in Theorem B; it seems to us that it should give one more component of  $\text{Bir}_{4,4}(\mathbb{P}_3, \mathbb{P}'_3)$ .

**Corollary 2.5.** *The morphisms  $F_S \rightarrow S$  and  $F_S^\sigma \rightarrow S$  are projective morphisms, moreover all the fibers of  $F_S \rightarrow S$  have the same dimension.*

- If  $q < p^2$ , we have for all  $\phi$  in  $S$ ,  $\dim F_\phi = 1$  (see Notation 3.4),
- If  $q = p^2$ , we have for all  $\phi$  in  $S$ ,  $\dim F_\phi = 0$ .

Let us recall the following application of the semicontinuity of the dimension of the fibers of a coherent sheaf.

**Lemma 2.6.** *Let  $f : X \rightarrow Y$  be a finite morphism of Noetherian schemes over  $\mathbb{C}$ . The map*

$$Y \rightarrow \mathbb{Z}, \quad y \mapsto \deg(X_y)$$

*is upper semicontinuous.*

**Proof.** Let  $\mathbb{C}(y)$  be the residual field of  $Y$  at  $y$ . Remark that the degree of  $X_y$  is the dimension of the complex vector space  $f_*(\mathcal{O}_X) \otimes \mathbb{C}(y)$  because  $f$  is finite. By [9, Theorem III-8.8], the sheaf  $f_*(\mathcal{O}_X)$  is coherent on  $Y$ , so we conclude by semicontinuity of the dimension of the fibers of a coherent sheaf (see [9, Example III, 12.7.2]). □

**Corollary 2.7.** *With the previous notation, the following functions are upper semicontinuous:*

- (1)  $\alpha : S \rightarrow \mathbb{Z}, \quad \phi \mapsto \deg(F_\phi)$ .
- (2)  $\beta : S \rightarrow \mathbb{Z}, \quad \phi \mapsto \dim(F_\phi^\sigma)$ .
- (3)  $\eta : \beta^{-1}(\{1\}) \rightarrow \mathbb{Z}, \quad \phi \mapsto \deg(F_\phi^\sigma)$ .

**Proof.** The semicontinuity of  $\beta$  is from Chevalley’s theorem (see [8], 13.1.5) because  $F_S^\sigma \rightarrow S$  is proper. Let us prove the semicontinuity of  $\alpha$  by showing that for any  $n$  in  $\mathbb{Z}$  the set  $\alpha^{-1}(]-\infty, n])$  is open. If it is not empty, then consider an arbitrary element  $\psi$  such that  $\alpha(\psi) \leq n$ . We can chose a hyperplane  $H$  of  $\mathbb{P}_3$  such that  $F_\psi \cap H$  is artinian of length  $\alpha(\psi) = \deg(F_\psi)$ . By Corollary 2.5, the projective morphism  $\pi_S : F_S \cdot H \rightarrow S$  is generically finite. According to Chevalley’s theorem, the set

$$U = \{\phi \in S, \dim(\pi_S^{-1}(\{\phi\})) \leq 0\}$$

is an open set of  $S$  containing  $\psi$ . Hence, over  $U$ , we have a finite morphism  $\pi_U : F_U \cdot H \rightarrow U$ . Applying Lemma 2.6 to  $\pi_U$ , we have an open set  $U'$  such that

$$\forall \phi \in U', \quad \alpha(\phi) = \deg(F_\phi) \leq \text{length}(F_\phi \cap H) \leq \text{length}(F_\psi \cap H) = \alpha(\psi) \leq n.$$

So, for any element  $\psi$  of  $\alpha^{-1}(]-\infty, n])$ , there is an open set  $U'$  of  $S$  such that

$$\psi \in U', \quad U' \subset \alpha^{-1}(]-\infty, n]).$$

In conclusion,  $\alpha^{-1}(]-\infty, n])$  is open and  $\alpha$  is upper semicontinuous on  $S$ . The proof is the same for  $\eta$  because it is also a projective morphism such that all its fibers are one-dimensional. □

Let us also recall the following:

**Lemma 2.8.** *Let  $\pi : X \rightarrow Y$  be a projective morphism of noetherian schemes, and  $\Omega_\pi$  be the sheaf of relative differentials, then for all  $n \in \mathbb{N}$ , the set*

$$\{x \in X, \text{rank}((\Omega_\pi)_x) \geq n\}$$

*is closed in  $X$ , so its image by  $\pi$  is closed in  $Y$ .*

**Proof.** From [9, Remark II, 8.9.1], the sheaf  $\Omega_\pi$  is coherent on  $X$ . As a result, the rank of its fiber is upper semicontinuous by [9, Example III, 12.7.2] and  $\{x \in X, \text{rank}((\Omega_\pi)_x) \geq n\}$  is closed in  $X$ . But,  $\pi$  is projective, hence the image of this closed set is closed in  $Y$ . □

### 3. Two Classical Families in $\text{Bir}_{d,d}(\mathbb{P}_3, \mathbb{P}'_3)$

It is a classical construction to obtain birational transformations over projective spaces from a factorization through the blow up of a linear space and a birational transformation between projective spaces of lower dimension [5, Definition 7.2.9]. The following one is a typical example of this construction.

#### 3.1. The Monoidal de Jonquière's family $\mathcal{J}_{d,d}$

Many equivalent properties are detailed in [13] or [14] so let us choose the following definition.

**Definition 3.1.** Let  $d$  be a nonzero integer, and let  $p$  be a point of  $\mathbb{P}_3$ . Choose coordinates such that the ideal of  $p$  in  $\mathbb{P}_3$  is  $\mathcal{J}_p = (z_0, z_1, z_2)$ . Consider  $S_d = z_3 P_{d-1} + P_d$  and  $S_{d-1} = z_3 Q_{d-2} + Q_{d-1}$  with  $P_i, Q_i$  in  $\mathbb{C}[z_0, z_1, z_2]$  homogeneous of degree  $i$  such that  $\text{gcd}(S_{d-1}, S_d) = 1$  and  $P_{d-1} Q_{d-1} \neq P_d Q_{d-2}$ . The rational map  $\phi : \mathbb{P}_3 \dashrightarrow \mathbb{P}'_3$  defined by an isomorphism  $\mathbb{P}'_3 \simeq |\mathcal{J}_\phi(dH)|^\vee$  with

$$\mathcal{J}_\phi = S_{d-1} \cdot \mathcal{J}_p + (S_d)$$

is birational of bidegree  $(d, d)$ . The base scheme  $F_\phi$  is defined by the monoidal complete intersection  $(S_{d-1}, S_d)$  with an immersed point at  $p$ . Denote by  $\mathcal{J}_{d,d}$  the corresponding family of  $\text{Bir}_{d,d}(\mathbb{P}_3, \mathbb{P}'_3)$ .

**Remark 3.2.** With the notation of Definition 3.1, we can find coordinates on  $\mathbb{P}'_3$  such that the map  $\phi$  can be written as

$$\left( z_0 : z_1 : z_2 : \frac{P_{d-1}z_3 + P_d}{Q_{d-2}z_3 + Q_{d-1}} \right),$$

it is then clear that the inverse of  $\phi$

$$\left( z_0 : z_1 : z_2 : \frac{Q_{d-1}z_3 - P_d}{-Q_{d-2}z_3 + P_{d-1}} \right)$$

is also a monoidal de Jonquière's element of  $\text{Bir}_{d,d}(\mathbb{P}'_3, \mathbb{P}_3)$ .

**Corollary 3.3 ([13]).** *The family  $\mathcal{J}_{d,d} \subset \text{Bir}_{d,d}(\mathbb{P}_3, \mathbb{P}'_3)$  is irreducible of dimension  $2d^2 + 2d + 14$ .*

**Proof.** The choice of  $S_{d-1}$  up to factor is  $\binom{d}{d-2} + \binom{d+1}{d-1} - 1$  dimensional. The choice of  $S_d$  up to factor with  $S_d \notin S_{d-1} \cdot \mathbb{C}[z_0, z_1, z_2]$  has dimension  $\binom{d+2}{d} + \binom{d+1}{d-1} - 3 - 1$ . With the choices of an automorphism of  $\mathbb{P}'_3$  and of a point  $p \in \mathbb{P}_3$ , we have  $\dim \mathcal{J}_{d,d} = \binom{d}{d-2} + 2\binom{d+1}{d-1} + \binom{d+2}{d} + 13$ .  $\square$

Let us recall that  $\mathcal{J}_{3,3}$  is not an irreducible component of  $\text{Bir}_{3,3}(\mathbb{P}_3, \mathbb{P}'_3)$  (see [4, §4.2.2]), so the next Lemma is quite unexpected.

**Notation 3.4.** Let  $\phi : \mathbb{P}_3 \dashrightarrow \mathbb{P}'_3$  be a birational transformation of bidegree  $(p, q)$ . For a general line  $l'$  in  $\mathbb{P}'_3$ , we will denote by

- $\phi^{-1}(l')$  the complete intersection in  $\mathbb{P}_3$  of degree  $p^2$  defined by  $l'$ .
- $C_1 = \phi_*^{-1}(l') \subset \mathbb{P}_3$  the closure of the image by  $\phi^{-1}$  of points of  $l'$  where  $\phi^{-1}$  is regular. As a result  $C_1$  is an irreducible curve of degree  $q$  and geometric genus 0.
- $C_2$  the residual scheme of  $C_1$  in  $\phi^{-1}(l')$ . (Note that  $F_\phi^1$  and  $C_2$  are set theoretically identical, but in general  $F_\phi^1$  is just a subscheme of  $C_2$ .)

Furthermore, for any scheme  $Z$ , we denote by  $\mathcal{J}_Z$  its ideal.

**Lemma 3.5.** *Let  $\phi : \mathbb{P}_3 \dashrightarrow \mathbb{P}'_3$  be a  $(4, 4)$ -birational map. Let  $H'$  be a general hyperplane of  $\mathbb{P}'_3$  and denote by  $S$  the surface  $\phi^{-1}(H')$ . If  $S$  is normal, then  $C_1$  is a plane rational quartic and  $\phi$  belongs to  $\mathcal{J}_{4,4}$ .*

**Proof.** The curves  $C_1$  and  $C_2$  are geometrically linked by quartic surfaces, so the dualizing sheaf of  $C_1 \cup C_2$  is  $\omega_{C_1 \cup C_2} = \mathcal{O}_{C_1 \cup C_2}(4H)$  and we have the liaison exact sequence

$$0 \longrightarrow \mathcal{J}_{C_1 \cup C_2}(4H) \longrightarrow \mathcal{J}_{C_2}(4H) \longrightarrow \mathcal{H}om(\mathcal{O}_{C_1}, \mathcal{O}_{C_1 \cup C_2}(4H)) \longrightarrow 0.$$

The surface  $S$  is normal and contains  $F_\phi^1$ . Hence,  $F_\phi^1$  is generically locally complete intersection. So, we have the scheme equality  $F_\phi^1 = C_2$ . Thus,  $h^0(\mathcal{J}_{C_2}(4H)) = h^0(\mathcal{J}_{F_\phi^1}(4H)) \geq 4$ . Therefore, from the liaison exact sequence, the dualizing sheaf  $\omega_{C_1} = \mathcal{H}om(\mathcal{O}_{C_1}, \mathcal{O}_{C_1 \cup C_2}(4H))$  has at least two independent sections. In other words the arithmetic genus of  $C_1$  is greater or equal to 2. By Castelnuovo inequality (cf. [10, Theorem 3.3]), the arithmetic genus of a nondegenerate locally Cohen-Macaulay space curve of degree 4 is at most 1. Hence,  $C_1$  is a plane rational quartic. By liaison,  $C_2$  is a complete intersection  $(3, 4)$ . Moreover, it gives the equality  $\mathcal{H}om(\mathcal{O}_{C_1}, \mathcal{O}_{C_1 \cup C_2}(4H)) = \mathcal{O}_{C_1}(H)$ , so the restriction of  $\phi$  to  $C_1$  is a pencil of sections of  $\mathcal{O}_{C_1}(H)$ . But,  $\phi$  sends  $C_1$  birationally to a line of  $\mathbb{P}'_3$ , hence this pencil must be the lines through a triple point  $p$  of the plane curve  $C_1$ . As  $S$  has a finite singular locus, the point  $p$  is independent of the choice of  $l'$  and it is a triple point for all the quartics in the linear system  $|\mathcal{J}_{F_\phi}(4H)|$ . In conclusion,  $F_\phi$  is a monoidal complete intersection  $(3, 4)$  with  $p$  as immersed point. Besides,  $\phi$  belongs to  $\mathcal{J}_{4,4}$

because the irreducibility of  $S = S_d$  implies  $\gcd(S_{d-1}, S_d) = 1$  and  $P_{d-1}Q_{d-1} \neq Q_{d-2}P_d$ . □

**Corollary 3.6.** *The closure of  $\mathcal{J}_{4,4}$  is an irreducible component of  $\text{Bir}_{4,4}(\mathbb{P}_3, \mathbb{P}'_3)$ .*

### 3.2. The ruled family $\mathcal{R}_{d,d}$

In this paragraph, we will prove the following direct application of Lemma 3.10:

**Proposition 3.7.** *Consider a line  $\delta$  in  $\mathbb{P}_3$  and an integer  $d$  with  $d \geq 2$ . Let  $\Delta_1, \dots, \Delta_{d-1}$  be  $d - 1$  general disjoint lines that intersect  $\delta$  in length one, and let  $p_1, \dots, p_{d-1}$  be  $(d - 1)$  general points in  $\mathbb{P}_3$ . Then the linear system  $|\mathcal{J}_\delta^{d-1} \cap \mathcal{J}_{\Delta_1} \cap \dots \cap \mathcal{J}_{\Delta_{d-1}} \cap \mathcal{J}_{p_1} \cap \dots \cap \mathcal{J}_{p_{d-1}}(d \cdot H)|$  gives a birational map*

$$\mathbb{P}_3 \dashrightarrow |\mathcal{J}_\delta^{d-1} \cap \mathcal{J}_{\Delta_1} \cap \dots \cap \mathcal{J}_{\Delta_{d-1}} \cap \mathcal{J}_{p_1} \cap \dots \cap \mathcal{J}_{p_{d-1}}(d \cdot H)|^\vee.$$

Moreover, the base scheme is defined by the ideal  $\mathcal{J}_\delta^{d-1} \cap \mathcal{J}_{\Delta_1} \cap \dots \cap \mathcal{J}_{\Delta_{d-1}} \cap \mathcal{J}_{p_1} \cap \dots \cap \mathcal{J}_{p_{d-1}}$  and has degree  $\frac{(d+2)(d-1)}{2}$ .

**Definition 3.8.** Birational maps of Proposition 3.7 composed with an isomorphism

$$|\mathcal{J}_\delta^{d-1} \cap \mathcal{J}_{\Delta_1} \cap \dots \cap \mathcal{J}_{\Delta_{d-1}} \cap \mathcal{J}_{p_1} \cap \dots \cap \mathcal{J}_{p_{d-1}}(d \cdot H)|^\vee \simeq \mathbb{P}'_3$$

form the family  $\mathcal{R}_{d,d} \subset \text{Bir}_{d,d}(\mathbb{P}_3, \mathbb{P}'_3)$ .

**Remark 3.9.** Note that surfaces of degree  $d$  in such linear systems have a line of multiplicity  $d - 1$ , so they are ruled. For  $d > 3$ , there may exist ruled surfaces of degree  $d$  without a line of multiplicity  $d - 1$ . Nevertheless, we keep the terminology “ruled” for  $\mathcal{R}_{d,d}$  to emphasize the analogy with the traditional naming convention when  $d = 3$ .

**Lemma 3.10.** *With notation of Proposition 3.7, the image of  $\mathbb{P}_3$  by the linear system*

$$|\mathcal{J}_\delta^{d-1} \cap \mathcal{J}_{\Delta_1} \cap \dots \cap \mathcal{J}_{\Delta_{d-1}}(d \cdot H)|$$

is a nondegenerate three-dimensional variety  $X_d$  of degree  $d$  of  $\mathbb{P}_{d+2}$ .

**Remark 3.11.** Let us note that  $X_d$  is of minimal degree (see [6]).

**Proof of Lemma 3.10.** Let us choose coordinates such that the ideal of  $\delta$  in  $\mathbb{P}_3$  is  $\mathcal{J}_\delta = (z_0, z_1)$ . Let  $\mathcal{J}_{\Delta_j}$  be  $(\ell_j, \ell'_j)$  with

- $\gcd(\ell_j, \ell'_k) = 1$  for  $1 \leq k, j < d$ ;
- $\gcd(\ell_j, \ell_k) = 1, \gcd(\ell'_j, \ell'_k) = 1$  for  $1 \leq k, j < d, k \neq j$ ;
- $\ell_j \in \mathcal{J}_\delta, \gcd(\ell_j, z_0) = 1$ ;
- $\ell'_j \notin \mathcal{J}_\delta$ .



Set  $\mathcal{F}_0 = \mathcal{J}_\delta^{d-1} \cap \mathcal{J}_{\Delta_1} \cap \dots \cap \mathcal{J}_{\Delta_{d-1}}$ . The ideal  $\mathcal{F}_0$  is defined by the  $(d-1) \times (d-1)$  minors of the matrix

$$M = \begin{pmatrix} z_0 \ell'_1 & z_0 \ell'_2 & \dots & z_0 \ell'_{d-1} \\ \ell_1 & 0 & \dots & 0 \\ 0 & \ell_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \ell_{d-1} \end{pmatrix}.$$

Let  $\widetilde{\mathbb{P}}_3 \subset \mathbb{P}_1 \times \mathbb{P}_3$  be the blow up of  $\mathbb{P}_3$  at  $\delta$ . Denote by  $s$  (respectively  $H$ ) the hyperplane class of  $\mathbb{P}_1$  (respectively  $\mathbb{P}_3$ ) and by  $\widetilde{M}$  the strict transform of the entries of  $M$ . The resolution of the ideal of the proper transform of  $\cup_{i=1}^{d-1} \Delta_i$  is

$$0 \rightarrow \mathcal{O}_{\widetilde{\mathbb{P}}_3}^{d-1}(-s) \xrightarrow{\widetilde{M}} \mathcal{O}_{\widetilde{\mathbb{P}}_3}(H) \oplus \mathcal{O}_{\widetilde{\mathbb{P}}_3}^{d-1} \rightarrow \mathcal{J}_Z((d-1)s + H) \rightarrow 0. \tag{3.1}$$

This surjection gives a rational map

$$\widetilde{\mathbb{P}}_3 \dashrightarrow X_d \subset \text{Proj}(\text{Sym}(\mathcal{O}_{\widetilde{\mathbb{P}}_3}^{d-1} \oplus \mathcal{O}_{\widetilde{\mathbb{P}}_3}(H))) \subset \widehat{\mathbb{P}}_3 \times \mathbb{P}_{d+2}$$

with  $\mathbb{P}_{d+2} = |\mathcal{J}_Z((d-1)s + H)|^\vee$ . Denote by  $H'$  the hyperplane class of  $\mathbb{P}_{d+2}$ , so the class of  $X_d$  in  $\text{Proj}(\text{Sym}(\mathcal{O}_{\widetilde{\mathbb{P}}_3}^{d-1} \oplus \mathcal{O}_{\widetilde{\mathbb{P}}_3}(H)))$  is  $(H' + s)^{d-1}$  with the relation  $H'^d = H'^{d-1} \cdot H$  [7, Example 8.3.4]. Since we have the equality

$$(H' + s)^{d-1} \cdot H'^3 = d,$$

the image of  $\widetilde{\mathbb{P}}_3$  by  $|\mathcal{J}_Z((d-1)s + H)|$  is of degree  $d$  in  $\mathbb{P}_{d+2}$ . Moreover, from resolution (3.1), it is not in a hyperplane of  $\mathbb{P}_{d+2}$ .  $\square$

**Proof of Proposition 3.7.** With notation of Lemma 3.10, choose  $d-1$  points  $(p_i)_{1 \leq i < d}$  in general position in  $X_d$ . They span in  $\mathbb{P}_{d+2}$  a projective space  $\pi_{d-2}$  of dimension  $d-2$ . The projection from  $\pi_{d-2}$  restricts to a birational map from  $X_d$  to a three-dimensional projective space. To see that the intersection  $\pi_{d-2} \cap X_d$  is the reduced scheme defined by the  $(p_i)_{1 \leq i < d}$ , let us consider an additional general point  $p_d$  of  $X_d$  and denote by  $\pi_{d-1}$  the projective space spanned by the  $(p_i)_{1 \leq i \leq d}$ . The threefold  $X_d$  is nondegenerate of degree  $d$  in  $\mathbb{P}_{d+2}$ , so  $X_d \cap \pi_{d-1}$  is the intersection of  $X_d$  by a general linear space of codimension three, hence it is just the  $d$  points  $(p_i)_{1 \leq i \leq d}$ . In conclusion,  $\pi_{d-2} \cap X_d$  is the reduced scheme defined by the  $(p_i)_{1 \leq i < d}$ . Moreover, we can assume that these points are in the locus where  $\mathbb{P}_3 \dashrightarrow X_d$  is an isomorphism. As a result, the base scheme in  $\mathbb{P}_3$  of the composition  $\mathbb{P}_3 \dashrightarrow X_d$  with the linear projection from  $\pi_{d-2}$  is defined by the ideal  $\mathcal{J}_\delta^{d-1} \cap \mathcal{J}_{\Delta_1} \cap \dots \cap \mathcal{J}_{\Delta_{d-1}} \cap \mathcal{J}_{p_1} \cap \dots \cap \mathcal{J}_{p_{d-1}}$ , it thus has degree  $\frac{d(d-1)}{2} + (d-1)$ .  $\square$

**Corollary 3.12.** *The family  $\mathcal{R}_{d,d} \subset \text{Bir}_{d,d}(\mathbb{P}_3, \mathbb{P}'_3)$  is irreducible of dimension*

$$4 + 3(d - 1) + 3(d - 1) + 15 = 6d + 13.$$

**Proposition 3.13.** *The closure of  $\mathcal{R}_{4,4}$  is an irreducible component of  $\text{Bir}_{4,4}(\mathbb{P}_3, \mathbb{P}'_3)$ .*

**Proof.** Let  $\mathcal{B}$  be an irreducible component of  $\text{Bir}_{4,4}(\mathbb{P}_3, \mathbb{P}'_3)$  containing  $\mathcal{R}_{4,4}$ . Let  $\phi$  be a general element of  $\mathcal{B}$ , and let  $\psi$  be a general element of  $\mathcal{R}_{4,4}$ . According to Proposition 3.7, the scheme  $F_\psi^1$  has degree 9 and  $\psi$  is in  $\mathcal{B}$ , thus by Corollary 2.7, the set  $\alpha^{-1}(] - \infty, 9])$  is nonempty and open in  $\mathcal{B}$  so the general element  $\phi$  satisfies  $\deg(F_\phi^1) \leq 9$ . With Notation 3.4, we have  $\deg C_2 = 12$ , so  $F_\phi^1$  has a component that is not locally complete intersection. Let  $H$  be a general hyperplane of  $\mathbb{P}_3$ , and let  $\mathcal{S}$  be the set of points  $p$  such that  $F_\phi^1 \cap H$  is not locally complete intersection at  $p$ . Denote by  $Z$  (respectively  $Y$ ) the union of the irreducible components of  $F_\phi^1 \cap H$  (respectively  $C_2 \cap H$ ) supported on points  $p$  of  $\mathcal{S}$ . We can now bound the degree of  $Z$ . By Corollary 2.5 and Chevalley’s semicontinuity theorem, there is an open subset  $U$  of  $\alpha^{-1}(] - \infty, 9])$  (hence of  $\mathcal{B}$ ) such that  $\psi \in U$  and  $\pi_U : F_U \cdot H \rightarrow U$  is finite (with notation of Definition 2.4). By Lemma 2.8, the set

$$\Sigma = \{(u, p) \in F_U \cdot H \subset U \times H, \text{rank}((\Omega_{\pi_U})_{(u,p)}) \geq 2\}$$

is closed in  $F_U \cdot H$ . As a consequence  $\pi_U : \Sigma \rightarrow U$  is also finite. Moreover, it is onto because  $U \subset \alpha^{-1}(] - \infty, 9])$ , so any element of  $U$  must have a base scheme that is not locally complete intersection. But, the base scheme of  $\psi$  have three smooth points so the degree of the fiber  $\Sigma_\psi$  is at most  $9 - 3 = 6$ . Lemma 2.6 implies the inequality  $\deg(\Sigma_\phi) \leq 6$  for the general element  $\phi$  of  $U$ , hence of  $\mathcal{B}$ . But,  $Z \subset \Sigma_\phi$  hence  $\deg(Z) \leq 6$  and  $\deg(Y) - \deg(Z) = \deg(C_2) - \deg(F_\phi^1) \geq 3$ . The cardinal of  $\mathcal{S}$  is at most 2 because  $\deg(Z) \leq 6$  and schemes of length at most 2 are locally complete intersection. We thus have the following possibilities:

- if  $\mathcal{S} = \{p_1, p_2\}$ ,  $p_1 \neq p_2$ , then  $J_Z = J_{p_1}^2 \cap J_{p_2}^2$  because  $\deg(Z) \leq 6$ . Therefore, each component of  $Y$  has length 4 and  $\deg(Y) - \deg(Z) = 8 - 6$  cannot be 3.
- if  $\mathcal{S} = \{p\}$  with  $J_p = (x, y)$ , then we have the following analytic classification of all non locally complete intersection ideals of  $\mathbb{C}\{x, y\}$  of colength at most 6 (see [3, §4.2])

Length ( $Z$ )	Ideal of $Z$	Length of a general complete intersection $Y$
$n + 1$	$(y^2, xy, x^n), 2 \leq n \leq 5$	$n + 2$
$n + 2$	$(y^2 + x^{n-1}, x^2y, x^n), 3 \leq n \leq 4$	$n + 3$
$n + 2$	$(y^2, x^2y, x^n), 3 \leq n \leq 4$	$2n$
6	$(y(y + x), x^2y, x^4)$	7
6	$(x^3, x^2y, xy^2, y^3)$	9

and note that the only solution to  $\deg(Y) - \deg(Z) \geq 3$ ,  $\deg(Z) \leq 6$  is the last case.

Hence,  $F_\phi^1$  has a triple line  $\Delta$  and a smooth curve of degree 3 denoted by  $\mathcal{K}$ . Any bisecant line to  $\mathcal{K}$  that intersects  $\Delta$  must be in all the quartics defining  $\phi$ , so it is in  $F_\phi^1$ . As a result,  $\mathcal{K}$  is a union of 3 disjoint lines that intersect  $\Delta$ , and  $\phi$  belongs to  $\mathcal{R}_{4,4}$ . □

#### 4. The Determinantal Family $\mathcal{D}_{4,4}$

**Notation 4.1.** Let  $\mathbb{P}_1, \mathbb{P}_3, \mathbb{P}'_3$  be three complex projective spaces of dimension 1, 3, 3; denote by  $s, H, H'$  their hyperplane class and by  $L, A, A'$  the following vector spaces:

$$L = H^0(\mathcal{O}_{\mathbb{P}_1}(s)), \quad A = H^0(\mathcal{O}_{\mathbb{P}_3}(H)), \quad A' = H^0(\mathcal{O}_{\mathbb{P}'_3}(H')).$$

##### 4.1. Construction of $\mathcal{D}_{4,4}$

###### 4.1.1. Description via $\tilde{\mathbb{P}}_3$

Let  $X$  be a complete intersection in  $\mathbb{P}_1 \times \mathbb{P}_3 \times \mathbb{P}'_3$  given by the vanishing of a general section of the bundle

$$\begin{aligned} &\mathcal{O}_{\mathbb{P}_1 \times \mathbb{P}_3 \times \mathbb{P}'_3}(s + H) \oplus \mathcal{O}_{\mathbb{P}_1 \times \mathbb{P}_3 \times \mathbb{P}'_3}(s + H') \oplus \mathcal{O}_{\mathbb{P}_1 \times \mathbb{P}_3 \times \mathbb{P}'_3}(H + H') \\ &\oplus \mathcal{O}_{\mathbb{P}_1 \times \mathbb{P}_3 \times \mathbb{P}'_3}(s + H + H'). \end{aligned}$$

**Lemma 4.2.** *The projections from  $X$  to  $\mathbb{P}_3$  and  $\mathbb{P}'_3$  are birational.*

**Proof.** It is an immediate computation from the class of

$$X \sim (s + H) \cdot (s + H') \cdot (H + H') \cdot (s + H + H'),$$

so  $X \cdot H^3 = s \cdot H^3 \cdot H'^3 = X \cdot H'^3$ . □

**Notation 4.3.** Let  $n_0$  (respectively  $n_1$ ) be a nonzero section of  $\mathcal{O}_{\mathbb{P}_1 \times \mathbb{P}_3 \times \mathbb{P}'_3}(s + H)$  (respectively  $\mathcal{O}_{\mathbb{P}_1 \times \mathbb{P}_3 \times \mathbb{P}'_3}(s + H')$ ) vanishing on  $X$ . The section  $n_0$  defines in  $\mathbb{P}_1 \times \mathbb{P}_3$  a divisor  $\tilde{\mathbb{P}}_3$  isomorphic to the blow up of  $\mathbb{P}_3$  along a line  $\Delta$ .

**Lemma 4.4.** *The complete intersection  $X$  is the blow up of  $\tilde{\mathbb{P}}_3$  along the curve  $\Gamma$  of ideal  $\mathcal{J}_{\Gamma|\tilde{\mathbb{P}}_3}$  with resolution defined by a general map  $\tilde{G}$*

$$0 \rightarrow \mathcal{O}_{\tilde{\mathbb{P}}_3}(-s) \oplus \mathcal{O}_{\tilde{\mathbb{P}}_3}(-H) \oplus \mathcal{O}_{\tilde{\mathbb{P}}_3}(-s - H) \xrightarrow{\tilde{G}} A' \otimes \mathcal{O}_{\tilde{\mathbb{P}}_3} \rightarrow \mathcal{J}_{\Gamma|\tilde{\mathbb{P}}_3}(2s + 2H) \rightarrow 0. \tag{4.1}$$

Moreover,

- the curve  $\Gamma$  is smooth irreducible of genus 5 and  $|\mathcal{O}_\Gamma(s)|$  is a  $g_3^1$ ,
- the sheaf  $\mathcal{O}_\Gamma(H)$  has degree 8.

**Proof.** The resolution of  $\mathcal{J}_{\Gamma|\tilde{\mathbb{P}}_3}(2s+2H)$  claimed in (4.1) is just the direct image by the first projection of the resolution of  $\mathcal{O}_X(H')$  as  $\mathcal{O}_{\tilde{\mathbb{P}}_3 \times \mathbb{P}'_3}$ -module. So, the map  $\tilde{G}$  is general. Since the bundle  $A' \otimes (\mathcal{O}_{\tilde{\mathbb{P}}_3}(s) \oplus \mathcal{O}_{\tilde{\mathbb{P}}_3}(H) \oplus \mathcal{O}_{\tilde{\mathbb{P}}_3}(s+H))$  is globally generated, the curve  $\Gamma$  is smooth because it is the degeneracy locus of the general map  $\tilde{G}$  [2, § 4.1], see [16, Theorem 1]). Moreover, the resolution (4.1) gives the vanishing  $h^1(\mathcal{J}_{\Gamma|\tilde{\mathbb{P}}_3}) = 0$ . As a consequence  $h^0(\mathcal{O}_\Gamma) = 1$  and  $\Gamma$  is connected and smooth, so irreducible. The dualizing sheaf of  $\tilde{\mathbb{P}}_3$  is  $\omega_{\tilde{\mathbb{P}}_3} = \mathcal{O}_{\tilde{\mathbb{P}}_3}(-s-3H)$ , so  $\text{Ext}^1(\mathcal{J}_{\Gamma|\tilde{\mathbb{P}}_3}(2s+2H), \mathcal{O}_{\tilde{\mathbb{P}}_3}(s-H)) = \omega_\Gamma$ . Besides the functor  $\mathcal{H}om(\cdot, \mathcal{O}_{\tilde{\mathbb{P}}_3}(s-H))$  applied to sequence (4.1) gives the following exact sequence:

$$\begin{aligned} 0 \rightarrow \mathcal{O}_{\tilde{\mathbb{P}}_3}(-s-3H) \rightarrow A'^{\vee} \otimes \mathcal{O}_{\tilde{\mathbb{P}}_3}(s-H) \rightarrow \mathcal{O}_{\tilde{\mathbb{P}}_3}(2s-H) \oplus \mathcal{O}_{\tilde{\mathbb{P}}_3}(s) \\ \oplus \mathcal{O}_{\tilde{\mathbb{P}}_3}(2s) \rightarrow \omega_\Gamma \rightarrow 0. \end{aligned} \tag{4.2}$$

As a result  $h^0(\omega_\Gamma) = 5$  and  $\Gamma$  has genus 5. The class of  $\Gamma$  is obtained from the degeneracy locus formula for  $\tilde{G}$  by computing the coefficient of  $t^2$  in the series  $\frac{1}{(1-s \cdot t)(1-H \cdot t)(1-(s+H) \cdot t)}$ . Therefore, in the Chow ring of  $\tilde{\mathbb{P}}_3$ , one has

$$\Gamma \sim 5s \cdot H + 3H^2.$$

Hence,  $\Gamma \cdot H$  has degree 8,  $\Gamma \cdot s$  has degree 3 and  $|\mathcal{O}_\Gamma(s)|$  is a  $g_3^1$ . □

#### 4.1.2. Description in $\mathbb{P}_3$

Let  $G$  be the composition  $\tilde{G} \circ \begin{pmatrix} (E_\Delta) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (E_\Delta) \end{pmatrix}$ , where  $(E_\Delta)$  is an equation of the exceptional divisor  $E_\Delta$  of  $\tilde{\mathbb{P}}_3$ .

**Proposition 4.5.** *Let  $\mathcal{J}_{\tilde{\Gamma}}$  be the ideal of the projection  $\tilde{\Gamma}$  of  $\Gamma$  in  $\mathbb{P}_3$ , and let  $\mathcal{J}_\Delta$  be the ideal of the line  $\Delta$  defined in Notation 4.3. Then,*

- the line  $\Delta$  is 5-secant to  $\tilde{\Gamma}$ ,
- one has the following exact sequence:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}_3}^2(-H) \oplus \mathcal{O}_{\mathbb{P}_3}(-2H) \xrightarrow{G} A' \otimes \mathcal{O}_{\mathbb{P}_3} \rightarrow (\mathcal{J}_\Delta^2 \cap \mathcal{J}_{\tilde{\Gamma}})(4H) \rightarrow 0, \tag{4.3}$$

- the linear system  $|(\mathcal{J}_\Delta^2 \cap \mathcal{J}_{\tilde{\Gamma}})(4H)|$  gives a birational map from  $\mathbb{P}_3$  to  $\mathbb{P}'_3$  that factors through  $X$ .

**Proof.** The intersection of  $\Delta \cap \bar{\Gamma}$  is computed in  $\tilde{\mathbb{P}}_3$  from Lemma 4.4, by  $\Gamma \cdot E_\Delta = \Gamma \cdot H - \Gamma \cdot s = 8 - 3$ . From Lemma 4.4, one gets the commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \downarrow & & & & \\
 0 & \longrightarrow & \mathcal{O}_{\tilde{\mathbb{P}}_3}^2(-H) \oplus \mathcal{O}_{\tilde{\mathbb{P}}_3}(-2H) & \xrightarrow{G} & A' \otimes \mathcal{O}_{\tilde{\mathbb{P}}_3} & \longrightarrow & \text{coker}(G) \longrightarrow 0 \\
 & & \downarrow & & \parallel & & \downarrow \\
 0 & \longrightarrow & \mathcal{O}_{\tilde{\mathbb{P}}_3}(-s) \oplus \mathcal{O}_{\tilde{\mathbb{P}}_3}(-H) \oplus \mathcal{O}_{\tilde{\mathbb{P}}_3}(-s-H) & \xrightarrow{\tilde{G}} & A' \otimes \mathcal{O}_{\tilde{\mathbb{P}}_3} & \longrightarrow & \mathbb{J}_{\Gamma|\tilde{\mathbb{P}}_3}(2s+2H) \longrightarrow 0 \\
 & & \downarrow & & & & \downarrow \\
 & & \mathcal{O}_{E_\Delta}(-s) \oplus \mathcal{O}_{E_\Delta}(-s-H) & & & & 0 \\
 & & \downarrow & & & & \\
 & & 0 & & & & 
 \end{array}$$

so,  $\text{coker}(G)$  is in the extension

$$0 \rightarrow \mathcal{O}_{E_\Delta}(-s) \oplus \mathcal{O}_{E_\Delta}(-s-H) \rightarrow \text{coker}(G) \rightarrow \mathbb{J}_{\Gamma|\tilde{\mathbb{P}}_3}(2s+2H) \rightarrow 0$$

and we have the isomorphism  $\rho_*(\text{coker}(G)) = \rho_*(\mathbb{J}_{\Gamma|\tilde{\mathbb{P}}_3}(2s+2H))$ , where  $\rho$  is the projection from  $\tilde{\mathbb{P}}_3$  to  $\mathbb{P}_3$ . This gives the resolution (4.3) after applying  $\rho$  to the exact sequence

$$0 \rightarrow \mathbb{J}_{\Gamma|\tilde{\mathbb{P}}_3}(2s+2H) \rightarrow \mathcal{O}_{\tilde{\mathbb{P}}_3}(4H) \rightarrow \mathcal{O}_{2E_\Delta \cup \Gamma}(4H) \rightarrow 0. \quad \square$$

In the next section, we will prove that any trigonal curve of genus 5 embedded in  $\mathbb{P}_3$  by a general line bundle of degree 8 is the curve  $\bar{\Gamma}$  for some choice of  $X \subset \mathbb{P}_1 \times \mathbb{P}_3 \times \mathbb{P}'_3$ .

**Remark 4.6.** The inverse of this rational map is obtained by the same construction from  $n_1$  and  $\mathbb{P}'_3$  (Notation 4.3). Hence, the inverse map is also given by quartic polynomials.

So, let us introduce the following:

**Definition 4.7.** The birational maps constructed in Proposition 4.5 will be called determinantal quarto-quartic birational maps. They form an irreducible family denoted by  $\mathcal{D}_{4,4}$ .

**Proposition 4.8.** *The family  $\mathcal{D}_{4,4} \subset \text{Bir}_{4,4}(\mathbb{P}_3, \mathbb{P}'_3)$  is irreducible of dimension 46.*

**Remark 4.9.** In Corollary 4.15, we will prove that this family turns out to be an irreducible component of  $\text{Bir}_{4,4}(\mathbb{P}_3, \mathbb{P}'_3)$ .

**Proof of Proposition 4.8.** First, let us detail the choices made to get  $X$ . The choice of a complete intersection of this type is equivalent to the choices of

- (i)  $\overline{n}_0 \in \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}_1 \times \mathbb{P}_3 \times \mathbb{P}'_3}(s + H)))$ ,
- (ii)  $\overline{n}_1 \in \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}_1 \times \mathbb{P}_3 \times \mathbb{P}'_3}(s + H')))$ ,
- (iii)  $\overline{n}_2 \in \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}_1 \times \mathbb{P}_3 \times \mathbb{P}'_3}(H + H')))$ ,
- (iv)  $\overline{n}_3 \in \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}_1 \times \mathbb{P}_3 \times \mathbb{P}'_3}(s + H + H'))/(\overline{n}_0 \cdot A' + \overline{n}_1 \cdot A + \overline{n}_2 \cdot L))$ .

So, the choice of  $X$  is made in an irreducible variety of dimension  $7 + 7 + 15 + 21 = 50$ . Let  $I_2 \subset \mathbb{P}_3 \times \mathbb{P}'_3$  be the divisor defined by  $\overline{n}_2$ . Pushing down  $\mathcal{O}_X(s)$  to  $I_2$ , one gets the following resolution of the ideal  $\mathcal{G}_X$  of the graph of the birational transformation constructed from  $X$ :

$$0 \rightarrow L^\vee \otimes \mathcal{O}_{I_2} \xrightarrow{\begin{pmatrix} n_0 \\ n_1 \\ n_3 \end{pmatrix}} \mathcal{O}_{I_2}(H) \oplus \mathcal{O}_{I_2}(H') \oplus \mathcal{O}_{I_2}(H + H') \rightarrow \mathcal{G}_X(2H + 2H') \rightarrow 0$$

so  $\dim(\mathcal{D}_{4,4}) = 50 - \dim(GL(L^\vee)) = 46$ . □

In Sec. 4.3, we provide an explicit construction of this map and its inverse and give more geometric properties.

### 4.2. Trigonal curves of genus 5

In the previous section, we obtained a trigonal curve of genus 5 naturally embedded in  $\tilde{\mathbb{P}}_3$ . To understand how this construction is general, we start in this section with an abstract trigonal curve of genus 5, then we choose some line bundle and explain how to obtain the previous construction. Notation of the various line bundles introduced here will turn out to be compatible with notation in the previous section.

#### 4.2.1. Model in $\tilde{\mathbb{P}}_2$

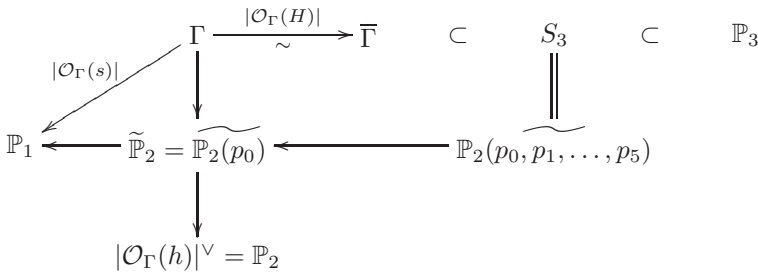
The following result is detailed in [1, Chap. 5]. Let  $\Gamma$  be any trigonal curve of genus 5, let  $\omega_\Gamma$  be its canonical bundle, and denote by  $\mathcal{O}_\Gamma(s)$  the line bundle of degree 3 generated by two sections. Without any extra choice, we have the following embedding of  $\Gamma$ :

- The linear system  $|\omega_\Gamma(-s)|$  sends  $\Gamma$  to a plane curve of degree 5. This plane curve has a unique singular point  $p_0$  (a node or an ordinary cusp).
- Let  $\tilde{\mathbb{P}}_2$  be the blow up of this plane at  $p_0$ , denote by  $E_0$  the exceptional divisor and by  $h$  the hyperplane class of this plane:

$$|\mathcal{O}_{\tilde{\mathbb{P}}_2}(h)| : \tilde{\mathbb{P}}_2 \longrightarrow \mathbb{P}_2 = |\omega_\Gamma(-s)|^\vee.$$

4.2.2. *Embeddings of degree 8 in  $\mathbb{P}_3$*

With notation of Sec. 4.2.1, let  $\mathcal{O}_\Gamma(H)$  be a general line bundle of degree 8 on  $\Gamma$ . One has  $h^0(\mathcal{O}_\Gamma(H - s)) = 1$ . The vanishing locus of this section defines in  $\Gamma$  a unique effective divisor  $D'_5$  of degree 5 such that  $\mathcal{O}_\Gamma(H) = \mathcal{O}_\Gamma(D'_5 + s)$ . Moreover,  $h^0(\mathcal{O}_\Gamma(2h - D'_5))$  is also 1, so there is a unique effective divisor  $D_5$  on  $\Gamma$  such that  $\mathcal{O}_\Gamma(2h) = \mathcal{O}_\Gamma(D_5 + D'_5)$ . Note that both  $D_5$  and  $D'_5$  have degree 5. So, the picture in  $\mathbb{P}_2$  looks like this. The curve  $\Gamma$  is sent via  $|\mathcal{O}_\Gamma(h)|$  to a quintic curve singular at  $p_0$ . This curve contains two divisors  $D_5, D'_5$  of degree 5 such that  $D_5 + D'_5$  is the complete intersection of the plane quintic with a conic. From the general assumptions on  $H$  (hence on  $D'_5$ ), the conic is smooth, it does not contain the point  $p_0$  and both  $D_5$  and  $D'_5$  are supported by five distinct points. Let  $p_1, p_2, \dots, p_5$  be the five points of  $D_5$ , and let  $S_3$  be the blow up of  $\mathbb{P}_2$  at the six points  $p_0, p_1, \dots, p_5$ . Let  $(E_i)_{0 \leq i \leq 5}$  be the corresponding exceptional divisors. Remark that  $\mathcal{O}_\Gamma(H)$  is the restriction to  $\Gamma$  of  $\mathcal{O}_{S_3}(3h - \sum_{i=0}^5 E_i)$ ; the linear system  $|\mathcal{O}_\Gamma(H)|$  embeds  $\Gamma$  on the cubic surface  $S_3$  in  $\mathbb{P}_3$ . Denote by  $\overline{\Gamma}$  this curve of degree 8 in  $\mathbb{P}_3$  and keep the notation  $H$  for the class  $3h - \sum_{i=0}^5 E_i$  on  $S_3$ . We can summarize these data in the following diagram:



**Remark 4.10.** Any irreducible smooth curve of degree 8 and genus 5 in  $\mathbb{P}_3$  with a 5-secant line  $\Delta$  is trigonal.

**Proof.** Indeed, the planes containing  $\Delta$  give a base point free linear system of degree 3. □

**Notation 4.11.** Let  $\Delta$  be the line of  $S_3$  equivalent to  $2h - \sum_{i=1}^5 E_i$ . One has  $\Delta \cap \overline{\Gamma} = D'_5$  and  $\Delta$  is the unique line 5-secant to  $\overline{\Gamma}$ . Denote by  $\widetilde{\mathbb{P}}_3$  the blow up of  $\mathbb{P}_3$  in  $\Delta$  and by  $\mathcal{J}_{\Gamma|\widetilde{\mathbb{P}}_3}$  the ideal of the proper transform of  $\overline{\Gamma}$ .

Keeping notation  $s, H$  for the pull back of the hyperplane classes of  $\mathbb{P}_1$  and  $\mathbb{P}_3$  on  $\widetilde{\mathbb{P}}_3$ , one gets the following:

**Lemma 4.12.** *We have an exact sequence*

$$0 \longrightarrow \mathcal{O}_{\widetilde{\mathbb{P}}_3}(s) \longrightarrow \mathcal{J}_{\Gamma|\widetilde{\mathbb{P}}_3}(2s + 2H) \longrightarrow \mathcal{O}_{S_3}(H - E_0) \longrightarrow 0.$$

Hence,  $h^0(\mathcal{J}_{\Gamma|\widetilde{\mathbb{P}}_3}(2s + 2H)) = 4$  and the sheaf  $\mathcal{J}_{\Gamma|\widetilde{\mathbb{P}}_3}(2s + 2H)$  is globally generated.

**Proof.** As  $\Delta \subset S_3$ , the ideal of  $S_3$  in  $\tilde{\mathbb{P}}_3$  is  $\mathcal{J}_{S_3|\tilde{\mathbb{P}}_3} = \mathcal{O}_{\tilde{\mathbb{P}}_3}(-s - 2H)$ . The class of  $\bar{\Gamma}$  in  $S_3$  is  $5h - 2E_0 - \sum_{i=1}^5 E_i$  and  $s \sim H - \Delta \sim h - E_0$ ; the ideal of  $\bar{\Gamma}$  in  $S_3$  is thus  $\mathcal{J}_{\Gamma|S_3} = \mathcal{O}_{S_3}(-H - 2s - E_0)$ . Hence, the following exact sequence of ideals

$$0 \longrightarrow \mathcal{J}_{S_3|\tilde{\mathbb{P}}_3} \longrightarrow \mathcal{J}_{\Gamma|\tilde{\mathbb{P}}_3} \longrightarrow \mathcal{J}_{\Gamma|S_3} \longrightarrow 0$$

twisted by  $2s + 2H$  gives the exact sequence of the statement. To conclude, just remark that  $h^1(\mathcal{O}_{\tilde{\mathbb{P}}_3}(s)) = 0$  and that both  $\mathcal{O}_{\tilde{\mathbb{P}}_3}(s)$  and  $\mathcal{O}_{S_3}(H - E_0)$  have two sections and are globally generated.  $\square$

**Proposition 4.13.** *Let  $A'$  be the four-dimensional vector space  $H^0(\mathcal{J}_{\Gamma|\tilde{\mathbb{P}}_3}(2s+2H))$ ; let us consider  $\mathcal{N}$  defined by the following exact sequence:*

$$0 \longrightarrow \mathcal{N} \longrightarrow A' \otimes \mathcal{O}_{\tilde{\mathbb{P}}_3} \longrightarrow \mathcal{J}_{\Gamma|\tilde{\mathbb{P}}_3}(2s + 2H) \longrightarrow 0.$$

We have

$$\mathcal{N} \simeq \mathcal{O}_{\tilde{\mathbb{P}}_3}(-s) \oplus \mathcal{O}_{\tilde{\mathbb{P}}_3}(-H) \oplus \mathcal{O}_{\tilde{\mathbb{P}}_3}(-s - H).$$

**Proof.** First, remark that  $\mathcal{N}$  is locally free. Indeed, the sheaf  $\mathcal{J}_{\Gamma|\tilde{\mathbb{P}}_3}$  is the ideal of a one-dimensional scheme of  $\tilde{\mathbb{P}}_3$  without embedded nor isolated points. As a consequence, for all closed point  $x$  of  $\tilde{\mathbb{P}}_3$ , we have the vanishing of the following localizations:

$$\forall i \geq 2, \quad \text{Ext}^i(\mathcal{J}_{\Gamma|\tilde{\mathbb{P}}_3,x}, \mathcal{O}_{\tilde{\mathbb{P}}_3,x}) = 0$$

so

$$\forall i \geq 1, \quad \text{Ext}^i(\mathcal{N}_x, \mathcal{O}_{\tilde{\mathbb{P}}_3,x}) = 0$$

and  $\mathcal{N}$  is locally free. Now, construct an injection from  $\mathcal{O}_{\tilde{\mathbb{P}}_3}(-s) \oplus \mathcal{O}_{\tilde{\mathbb{P}}_3}(-H) \oplus \mathcal{O}_{\tilde{\mathbb{P}}_3}(-s - H)$  to  $\mathcal{N}$ .

- Since  $h^0(\mathcal{J}_{\Gamma|\tilde{\mathbb{P}}_3}(3s+2H)) = h^0(\mathcal{O}_{\tilde{\mathbb{P}}_3}(3s) + h^0(\mathcal{O}_{S_3}(2H - E_0 - \Delta))) = 7$  and  $h^0(A' \otimes \mathcal{O}_{\tilde{\mathbb{P}}_3}(s)) = 8$ , we have  $h^0(\mathcal{N}(s)) \geq 1$ .
- $h^0(\mathcal{N}(H)) \geq 2$  because  $h^0(\mathcal{J}_{\Gamma|\tilde{\mathbb{P}}_3}(2s + 3H)) = h^0(\mathcal{O}_{\tilde{\mathbb{P}}_3}(s + H)) + h^0(\mathcal{O}_{S_3}(2H - E_0)) = 14$  and  $h^0(A' \otimes \mathcal{O}_{\tilde{\mathbb{P}}_3}(H)) = 16$ .
- As  $h^0(\mathcal{J}_{\Gamma|\tilde{\mathbb{P}}_3}(3s + 3H)) = h^0(\mathcal{O}_{\tilde{\mathbb{P}}_3}(2s + H)) + h^0(\mathcal{O}_{S_3}(3H - E_0 - \Delta)) = 21$  and  $h^0(A' \otimes \mathcal{O}_{\tilde{\mathbb{P}}_3}(s + H)) = 28$ , we have  $h^0(\mathcal{N}(s + H)) \geq 7$ .

We are thus able to find a nonzero section  $\sigma_0$  of  $\mathcal{N}(s)$ , then a section  $\sigma_1$  of  $\mathcal{N}(H)$  independent with  $E_\Delta \cdot \sigma_0$  (with notation of Sec. 4.1.2), then a section  $\sigma_2$  of  $\mathcal{N}(s + H)$  not in  $A \otimes \sigma_0 \oplus L \otimes \sigma_1$  (Notation 4.1). As a result  $\sigma_0, \sigma_1$  and  $\sigma_2$  give an injection of  $\mathcal{O}_{\tilde{\mathbb{P}}_3}(-s) \oplus \mathcal{O}_{\tilde{\mathbb{P}}_3}(-H) \oplus \mathcal{O}_{\tilde{\mathbb{P}}_3}(-s - H)$  into  $\mathcal{N}$ . Moreover, these two vector bundles have the same first Chern class, so it is an isomorphism.  $\square$



In other words, Proposition 4.13 gives the existence of the map  $\tilde{G}$  from Lemma 4.4 for any trigonal curve  $\Gamma$  of genus 5, therefore we have the following:

**Corollary 4.14.** *Let  $\bar{\Gamma}$  be a trigonal curve of genus 5 embedded in  $\mathbb{P}_3$  by a general linear system of degree 8. The quartics containing  $\bar{\Gamma}$  and singular along its unique 5-secant line give an element of  $\mathcal{D}_{4,4}$ .*

**Corollary 4.15.** *The closure of  $\mathcal{D}_{4,4}$  is an irreducible component of  $\text{Bir}_{4,4}(\mathbb{P}_3, \mathbb{P}'_3)$ .*

**Proof.** From Corollary 4.8, the family  $\mathcal{D}_{4,4}$  is already irreducible. Choose an irreducible component  $\mathcal{B}$  of  $\text{Bir}_{4,4}(\mathbb{P}_3, \mathbb{P}'_3)$  containing  $\mathcal{D}_{4,4}$ . Let  $\phi$  be a general element of  $\mathcal{B}$ , and let  $\psi$  be a general element of  $\mathcal{D}_{4,4}$ . With notation of Sec. 2, the degree of the one-dimensional scheme  $F_\psi$  is 11 and  $\psi$  is in  $\mathcal{B}$ . So, by Corollary 2.7 for the family  $\mathcal{B}$ , the degree of the one-dimensional scheme  $F_\phi$  is at most 11 because  $\phi$  is general in  $\mathcal{B}$ . For a general hyperplane  $H'$  of  $\mathbb{P}'_3$ , the quartic surface  $\phi^{-1}(H')$  is thus not normal. Indeed, we proved in Lemma 3.5 that if  $\phi^{-1}(H')$  is normal, then the schemes  $F_\phi^1$  and  $C_2$  are equal and  $\deg F_\phi = 12$ . With Definition 2.4, we have  $\dim(F_\phi^\sigma) = 1$ , hence  $\deg(F_\phi^\sigma) \geq 1$ . But, with notation of Corollary 2.7, we have  $\beta(\phi) = 1, \beta(\psi) = 1$  and  $\eta(\psi) = 1$ . As a consequence by semicontinuity of  $\eta$ , we have  $\deg(F_\phi^\sigma) = 1$  for the general element  $\phi$  of  $\mathcal{B}$ . There is thus a line  $\Delta$  in  $\mathbb{P}_3$  such that  $\phi^{-1}(H')$  is singular on  $\Delta$ . Denote again by  $\tilde{\mathbb{P}}_3$  the blow up of  $\mathbb{P}_3$  in  $\Delta$ , then  $\phi$  factors through a rational map  $\tilde{\phi} : \tilde{\mathbb{P}}_3 \dashrightarrow \mathbb{P}'_3$  via a linear subsystem  $W_\phi$  of  $|\mathcal{O}_{\tilde{\mathbb{P}}_3}(2s + 2H)|$ . Now, adapt Notation 3.4 to geometric liaison in  $\tilde{\mathbb{P}}_3$  instead of  $\mathbb{P}_3$ . Let  $\ell'$  be a general line of  $\mathbb{P}'_3$ , set  $\tilde{C}_1 = \tilde{\phi}_*^{-1}(\ell')$ , and let  $\tilde{C}_2$  be the residual of  $\tilde{C}_1$  in the complete intersection  $\tilde{\phi}^{-1}(\ell')$ . The curves  $\tilde{C}_1$  and  $\tilde{C}_2$  are geometrically linked by two hypersurfaces of class  $2s + 2H$ . The liaison exact sequence gives

$$0 \longrightarrow \mathcal{J}_{\tilde{C}_1 \cup \tilde{C}_2}(2s + 2H) \longrightarrow \mathcal{J}_{\tilde{C}_2}(2s + 2H) \longrightarrow \omega_{\tilde{C}_1}(H - s) \longrightarrow 0.$$

By Lemma 2.8, the curves  $C_1, \tilde{C}_1$  and  $\tilde{C}_2$  are smooth because it is already true for  $\tilde{\psi}$ . Thus  $\tilde{C}_2$  is the base scheme of  $\tilde{\phi}$  and all the elements of  $W_\phi$  vanish on  $\tilde{C}_2$ . Hence,  $h^0(\mathcal{J}_{\tilde{C}_2}(2s + 2H)) \geq 4$ , so we have  $h^0(\omega_{\tilde{C}_1}(H - s)) \geq 2$  on the smooth rational curve  $\tilde{C}_1$ . Hence,  $\mathcal{O}_{\tilde{C}_1}(H - s)$  has degree at least 3. The class of  $\tilde{C}_1$  in the Chow ring of  $\tilde{\mathbb{P}}_3$  is thus  $\tilde{C}_1 \sim (4 - a) \cdot H^2 + aH \cdot s$  with  $a \geq 3$ . But,  $\tilde{C}_1 \cdot s = 4 - a \geq 1$  because  $C_1$  is irreducible of degree 4 and not in a plane containing  $\Delta$ . So  $a = 3$  and the class of  $\tilde{C}_2$  is  $3H^2 + 5H \cdot s$ . As a consequence,  $|\mathcal{O}_{\tilde{C}_2}(s)|$  is a  $g_3^1$ . The restrictions to  $\tilde{C}_1$  and  $\tilde{C}_2$  of the exact sequences

$$0 \longrightarrow \omega_{\tilde{C}_3-i} \longrightarrow \omega_{\tilde{C}_1 \cup \tilde{C}_2} \longrightarrow \omega_{\tilde{C}_1 \cup \tilde{C}_2} \otimes \mathcal{O}_{\tilde{C}_i} \longrightarrow 0, \quad i \in \{1, 2\}$$

show that  $\tilde{C}_1 \cap \tilde{C}_2$  has length 9 and that  $\tilde{C}_2$  has genus 5. As a result, the base locus of  $\tilde{\phi}$  is a trigonal curve of genus 5, so  $\phi$  belongs to  $\mathcal{D}_{4,4}$  by Corollary 4.14.  $\square$

### 4.3. Explicit construction of the birational map and its inverse

In the next paragraphs, we want to expose properties similar to the usual properties of the classical cubo-cubic transformation such as explicit computation of the inverse and isomorphism between the base loci [5, §7.2.2].

#### 4.3.1. Explicit construction of $\bar{\tau} : \mathbb{P}_3 \dashrightarrow \mathbb{P}'_3$

In the proof of Proposition 4.8, we detailed the choices needed to construct the complete intersection  $X \subset \mathbb{P}_1 \times \mathbb{P}_3 \times \mathbb{P}'_3$ . Let us replace them by the choice of some linear maps.

**Definition 4.16.** With Notation 4.1, choose general linear maps  $N_0, N_1, M, T$  between the following vector spaces:

$$N_0 : L^\vee \longrightarrow A, \quad N_1 : L^\vee \longrightarrow A', \quad M : A^\vee \longrightarrow A', \quad T : L^\vee \longrightarrow \text{Hom}(A^\vee, A').$$

$$\lambda \longmapsto T_\lambda$$

For any  $\lambda \in L^\vee$ , the image of  $\lambda$  by  $T$  will be denoted by  $T_\lambda$ . Choose also generators for  $\wedge^2 L, \wedge^4 A, \wedge^4 A'$  and denote the corresponding isomorphisms by

$$B : L \xrightarrow{\sim} L^\vee, \quad \alpha : \wedge^3 A \xrightarrow{\sim} A^\vee, \quad \alpha' : \wedge^3 A' \xrightarrow{\sim} A'^\vee.$$

The duality bracket between a vector space and its dual will be denoted by  $\langle \cdot, \cdot \rangle$ .

For any  $z$  in  $A^\vee$ , let us define the elements  $g_0, g_1, g_2$  of  $A'$  by

$$g_0 = N_1 \circ B \circ {}^t N_0(z), \quad g_1 = M(z), \quad g_2 = T_{B \circ {}^t N_0(z)}(z).$$

Now, consider the quartic map

$$\begin{aligned} \tau : A^\vee &\longrightarrow A'^\vee \\ z &\longmapsto \alpha'(g_0 \wedge g_1 \wedge g_2) \end{aligned}$$

and also the rational map  $\bar{\tau} : \mathbb{P}_3 \dashrightarrow \mathbb{P}'_3$  induced by  $\tau$ .

**Example 4.17.** Take basis for  $L, A, A'$  and choose the identity matrix for  $M$ . With

$$N_0 = N_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad U_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad U_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

define  $T : (\lambda_0, \lambda_1) \mapsto \lambda_0 U_0 + \lambda_1 U_1$ . Then, we have

$$G = \begin{pmatrix} -z_1 & z_0 & -z_1^2 + z_0 z_3 \\ z_0 & z_1 & z_0^2 - z_1 z_2 \\ 0 & z_2 & z_0 z_1 - z_1 z_3 \\ 0 & z_3 & -z_0 z_1 + z_0 z_2 \end{pmatrix}.$$

Denote by  $\Delta_k(G)$  the determinant of the matrix  $G$  with its  $k$ th line removed. The map  $\bar{\tau}$  defined by

$$(z_0 : z_1 : z_2 : z_3) \dashrightarrow (\Delta_1(G) : -\Delta_2(G) : \Delta_3(G) : -\Delta_4(G))$$

is birational (see Sec. 4.3.2) and is in the closure of  $\mathcal{D}_{4,4}$ .

#### 4.3.2. Explicit construction of $\bar{\tau}^{-1}$

For any  $y$  in  $A^\vee$  denote by  $g'_0, g'_1, g'_2$ , the following are the elements of  $A$ .

$$g'_0 = -N_0 \circ B \circ {}^tN_1(y), \quad g'_1 = {}^tM(y), \quad g'_2 = {}^t(T_{B \circ {}^tN_1(y)})(y).$$

Now, consider the quartic map

$$\begin{aligned} \tau' : A^\vee &\longrightarrow A^\vee \\ y &\longmapsto \alpha(g'_0 \wedge g'_1 \wedge g'_2) \end{aligned}$$

and also the rational map  $\bar{\tau}' : \mathbb{P}_3 \dashrightarrow \mathbb{P}_3$  induced by  $\tau'$ .

**Lemma 4.18.** *For  $z \in A^\vee$  and  $y \in A^\vee$ , we have*

$$y \wedge \tau(z) = 0 \iff \forall i \in \{0, 1, 2\}, \quad \langle y, g_i \rangle = 0 \iff z \wedge \tau'(y) = 0.$$

**Proof.** By definition of  $\alpha$  and  $\alpha'$ , we have

$$\begin{aligned} y \wedge \tau(z) = 0 &\iff \forall i \in \{0, 1, 2\} \langle y, g_i \rangle = 0, \\ z \wedge \tau'(y) = 0 &\iff \forall i \in \{0, 1, 2\} \langle z, g'_i \rangle = 0. \end{aligned}$$

Now, remark that  $\langle y, g_0 \rangle = 0 \iff \langle {}^tN_1(y), B \circ {}^tN_0(z) \rangle = 0 \iff \langle g'_0, z \rangle = 0$ . Note that it is also equivalent to the proportionality of  ${}^tN_1(y)$  and  ${}^tN_0(z)$ . Furthermore, remind that  $T$  is general in  $\text{Hom}(L^\vee, \text{Hom}(A^\vee, A'))$ , so it is injective. Thus, we have  $\langle y, g_0 \rangle = 0$  if and only if  $T_{B \circ {}^tN_1(y)}$  and  $T_{B \circ {}^tN_0(z)}$  are proportional in  $\text{Hom}(A^\vee, A')$ . Therefore, we have

$$\begin{cases} \langle y, g_0 \rangle = 0 \\ \langle y, g_2 \rangle = 0 \end{cases} \iff \begin{cases} T_{B \circ {}^tN_1(y)} \wedge T_{B \circ {}^tN_0(z)} = 0 \\ \langle y, T_{B \circ {}^tN_1(y)}(z) \rangle = 0 \end{cases} \iff \begin{cases} \langle g'_0, z \rangle = 0 \\ \langle g'_2, z \rangle = 0 \end{cases}.$$

We achieve the proof because  $\langle y, g_1 \rangle = \langle g'_1, z \rangle$ . □

**Corollary 4.19.** *For  $(\lambda, z) \in L^\vee \times A^\vee$ , consider the elements of  $A^\vee$  given by*

$$\tilde{g}_0 = N_1(\lambda), \quad \tilde{g}_1 = M(z), \quad \tilde{g}_2 = T_\lambda(z).$$

*Thus,  $\bar{\tau}' \circ \bar{\tau} = \text{id}_{\mathbb{P}_3}$  and  $\bar{\tau}$  is the element of  $\mathcal{D}_{4,4}$  constructed with*

$$\tilde{G}_{(\lambda, z)} = (\tilde{g}_0, \tilde{g}_1, \tilde{g}_2) \quad \text{and} \quad G_z = (g_0, g_1, g_2).$$

**Proof.** Lemma 4.18 directly implies that  $\bar{\tau}' \circ \bar{\tau}$  is the identity map of  $\mathbb{P}_3$ . Remind that for  $(\lambda, z) \in L^\vee \times A^\vee$ , the corresponding point of  $\mathbb{P}_1 \times \mathbb{P}_3$  is in  $\tilde{\mathbb{P}}_3$  if and only if  $\langle \lambda, {}^tN_0(z) \rangle = 0$ . But, it is equivalent to the proportionality of  $\lambda$  and  $B \circ {}^tN_0(z)$ . So, for all  $y \in A^\vee$  and  $(\tilde{\lambda}, \tilde{z}) \in \tilde{\mathbb{P}}_3$ , we have  $y \wedge \tau(z) = 0$  if and only if  $\langle y, \tilde{g}_i \rangle = 0$  for any  $i \in \{0, 1, 2\}$ . Thus,  $\bar{\tau}$  is the element of  $\mathcal{D}_{4,4}$  constructed from  $\tilde{G}$ . □

4.3.3. The isomorphism between  $\Gamma$  and  $\Gamma'$

**Proposition 4.20.** *Let  $\bar{\tau} : \mathbb{P}_3 \dashrightarrow \mathbb{P}'_3$  be a element of  $\mathcal{D}_{4,4}$ . Let  $\Gamma$  (respectively  $\Gamma'$ ) be the trigonal curve of genus 5 defining  $\bar{\tau}$  (respectively  $\bar{\tau}^{-1}$ ). Then,  $\Gamma'$  is isomorphic to  $\Gamma$  and the embedding of  $\Gamma$  in  $\mathbb{P}'_3$  is given by*

$$\mathcal{O}_\Gamma(H') = \omega_\Gamma^{\otimes 2}(-H).$$

We will prove this result after the construction in Lemma 4.23 of two exact sequences over  $\tilde{\mathbb{P}}_2$ .

**Notation 4.21.** In the context of Sec. 4.2.1, we have

$$\tilde{\mathbb{P}}_2 \subset \mathbb{P}_1 \times \mathbb{P}_2, \quad U = H^0(\mathcal{O}_{\mathbb{P}_2}(h)), \quad L = H^0(\mathcal{O}_{\mathbb{P}_1}(s)).$$

The equation of  $\tilde{\mathbb{P}}_2$  in  $\mathbb{P}_1 \times \mathbb{P}_2$  gives a linear map

$$N_2 : L^\vee \longrightarrow U.$$

Let  $\mathcal{O}_{\tilde{\mathbb{P}}_2}(-s) \xrightarrow{\Lambda} L^\vee \otimes \mathcal{O}_{\tilde{\mathbb{P}}_2}$  be the pull back of the tautological injection on  $\mathbb{P}_1$ , and let  $Q, Q'$  be the cokernels of  $N_0 \circ \Lambda$  and  $N_1 \circ \Lambda$ . Hence, we have the exact sequences defining  $\pi$  and  $\pi'$ ,

$$\begin{aligned} 0 \rightarrow \mathcal{O}_{\tilde{\mathbb{P}}_2}(-s) &\xrightarrow{N_0 \circ \Lambda} A \otimes \mathcal{O}_{\tilde{\mathbb{P}}_2} \xrightarrow{\pi} Q \rightarrow 0, \\ 0 \rightarrow \mathcal{O}_{\tilde{\mathbb{P}}_2}(-s) &\xrightarrow{N_1 \circ \Lambda} A' \otimes \mathcal{O}_{\tilde{\mathbb{P}}_2} \xrightarrow{\pi'} Q' \rightarrow 0. \end{aligned}$$

**Remark 4.22.** Both  $Q$  and  $Q'$  are isomorphic to  $\mathcal{O}_{\tilde{\mathbb{P}}_2}^2 \oplus \mathcal{O}_{\tilde{\mathbb{P}}_2}(s)$ , but we prefer not to choose one of these isomorphisms to avoid confusions.

— Choose a linear map  $\gamma : U^\vee \rightarrow \mathbb{C}$  such that the map

$$\begin{aligned} U^\vee &\rightarrow \mathbb{C} \oplus L \\ u &\mapsto (\gamma(u), {}^tN_2(u)) \end{aligned}$$

has rank 3.

— Define

$$\begin{aligned} \tilde{R} : U^\vee &\longrightarrow \text{Hom}(A^\vee, A') \\ u &\longmapsto \tilde{R}_u = \gamma(u) \cdot M + T_{B \circ {}^tN_2}(u) \end{aligned}$$

and denote by  $\tilde{R}_u$  the image of  $u$  by  $\tilde{R}$ .

— Set  $R = \pi' \circ \tilde{R} \circ \iota_\pi$ .

We have the commutative diagram

$$\begin{array}{ccc} A^\vee(-h) & \xrightarrow{\tilde{R}} & A' \otimes \mathcal{O}_{\tilde{\mathbb{P}}_2} \\ \uparrow \iota_\pi & & \downarrow \pi' \\ Q^\vee(-h) & \xrightarrow{R} & Q' \end{array}$$

**Lemma 4.23.** *We have the following exact sequences of  $\mathcal{O}_{\tilde{\mathbb{P}}_2}$ -modules:*

$$\begin{aligned} 0 \longrightarrow Q^\vee(-h) \xrightarrow{R} Q' \longrightarrow \mathcal{O}_\Gamma(H') \longrightarrow 0, \\ 0 \longrightarrow Q'^\vee(-h) \xrightarrow{tR} Q \longrightarrow \mathcal{O}_\Gamma(H) \longrightarrow 0. \end{aligned}$$

**Proof.** Remark first that the class in  $\text{Pic}(\tilde{\mathbb{P}}_2)$  of the determinant of  $R$  is  $2s + 3h$  and that it is also the class of  $\Gamma$  in  $\tilde{\mathbb{P}}_2$ . Nevertheless, we need to explicitly relate points  $p$  of  $\tilde{\mathbb{P}}_2$  such that  $R_p$  is not injective to points  $p$  of  $\tilde{\mathbb{P}}_3$  such that  $\tilde{G}_p$  is not injective. So, consider  $(\lambda, u)$  in  $L^\vee \times U^\vee$  such that the corresponding point  $(\bar{\lambda}, \bar{u})$  of  $\mathbb{P}_1 \times \mathbb{P}_2$  is in  $\tilde{\mathbb{P}}_2$ . In other words, we have

$$\langle \lambda, {}^tN_2(u) \rangle = 0.$$

The map  $R_{(\bar{\lambda}, \bar{u})}$  is not injective if and only if there exists  $z$  in the image of  ${}^t\pi_{(\bar{\lambda}, \bar{u})}$  such that  $\tilde{R}_u(z)$  and  $N_1(\lambda)$  are proportional. Therefore,

$$\begin{aligned} (R_{(\bar{\lambda}, \bar{u})} \text{ not injective}) \\ \iff (\exists z \in A^\vee - \{0\}, \langle z, N_0(\lambda) \rangle = 0 \quad \text{and} \quad \tilde{R}_u(z) \wedge N_1(\lambda) = 0). \end{aligned}$$

By definition  $\tilde{R}_u(z) = \gamma(u) \cdot M(z) + T_\lambda(z)$ , so  $\tilde{R}_u(z) \wedge N_1(\lambda) = 0$  implies that  $N_1(\lambda), M(z), T_\lambda(z)$  are linearly dependent elements of  $A'$ . Now, remind that  $\langle z, N_0(\lambda) \rangle = 0$  is just the equation of  $\tilde{\mathbb{P}}_3$  in  $\mathbb{P}_1 \times \mathbb{P}_3$ , we thus have

$$\begin{aligned} ((\bar{\lambda}, \bar{u}) \in \tilde{\mathbb{P}}_2 \text{ and } R_{(\bar{\lambda}, \bar{u})} \text{ not injective}) \Rightarrow (\exists \bar{z} \in \mathbb{P}_3 \text{ such that } (\bar{\lambda}, \bar{z}) \in \tilde{\mathbb{P}}_3 \text{ and} \\ \tilde{G}_{(\bar{\lambda}, \bar{z})} \text{ not injective}). \end{aligned}$$

Conversely, if  $(\bar{\lambda}, \bar{z}) \in \tilde{\mathbb{P}}_3$  and  $N_1(\lambda), M(z), T_\lambda(z)$  are linearly dependent, then there exists by definition of  $\gamma$  an element  $u \in U^\vee - \{0\}$  such that  $(\gamma(u) \cdot M(z) + T_\lambda(z)) \wedge N_1(\lambda) = 0$  and  $\langle \lambda, {}^tN_2(u) \rangle = 0$ . As a result,

$$\begin{aligned} ((\bar{\lambda}, \bar{z}) \in \tilde{\mathbb{P}}_3 \text{ and } \tilde{G}_{(\bar{\lambda}, \bar{z})} \text{ not injective}) \\ \Rightarrow (\exists \bar{u} \in \mathbb{P}_2 \text{ such that } (\bar{\lambda}, \bar{u}) \in \tilde{\mathbb{P}}_2 \text{ and } R_{(\bar{\lambda}, \bar{u})} \text{ not injective}). \end{aligned}$$

These two implications show that the cokernel of  ${}^tR$  is  $\mathcal{O}_\Gamma(H)$ . The statement follows by exchanging  $\mathbb{P}_3$  and  $\mathbb{P}'_3$ . □

**Proof of Proposition 4.20.** Lemma 4.23 implies

$$\mathcal{O}_\Gamma(H') = \mathcal{E}xt^1(\mathcal{O}_\Gamma(H), \mathcal{O}_{\tilde{\mathbb{P}}_2}(-h)) = \omega_\Gamma \otimes \mathcal{O}_{\tilde{\mathbb{P}}_2}(s+h) = \omega_\Gamma^{\otimes 2}(-H). \quad \square$$

#### 4.4. Geometric description of the $P$ -locus

##### 4.4.1. The ruled surface $S_9$

Remind from Sec. 4.1.1 that  $\bar{\tau}' : \mathbb{P}'_3 \dashrightarrow \mathbb{P}_3$  factors through  $X \rightarrow \tilde{\mathbb{P}}_3$ . Therefore, if  $N_\Gamma$  denotes the normal bundle of  $\Gamma$  in  $\tilde{\mathbb{P}}_3$ , Sequence (4.1) shows that  $\bar{\tau}'$  contracts

the ruled surface  $\tilde{S}'_9 = \text{Proj}(\text{Sym}(N_\Gamma^\vee(2s+2H)))$  to  $\Gamma$ . Moreover, the exact sequence of Lemma 4.12 restricted to  $\Gamma$  gives

$$0 \longrightarrow \mathcal{O}_\Gamma(s) \longrightarrow N_\Gamma^\vee(2s+2H) \longrightarrow \mathcal{O}_\Gamma(H-E_0) \longrightarrow 0.$$

Sections of  $\mathcal{O}_\Gamma(s)$  are thus hyperplane sections of  $\text{Proj}(\text{Sym}(N_\Gamma^\vee(2s+2H)))$  and each of them contains three rules of this surface. So, the projection of this surface in  $\mathbb{P}'_3$  is the union of the “triangles” with vertices given by the  $g_3^1$ . In a symmetric way, the  $P$ -locus of  $\bar{\tau}$  contains a ruled surface  $S_9 \subset \mathbb{P}_3$ . The basis of  $S_9$  is isomorphic to  $\Gamma$  and its rules are the lines of  $\mathbb{P}_3$  that cut an element of the  $g_3^1$  in length 2. In other words,  $S_9$  is the union of the bisecant lines of the intersection of  $\bar{\Gamma} \subset \mathbb{P}_3$  with any plane containing the 5-secant  $\Delta$ . The smooth model  $\tilde{S}_9$  of  $S_9$  is an extension of  $\mathcal{O}_\Gamma(H'-E_0)$  by  $\mathcal{O}_\Gamma(s)$ ; hence  $S_9$  has degree 9 and  $\bar{\tau}(S_9)$  is the curve  $\Gamma$  embedded in  $\mathbb{P}'_3$  by  $\mathcal{O}_\Gamma(H') = \omega_\Gamma^{\otimes 2}(-H)$  (Proposition 4.20). It is easy to explain geometrically why  $S_9$  is contracted by  $\bar{\tau}$ . Indeed, by the previous description, the rules of  $\Gamma$  intersect the scheme defined by the ideal  $\mathcal{J}_\Delta^2 \cap \mathcal{J}_{\bar{\Gamma}}$  in length 4, so these lines are contracted by  $|\mathcal{J}_\Delta^2 \cap \mathcal{J}_{\bar{\Gamma}}(4H)|$ .

#### 4.4.2. The cubic surface $S_3$

It is interesting to understand geometrically why the cubic surface  $S_3$  defined in Sec. 4.2.2 is also contracted by  $\bar{\tau}$  because it cannot be explained anymore by 4-secant lines. This time, consider the planes of  $\mathbb{P}_3$  containing the line  $E_0 \subset S_3$ . The general element of the linear system  $|\mathcal{O}_{S_3}(H-E_0)|$  gives a conic of  $\mathbb{P}_3$  intersecting  $\Delta$  in a point and  $\bar{\Gamma}$  in six points. Any of these conics thus cuts the scheme defined by  $\mathcal{J}_\Delta^2 \cap \mathcal{J}_{\bar{\Gamma}}$  in length at least 8. So, these conics are contracted by  $|\mathcal{J}_\Delta^2 \cap \mathcal{J}_{\bar{\Gamma}}(4H)|$  and  $\bar{\tau}(S_3)$  is the line 5-secant to the image of  $\Gamma$  in  $\mathbb{P}'_3$ . We have achieved the description of the  $P$ -locus of  $\bar{\tau}$  with the following:

**Corollary 4.24.** *The jacobian of  $\bar{\tau}$  is the product of equations of  $S_9$  and  $S_3$ .*

**Proof.** Indeed, the jacobian of  $\bar{\tau}$  has degree 12 and vanishes on  $S_9$  and  $S_3$ . □

### 5. On the components of $\text{Bir}_{4,4}(\mathbb{P}_3, \mathbb{P}'_3)$

In the previous sections, we found elements  $\phi$  of  $\text{Bir}_{4,4}(\mathbb{P}_3, \mathbb{P}'_3)$  such that for a general hyperplane  $H'$  of  $\mathbb{P}'_3$ , the quartic surface  $\phi^{-1}(H')$  has the following properties:

- $\phi \in \mathcal{J}_{4,4} \iff \phi^{-1}(H')$  is normal (Lemma 3.5)  $\iff$  the genus of  $\phi$  is 3.
- If  $\phi$  belongs to  $\mathcal{D}_{4,4}$ , then  $\phi^{-1}(H')$  has a double line; furthermore  $\phi^{-1}(H')$  has a double line if and only if the genus of  $\phi$  is 2.
- The singular locus of  $\phi^{-1}(H')$  cannot contain two disjoint lines or a reduced cubic curve (Corollary 2.2).
- If  $\phi$  is in  $\mathcal{R}_{4,4}$ , then  $\phi^{-1}(H')$  has a triple line and the genus of  $\phi$  is 0.

So, it remains to provide a genus one example or to know if  $\phi^{-1}(H')$  can be singular along a conic. This is achieved with the following example.

**5.1. The family  $\mathcal{C}_{4,4}$**

**Proposition 5.1.** *Let  $p$  be a point of  $\mathbb{P}_3$ . Choose some coordinates such that the ideal of  $p$  in  $\mathbb{P}_3$  is  $\mathcal{J}_p = (z_0, z_1, z_2)$ . Take  $Q_1, Q_2, f$  general in the following spaces:*

$$Q_1 \in H^0(\mathcal{J}_p(2H)), Q_2 \in H^0(\mathcal{J}_p^2(2H)), f \in H^0(\mathcal{O}_{\mathbb{P}_3}(H))$$

and a general point  $p_1$  of  $\mathbb{P}_3$ . Then,

— The ideal

$$\mathcal{J} = (f, Q_1)^2 \cap (Q_1, Q_2) \cap \mathcal{J}_p^2 \cap \mathcal{J}_{p_1}$$

gives a three-dimensional linear system  $|\mathcal{J}(4H)|$ .

- Any isomorphism  $|\mathcal{J}(4H)|^\vee \simeq \mathbb{P}'_3$  defines a  $(4, 4)$ -birational map  $\phi$  from  $\mathbb{P}_3$  into  $\mathbb{P}'_3$  such that  $\phi^{-1}(H')$  is a quartic with a double conic.
- The base scheme  $F_\phi$  of  $\phi$  has degree 10. The subscheme  $F_\phi^1$  is the union of a multiple structure on a conic and a quartic curve singular at  $p$ .

**Proof.** First, consider the ideal

$$\mathcal{G} = (f, Q_1)^2 \cap (Q_1, Q_2) \cap \mathcal{J}_p^2.$$

In other words, we have

$$\mathcal{G} = (f^2Q_2, Q_1^2, fQ_1z_0, fQ_1z_1, fQ_1z_2).$$

From the inclusion  $H^0(\mathcal{J}_p^2(2H)) \subset \mathbb{C}[z_0, z_1, z_2]$ , we consider  $Q_2$  as a homogeneous polynomial of degree 2 in three variables. Therefore, the image of the rational map

$$|\mathcal{G}(4H)| : \mathbb{P}_3 \dashrightarrow \mathbb{P}_4$$

$$(z_0 : z_1 : z_2 : z_3) \mapsto (Z_0 : Z_1 : Z_2 : Z_3 : Z_4) = (fQ_1z_0 : fQ_1z_1 : fQ_1z_2 : f^2Q_2 : Q_1^2)$$

is the quadric of  $\mathbb{P}_4$  of equation  $Q_2(Z_0, Z_1, Z_2) - Z_3Z_4$  because we have by homogeneity of  $Q_2$

$$Q_2(fQ_1z_0, fQ_1z_1, fQ_1z_2) = (fQ_1)^2Q_2(z_0, z_1, z_2).$$

We have  $\mathcal{J} = \mathcal{G} \cap \mathcal{J}_{p_1}$  where  $\mathcal{J}_{p_1}$  is the ideal of a general point  $p_1$  of  $\mathbb{P}_3$ . Hence, the map  $\phi$  factors through the projection from a point of a quadric of  $\mathbb{P}_4$ , so it is birational with base scheme defined by  $\mathcal{J}$ . Let us compute the degree of  $\phi^{-1}$ . Let  $H'$  be a general hyperplane of  $\mathbb{P}'_3$ , then  $\phi^{-1}(H')$  is a quartic with a double conic. With Notation 3.4, the curve  $C_2$  is the union of a component supported on the conic of ideal  $(f, Q_1)$  and the singular quartic curve of ideal  $(Q_1, Q_2)$ . The multiple structure defined by  $C_2$  on the conic has degree 8, as a result  $\deg C_2 = 12$  and  $\deg C_1 = 4$ , then  $\phi \in \text{Bir}_{4,4}(\mathbb{P}_3, \mathbb{P}'_3)$ . On the other hand, the multiple structure defined by  $\mathcal{J}$  on the conic has degree 6, so  $\deg F_\phi = 10$ . □

**Definition 5.2.** Birational maps described in Proposition 5.1 form the family  $\mathcal{C}_{4,4}$ .

**Corollary 5.3.** *The family  $\mathcal{C}_{4,4} \subset \text{Bir}_{4,4}(\mathbb{P}_3, \mathbb{P}'_3)$  is irreducible of dimension 37.*

**Proof.** The ideal  $\mathcal{J}$  is determined by the singular quartic curve and the conic, but these two curves must be on a unique quadric defined by  $Q_1$ . Thus, to construct  $\mathcal{J}$ , we need a general quadric, then a singular biquadratic on this quadric, a general plane and the isolated point  $p_1$ . It gives an irreducible parameter space of dimension  $9 + 7 + 3 + 3$  to obtain  $\mathcal{J}$ , so

$$\dim(\mathcal{C}_{4,4}) = 22 + \dim(\mathrm{PGL}_4) = 37. \quad \square$$

**Proof of Theorem B.** From Proposition 3.13, Corollaries 3.6, 4.15 and 5.3 one has

- the closure of  $\mathcal{J}_{4,4}$  is an irreducible component of  $\mathrm{Bir}_{4,4}(\mathbb{P}_3, \mathbb{P}'_3)$ ,
- the closure of  $\mathcal{R}_{4,4}$  is an irreducible component of  $\mathrm{Bir}_{4,4}(\mathbb{P}_3, \mathbb{P}'_3)$ ,
- the closure of  $\mathcal{D}_{4,4}$  is an irreducible component of  $\mathrm{Bir}_{4,4}(\mathbb{P}_3, \mathbb{P}'_3)$ ,
- the family  $\mathcal{C}_{4,4}$  is irreducible.

Note that our definitions of  $\mathcal{R}_{4,4}$ ,  $\mathcal{D}_{4,4}$ ,  $\mathcal{C}_{4,4}$  contain general position assumptions, so we were able to compute degrees and dimensions for any element of these families (Propositions 3.7, 4.5 and 5.1). With notation of Definition 2.4, we have: if  $\phi$  is any element of  $\mathcal{J}_{4,4}$ , then by Definition 3.1  $\deg F_\phi^1 = 12$ , but this degree is 10 for an element of  $\mathcal{C}_{4,4}$ . So  $\mathcal{C}_{4,4}$  is not in the closure of  $\mathcal{J}_{4,4}$  by Corollary 2.7. Moreover,  $\dim F_\phi^\sigma = 0$ , but it is 1 for any element of  $\mathcal{C}_{4,4}$ , hence by Corollary 2.7  $\mathcal{J}_{4,4}$  is not in the closure of  $\mathcal{C}_{4,4}$ .

- If  $\phi$  is any element of  $\mathcal{D}_{4,4}$ , then  $\deg F_\phi = 11$ ,  $\dim F_\phi^\sigma = 1$  and  $\deg F_\phi^\sigma = 1$ .
- If  $\phi$  is any element of  $\mathcal{C}_{4,4}$ , then  $\deg F_\phi = 10$ ,  $\dim F_\phi^\sigma = 1$  and  $\deg F_\phi^\sigma = 2$ .
- If  $\phi$  is any element of  $\mathcal{R}_{4,4}$ , then  $\deg F_\phi = 9$ ,  $\dim F_\phi^\sigma = 1$  and  $\deg F_\phi^\sigma = 3$ .

By Corollary 2.7, since the map  $\alpha$  and  $\eta$  cannot decrease by specialization, one gets that none of the family  $\mathcal{R}_{4,4}$ ,  $\mathcal{C}_{4,4}$ ,  $\mathcal{D}_{4,4}$ ,  $\mathcal{J}_{4,4}$  is in the closure of another one. Let  $H$  (respectively  $H'$ ) be a general plane of  $\mathbb{P}_3$  (respectively,  $\mathbb{P}'_3$ ). The quartic surface  $\phi^{-1}(H')$  has the following property.

$\phi$ general in	$\mathcal{R}_{4,4}$	$\mathcal{C}_{4,4}$	$\mathcal{D}_{4,4}$	$\mathcal{J}_{4,4}$
$\phi^{-1}(H')$	has a triple line	has a double conic	has a double line	is normal

It gives the genus of  $H \cap \phi^{-1}(H')$ . □

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