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JONQUIÈRES MAPS AND SL(2;ℂ)-COCYCLES

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ABSTRACT. We started the study of the family of birational maps $(f_{\alpha,\beta})$ of $\mathbb{P}^2_{\mathbb{C}}$ in [12]. For "(α,β) well chosen" of modulus 1, the centraliser of $f_{\alpha,\beta}$ is trivial, the topological entropy of $f_{\alpha,\beta}$ is 0, and there exist two domains of linearisation: in the first one the closure of the orbit of a point is a torus, in the other one the closure of the orbit of a point is the union of two circles. On $\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}$, any $f_{\alpha,\beta}$ can be viewed as a cocyle; using recent results about SL(2;ℂ)-cocycles ([1]), we determine the Lyapunov exponent of the cocyle associated to $f_{\alpha,\beta}$.

INTRODUCTION

In this article we deal with a family of birational maps $(f_{\alpha,\beta})$ given by

$$f_{\alpha,\beta}\colon \mathbb{P}^2_{\mathbb{C}} \dashrightarrow \mathbb{P}^2_{\mathbb{C}} \quad (x:y:z) \dashrightarrow \big((\alpha x+y)z : \beta y(x+z) : z(x+z)\big),$$

where α, β denote two complex numbers of modulus 1, a case for which we know almost nothing about the dynamics. Let us consider the set Ω of pairs of complex numbers of modulus 1 that satisfy the Diophantine condition. The family $(f_{\alpha,\beta})$ satisfies the following properties ([12]):

- For $(\alpha,\beta)\in\Omega$ the centraliser of $f_{\alpha,\beta}$, that is the set of birational maps of $\mathbb{P}^2_{\mathbb{C}}$ that commutes with $f_{\alpha,\beta}$, is isomorphic to $\mathbb{Z}$.
- The topological entropy of $f_{\alpha,\beta}$ is 0.
- Rotation domains of ranks 1 and 2 coexist: there is a domain of linearisation where the orbit of a generic point under $f_{\alpha,\beta}$ is a torus, and there is another domain of linearisation where the orbit of a generic point under $f^2_{\alpha,\beta}$ is a circle.

We can also view $f_{\alpha,\beta}$ on $\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}$ (since all the computations of [12] have been done in an affine chart, they may all be carried on $\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}$); the sets $\mathbb{P}^1_{\mathbb{C}}\times\mathbb{S}^1_{\rho}$, where $\mathbb{S}^1_{\rho}=\{y\in\mathbb{C}\mid |y|=\rho\}$, are invariant.

Let us define $A^{\alpha,\rho}_n\colon \mathbb{S}^1_{\rho}\to \mathrm{M}(2;\mathbb{C})$, given in terms of $A^{\alpha,\rho}(y)=\begin{bmatrix}\alpha & y\\ 1 & 1\end{bmatrix}$, by

$$A^{\alpha,\rho}_n(\cdot)=A^{\alpha,\rho}(\beta^n\cdot)A^{\alpha,\rho}(\beta^{n-1}\cdot)\dots A^{\alpha,\rho}(\beta\cdot)A^{\alpha,\rho}(\cdot).$$

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To compute $f^n_{\alpha,\beta}(x,y)$ is equivalent to compute $A^{\alpha,\rho}_n(y)$ as soon as $f^k_{\alpha,\beta}(x,y)\neq(-1,\alpha)$, for any $1\leq k\leq n$.

Using [1] we are able to determine the Lyapunov exponent of the cocycle $(A^{\alpha,\rho},\beta)$.

THEOREM A. *The Lyapunov exponent of* $(A^{\alpha,\rho},\beta)$ *is positive if* $\rho>1$ *and zero if* $\rho\leq 1$. *More precisely,* $f_{\alpha,\beta}$ *is semi-conjugate to* $\left(\frac{\alpha x+y^2}{x+1},\beta^{1/2}y\right)$ *and the Lyapunov exponent of the cocycle* $\left(B^{\alpha,\rho},\beta^{1/2}\right)$, *where*

$$B^{\alpha,\rho}(y)=\begin{bmatrix}\alpha & y^2\\ 1 & 1\end{bmatrix},$$

is equal to $\max(0,\ln\rho)$.

In the next section we introduce the family $(f_{\alpha,\beta})$ and its properties (§1). Then we deal with the recent works of Avila on SL(2;$\mathbb{C}$)-cocyles. In the last section we give the proof of Theorem A (see §2). Let us explain the sketch of it. We associate to $\left(B^{\alpha,\rho},\beta^{1/2}\right)$ a cocycle $\left(\widetilde{B}^{\alpha,\rho},\beta^{1/2}\right)$ that belongs to SL(2;$\mathbb{C}$). We first determine

$$\lim_{\rho\to 0}L\left(\widetilde{B}^{\alpha,\rho},\beta^{1/2}\right)$$

and, then,

$$\lim_{\rho\to +\infty}L\left(\widetilde{B}^{\alpha,\rho},\beta^{1/2}\right),$$

where $L(C,\gamma)$ denotes the Lyapunov exponent of the SL(2;$\mathbb{C}$)-cocyle (C,γ). In both cases, we get 0. Using [1, Theorem 5] we obtain that $L\left(\widetilde{B}^{\alpha,\rho},\beta^{1/2}\right)$ vanishes everywhere; it allows us to determine $L\left(A^{\alpha,\rho},\beta\right)$ since

$$L\left(B^{\alpha,\rho}(y),\beta^{1/2}\right)=L\left(\widetilde{B}^{\alpha,\rho}(y),\beta^{1/2}\right)+\max(0,\ln\rho),$$

and since $\left(A^{\alpha,\rho},\beta\right)$ and $\left(\beta^{1/2},B^{\alpha,\rho}\right)$ are conjugate.

1. SOME PROPERTIES OF THE FAMILY $(f_{\alpha,\beta})$

A *rational map* ϕ from $\mathbb{P}^2_{\mathbb{C}}$ into itself is a map of the form

$$(x:y:z)\dashrightarrow\left(\phi_0(x,y,z):\phi_1(x,y,z):\phi_2(x,y,z)\right),$$

where the ϕ_i's are some homogeneous polynomials of the same degree without common factor; ϕ is *birational* if it admits an inverse of the same type. We will denote by $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ the group of birational maps of $\mathbb{P}^2_{\mathbb{C}}$, also called the *Cremona group*. The *degree* of ϕ, denoted $\deg\phi$, is the degree of the ϕ_i's. The degree is not a birational invariant: $\deg\psi\phi\psi^{-1}\neq\deg\phi$ for generic birational maps ϕ and ψ. The *first dynamical degree* of ϕ given by

$$\lambda(\phi)=\lim_{n\to+\infty}\left(\deg\phi^n\right)^{1/n}$$

is a birational invariant; it is strongly related to the topological entropy $h_{\mathrm{top}}(\phi)$ of ϕ (see [17, 20]),

$$h_{\mathrm{top}}(\phi)\leq\log\lambda(\phi).\tag{1.1}$$

Any birational map ϕ admits a resolution

$$\begin{array}{ccc} & \mathrm{S} & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ \mathbb{P}^2_{\mathbb{C}} & \overset{}{\underset{\phi}{\dashrightarrow}} & \mathbb{P}^2_{\mathbb{C}} \end{array}$$

where π_1, $\pi_2\colon \mathrm{S} \to \mathbb{P}^2_{\mathbb{C}}$ are sequences of blow-ups (see [3], for example). The resolution is *minimal* if and only if no (-1)-curve of S is contracted by both π_1 and π_2. The *base-points* of ϕ are the points blown-up in π_1, which can be points of $\mathbb{P}^2_{\mathbb{C}}$ or infinitely near points. We denote by $\mathfrak{b}(\phi)$ the number of such points, which is also equal to the difference of the ranks of $\mathrm{Pic}(\mathrm{S})$ and $\mathrm{Pic}(\mathbb{P}^2_{\mathbb{C}})$, and thus is equal to $\mathfrak{b}(\phi^{-1})$.

The *dynamical number of base-points of* ϕ introduced in [8] is by definition

$$\mu(\phi) = \lim_{n\to+\infty} \frac{\mathfrak{b}(\phi^n)}{n}.$$

It is a real positive number that satisfies $\mu(\psi\phi\psi^{-1}) = \mu(\phi)$ and, for any $n \in \mathbb{Z}$, $\mu(\phi^n) = |n\mu(\phi)|$. It allows us to give a characterization of birational maps conjugate to automorphisms.

THEOREM 1.1 ([8]). *Let* S *be a smooth projective surface; the birational map* $\phi \in \mathrm{Bir}(\mathrm{S})$ *is conjugate to an automorphism of a smooth projective surface if and only if* $\mu(\phi) = 0$.

The behavior of $\phi \in \mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ is strongly related to the behavior of $\big(\deg\phi^n\big)_{n\in\mathbb{N}}$ (see [16, 15, 8]); up to birational conjugacy exactly one of the following holds:

1. The sequence $\big(\deg\phi^n\big)_{n\in\mathbb{N}}$ is bounded and either ϕ is of finite order, or ϕ is an automorphism of $\mathbb{P}^2_{\mathbb{C}}$.
2. There exists an integer k such that
$$\lim_{n\to+\infty} \frac{\deg\phi^n}{n} = k^2\,\frac{\mu(\phi)}{2}$$
and ϕ is not an automorphism.
3. There exists an integer $k \geq 3$ such that
$$\lim_{n\to+\infty} \frac{\deg\phi^n}{n^2} = k^2\,\frac{\kappa(\phi)}{9},$$
where $\kappa(\phi) \in \mathbb{Q}$ is a birational invariant, and ϕ is an automorphism.
4. The sequence $\big(\deg\phi^n\big)_{n\in\mathbb{N}}$ grows exponentially (see [15] for more precise dynamical properties).

In the first three cases $\lambda(\phi) = 1$, in the last one $\lambda(\phi) > 1$. In case 2 (respectively, 3) the map ϕ preserves a unique fibration which is rational (respectively, elliptic).

In case 1 (respectively 2, 3, and 4) we say that ϕ is *elliptic* (respectively a *Jonquières twist*, an *Halphen twist*, and *hyperbolic*).

Let us give some examples. Let

$$\phi(x, y) = \left(\frac{a(y)x + b(y)}{c(y)x + d(y)}, \frac{\alpha y + \beta}{\gamma y + \delta}\right)$$

be an element of the *Jonquières group* $\mathrm{PGL}(2;\mathbb{C}(y)) \rtimes \mathrm{PGL}(2;\mathbb{C})$; either ϕ is elliptic (for instance, $\phi\colon (x:y:z) \dashrightarrow (yz:xz:xy)$), or ϕ is a Jonquières twist (for example, $\phi\colon (x:y:z) \dashrightarrow (xz:xy:z^2)$ for which the unique invariant fibration is $y/z = $ constant). The map

$$\phi\colon \mathbb{P}^2_{\mathbb{C}} \dashrightarrow \mathbb{P}^2_{\mathbb{C}} \quad (x:y:z) \dashrightarrow \big((2y+z)(y+z) : x(2y-z) : 2z(y+z)\big)$$

is an Halphen twist ([15, Proposition 9.5]). Hénon automorphisms give by homogenization examples of hyperbolic maps.

Clearly, elliptic birational maps have a poor dynamical behavior contrary to hyperbolic ones. The study of automorphisms of positive entropy is strongly related with birational maps of $\mathbb{P}^2_{\mathbb{C}}$.

THEOREM 1.2 ([9])**.** *Let* S *be a compact complex surface that carries an automorphism ϕ of positive topological entropy. Then, either*

- *the Kodaira dimension of* S *is zero and ϕ is conjugate to an automorphism on the unique minimal model of* S *that necessarily is a torus, a K3 surface, or an Enriques surface; or*
- *the surface* S *is a non-minimal rational one, isomorphic to $\mathbb{P}^2_{\mathbb{C}}$ blown up at n points, $n \geq 10$, and ϕ is conjugate to a birational map of $\mathbb{P}^2_{\mathbb{C}}$.*

This yields many examples of hyperbolic birational maps for which we can establish many dynamical properties ([18, 4, 5, 6, 7, 14, 13]).

Another way to measure chaos is to look at the size of centralisers. Let us give two examples. The polynomial automorphisms of $\mathbb{C}^2$ having rich dynamics are Hénon maps; furthermore, a polynomial automorphism of $\mathbb{C}^2$ is a Hénon one if and only if its centraliser is countable. Let us now consider rational maps on $\mathbb{S}^1$; if the centraliser of such maps is not trivial [1], then the Julia set is "special". The centraliser of an elliptic birational map of infinite order is uncountable ([8]). The centralisers of Halphen twists are described in [16]. The centraliser of an hyperbolic map is countable ([10]). In [11] we end the story by studying centralisers of Jonquières twists. If the fibration is fiberwise invariant, then the centraliser is uncountable; but if it isn't, then generically the centraliser is isomorphic to $\mathbb{Z}$. We don't know much about the dynamics of these maps, thus in this article we will focus on a family of such maps. We consider the Jonquières maps

$$f_{\alpha,\beta}\colon \mathbb{P}^2_{\mathbb{C}} \dashrightarrow \mathbb{P}^2_{\mathbb{C}} \quad (x:y:z) \dashrightarrow \big((\alpha x + y)z : \beta y(x+z) : z(x+z)\big),$$

where α, β denote two complex numbers of modulus 1. The base-points of $f_{\alpha,\beta}$ are

$$(1:0:0), \quad (0:1:0), \quad (-1:\alpha:1).$$

1. The centraliser of a map ϕ is trivial if it coincides with the iterates of ϕ.

Any $f_{\alpha,\beta}$ preserves a rational fibration (the fibration y = constant in the affine chart $z=1$). Each element of the family $(f_{\alpha,\beta})$ has first dynamical degree 1, hence topological entropy zero (1.1); more precisely, one has ([8, Example 4.3])

$$\mu(f_{\alpha,\beta})=\frac{1}{2},$$

so $f_{\alpha,\beta}$ is not conjugate to an automorphism (Theorem 1.1). The centralizer of $f_{\alpha,\beta}$ is isomorphic to $\mathbb{Z}$ (see [12, Theorem 1.6]). The idea of the proof is as follows: the point $p=(1:\alpha:1)$ is blown-up onto a fiber of the fibration $y=$ constant. Let ψ be an element of

$$\mathrm{Cent}(f_{\alpha,\beta})=\left\{g\in\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})\mid g\circ f_{\alpha,\beta}=f_{\alpha,\beta}\circ g\right\}.$$

Since ψ blows down a finite number of curves, there exists a positive integer k (chosen minimal) such that $f^k_{\alpha,\beta}(p)$ is not blown down by ψ. Replacing ψ by $\widetilde{\psi}=\psi f^{k-1}_{\alpha,\beta}$, one gets that $\widetilde{\psi}(p)$ is an indeterminacy point of $f_{\alpha,\beta}$. In other words, $\widetilde{\psi}$ permutes the indeterminacy points of $f_{\alpha,\beta}$. A more precise study allows us to establish that p is fixed by $\widetilde{\psi}$. The pair (α,β) being in Ω, the closure of the negative orbit of p under the action of $f_{\alpha,\beta}$ is Zariski dense; since $\widetilde{\psi}$ fixes any element of the orbit of p, one obtains $\widetilde{\psi}=\mathrm{id}$.

Let us recall that if ψ is an automorphism on a compact complex manifold M, then the *Fatou set* $\mathscr{F}(\psi)$ of ψ is the set of points that have a neighborhood $\mathscr{V}$ such that $\left\{f^n_{|\mathscr{V}}\mid n\in\mathbb{N}\right\}$ is a normal family. Set

$$\mathscr{G}(\mathscr{U})=\left\{\phi\colon\mathscr{U}\to\overline{\mathscr{U}}\mid\phi=\lim_{n_j\to+\infty}\psi^{n_j}\right\}.$$

We say that $\mathscr{U}$ is a *rotation domain* if $\mathscr{G}(\mathscr{U})$ is a subgroup of $\mathrm{Aut}(\mathscr{U})$. An equivalent definition is the following: a component $\mathscr{U}$ of $\mathscr{F}(\psi)$ which is invariant by ψ is a rotation domain if $\psi_{|\mathscr{U}}$ is conjugate to a linear rotation. If $\mathscr{U}$ is a rotation domain, then $\mathscr{G}(\mathscr{U})$ is a compact Lie group, and the action of $\mathscr{G}(\mathscr{U})$ on $\mathscr{U}$ is analytic real. Since $\mathscr{G}(\mathscr{U})$ is a compact, infinite, abelian Lie group, the connected component of the identity of $\mathscr{G}(\mathscr{U})$ is a torus of dimension $0\leq d\leq\dim_{\mathbb{C}}\mathrm{M}$. The integer d is the *rank of the rotation domain.* The rank coincides with the dimension of the closure of a generic orbit of a point in $\mathscr{U}$.

We can also view $f_{\alpha,\beta}$ on $\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}$ and that is what we will do in the sequel (since all the computations of [12] have been done in an affine chart, they may all be carried on $\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}$); the sets $\mathbb{P}^1_{\mathbb{C}}\times\mathbb{S}^1_{\rho}$ are invariant. In [12] we show that there are two rotation domains for $f^2_{\alpha,\beta}$, one of rank 1, and the other one of rank 2 [2]; for the first case, we give below a more precise statement than in [12].

THEOREM 1.3. *Assume that* (α,β) *belongs to* Ω. *There exists a strictly positive real number* r *such that* $f_{\alpha,\beta}$ *is conjugate to* $(\alpha x,\beta y)$ *on* $\mathbb{P}^1_{\mathbb{C}}\times\mathbb{D}(0,r)$, *where* $\mathbb{D}(0,r)$ *denotes the disk centered at the origin with radius* r. *There exists a strictly positive real number* $\widetilde{r}$ *such that* $f^2_{\alpha,\beta}$ *is conjugate to* $\left(\frac{x}{\beta},\frac{z}{\beta^2}\right)$ *on* $\mathbb{P}^1_{\mathbb{C}}\times\mathbb{D}(0,\widetilde{r})$.

2. There already exists an example of automorphism of positive entropy with rotation domains of rank 1 and 2 (*see* [5]), but $f_{\alpha,\beta}$ is not conjugate to an automorphism on a rational surface.

REMARK 1.4. The point $(\alpha-1,0)$ is also a fixed point of $f_{\alpha,\beta}$, where the behavior of $f_{\alpha,\beta}$ is the same as near $(0,0)$.

Proof. The first assertion is proved in [12].

Let us consider the map $\psi(x,z)=\left(\frac{a(z)x+b(z)}{c(z)x+1},z\right)$. The equation

$$\psi^{-1}f^2_{\alpha,\beta}\psi=\left(\frac{x}{\beta},\frac{z}{\beta^2}\right)$$

yields

$$\begin{aligned}(1.2)\quad &\beta\, a\big(\beta^{-2}z\big)c(z)+\beta\, a\big(\beta^{-2}z\big)a(z)-c\big(\beta^{-2}z\big)a(z)+\alpha\, a\big(\beta^{-2}z\big)a(z)\\ &+z\big(\alpha^2 a\big(\beta^{-2}z\big)c(z)-\alpha\, c\big(\beta^{-2}z\big)c(z)-c\big(\beta^{-2}z\big)c(z)-c\big(\beta^{-2}z\big)a(z)\big)=0,\end{aligned}$$

$$\begin{aligned}(1.3)\quad &\beta\, a\big(\beta^{-2}z\big)-\beta\, a(z)+z\big(\alpha^2 a\big(\beta^{-2}z\big)-\alpha\beta\, c(z)-\beta\, c(z)-\beta\, a(z)\\ &-\alpha\, c\big(\beta^{-2}z\big)-c\big(\beta^{-2}z\big)\big)+\beta(\alpha+\beta)\, a(z)b\big(\beta^{-2}z\big)+(\alpha+\beta)\, b(z)a\big(\beta^{-2}z\big)\\ &+\beta^2\, b\big(\beta^{-2}z\big)c(z)-b(z)c\big(\beta^{-2}z\big)+z\big(\alpha^2\beta b\big(\beta^{-2}z\big)c(z)-b(z)c\big(\beta^{-2}z\big)\big)=0,\end{aligned}$$

and

$$\begin{aligned}(1.4)\quad &(\alpha+1)\,z+b(z)-\beta\, b\big(\beta^{-2}z\big)-\alpha^2 zb\big(\beta^{-2}z\big)\\ &+zb(z)-(\alpha+\beta)\, b\big(\beta^{-2}z\big)b(z)=0.\end{aligned}$$

Let us set

$$a(z)=\sum_{i\geq 0}a_iz^i,\qquad b(z)=\sum_{i\geq 0}b_iz^i,\qquad c(z)=\sum_{i\geq 0}c_iz^i.$$

We easily get $a_0=1-\beta$, $b_0=0$ and $c_0=\alpha+\beta$.

Relation (1.4) implies that

$$b_1=\frac{\beta(1+\alpha)}{1-\beta}\quad\text{and}\quad \beta\, b_\nu\left(1-\beta^{1-2\nu}\right)+F_i(b_i\,|\,i<\nu)=0\,,\forall\,\nu>1,$$

(1.3) yields

$$a_\nu\left(\beta^{1-2\nu}-\beta\right)+b_\nu\left((\alpha+\beta)a_0\left(1+\beta^{1-2\nu}\right)+c_0\left(\beta^{2-2\nu}-1\right)\right)+G_i(a_i,\,b_i,\,c_i\,|\,i<\nu)=0,$$

and (1.2) implies

$$c_\nu a_0\left(\beta-\beta^{-2\nu}\right)+a_\nu\left((\alpha+\beta)a_0\left(1+\beta^{-2\nu}\right)+c_0\left(\beta^{1-2\nu}-1\right)\right)+H_i(a_i,\,b_i,\,c_i\,|\,i<\nu)=0,$$

where the F_i's, G_i's and H_i's denote universal polynomials; this allows to compute b_ν, a_ν and c_ν. Thus we get a formal conjugacy of $f^2_{\alpha,\beta}$ to its linear part. Since this linear part satisfies a Rüssmann condition (see [19, Theorem 2.1], condition (2)), according to [19, Theorem 2.1], any formal linearizing map conjugating $f^2_{\alpha,\beta}$ to its linear part is convergent on a polydisc. □

2. ABOUT SL(2;ℂ)-COCYCLES

A (one-frequency, analytic) *quasiperiodic* SL(2;ℂ)*-cocycle* is a pair (A,β), where $\beta\in\mathbb{R}$ and

$$A\colon \mathbb{S}^1_1\to \mathrm{SL}(2;\mathbb{C})$$

is analytic, and defines a linear skew product acting on $\mathbb{C}^2\times\mathbb{S}^1_1$ by

$$(x,y)\mapsto (A(y)\cdot x,\beta y).$$

The iterates of the cocyle are given by $(A_n,n\beta)$ where A_n is given by

$$A_n(y)=A\big(\beta^{n-1}y\big)\dots A(y)\quad n\geq 1,\quad A_0(y)=\mathrm{id},\quad A_{-n}(y)=A_n(\beta^{-n}y)^{-1}.$$

The *Lyapunov exponent* $L(A,\beta)$ of a quasiperiodic SL(2;ℂ)-cocycle (A,β) is given by

$$\lim_{n\to+\infty}\frac{1}{n}\int_{\mathbb{S}^1_1}\ln||A_n(y)||\,\mathrm{d}y.$$

A quasiperiodic SL(2;ℂ)-cocycle (A,β) is *uniformly hyperbolic* if there exist analytic functions

$$u,\,s\colon \mathbb{S}^1_1\to\mathbb{P}^2_{\mathbb{C}},$$

called the *unstable and stable directions*, and $n\geq 1$ such that for any $y\in\mathbb{S}^1_1$,

$$A(y)\cdot u(y)=u(\beta y),\quad A(y)\cdot s(y)=s(\beta y),$$

and for any unit vector $x\in s(y)$ (respectively, $x\in u(y)$) we have $||A_n(y)\cdot x||<1$ (respectively, $||A_n(y)\cdot x||>1$). The unstable and stable directions are uniquely characterized by those properties, and clearly $u(y)\neq s(y)$ for any $y\in\mathbb{S}^1_1$. If (A,β) is uniformly hyperbolic, then $L(A,\beta)>0$. Let us denote by

$$\mathscr{UH}\subset C^\omega\big(\mathrm{SL}(2;\mathbb{C}),\mathbb{S}^1_1\big)$$

the set of A such that (A,β) is uniformly hyperbolic. Uniform hyperbolicity is a stable property: $\mathscr{UH}$ is open, and $A\mapsto L(A,\beta)$ is analytic over $\mathscr{UH}$ (regularity properties of the Lyapunov exponent are consequence of the regularity of the unstable and stable directions which depend smoothly on both variables).

DEFINITION. Let (A,β) be a quasiperiodic SL(2;ℂ)-cocycle. If $L(A,\beta)>0$ but $(A,\beta)\not\in\mathscr{UH}$, then (A,β) is *nonuniformly hyperbolic*.

If $A\in C^\omega\big(\mathrm{SL}(2;\mathbb{C}),\mathbb{S}^1_1\big)$ admits a holomorphic extension to $|\mathrm{Im}\,y|<\delta$, then for $|\varepsilon|<\delta$ we can define $A_\varepsilon\in C^\omega\big(\mathrm{SL}(2;\mathbb{C}),\mathbb{S}^1_1\big)$ by

$$A_\varepsilon(y)=A(y+\mathbf{i}\varepsilon).$$

The Lyapunov exponent $L(A_\varepsilon,\beta)$ is a convex function of ε. We can thus introduce the following notion. The *acceleration* of a quasiperiodic SL(2;ℂ)-cocyle (A,β) is given by

$$\omega(A,\beta)=\lim_{\varepsilon\to 0^+}\frac{1}{2\pi\varepsilon}\big(L(A_\varepsilon,\beta)-L(A,\beta)\big).$$

REMARK 2.1. The convexity of the Lyapunov exponent as function of ε implies that the acceleration is decreasing.

Since the Lyapunov exponent is a convex and continuous function, the acceleration is an upper semi-continuous function in $\mathbb{R}\smallsetminus\mathbb{Q}\times C^\omega\big(\mathrm{SL}(2;\mathbb{C}),\mathbb{S}^1_1\big)$. The acceleration is quantized.

THEOREM 2.2 ([1]). *If (A,β) is a $\mathrm{SL}(2;\mathbb{C})$-cocycle with $\beta\in\mathbb{R}\smallsetminus\mathbb{Q}$, then $\omega(A,\beta)$ is always an integer.*

A direct consequence is the following.

COROLLARY 2.3. *The function $\varepsilon\mapsto L(A_\varepsilon,\beta)$ is a piecewise affine function of ε.*

It is thus natural to introduce the notion of regularity. A cocycle

$$(A,\beta)\in C^\omega\big(\mathrm{SL}(2;\mathbb{C}),\mathbb{S}^1_1\big)\times\mathbb{R}\smallsetminus\mathbb{Q}$$

is *regular* if $L(A_\varepsilon,\beta)$ is affine for ε in a neighborhood of 0. In other words, (A,β) is regular if the equality

$$L(A_\varepsilon,\beta)-L(A,\beta)=2\pi\varepsilon\omega(A,\beta)$$

holds for all ε small, and not only for the positive ones. Regularity is equivalent to the acceleration being locally constant near (A,β). It is an open condition in $C^\omega\big(\mathrm{SL}(2;\mathbb{C}),\mathbb{S}^1_1\big)\times\mathbb{R}\smallsetminus\mathbb{Q}$. The following statement gives a characterization of the dynamics of regular cocycles with positive Lyapunov exponent.

THEOREM 2.4 ([1]). *Let (A,β) be a $\mathrm{SL}(2;\mathbb{C})$-cocycle with $\beta\in\mathbb{R}\smallsetminus\mathbb{Q}$. Assume that $L(A,\beta)>0$; then (A,β) is regular if and only if (A,β) is in $\mathcal{UH}$.*

One striking consequence is the following:

COROLLARY 2.5 ([1]). *For any (A,β) in $C^\omega\big(\mathrm{SL}(2;\mathbb{C}),\mathbb{S}^1_1\big)\times\mathbb{R}\smallsetminus\mathbb{Q}$, there exists ε_0 such that*

- *$L(A_\varepsilon,\beta)=0$ (and $\omega(A,\beta)=0$) for every $0<\varepsilon<\varepsilon_0$, or*
- *$(A_\varepsilon,\beta)\in\mathcal{UH}$ for every $0<\varepsilon<\varepsilon_0$.*

REMARK 2.6. Let us mention that there is a link between $\mathrm{SL}(2;\mathbb{C})$-cocycles and Schrödinger operators (see [1] for more details).

3. PROOF OF THEOREM A

Suppose that $\rho\neq1$, and let us consider the cocycle $(B^{\alpha,\rho},\beta^{1/2})$, where

$$B^{\alpha,\rho}(y)=\begin{bmatrix}\alpha & y^2\\ 1 & 1\end{bmatrix}.$$

Since

$$\Big(\frac{\alpha x+y}{x+1},\beta y\Big)\,(x,y^2)=(x,y^2)\,\Big(\frac{\alpha x+y^2}{x+1},\beta^{1/2}y\Big),$$

the cocycles $(A^{\alpha,\rho},\beta)$ and $(B^{\alpha,\rho},\beta^{1/2})$ have the same behavior. Using two different arguments of monodromy (one for $\rho<1$, and the other one for $\rho>1$) we see that there is a continuous determination for the square root of $\det B^{\alpha,\rho}(y)=\alpha-y^2$. Let us set

$$\widetilde{B}^{\alpha,\rho}(y)=\frac{1}{\sqrt{\alpha-y^2}}B^{\alpha,\rho}(y)\in\mathrm{SL}(2;\mathbb{C})$$

that is thus defined on two different domains of analyticity. According to Theorem 1.3 one has $L\big(\widetilde{B}^{\alpha,\rho},\beta^{1/2}\big)=0$ when ρ is close to both 0 and ∞.

Assume that $L\big(\widetilde{B}^{\alpha,\rho},\beta^{1/2}\big)$ is nonconstant. When $\widetilde{B}^{\alpha,\rho}$ is holomorphic, in particular, when $\rho<1$ and $\rho>1$, the acceleration is decreasing (Remark 2.1); furthermore, the acceleration is positive for $\rho<1$ and negative for $\rho>1$ (because L is continuous). Theorem 2.2 thus implies

$$\omega\big(\widetilde{B}^{\alpha,1^+},\beta^{1/2}\big)-\omega\big(\widetilde{B}^{\alpha,1^-},\beta^{1/2}\big)\le -2.$$

By definition of $\widetilde{B}^{\alpha,\rho}$ we have

$$\begin{aligned}L\big(\widetilde{B}^{\alpha,\rho}(y),\beta^{1/2}\big)&=L\big(B^{\alpha,\rho}(y),\beta^{1/2}\big)-\int_{\mathbb{S}^1_\rho}\ln\sqrt{\alpha-y^2}\,\mathrm{d}y\\&=L\big(B^{\alpha,\rho}(y),\beta^{1/2}\big)-\max(0,\ln\rho).\end{aligned}$$

Even though $\big(B^{\alpha,\rho}(y),\beta^{1/2}\big)$ is not a SL(2;ℂ)-cocycle, the Lyapunov exponent is still a convex function of $\log\rho$ (see for example [2]). The jump of $\omega(B^{\alpha,\rho}(y),\beta^{1/2})$ is thus ≥ 0, and the jump for the second term of the right member is -1. Therefore the jump of $L\big(\widetilde{B}^{\alpha,\rho}(y),\beta^{1/2}\big)$ is ≥ -1, contradiction. □

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