Illinois Journal of Mathematics
Volume 59, Number 2, Summer 2015, Pages 345–358
S 0019-2082

ON SOLVABLE SUBGROUPS OF THE CREMONA GROUP

JULIE DÉSERTI

ABSTRACT. The Cremona group $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$ is the group of birational self-maps of $\mathbb{P}^2_\mathbb{C}$. Using the action of $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$ on the Picard-Manin space of $\mathbb{P}^2_\mathbb{C}$, we characterize its solvable subgroups. If $\mathrm{G}\subset\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$ is solvable, nonvirtually Abelian, and infinite, then up to finite index: either any element of G is of finite order or conjugate to an automorphism of $\mathbb{P}^2_\mathbb{C}$, or G preserves a unique fibration that is rational or elliptic, or G is, up to conjugacy, a subgroup of the group generated by one hyperbolic monomial map and the diagonal automorphisms.

We also give some corollaries.

1. Introduction

We know properties on finite subgroups ([16]), finitely generated subgroups ([6]), uncountable maximal Abelian subgroups ([13]), nilpotent subgroups ([14]) of the Cremona group. In this article, we focus on solvable subgroups of the Cremona group.

Let G be a group. Recall that $[g,h]=ghg^{-1}h^{-1}$ denotes the commutator of g and h. If Γ_1 and Γ_2 are two subgroups of G, then $[\Gamma_1,\Gamma_2]$ is the subgroup of G generated by the elements of the form $[g,h]$ with $g\in\Gamma_1$ and $h\in\Gamma_2$. We define the *derived series* of G by setting

$$\mathrm{G}^{(0)}=\mathrm{G},\qquad \mathrm{G}^{(n+1)}=\left[\mathrm{G}^{(n)},\mathrm{G}^{(n)}\right]\quad\forall n\geq 0.$$

The *soluble length* $\ell(\mathrm{G})$ of G is defined by

$$\ell(\mathrm{G})=\min\left\{k\in\mathbb{N}\cup\{0\}\,|\,\mathrm{G}^{(k)}=\{\mathrm{id}\}\right\}$$

with the convention: $\min\emptyset=\infty$. We say that G is *solvable* if $\ell(\mathrm{G})<\infty$. The study of solvable groups started a long time ago, and any linear solvable

Received April 9, 2015; received in final form February 3, 2016.
Author partially supported by ANR Grant “BirPol” ANR-11-JS01-004-01.
2010 *Mathematics Subject Classification.* 14E07, 14E05.

subgroup is up to finite index triangularizable (Lie–Kolchin theorem, [23, Theorem 21.1.5]). The assumption "up to finite index" is essential: for instance, the subgroup of $\mathrm{PGL}(2,\mathbb{C})$ generated by $\left[\begin{smallmatrix}1 & 0\\ 1 & -1\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}-1 & 1\\ 0 & 1\end{smallmatrix}\right]$ is isomorphic to $\mathfrak{S}_3$ so is solvable but is not triangularizable.

Theorem A. *Let* G *be an infinite, solvable, non virtually Abelian subgroup of* $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$. *Then, up to finite index, one of the following holds*:

(1) *any element of* G *is either of finite order, or conjugate to an automorphism of* $\mathbb{P}^2_\mathbb{C}$;

(2) G *preserves a unique fibration that is rational, in particular* G *is, up to conjugacy, a subgroup of* $\mathrm{PGL}(2,\mathbb{C}(y))\rtimes\mathrm{PGL}(2,\mathbb{C})$;

(3) G *preserves a unique fibration that is elliptic*;

(4) G *is, up to birational conjugacy, contained in the group generated by*

$$\big\{\big(x^py^q,x^ry^s\big),(\alpha x,\beta y)|\alpha,\beta\in\mathbb{C}^*\big\},$$

where $M=\left[\begin{smallmatrix}p & q\\ r & s\end{smallmatrix}\right]$ *denotes an element of* $\mathrm{GL}(2,\mathbb{Z})$ *with spectral radius* >1. *The group* G *preserves the two holomorphic foliations defined by the* 1*-forms* $\alpha_1x\mathrm{d}y+\beta_1y\mathrm{d}x$ *and* $\alpha_2x\mathrm{d}y+\beta_2y\mathrm{d}x$ *where* (α_1,β_1) *and* (α_2,β_2) *denote the eigenvectors of* ${}^tM$.

Furthermore if G *is uncountable, case* 3. *does not hold.*

Examples. • Denote by S_3 the group generated by $\left[\begin{smallmatrix}1 & 0\\ 1 & -1\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}-1 & 1\\ 0 & 1\end{smallmatrix}\right]$. As we recall before $\mathrm{S}_3\simeq\mathfrak{S}_3$. Consider now the subgroup G of $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$ whose elements are the monomial maps (x^py^q,x^ry^s) with $\left[\begin{smallmatrix}p & q\\ r & s\end{smallmatrix}\right]\in\mathrm{S}_3$. Then any element of G has finite order, and G is solvable; it gives an example of case 1.

- Other examples that illustrate case 1. are the following groups

$$\big\{(\alpha x+\beta y+\gamma,\delta y+\varepsilon)|\alpha,\delta\in\mathbb{C}^*,\beta,\gamma,\varepsilon\in\mathbb{C}\big\}\subset\mathrm{Aut}\big(\mathbb{P}^2_\mathbb{C}\big),$$

and

$$\mathrm{E}=\big\{\big(\alpha x+P(y),\beta y+\gamma\big)|\alpha,\beta\in\mathbb{C}^*,\gamma\in\mathbb{C},P\in\mathbb{C}[y]\big\}\subset\mathrm{Aut}\big(\mathbb{C}^2\big).$$

- The centralizer of a birational map of $\mathbb{P}^2_\mathbb{C}$ that preserves a unique fibration that is rational is virtually solvable ([9, Corollary C]); this example falls in case 2 (see Section 3.2).
- In [10, Proposition 2.2] Cornulier proved that the group

$$\big\langle(x+1,y),(x,y+1),(x,xy)\big\rangle$$

is solvable of length 3, and is not linear over any field; this example falls in case 2. The invariant fibration is given by $x=\mathrm{cst}$.

Remark. In case 1 if there exists an integer d such that $\deg\phi\leq d$ for any ϕ in G, then there exist a smooth projective variety M and a birational map $\psi\colon M\dashrightarrow\mathbb{P}^2_\mathbb{C}$ such that $\psi^{-1}\mathrm{G}\psi$ is a solvable subgroup of $\mathrm{Aut}(M)$ (see Section 3.3). But there is some solvable subgroups G with only elliptic elements that do not satisfy this property: the group E introduced in Examples. Let us mention an other example: Wright constructs Abelian subgroups H of $\mathrm{Aut}(\mathbb{C}^2)$ such that any element of H is of finite order, H is unbounded and does not preserve any fibration ([28]).

In Section 3, we prove Theorem A: we first assume that our solvable, infinite and non virtually Abelian subgroup G contains a hyperbolic map, then that it contains a twist and no hyperbolic map, and finally that all elements of G are elliptic. In the last section (Section 4), we also

- recover the following fact: if G is an infinite nilpotent subgroup of $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$, then G does not contain a hyperbolic map;
- remark that we can bound the soluble length of a nilpotent subgroup of $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$ by the dimension of $\mathbb{P}^2_\mathbb{C}$ as Epstein and Thurston did in the context of Lie algebras of rational vector fields on a connected complex manifold;
- give a negative answer to the following question of Favre: does any solvable and finitely generated subgroup G of $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$ contain a subgroup of finite index whose commutator subgroup is nilpotent? if we assume that $[\mathrm{G},\mathrm{G}]$ is not a torsion group;
- give a description of the embeddings of the solvable Baumslag–Solitar groups into the Cremona group.

2. Some properties of the birational maps

First definitions. Let $\mathcal{S}$ be a projective surface. We will denote by $\mathrm{Bir}(\mathcal{S})$ the group of birational self-maps of $\mathcal{S}$; in the particular case of the complex projective plane the group $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$ is called *Cremona group*. Take ϕ in $\mathrm{Bir}(\mathcal{S})$, we will denote by $\mathrm{Ind}\,\phi$ the set of points of indeterminacy of ϕ; the codimension of $\mathrm{Ind}\,\phi$ is ≥ 2.

A birational map from $\mathbb{P}^2_\mathbb{C}$ into itself can be written

$$(x:y:z) \dashrightarrow \big(\phi_0(x,y,z):\phi_1(x,y,z):\phi_2(x,y,z)\big),$$

where the ϕ_i's denote some homogeneous polynomials of the same degree and without common factors of positive degree. The *degree* of ϕ is equal to the degree of the ϕ_i's. Let ϕ be a birational map of $\mathbb{P}^2_\mathbb{C}$. One can define the *dynamical degree* of ϕ as

$$\lambda(\phi) = \lim_{n\to+\infty}\big(\deg\phi^n\big)^{1/n}.$$

More generally, let $\mathcal{S}$ be a projective surface, and $\phi\colon \mathcal{S} \dashrightarrow \mathcal{S}$ be a birational map. Take any norm $\|\cdot\|$ on the Néron–Severi real vector space $\mathrm{N}^1(\mathcal{S})$. If ϕ^* is the induced action by ϕ on $\mathrm{N}^1(\mathcal{S})$, we can define

$$\lambda(\phi) = \lim_{n\to+\infty}\big\|\big(\phi^n\big)^*\big\|^{1/n}.$$

Remark that this quantity is a birational invariant: if $\psi\colon \mathcal{S} \dashrightarrow \mathcal{S}'$ is a birational map, then $\lambda(\psi\phi\psi^{-1}) = \lambda(\phi)$.

Classification of birational maps. The algebraic degree is not a birational invariant, but the first dyna mical degree is; more precisely one has a classification of birational maps based on the degree growth. Before stating it, let us first introduce the following definitions. Let ϕ be an element of $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$. If

- $(\deg\phi^n)_{n\in\mathbb{N}}$ is bounded, we say that ϕ is an *elliptic map*,
- $(\deg\phi^n)_{n\in\mathbb{N}}$ grows linearly, we say that ϕ is a *Jonquières twist*,
- $(\deg\phi^n)_{n\in\mathbb{N}}$ grows quadratically, we say that ϕ is a *Halphen twist*,
- $(\deg\phi^n)_{n\in\mathbb{N}}$ grows exponentially, we say that ϕ is a *hyperbolic map*.

THEOREM 2.1 ([15], [20], [3]). *Let ϕ be an element of* $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$. *Then one and only one of the following cases holds*

- *ϕ is elliptic, furthermore if ϕ is of infinite order, then ϕ is up to birational conjugacy an automorphism of $\mathbb{P}^2_{\mathbb{C}}$,*
- *ϕ is a Jonquières twist, ϕ preserves a unique fibration that is rational and every conjugate of ϕ is not an automorphism of a projective surface,*
- *ϕ is a Halphen twist, ϕ preserves a unique fibration that is elliptic and ϕ is conjugate to an automorphism of a projective surface,*
- *ϕ is a hyperbolic map.*

In the three first cases, $\lambda(\phi)=1$, *in the last one* $\lambda(\phi)>1$.

The Picard–Manin and bubble spaces. Let $\mathcal{S}$, and $\mathcal{S}_i$ be complex projective surfaces. If $\pi\colon\mathcal{S}_1\to\mathcal{S}$ is a birational morphism, one gets $\pi^*\colon \mathrm{N}^1(\mathcal{S})\to \mathrm{N}^1(\mathcal{S}_1)$ an embedding of Néron–Severi groups. Take two birational morphisms $\pi_1\colon\mathcal{S}_1\to\mathcal{S}$ and $\pi_2\colon\mathcal{S}_2\to\mathcal{S}$; the morphism π_2 is *above* π_1 if $\pi_1^{-1}\pi_2$ is regular. Starting with two birational morphisms one can always find a third one that covers the two first. Therefore, the inductive limit of all groups $\mathrm{N}^1(\mathcal{S}_i)$ for all surfaces $\mathcal{S}_i$ above $\mathcal{S}$ is well-defined; it is the *Picard-Manin space* $\mathcal{Z}_{\mathcal{S}}$ of $\mathcal{S}$. For any birational map π, π^* preserves the intersection form and maps nef classes to nef classes hence the limit space $\mathcal{Z}_{\mathcal{S}}$ is endowed with an intersection form of signature $(1,\infty)$ and a nef cone.

Let $\mathcal{S}$ be a complex projective surface. Consider all complex and projective surfaces $\mathcal{S}_i$ above $\mathcal{S}$, that is all birational morphisms $\pi_i\colon\mathcal{S}_i\to\mathcal{S}$. If p (resp. q) is a point of a complex projective surface $\mathcal{S}_1$ (resp. $\mathcal{S}_2$), and if $\pi_1\colon\mathcal{S}_1\to\mathcal{S}$ (resp. $\pi_2\colon\mathcal{S}_2\to\mathcal{S}$) is a birational morphism, then p is identified with q if $\pi_1^{-1}\pi_2$ is a local isomorphism in a neighborhood of q that maps q onto p. The *bubble space* $\mathcal{B}(\mathcal{S})$ is the union of all points of all surfaces above $\mathcal{S}$ modulo the equivalence relation induced by this identification. If p belongs to $\mathcal{B}(\mathcal{S})$ represented by a point p on a surface $\mathcal{S}_i\to\mathcal{S}$, denote by E_p the exceptional divisor of the blow-up of p and by e_p its divisor class viewed as a point in $\mathcal{Z}_{\mathcal{S}}$. The following properties are satisfied

$$\begin{cases} e_p\cdot e_q=0 & \text{if } p\neq q,\\ e_p\cdot e_p=-1. \end{cases}$$

Hyperbolic space $\mathbb{H}_{\mathcal{S}}$**.** Embed $\mathrm{N}^1(\mathcal{S})$ as a subgroup of $\mathcal{Z}_{\mathcal{S}}$; this finite dimensional lattice is orthogonal to e_p for any $p \in \mathcal{B}(\mathcal{S})$, and

$$\mathcal{Z}_{\mathcal{S}} = \left\{ D + \sum_{p \in \mathcal{B}(\mathcal{S})} a_p e_p \middle| D \in \mathrm{N}^1(\mathcal{S}), a_p \in \mathbb{R} \right\}.$$

The *completed Picard-Manin space* $\overline{\mathcal{Z}}_{\mathcal{S}}$ of $\mathcal{S}$ is the L^2-completion of $\mathcal{Z}_{\mathcal{S}}$; in other words

$$\overline{\mathcal{Z}}_{\mathcal{S}} = \left\{ D + \sum_{p \in \mathcal{B}(\mathcal{S})} a_p e_p \middle| D \in \mathrm{N}^1(\mathcal{S}), a_p \in \mathbb{R}, \sum a_p^2 < +\infty \right\}.$$

The intersection form extends as an intersection form with signature $(1,\infty)$ on $\overline{\mathcal{Z}}_{\mathcal{S}}$. Let

$$\overline{\mathcal{Z}}_{\mathcal{S}}^+ = \{ d \in \overline{\mathcal{Z}}_{\mathcal{S}} | d \cdot c \geq 0 \ \forall c \in \overline{\mathcal{Z}}_{\mathcal{S}} \}$$

be the nef cone of $\overline{\mathcal{Z}}_{\mathcal{S}}$ and

$$\mathcal{L}\overline{\mathcal{Z}}_{\mathcal{S}} = \{ d \in \overline{\mathcal{Z}}_{\mathcal{S}} | d \cdot d = 0 \}$$

be the light cone of $\overline{\mathcal{Z}}_{\mathcal{S}}$.

The *hyperbolic space* $\mathbb{H}_{\mathcal{S}}$ of $\mathcal{S}$ is then defined by

$$\mathbb{H}_{\mathcal{S}} = \left\{ d \in \overline{\mathcal{Z}}_{\mathcal{S}}^+ | d \cdot d = 1 \right\}.$$

Let us remark that $\mathbb{H}_{\mathcal{S}}$ is an infinite dimensional analogue of the classical hyperbolic space $\mathbb{H}^n$. The *distance* on $\mathbb{H}_{\mathcal{S}}$ is defined by

$$\cosh\big(\mathrm{dist}\big(d, d'\big)\big) = d \cdot d' \quad \forall d, d' \in \mathbb{H}_{\mathcal{S}}.$$

The *geodesics* are intersections of $\mathbb{H}_{\mathcal{S}}$ with planes. The projection of $\mathbb{H}_{\mathcal{S}}$ onto $\mathbb{P}(\overline{\mathcal{Z}}_{\mathcal{S}})$ is one-to-one, and the boundary of its image is the projection of the cone of isotropic vectors of $\overline{\mathcal{Z}}_{\mathcal{S}}$. Hence

$$\partial\mathbb{H}_{\mathcal{S}} = \left\{ \mathbb{R}_+ d | d \in \overline{\mathcal{Z}}_{\mathcal{S}}^+, d \cdot d = 0 \right\}.$$

Isometries of $\mathbb{H}_{\mathcal{S}}$**.** If $\pi \colon \mathcal{S}' \to \mathcal{S}$ is a birational morphism, we get a canonical isometry π^* (and not only an embedding) between $\mathbb{H}_{\mathcal{S}}$ and $\mathbb{H}_{\mathcal{S}'}$. This allows to define an action of $\mathrm{Bir}(\mathcal{S})$ on $\mathbb{H}_{\mathcal{S}}$. Consider a birational map ϕ on a complex projective surface $\mathcal{S}$. There exists a surface $\mathcal{S}'$, and $\pi_1 \colon \mathcal{S}' \to \mathcal{S}$, $\pi_2 \colon \mathcal{S}' \to \mathcal{S}$ two morphisms such that $\phi = \pi_2 \pi_1^{-1}$. One can define $\phi_\bullet$ by

$$\phi_\bullet = \big(\pi_2^*\big)^{-1} \pi_1^*;$$

in fact, one gets a faithful representation of $\mathrm{Bir}(\mathcal{S})$ into the group of isometries of $\mathbb{H}_{\mathcal{S}}$ (see [6]).

The isometries of $\mathbb{H}_S$ are classified in three types ([4], [19]). The *translation length* of an isometry $\phi_\bullet$ of $\mathbb{H}_S$ is defined by

$$L(\phi_\bullet) = \inf\left\{ \mathrm{dist}\big(p, \phi_\bullet(p)\big) | p \in \mathbb{H}_S \right\}.$$

If the infimum is a minimum, then

— either it is equal to 0 and $\phi_\bullet$ has a fixed point in $\mathbb{H}_S$, $\phi_\bullet$ is thus *elliptic*,

— or it is positive and $\phi_\bullet$ is *hyperbolic*. Hence, the set of points $p\in\mathbb{H}_S$ such that $\mathrm{dist}(p,\phi_\bullet(p))$ is equal to $L(\phi_\bullet)$ is a geodesic line $\mathrm{Ax}(\phi_\bullet)\subset\mathbb{H}_S$. Its boundary points are represented by isotropic vectors $\omega(\phi_\bullet)$ and $\alpha(\phi_\bullet)$ in $\overline{\mathcal{Z}}_S$ such that

$$\phi_\bullet\big(\omega(\phi_\bullet)\big)=\lambda(\phi)\omega(\phi_\bullet),\qquad \phi_\bullet\big(\alpha(\phi_\bullet)\big)=\frac{1}{\lambda(\phi)}\alpha(\phi_\bullet).$$

The axis $\mathrm{Ax}(\phi_\bullet)$ of $\phi_\bullet$ is the intersection of $\mathbb{H}_S$ with the plane containing $\omega(\phi_\bullet)$ and $\alpha(\phi_\bullet)$; furthermore, $\phi_\bullet$ acts as a translation of length $L(\phi_\bullet)=\log\lambda(\phi)$ along $\mathrm{Ax}(\phi_\bullet)$ (see [8, Remark 4.5]). For all p in $\mathbb{H}_S$ one has

$$\lim_{k\to+\infty}\frac{\phi_\bullet^{-k}(p)}{\lambda(\phi)}=\alpha(\phi_\bullet),\qquad \lim_{k\to+\infty}\frac{\phi_\bullet^{k}(p)}{\lambda(\phi)}=\omega(\phi_\bullet).$$

When the infimum is not realized, $L(\phi_\bullet)=0$ and $\phi_\bullet$ is *parabolic*: $\phi_\bullet$ fixes a unique line in $\mathcal{L}\overline{\mathcal{Z}}_S$; this line is fixed pointwise, and all orbits $\phi_\bullet^n(p)$ in $\mathbb{H}_S$ accumulate to the corresponding boundary point when n goes to $\pm\infty$.

There is a strong relationship between this classification and the classification of birational maps of the complex projective plane ([6, Theorem 3.6]): if ϕ is an element of $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$, then

- $\phi_\bullet$ is an elliptic isometry if and only if ϕ is an elliptic map;
- $\phi_\bullet$ is a parabolic isometry if and only if ϕ is a twist;
- $\phi_\bullet$ is a hyperbolic isometry if and only if ϕ is a hyperbolic map.

Tits alternative. Cantat proved the Tits alternative for the Cremona group ([6, Theorem C]): let G be a finitely generated subgroup of $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$, then

- either G contains a free non-Abelian subgroup,
- or G contains a subgroup of finite index that is solvable.

As a consequence, he studied finitely generated and solvable subgroups of $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$ without torsion ([6, Theorem 7.3]): let G be such a group, there exists a subgroup G_0 of G of finite index such that

- either G_0 is Abelian,
- or G_0 preserves a foliation.

3. Proof of Theorem A

3.1. Solvable groups of birational maps containing a hyperbolic map. Let us recall the following criterion (for its proof see, for example, [11]) used on many occasions by Klein, and also by Tits ([26]) known as Ping-Pong Lemma: *let* H *be a group acting on a set* X, *let* Γ_1, Γ_2 *be two subgroups of* H, *and let* Γ *be the subgroup generated by* Γ_1 *and* Γ_2. *Assume that* Γ_1 *contains at least three elements, and* Γ_2 *at least two elements. Suppose that there exist two non-empty subsets* X_1, X_2 *of* X *such that*

$$X_2\not\subset X_1,\qquad \gamma(X_2)\subset X_1\quad\forall\gamma\in\Gamma_1\setminus\{\mathrm{id}\},\qquad \gamma'(X_1)\subset X_2\quad\forall\gamma'\in\Gamma_2\setminus\{\mathrm{id}\}.$$

Then Γ *is isomorphic to the free product* $\Gamma_1 * \Gamma_2$. The Ping-Pong argument allows us to prove the following.

LEMMA 3.1. *A solvable, non-Abelian subgroup of* $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ *cannot contain two hyperbolic maps* ϕ *and* ψ *such that* $\{\omega(\phi_\bullet), \alpha(\phi_\bullet)\} \neq \{\omega(\psi_\bullet), \alpha(\psi_\bullet)\}$.

Proof. Assume by contradiction that $\{\omega(\phi_\bullet), \alpha(\phi_\bullet)\} \neq \{\omega(\psi_\bullet), \alpha(\psi_\bullet)\}$. Then the Ping-Pong argument implies that there exist two integers n and m such that ψ^n and ϕ^m generate a subgroup of G isomorphic to the free group F_2 (see [6]). But $\langle \phi, \psi \rangle$ is a solvable group: contradiction. □

Let G be an infinite solvable, non-virtually Abelian, subgroup of $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$. Assume that G contains a hyperbolic map ϕ. Let $\alpha(\phi_\bullet)$ and $\omega(\phi_\bullet)$ be the two fixed points of $\phi_\bullet$ on $\partial\mathbb{H}_{\mathbb{P}^2_{\mathbb{C}}}$, and $\mathrm{Ax}(\phi_\bullet)$ be the geodesic passing through these two points. As G is solvable there exists a subgroup of G of index ≤ 2 that preserves $\alpha(\phi_\bullet)$, $\omega(\phi_\bullet)$, and $\mathrm{Ax}(\phi_\bullet)$ (see [6, Theorem 6.4]); let us still denote by G this subgroup. Note that there is no twist in G since a parabolic isometry has a unique fixed point on $\partial\mathbb{H}_{\mathbb{P}^2_{\mathbb{C}}}$. One has a morphism $\kappa\colon \mathrm{G} \to \mathbb{R}_{>0}$ such that

$$\psi_\bullet(\ell) = \kappa(\psi)\ell$$

for any ℓ in $\overline{\mathcal{Z}}_{\mathbb{P}^2_{\mathbb{C}}}$ lying on $\mathrm{Ax}(\phi_\bullet)$.

The kernel of κ is an infinite subgroup that contains only elliptic maps. Indeed the set of elliptic elements of G coincides with $\ker\kappa$; and $[\mathrm{G},\mathrm{G}] \subset \ker\kappa$ so if $\ker\kappa$ is finite, G is Abelian up to finite index which is by assumption impossible.

Gap property. If ψ is an hyperbolic birational map of G, then $\kappa(\psi) = L(\psi_\bullet) = \log\lambda(\psi)$. Recall that $\lambda(\phi)$ is an algebraic integer with all Galois conjugates in the unit disk, that is a Salem number, or a Pisot number. The smallest known number is the Lehmer number $\lambda_L \simeq 1.176$ which is a root of $X^{10}+X^9-X^7-X^6-X^5-X^4-X^3+X+1$. Blanc and Cantat prove in [2, Corollary 2.7] that there is a gap in the dynamical spectrum $\Lambda = \{\lambda(\phi) \,|\, \phi \in \mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})\}$: there is no dynamical degree in $]1, \lambda_L[$.

The gap property implies that in fact κ is a morphism from G to a subgroup of $\mathbb{R}_{>0}$ isomorphic to $\mathbb{Z}$.

Elliptic subgroups of the Cremona group with a large normalizer. Consider in $\mathbb{P}^2_{\mathbb{C}}$ the complement of the union of the three lines $\{x=0\}$, $\{y=0\}$ and $\{z=0\}$. Denote by $\mathcal{U}$ this open set isomorphic to $\mathbb{C}^*\times\mathbb{C}^*$. One has an action of $\mathbb{C}^*\times\mathbb{C}^*$ on $\mathcal{U}$ by translation. Furthermore $\mathrm{GL}(2,\mathbb{Z})$ acts on $\mathcal{U}$ by monomial maps

$$\begin{bmatrix} p & q \\ r & s \end{bmatrix} \mapsto \big((x,y) \mapsto \big(x^py^q, x^ry^s\big)\big).$$

One thus has an injective morphism from $(\mathbb{C}^*\times\mathbb{C}^*) \rtimes \mathrm{GL}(2,\mathbb{Z})$ into $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$. Let $\mathrm{G}_{\mathrm{toric}}$ be its image.

One can now apply [12, Theorem 4] that says that if there exists a short exact sequence

$$1 \longrightarrow \mathrm{A} \longrightarrow \mathrm{N} \longrightarrow \mathrm{B} \longrightarrow 1,$$

where $\mathrm{N}\subset \mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$ contains at least one hyperbolic element, and $\mathrm{A}\subset\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$ is an infinite and elliptic[1] group, then N is up to conjugacy a subgroup of $\mathrm{G}_{\mathrm{toric}}$. Hence, up to birational conjugacy $\mathrm{G}\subset\mathrm{G}_{\mathrm{toric}}$. Recall now that if ψ is a hyperbolic map of the form (x^ay^b,x^cy^d), then to preserve $\alpha(\psi_\bullet)$ and $\omega(\psi_\bullet)$ is equivalent to preserve the eigenvectors of the matrix $\left[\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right]$. We can now thus state:

Proposition 3.2. *Let* G *be an infinite solvable, non-virtually Abelian, subgroup of* $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$. *If* G *contains a hyperbolic birational map, then* G *is, up to conjugacy and finite index, a subgroup of the group generated by*

$$\big\{\big(x^py^q,x^ry^s\big),(\alpha x,\beta y)|\alpha,\beta\in\mathbb{C}^*\big\},$$

where $\left[\begin{smallmatrix} p & q\\ r & s\end{smallmatrix}\right]$ *denotes an element of* $\mathrm{GL}(2,\mathbb{Z})$ *with spectral radius* >1.

3.2. Solvable groups with a twist. Consider a solvable, non-Abelian subgroup G of $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$. Let us assume that G contains a twist ϕ; the map ϕ preserves a unique fibration $\mathcal{F}$ that is rational or elliptic. Let us prove that any element of G preserves $\mathcal{F}$. Denote by $\alpha(\phi_\bullet)\in\partial\mathbb{H}_{\mathbb{P}^2_\mathbb{C}}$ the fixed point of $\phi_\bullet$. Take one element in $\mathcal{L}\overline{\mathcal{Z}}_{\mathbb{P}^2_\mathbb{C}}$ still denoted $\alpha(\phi_\bullet)$ that represents $\alpha(\phi_\bullet)$. Take $\varphi\in\mathrm{G}$ such that $\varphi(\alpha(\phi_\bullet))\neq\alpha(\phi_\bullet)$. Then $\psi=\varphi\phi\varphi^{-1}$ is parabolic and fixes the unique element $\alpha(\psi_\bullet)$ of $\mathcal{L}\overline{\mathcal{Z}}_{\mathbb{P}^2_\mathbb{C}}$ proportional to $\varphi(\alpha(\phi_\bullet))$. Take $\varepsilon>0$ such that $\mathcal{U}(\alpha(\phi_\bullet),\varepsilon)\cap\mathcal{U}(\alpha(\psi_\bullet),\varepsilon)=\emptyset$ where

$$\mathcal{U}(\alpha,\varepsilon)=\{\ell\in\mathcal{L}\overline{\mathcal{Z}}_{\mathbb{P}^2_\mathbb{C}}|\alpha\cdot\ell<\varepsilon\}.$$

Since $\psi_\bullet$ is parabolic, then for n large enough $\psi_\bullet^n(\mathcal{U}(\alpha(\phi_\bullet),\varepsilon))$ is included in a $\mathcal{U}(\alpha(\psi_\bullet),\varepsilon)$. For m sufficiently large $\phi_\bullet^m\psi_\bullet^n(\mathcal{U}(\alpha(\phi_\bullet),\varepsilon))\subset(\mathcal{U}(\alpha(\phi_\bullet),\varepsilon/2))\subsetneq(\mathcal{U}(\alpha(\phi_\bullet),\varepsilon))$; hence $\phi_\bullet^m\psi_\bullet^n$ is hyperbolic. You can by this way build two hyperbolic maps whose sets of fixed points are distinct: this gives a contradiction with Lemma 3.1. So for any $\varphi\in\mathrm{G}$ one has: $\alpha(\phi_\bullet)=\alpha(\varphi_\bullet)$; one can thus state the following result.

Proposition 3.3. *Let* G *be a solvable, non-Abelian subgroup of* $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$ *that contains a twist* ϕ. *Then*

- *if* ϕ *is a Jonquières twist, then* G *preserves a rational fibration, that is up to birational conjugacy* G *is a subgroup of* $\mathrm{PGL}(2,\mathbb{C}(y))\rtimes\mathrm{PGL}(2,\mathbb{C})$,
- *if* ϕ *is a Halphen twist, then* G *preserves an elliptic fibration.*

Remark 3.4. Both cases are mutually exclusive.

[1] A subgroup of $\mathrm{Bir}(\mathbb{P}^2_\mathbb{C})$ is elliptic if it fixes a point in $\mathbb{H}_{\mathbb{P}^2_\mathbb{C}}$.

Note that if G is a subgroup of $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ that preserves an elliptic fibration, then G is countable ([5]). Let us explain briefly why. A smooth rational projective surface $\mathcal{S}$ is a *Halphen surface* if there exists an integer $m>0$ such that the linear system $|-mK_{\mathcal{S}}|$ is of dimension 1, has no fixed component, and has no base point. The smallest positive integer for which $\mathcal{S}$ satisfies such a property is the *index* of $\mathcal{S}$. If $\mathcal{S}$ is a Halphen surface of index m, then $K^2_{\mathcal{S}}=0$ and, by the genus formula, the linear system $|-mK_{\mathcal{S}}|$ defines a genus 1 fibration $\mathcal{S}\to\mathbb{P}^1_{\mathbb{C}}$. This fibration is *relatively minimal* in the sense that there is no (-1)-curve contained in a fiber. The following properties are equivalent:

- $\mathcal{S}$ is a Halphen surface of index m,
- there exists an irreducible pencil of curves of degree $3m$ with 9 base points of multiplicity m in $\mathbb{P}^2_{\mathbb{C}}$ such that $\mathcal{S}$ is the blow-up of the 9 base points and $|-mK_{\mathcal{S}}|$ is the proper transform of this pencil (the base points set may contain infinitely near points).

As a corollary of the classification of relatively minimal elliptic surfaces the relative minimal model of a rational elliptic surface is a Halphen surface of index m ([22, Chapter 2, Section 10]). Up to conjugacy G is a subgroup of $\mathrm{Aut}(\mathcal{S})$ where $\mathcal{S}$ denotes a Halphen surface of index m. The action of G on $\mathrm{NS}(\mathcal{S})$ is almost faithful, and G is a discrete (it preserves the integral structure of $\mathrm{NS}(\mathcal{S})$) and virtually Abelian (it preserves the intersection form and the class of the elliptic fibration) subgroup of $\mathrm{Aut}(\mathcal{S})$. So one has the following.

Corollary 3.5. *If* G *is an uncountable, solvable, non-Abelian subgroup of* $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$, *then* G *doesn't contain a Halphen twist.*

Example 3.6. Let us come back to the example given in Section 1. If $\phi\in\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ preserves a unique fibration that is rational then one can assume that up to birational conjugacy this fibration is given, in the affine chart $z=1$, by $y=\mathrm{cst}$. If ϕ preserves $y=\mathrm{cst}$ fiberwise, then

- ϕ is contained in a maximal Abelian subgroup denoted $\mathrm{Ab}(\phi)$ that preserves $y=\mathrm{cst}$ fiberwise ([13]),
- the centralizer of ϕ is a finite extension of $\mathrm{Ab}(\phi)$ (see [9, Theorem B]).

This allows us to establish that if ϕ preserves a fibration not fiberwise, then the centralizer of ϕ is virtually solvable. For instance, if $\phi=(x+a(y),y+1)$ (resp. $(b(y)x,\beta y)$ or $(x+a(y),\beta y)$ with $\beta\in\mathbb{C}^*$ of infinite order) preserves a unique fibration, then the centralizer of ϕ is solvable and metabelian ([9, Propositions 5.1 and 5.2]).

3.3. Solvable groups with no hyperbolic map, and no twist. Let M be a smooth, irreducible, complex, projective variety of dimension n. Fix a Kähler form κ on M. If ℓ is a positive integer, denote by $\pi_i\colon M^\ell\to M$ the projection onto the ith factor. The manifold M^ℓ is then endowed with the Kähler form $\sum_{i=1}^{\ell}\pi_i^*\kappa$ which induces a Kähler metric. To any $\phi\in\mathrm{Bir}(M)$

one can associate its graph $\Gamma_\phi \subset M \times M$ defined as the Zariski closure of

$$\big\{\big(z,\phi(z)\big) \in M \times M | z \in M \setminus \operatorname{Ind}\phi\big\}.$$

By construction Γ_ϕ is an irreducible subvariety of $M \times M$ of dimension n. Both projections $\pi_1, \pi_2 \colon M \times M \to M$ restrict to a birational morphism $\pi_1, \pi_2 \colon \Gamma_\phi \to M$.

The *total degree* $\operatorname{tdeg}\phi$ of $\phi \in \operatorname{Bir}(M)$ is defined as the volume of Γ_ϕ with respect to the fixed metric on $M \times M$:

$$\operatorname{tdeg}\phi = \int_{\Gamma_\phi} \big(\pi_1^*\kappa + \pi_2^*\kappa\big)^n = \int_{M\setminus\operatorname{Ind}\phi} \big(\kappa + \phi^*\kappa\big)^n.$$

Let $d \geq 1$ be a natural integer, and set

$$\operatorname{Bir}_d(M) = \big\{\phi \in \operatorname{Bir}(M) | \operatorname{tdeg}\phi \leq d\big\}.$$

A subgroup G of $\operatorname{Bir}(M)$ has *bounded degree* if it is contained in $\operatorname{Bir}_d(M)$ for some $d \in \mathbb{N}^*$.

Any subgroup G of $\operatorname{Bir}(M)$ that has bounded degree can be regularized, that is up to birational conjugacy all indeterminacy points of all elements of G disappear simultaneously.

THEOREM 3.7 ([27]). *Let M be a complex projective variety, and let G be a subgroup of $\operatorname{Bir}(M)$. If G has bounded degree, there exists a smooth, complex, projective variety M', and a birational map $\psi \colon M' \dashrightarrow M$ such that $\psi^{-1}\mathrm{G}\psi$ is a subgroup of $\operatorname{Aut}(M')$.*

The proof of this result can be found in [21], [29]; an heuristic idea appears in [7].

4. Applications

4.1. Nilpotent subgroups of $\operatorname{Bir}(\mathbb{P}^2_\mathbb{C})$. Let us recall that if G is a group, the *descending central series* of G is defined by

$$C^0\mathrm{G} = \mathrm{G}, \qquad C^{n+1}\mathrm{G} = \big[\mathrm{G}, C^n\mathrm{G}\big] \quad \forall n \geq 0.$$

We say that G is *nilpotent* if there exists $j \geq 0$ such that $C^j\mathrm{G} = \{\mathrm{id}\}$. If j is the minimum non-negative number with such a property, we say that G is of *nilpotent class j*. Nilpotent subgroups of the Cremona group have been described:

THEOREM 4.1 ([14]). *Let G be a nilpotent subgroup of $\operatorname{Bir}(\mathbb{P}^2_\mathbb{C})$. Then*

- *either G is up to finite index metabelian,*
- *or G is a torsion group.*

We find an alternative proof of [14, Lemma 4.2] for G infinite:

LEMMA 4.2. *Let G be an infinite, nilpotent, non-virtually Abelian subgroup of $\operatorname{Bir}(\mathbb{P}^2_\mathbb{C})$. Then G does not contain a hyperbolic map.*

Proof. The group G is also solvable. Assume by contradiction that G contains a hyperbolic map; then according to Theorem A up to birational conjugacy and finite index there exists $\Upsilon \subset \mathbb{C}^* \times \mathbb{C}^*$ infinite such that G is generated by $\phi = (x^p y^q, x^r y^s)$ and

$$\{(\alpha x, \beta y) | (\alpha, \beta) \in \Upsilon\}.$$

The group C^1G contains

$$\{[\phi, (\alpha x, \beta y)] | (\alpha, \beta) \in \Upsilon\} = \{(\alpha^{p-1}\beta^q x, \alpha^r \beta^{s-1} y) | (\alpha, \beta) \in \Upsilon\}$$

that is infinite since Υ is infinite. Suppose that C^iG contains the infinite set

$$\{(\alpha^{\ell_i}\beta^{n_i} x, \alpha^{k_i}\beta^{m_i} y) | (\alpha, \beta) \in \Upsilon\}$$

(ℓ_i, n_i, k_i and m_i are some functions in p, q, r and s); then C^{i+1}G contains

$$\{(\alpha^{(p-1)\ell_i + qm_i}\beta^{(p-1)k_i + qn_i} x, \alpha^{r\ell_i + qm_i}\beta^{rk_i + (s-1)n_i} y) | (\alpha, \beta) \in \Upsilon\}$$

that is still infinite. □

So any nilpotent and infinite subgroup of $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ falls in case (1), (2), (3) of Theorem A. If it falls in case (2) or (3) then G is virtually metabelian ([14, Proof of Theorem 1.1]). Finally if G falls in case (1), we can prove as in [14] that either G is a torsion group, or G is virtually metabelian.

4.2. Soluble length of a nilpotent subgroup of $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$**.** Let us recall the following statement due to Epstein and Thurston ([17]): let M be a connected complex manifold. Let $\mathfrak{h}$ be a nilpotent Lie subalgebra of the complex vector space of rational vector fields on M. Then $\mathfrak{h}^{(n)} = \{0\}$ if $n \geq \dim M$; hence, the solvable length of $\mathfrak{h}$ is bounded by the dimension of M. We have a similar statement in the context of birational maps; indeed a direct consequence of Theorem 4.1 is the following property: let $\mathrm{G} \subset \mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ be a nilpotent subgroup of $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ that is not a torsion group, then the soluble length of G is bounded by the dimension of $\mathbb{P}^2_{\mathbb{C}}$.

4.3. Favre's question. In [18], Favre asked few questions; among them there is the following: does any solvable, finitely generated subgroup G of $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ contain a subgroup H of finite index such that $[\mathrm{H}, \mathrm{H}]$ is nilpotent? We will prove that the answer is no if $[\mathrm{G}, \mathrm{G}]$ is not a torsion group.

Take G a solvable and finitely generated subgroup of the Cremona group; besides suppose that $[\mathrm{G}, \mathrm{G}]$ is not a torsion group. Assume that the answer of Favre's question is yes. Up to finite index one can assume that $[\mathrm{G}, \mathrm{G}]$ is nilpotent. According to Theorem 4.1 the group $\mathrm{G}^{(1)} = [\mathrm{G}, \mathrm{G}]$ is up to finite index metabelian; in other words up to finite index $\mathrm{G}^{(2)} = [\mathrm{G}^{(1)}, \mathrm{G}^{(1)}]$ is Abelian and so $\mathrm{G}^{(3)} = [\mathrm{G}^{(2)}, \mathrm{G}^{(2)}] = \{\mathrm{id}\}$, *that is*, the soluble length of G is bounded by 3 up to finite index. Consider the subgroup

$$\left\langle (x + y^2, y), (x(1+y), y), \left(x, \frac{y}{1+y}\right), (x, 2y) \right\rangle$$

of $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$. It is solvable of length 4 (see [24]): contradiction.

4.4. Baumslag–Solitar groups. For any integers m, n such that $mn \neq 0$, the Baumslag–Solitar group $\mathrm{BS}(m;n)$ is defined by the following presentation

$$\mathrm{BS}(m;n) = \langle r, s | rs^m r^{-1} = s^n \rangle.$$

In [3], we prove that there is no embedding of $\mathrm{BS}(m;n)$ into $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ as soon as $|n|$, $|m|$, and 1 are distinct; it corresponds exactly to the case $\mathrm{BS}(m;n)$ is not solvable. Indeed $\mathrm{BS}(m;n)$ is solvable if and only if $|m| = 1$ or $|n| = 1$ (see [25, Proposition A.6]).

Proposition 4.3. *Let ρ be an embedding of* $\mathrm{BS}(1;n) = \langle r, s | rsr^{-1} = s^n \rangle$, *with $n \neq 1$, into the Cremona group. Then*

- *the image of ρ doesn't contain a hyperbolic map,*
- *and*
$$\rho(s) = (x, y+1), \qquad \rho(r) = \big(\nu(x), n\big(y + a(x)\big)\big)$$
with $\nu \in \mathrm{PGL}(2,\mathbb{C})$ and $a \in \mathbb{C}(x)$.

Proof. According to [3, Proposition 6.2, Lemma 6.3] one gets that $\rho(s) = (x, y+1)$ and $\rho(r) = (\nu(x), n(y + a(x)))$ for some $\nu \in \mathrm{PGL}(2,\mathbb{C})$ and $a \in \mathbb{C}(x)$.

Furthermore, $\rho(s)$ can neither be conjugate to an automorphism of the form $(\alpha x, \beta y)$ (see [1]), nor to a hyperbolic birational map of the form $(\gamma x^p y^q, \delta x^r y^s)$ with $\left[\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right] \in \mathrm{GL}(2,\mathbb{Z})$ of spectral radius > 1. As a consequence, Proposition 3.2 implies that $\rho(\mathrm{BS}(1;n))$ does not contain a hyperbolic birational map. □

Acknowledgments. I would like to thank D. Cerveau for always having time to listen to me, for his precious advices, and for pointing out [24]. Thanks to S. Cantat for helpful remarks and comments, and also to J. Ribon for the example of Section 4.3. I would like also thank P. Sad for his permanent invitation at IMPA where I found the quietness to put my ideas in order. Thanks to the referee whose comments and suggestions help me to improve the presentation.

References

[1] J. Blanc, *Conjugacy classes of affine automorphisms of $\mathbb{K}^n$ and linear automorphisms of $\mathbb{P}^n$ in the Cremona groups*, Manuscripta Math. **119** (2006), no. 2, 225–241. MR 2215969

[2] J. Blanc and S. Cantat, *Dynamical degrees of birational transformations of projective surfaces*, J. Amer. Math. Soc. **29** (2016), no. 2, 415–471. MR 3454379

[3] J. Blanc and J. Déserti, *Degree growth of birational maps of the plane*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **14** (2015), no. 2, 507–533. MR 3410471

[4] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften, vol. 319, Springer-Verlag, Berlin, 1999. MR 1744486

[5] S. Cantat, *Dynamique des automorphismes des surfaces complexes compactes*, Ph.D. thesis, École Normale Supérieure de Lyon, 1999.

[6] S. Cantat, *Sur les groupes de transformations birationnelles des surfaces*, Ann. of Math. (2) **174** (2011), no. 1, 299–340. MR 2811600

[7] S. Cantat, *Morphisms between Cremona groups, and characterization of rational varieties*, Compos. Math. **150** (2014), no. 7, 1107–1124. MR 3230847

[8] S. Cantat and S. Lamy, *Normal subgroups in the Cremona group*, Acta Math. **210** (2013), no. 1, 31–94. With an appendix by Yves de Cornulier. MR 3037611

[9] D. Cerveau and J. Déserti, *Centralisateurs dans le groupe de Jonquières*, Michigan Math. J. **61** (2012), no. 4, 763–783. MR 3049289

[10] Y. Cornulier, *Nonlinearity of some subgroups of the planar Cremona group*, unpublished manuscript, 2013, `http://www.normalesup.org/~cornulier/crelin.pdf`.

[11] P. de la Harpe, *Topics in geometric group theory*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 2000. MR 1786869

[12] T. Delzant and P. Py, *Kähler groups, real hyperbolic spaces and the Cremona group*, Compos. Math. **148** (2012), no. 1, 153–184. With an appendix by S. Cantat. MR 2881312

[13] J. Déserti, *Sur les automorphismes du groupe de Cremona*, Compos. Math. **142** (2006), no. 6, 1459–1478. MR 2278755

[14] J. Déserti, *Sur les sous-groupes nilpotents du groupe de Cremona*, Bull. Braz. Math. Soc. (N.S.) **38** (2007), no. 3, 377–388. MR 2344204

[15] J. Diller and C. Favre, *Dynamics of bimeromorphic maps of surfaces*, Amer. J. Math. **123** (2001), no. 6, 1135–1169. MR 1867314

[16] I. V. Dolgachev and V. A. Iskovskikh, *Finite subgroups of the plane Cremona group*, Algebra, arithmetic, and geometry: In honor of Yu. I. Manin. Vol. I, Progr. Math., vol. 269, Birkhäuser Boston, Inc., Boston, MA, 2009, pp. 443–548. MR 2641179

[17] D. B. A. Epstein and W. P. Thurston, *Transformation groups and natural bundles*, Proc. Lond. Math. Soc. (3) **38** (1979), no. 2, 219–236. MR 0531161

[18] C. Favre, *Le groupe de Cremona et ses sous-groupes de type fini*, Astérisque **332** (2010), Exp. no. 998, vii, 11–43. Séminaire Bourbaki. Volume 2008/2009. Exposés 997–1011. MR 2648673

[19] É. Ghys and P. de la Harpe, eds., *Sur les groupes hyperboliques d'après Mikhael Gromov*, Progress in Mathematics, vol. 83, Birkhäuser Boston, Inc., Boston, MA, 1990. Papers from the Swiss Seminar on Hyperbolic Groups held in Bern, 1988. MR 1086648

[20] M. H. Gizatullin, *Rational G-surfaces*, Izv. Akad. Nauk SSSR Ser. Mat. **44** (1980), no. 1, 110–144, 239. MR 0563788

[21] A. Huckleberry and D. Zaitsev, *Actions of groups of birationally extendible automorphisms*, Geometric complex analysis (Hayama, 1995), World Sci. Publ., River Edge, NJ, 1996, pp. 261–285. MR 1453608

[22] V. A. Iskovskikh and I. R. Shafarevich, *Algebraic surfaces*, Algebraic geometry, II, Encyclopaedia Math. Sci., vol. 35, Springer, Berlin, 1996, pp. 127–262. MR 1392959

[23] M. I. Kargapolov and J. I. Merzljakov, *Fundamentals of the theory of groups*, Graduate Texts in Mathematics, vol. 62, Springer-Verlag, New York-Berlin, 1979. Translated from the second Russian edition by Robert G. Burns. MR 0551207

[24] M. Martelo and J. Ribón, *Derived length of solvable groups of local diffeomorphisms*, Math. Ann. **358** (2014), no. 3–4, 701–728. MR 3175138

[25] E. Souche, *Quasi-isométrie et quasi-plans dans l'étude des groupes discrets*, Ph.D. thesis, Université de Provence, 2001.

[26] J. Tits, *Free subgroups in linear groups*, J. Algebra **20** (1972), 250–270. MR 0286898

[27] A. Weil, *On algebraic groups of transformations*, Amer. J. Math. **77** (1955), 355–391. MR 0074083

[28] D. Wright, *Abelian subgroups of* $\mathrm{Aut}_k(k[X,Y])$ *and applications to actions on the affine plane*, Illinois J. Math. **23** (1979), no. 4, 579–634. MR 0540400

[29] D. Zaitsev, *Regularization of birational group operations in the sense of Weil*, J. Lie Theory **5** (1995), no. 2, 207–224. MR 1389430

Julie Déserti, Institut de Mathématiques de Jussieu-Paris Rive gauche, UMR 7586, Université Paris 7, Bâtiment Sophie Germain, Case 7012, 75205 Paris Cedex 13, France

E-mail address: deserti@math.univ-paris-diderot.fr