THE ACTION OF THE CREMONA GROUP ON RATIONAL CURVES OF $\mathbb{P}^3$

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Abstract. A Cremona transformation is a birational self–map of the projective space $\mathbb{P}^n$. Cremona transformations of $\mathbb{P}^n$ form a group and this group has a rational action on subvarieties of $\mathbb{P}^n$ and hence on its Hilbert scheme. We study this action on the family of rational curves of $\mathbb{P}^3$ and we prove the rectifiability of any one dimensional family. This shows that any uniruled surface is Cremona equivalent to a scroll and it answers a question of Bogomolov–Böhning related to the study of uniformly rational varieties. We provide examples of infinitely many scrolls in the same Cremona orbit and we show that a "general" scroll is not in the Cremona orbit of a "general" rational surface.

1. Introduction

A Cremona transformation is a birational self–map of the projective space $\mathbb{P}^n$. Cremona transformations of $\mathbb{P}^n$ form a group and this group has a rational action on subvarieties of $\mathbb{P}^n$ and hence on its Hilbert scheme. The problem of studying the orbit of a given subvariety under this action is classical, an introduction to classical results may be found in [Coo] and [Con]. Two subvarieties are called Cremona equivalent, see Definition 2.1, if they are in the same orbit. Quite surprisingly for an irreducible subvariety $V \subset \mathbb{P}^n$ of codimension at least 2 the Cremona equivalence is birational equivalence, [MP1]. That is any subvariety birational to V is in the orbit of V. The case of divisors is completely different and it is well known that, in any dimension, there are birational divisors that are not Cremona equivalent, [AM],[Je], [MP1]. As a matter of facts the detailed classification of orbits of the Cremona action for divisors is really intricate and only few cases are well understood. In modern times, the Cremona equivalence of irreducible plane curves has been considered again starting from the 1960s by Nagata, Kumar–Murthy, Dicks, Reid, Iitaka, Matsuda, Mella–Polastri, and Calabri–Ciliberto, see [CC1] for a full bibliography. Away from irreducible plane

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curves very few is known: rational surfaces Cremona equivalent to a plane, [MP2], cones, [Me], and arrangement of lines, [CC2].

When considering the action on the Hilbert scheme it is natural to ask about families of irreducible subvarieties. By the mentioned result as soon as the locus of these families in $\mathbb{P}^n$ is of codimension at least two then Cremona action is equivalent to birational equivalence. In this paper we concentrate on the next case: divisorial families of subvarieties. In particular we start with the family of rational curves of $\mathbb{P}^3$.

Theorem. *Let $cr_3 : Cr_3 \times \mathrm{Hilb}(\mathbb{P}^3) \dashrightarrow \mathrm{Hilb}(\mathbb{P}^3)$ be the natural action induced by the group of Cremona transformations. Then any one dimensional family of rational curves is rectifiable by cr_3, that is it can be mapped into the Klein quadric (the irreducible variety representing lines of $\mathbb{P}^3$) by cr_3.*

The Cremona action on rational curves of $\mathbb{P}^3$ is related to the rationality of conic bundles via Cantor Conjecture, [Ka]. The latter predicts that any congruence of rational curves, that is an irreducible surface in the Hilbert scheme of rational curves, with a unique element trough the general point of $\mathbb{P}^3$ can always be transformed into a congruence of conics (or lines through a fixed point) by a Cremona transformation. Iskovskikh proved, [Isk87], that this conjecture is equivalent to the rationality criterion of 3-fold conic bundles.

From a slightly different point of view we may reformulate the above statement in terms of Cremona equivalence of rational surfaces.

Theorem 1.1. *Let $S \subset \mathbb{P}^3$ be a uniruled surface. Then S is Cremona equivalent to a scroll.*

In the above form the result seems to be of interest also in the study of uniformly rational varieties, [BB]. In particular it answers positively the question posed by the authors after [BB, Proposition 2.9], and it suggests a positive answer to the rectifiability of divisorial images of $\mathbb{P}^r \times Y$ studied in [BB].

The approach we adopt to prove our main theorem is different from the standard one in Cremona equivalences. In general one tries to study elements of minimal degree in the orbit. To do this one needs to: produce Cremona transformations that lower the degree of a fixed subvariety, and define a class of singularities for which it is impossible to lower the degree via Cremona modifications. This time we are in a different position. We are interested in rectifiability of a one dimensional family of rational curves. For this reason we discard the degree of the surface spanned by the family and produce Cremona modifications that are able to lower the degree of the family at the expense of both increasing the degree of the surface and introducing worse singularities. These Cremona modifications are produced using monoids, see Definition 2.4, that, in the spirit of [MP1] and [CCMRZ], in a recursive way lower the degree of the family. We complete the discussion providing examples of infinitely many scrolls in the same Cremona orbit and showing that a "general" scroll is not in the Cremona orbit of a "general" rational surface.

The paper is organized as follows. Section 1 is concerned with notation and numerical computations about monoids. In Section 2 the main result is proven and in Section 3 we collect examples and remarks.

2. Preliminary definitions and notations

We work over the complex field.

Definition 2.1. Let $S_1, S_2 \subset \mathbb{P}^3$ be irreducible surfaces. S_1 and S_2 are said to be *Cremona equivalent* if there exists a Cremona transformation $\omega : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ such that ω (resp. ω^{-1}) is defined at the general point of S_1 (resp. of S_2) and $\omega(S_1) = S_2$.

Let us fix some notation to work with uniruled surfaces.

Definition 2.2. Let C be a non singular curve of genus $g \geq 0$ and $X = \mathbb{P}^1 \times C = \mathbb{P}(\mathcal{O}_C \oplus \mathcal{O}_C)$. Let $\pi : X \to C$ be the ruled structure, C_0 and F a section and a fiber of π. We say that a uniruled surface $S \subset \mathbb{P}^3$ is realized by a linear system $\mathcal{L} := \{L_0, L_1, L_2, L_3\} \subset |aC_0 + \pi^* B|$, with $B \in \mathrm{Pic}(C)$, if $\mathcal{L}$ is without fixed components and $\varphi_{\mathcal{L}}(X) = S \subset \mathbb{P}^3$, where $\varphi_{\mathcal{L}}$ is the map associated to $\mathcal{L}$. In particular such an S is generically ruled by rational curves of degree a, the image of the fibers of π. If $a = 1$ then S is covered by lines and it is called a scroll.

Remark 2.3. *Let $X \subset \mathbb{P}^3$ be a uniruled surface of irregularity g. Then X is birational to $\mathbb{P}_C(\mathcal{E})$ for some vector bundle $\mathcal{E}$ of rank 2 over a genus g curve C. Hence by* [Ma] *via a chain of elementary transformations X is birational to $\mathbb{P}^1 \times C$. In particular any uniruled surface of $\mathbb{P}^3$ is realized by some linear system $\mathcal{L}_X$ on $\mathbb{P}^1 \times C$, for some C.*

Quadric surfaces and cubics with a double line are examples of scrolls. It is well known, see for instance [MP2]*, that any rational surface of degree at most 3 is Cremona equivalent to a plane. Therefore any rational surface of degree at most 3 is Cremona equivalent to a scroll and such a scroll is not unique. We will see, Example 4.1 , that this is the case of any uniruled surface. This behaviour makes very difficult to choose a representative scroll in a given orbit.*

To study the action of Cremona group it is useful to have an easy way to produce Cremona transformations of $\mathbb{P}^3$. For this purpose we introduce the following.

Definition 2.4. Let $\Xi \subset \mathbb{P}^4$ be a degree-d hypersurface, with $d > 1$. Ξ is said to be a *monoid* with *vertex* $p \in \mathbb{P}^4$ if Ξ is irreducible and $p \in \Xi$ is a point of multiplicity $d-1$. Let $M_d(p)$ be the linear system spanned by monoids with vertex p and $M_d(p,q) \subset M_d(p)$ the linear system spanned by monoids with vertexes in both p and q.

Remark 2.5. *Let $\pi : V \to \mathbb{P}^4$ be the blow up of p in $\mathbb{P}^4$ with exceptional divisor E and H the pullback of a hyperplane. Then we have*

$$M_d(p) = |dH - (d-1)E| = |(d-1)(H-E) + H|.$$

Remark 2.6. *The reason we are interested in monoids is this simple construction that allows to produce Cremona transformations of $\mathbb{P}^3$ from monoids in $M_d(q_1, q_2)$. Let $\Xi \in M_d(q_1, q_2)$ be a monoid. In particular Ξ is irreducible and $\operatorname{mult}_{q_1} \Xi = \operatorname{mult}_{q_2} \Xi = d-1$. Let $\varphi_i : \mathbb{P}^4 \dashrightarrow \mathbb{P}^3$ be the projection from the point p_i and $f_i := \varphi_{i|\Xi}$ its restriction to the monoid. Then the composition $\omega := f_2 \circ f_1^{-1}$ is a Cremona transformation of $\mathbb{P}^3$.*

Let $Z \subset \mathbb{P}^4$ be an irreducible surface and $q_1, q_2 \in \mathbb{P}^4$ two points, with projections $\varphi_i : \mathbb{P}^4 \dashrightarrow \mathbb{P}^3$. Assume that $\varphi_{i|Z}$ is birational, and there is a monoid $\Xi \in M_d(q_1, q_2)$ containing Z. Then the Cremona transformation ω_Ξ is a Cremona equivalence between $\varphi_1(Z)$ and $\varphi_2(Z)$ if and only if Ξ does not contain the cones over Z with vertex q_i.

To produce a Cremona equivalence via monoids we are therefore interested in studying monoids in $M_d(p_1, p_2)$ containing assigned surfaces but not their cones. The latter condition is not closed. Hence to produce the required monoid we cannot do a simple dimensional computation. To avoid this problem we introduce the following notation.

Definition 2.7. Let $W \subset \mathbb{P}^4$ be a subvariety. Then

$$M_d(p, W) \subset M_d(p)$$

is the sublinear system spanned by monoids containing W. Let Z be a surface and $C_p(Z)$ be the cone over Z with vertex p then

$$M_d(p, Z)^{\not{C}} \subset M_d(p, Z)$$

is a linear system disjoint from $M_d(p, C_p(Z))$.

Remark 2.8. *Despite the fact that $M_d(p, Z)^{\not{C}}$ is not a well defined linear system we prefer, to simplify notations, to introduce a unique name for all linear systems in $M_d(p, Z)$ that do not intersect $M_d(p, C_p(Z))$. We hope this won't confuse the reader.*

The following Lemma is the result we need to reduce the existence of a Cremona equivalence to a dimensional computation of linear systems.

Lemma 2.9. *Let $Z \subset \mathbb{P}^4$ be a non degenerate irreducible surface and $q_1, q_2 \in \mathbb{P}^4$ two points. Let $\varphi_i : \mathbb{P}^4 \dashrightarrow \mathbb{P}^3$ be the projection from the point q_i. Assume that $\varphi_{i|Z}$ is birational and, for some d,*

$$M_d(q_1, Z)^{\not{C}} \cap M_d(q_2, Z)^{\not{C}} \neq \emptyset.$$

Then the surfaces $\varphi_1(Z)$ and $\varphi_2(Z)$ are Cremona equivalent.

Proof. Let $\Xi \in M_d(q_1, Z)^{\not{C}} \cap M_d(q_2, Z)^{\not{C}}$ be an element. Ξ contains the surface Z and does not contain the cones. Thus $\operatorname{mult}_{q_i} \Xi = d-1$. Assume that $\Xi = A + B$ is reducible and $A \supset Z$ is an irreducible component. Let $\alpha = \deg A$, $\beta = \deg B$ be the degrees and $a_i = \mathrm{mult}_{q_i} A$, $b_i = \mathrm{mult}_{q_i} B$ the multiplicities, for $i = 1, 2$. Since Z is not degenerate then $\alpha > 1$. Moreover A does not contain the cones over Z therefore $a_i \leq \alpha - 1$. On the other hand we know that

$$a_i + b_i = d - 1 = \alpha + \beta - 1,$$

hence

$$\alpha - 1 \geq a_i = \alpha - 1 + (\beta - b_i) \geq \alpha - 1.$$

This proves that A is a monoid in $M_\alpha(q_1, q_2)$. Then, as observed in Remark 2.6, the map $\omega_A := \varphi_{2|A} \circ \varphi_{1|A}^{-1}$ is a Cremona transformation. Moreover by hypothesis

$$A \in M_\alpha(q_1, Z)^{\not C} \cap M_\alpha(q_2, Z)^{\not C},$$

then ω_A is well defined on the general point of $\varphi_1(Z)$ and ω_Ξ^{-1} is well defined on the general point of $\varphi_2(Z)$. Hence ω_A is a Cremona equivalence between $\varphi_1(Z)$ and $\varphi_2(Z)$. □

To apply Lemma 2.9 we have to efficiently bound the dimension of $M_d(p, Z)^{\not C}$. Note that by construction we may always choose $M_d(p, Z)^{\not C}$ in such a way that

$$\dim M_d(p, Z)^{\not C} = \dim M_d(p, Z) - \dim M_d(p, C_p(Z)) - 1. \tag{1}$$

Let us start computing the dimension of the linear spaces we are interested in. Without loss of generality we may assume that $p = [1, 0, 0, 0, 0]$ and so a general Ξ in $M_d(p)$ is defined by

$$F(x_0, x_1, x_2, x_3, x_4) = x_0 F_{d-1}(x_1, x_2, x_3, x_4) + F_d(x_1, x_2, x_3, x_4) = 0$$

where F_{d-1} and F_d are homogeneous polynomials, respectively, of degree $d - 1$ and d. Moreover, assuming that p is as above and $q = [0, 0, 0, 0, 1]$, a general $\Xi \in M_d(p, q)$ is given by

$$G(x_0, x_1, x_2, x_3, x_4) = x_0 x_4 G_{d-2}(x_1, x_2, x_3) + x_0 G'_{d-1}(x_1, x_2, x_3)$$
$$+ x_4 G''_{d-1}(x_1, x_2, x_3) + G_d(x_1, x_2, x_3) = 0$$

where $G_{d-2}, G'_{d-1}, G''_{d-1}, G_d$ are homogeneous polynomials, respectively, of degree $d - 2, d - 1, d - 1$ and d. These facts yield

$$\dim M_d(p) = \binom{3 + d - 1}{d - 1} + \binom{3 + d}{d} - 1 = \frac{1}{3}d^3 + \frac{3}{2}d^2 + \frac{13}{6}d, \tag{2}$$

$$\dim M_d(p, q) = \binom{d}{d - 2} + 2\binom{d + 1}{d - 1} + \binom{d + 2}{d} - 1 = 2d^2 + 2d. \tag{3}$$

Next we need to estimate $\dim M_d(p, Z)$. Accordingly to Remark 2.5 let $\pi_p : V \to \mathbb{P}^4$ be the blow up of p and Z_V the strict transform of Z. Then the structure sequence of $Z_V \subset V$ yields

$$\dim M_d(p, Z) \geq \dim M_d(p) - h^0(Z_V, \mathcal{O}((d - 1)(H - E) + \pi_p^* \mathcal{O}(1))). \tag{4}$$

Now, let δ be the degree of the projection of Z from p. Then $C_p(Z)$ is a divisor of degree δ, hence

$$M_d(p, C_p(Z)) \subseteq |(d-\delta-1)(H-E)+H|$$

and Equation (2) yields

$$\dim M_d(p, C_p(Z)) \leq \frac{1}{3}d^3 + \left(\frac{3}{2}-\delta\right) d^2 + \left(\delta^2 - 3\delta + \frac{13}{6}\right) d + \left(-\frac{\delta^3}{3} + \frac{3}{2}\delta^2 - \frac{13}{6}\delta\right). \tag{5}$$

Plugging these computations in Equation (1) we get the following Lemma.

Lemma 2.10.

$$\dim M_d(p,Z)^{\not{C}} \geq \delta d^2 - (\delta^2-3\delta)d - \left(-\frac{\delta^3}{3} + \frac{3}{2}\delta^2 - \frac{13}{6}\delta\right) + \\ -h^0(Z_V, \mathcal{O}((d-1)(H-E)+\pi_p^*\mathcal{O}(1)))$$

3. Cremona equivalence for uniruled surfaces

The aim of this section is to prove Theorem 1.1. Let $S \subset \mathbb{P}^3$ be a uniruled surface generically ruled by rational curves of degree a. If $a = 1$ there is nothing to prove. So we may assume that $a \geq 2$ and prove the statement by induction on a.

Set $X = \mathbb{P}^1 \times C$ and let $\pi : X \to C$ be the ruled structure. Then, for some divisor $B \in \operatorname{Pic}(C)$, with $\deg B = b \geq 1$, the surface S is realized by a linear system $\mathcal{L} := \{L_0, L_1, L_2, L_3\} \subset |aC_0 + \pi^*B|$, that is $\varphi_{\mathcal{L}}(X) = S \subset \mathbb{P}^3$.

Fix a general section $\Gamma \in |C_0 + \pi^*G|$, with $\deg G = 2g(C)+1$. Then for positive enough $B_1 \in \operatorname{Pic}(C)$, with $\deg B_1 = \beta$, we may consider $M \in |(a-1)C_0 + \pi^*(B+B_1-G)|$ and the linear system

$$\Lambda_M =: \Lambda_{\Gamma,M} := \left\{ L_0 + \sum_{i=1}^{\beta} \overline{F_i}, L_1 + \sum_{i=1}^{\beta} \overline{F_i}, L_2 + \sum_{i=1}^{\beta} \overline{F_i}, L_3 + \sum_{i=1}^{\beta} \overline{F_i}, \Gamma + M \right\}$$

with $\overline{F_i}$ general fibers of X. In particular $\Lambda_M \subset |aC_0 + \pi^*(B+B_1)|$.
For a general M the map induced by Λ_M

$$\varphi_{\Lambda_M} : X \dashrightarrow \mathbb{P}^4$$

is birational and $S_M = \varphi_{\Lambda_M}(X)$ is the image. For Γ and M general the scheme base locus of Λ_M is

$$\operatorname{Bs} \Lambda_M = \sum_{i=1}^{\beta} \overline{F_i} \cap (\Gamma + M).$$

In particular $\operatorname{Bs} \Lambda_M$ is a reduced 0-dimensional scheme of length $a\beta$. Let

$$\nu : Y \to X$$

be the blow up of these points in X with exceptional divisors E_i, and

$$\Lambda_Y := \nu^*\Lambda_M - \sum_{i=1}^{a\beta} E_i \sim \nu^*(aC_0 + \pi^*(B + B_1)) - \sum_{i=1}^{a\beta} E_i,$$

the strict transform linear system. Then

$$\begin{aligned}\deg S_M &= \left(\nu^*(aC_0 + \pi^*(B + B_1)) - \sum_{i=1}^{a\beta} E_i\right)^2 \\ &= 2a(b + \beta) - a\beta = a(2b + \beta).\end{aligned}$$

Let p_i be the i^{th} coordinate point in $\mathbb{P}^4$. Then by construction $p_i \notin S_M$ for $i \in \{0, \dots, 3\}$, while $p_4 \in S_M$ and the fiber $\varphi_M^{-1}(p_4) = \operatorname{Bs}\mathcal{L} \cup \sum_i \overline{F_i}$.

By construction all projections $\varphi_i : \mathbb{P}^4 \dashrightarrow \mathbb{P}^3$ from p_i are birational when restricted to S_M. Our first aim is to produce a monoid with vertexes p_0 and p_4 that contains S_M.

Let

$$\pi_i : V \to \mathbb{P}^4$$

be the blow up of the point p_i, $i \in \{0, 4\}$, in $\mathbb{P}^4$ with exceptional divisor E and let $S_i := \pi_{i*}^{-1} S_M$. By construction

$$\delta_0 := \deg S_0 = \deg S_M = a(2b + \beta). \tag{6}$$

The following Lemma bounds the dimension of the restricted linear system, in view of Lemma 2.10.

Lemma 3.1. *For $d \gg 0$ we have*

$$h^0(S_0, \mathcal{O}_{S_0}((d-1)(H-E) + \pi_0^*(\mathcal{O}(1)))) \le \frac{a(2b+\beta)}{2}d^2 - \frac{a-2}{2}\beta d + \ell d + m,$$

with ℓ and m independent from β and d.

Proof. By construction $S_0 \cong S_M$ and $(H - E)_{|S_0} \sim \mathcal{O}(1)_{S_M}$. Therefore $\mathcal{O}_{S_0}((d-1)(H-E) + \pi_0^*(\mathcal{O}(1))) = \mathcal{O}_{S_0}(d)$, $\mu : Y \to S_M$ is the embedding given by Λ_Y, and

$$\mu^*(\mathcal{O}_{S_M}(d)) \sim d\left(\nu^*(aC_0 + \pi^*(B + B_1)) - \sum_{i=1}^{a\beta} E_i\right).$$

This yields

$$\begin{aligned}&h^0(S_0, \mathcal{O}_{S_0}((d-1)(H-E) + \pi_0^*\mathcal{O}(1))) \\ &\le h^0\left(Y, d\left(\nu^*(aC_0 + \pi^*(B + B_1)) - \sum_{i=1}^{a\beta} E_i\right)\right).\end{aligned}$$

Let $L = \nu^*(aC_0 + \pi^*(B + B_1)) - \sum_{i=1}^{a\beta} E_i$, then Riemann-Roch Theorem yields

$$h^0(Y, dL) = \chi(Y, dL) + h^1(Y, dL) = \frac{L^2}{2}d^2 - \frac{K_Y.L}{2}d + 1 - g + h^1(Y, dL)$$

$$= \frac{a(2b+\beta)}{2}d^2 - \frac{\left(\nu^*(K_X)+\sum_{i=1}^{a\beta}E_i\right).L}{2}d+1-g+h^1(Y,dL) = \frac{a(2b+\beta)}{2}d^2 +$$

$$-\frac{\left(\nu^*(-2C_0+(2g-2)F)+\sum_{i=1}^{a\beta}E_i\right)\cdot\left(\nu^*(aC_0+\pi^*(B+B_1))-\sum_{i=1}^{a\beta}E_i\right)}{2}d$$

$$+1-g+h^1(Y,dL) = \frac{a(2b+\beta)}{2}d^2 - \frac{(a-2)}{2}\beta d + \ell' d + 1 - g + h^1(Y,dL),$$

for some ℓ' constant with respect to β and d. Let $A \in \Lambda_Y$ be a general element. Then A is irreducible and $L \cdot A = \delta_0 > 0$. Consider the structure sequence of A in Y tensored by dL. That is

$$0 \longrightarrow \mathcal{O}_Y((d-1)L) \longrightarrow \mathcal{O}_Y(dL) \longrightarrow \mathcal{O}_A(dL) \longrightarrow 0.$$

Passing to cohomology we get

$$H^1(Y,(d-1)L) \longrightarrow H^1(Y,dL) \longrightarrow H^1(A, dL_{|A}).$$

Since $A \cdot L > 0$ the last term vanishes for $d \gg 0$ and therefore for $d \gg 0$

$$h^1(Y,dL) \leq h^1(Y,(d-1)L).$$

In particular $h^1(Y,dL)$ does not depend on d, for $d \gg 0$. In a similar fashion considering the structure sequence of $\nu^*(\pi^*B_1) - \sum_1^{a\beta} E_i$ we prove that

$$h^1(Y,dL) \leq h^1\left(Y, d\left(\nu^*(aC_0+\pi^*B)\right)\right) = h^1\left(S_M, d\left(aC_0+\pi^*B\right)\right).$$

Therefore $h^1(Y,dL)$ does not depend on β, for $d \gg 0$.

Putting all together we have, for $d \gg 0$,

$$h^0(Y,dL) \leq \frac{a(2b+\beta)}{2}d^2 - \frac{a-2}{2}\beta d + \ell' d + m$$

where ℓ' and m don't depend on β and d. □

The bound in Lemma 3.1 allows to prove the following.

Lemma 3.2. *Let $S \subset \mathbb{P}^3$ be a uniruled surface. Let $\mathcal{L}$ and $\Lambda_M \subset |aC_0 + \pi^*(B+B_1)|$ be as above. Then there are positive integers β and d such that*

$$M_d(p_0, S_M)^{\not\subset} \cap M_d(p_0, p_4) \neq \emptyset.$$

Proof. Both $M_d(p_0,S_M)^{\not\subset}$ and $M_d(p_0,p_4)$ are sublinear spaces of $M_d(p_0)$. Therefore to prove the statement it is enough to show that

$$\dim M_d(p_0,S_M))^{\not\subset} + \dim M_d(p_0,p_4) > \dim M_d(p_0) \tag{7}$$

for $d \gg 0$. Combining Lemma 3.1, Lemma 2.10, and Equations (6) we obtain

$$\dim M_d(p_0,S_M)^{\not\subset} \geq \frac{a(2b+\beta)}{2}d^2 \tag{8}$$

$$+\left(\frac{(a-2)\beta}{2} - a^2(2b+\beta)^2 + 3a(2b+\beta) - \ell\right)d$$

$$+\left(\frac{a^3(2b+\beta)^3}{3}-\frac{3}{2}a^2(2b+\beta)^2+\frac{13}{6}a(2b+\beta)-m\right).$$

The comparison of cubic terms in the variables β, d appearing in Equations (8), (3), and (2) gives:

$$\frac{ad^2}{2}\beta - a^2 d\beta^2 + \frac{a^3}{3}\beta^3 = \frac{d^3}{3}. \tag{9}$$

The unique real root of Equation (9), that is of

$$(2a^3)\beta^3 - (6a^2 d)\beta^2 + (3ad^2)\beta - (2d^3) = 0,$$

solved with respect to β, is

$$\beta(d) = \frac{d}{a} + \sqrt[3]{\sqrt{\frac{7d^6}{16a^6}} + \frac{3d^3}{4a^3}} + \frac{1}{2a^2}\frac{d^2}{\sqrt[3]{\sqrt{\frac{7d^6}{16a^6}} + \frac{3d^3}{4a^3}}} \tag{10}$$

$$= \frac{d}{a} + \frac{d}{2^{\frac{2}{3}}a}\sqrt[3]{\sqrt{7}+3} + \frac{d}{2^{\frac{1}{3}}a\sqrt[3]{\sqrt{7}+3}} = \frac{d}{a}\left(1+\sqrt[3]{\frac{\sqrt{7}+3}{4}}+\frac{1}{\sqrt[3]{2(\sqrt{7}+3)}}\right) > 0$$

for all $a,\, d \in \mathbb{N}$. In particular, $\beta(d) = \xi\frac{d}{a} \notin \mathbb{N}$ for all $a, d \in \mathbb{N}$, being $\xi \notin \mathbb{Q}$ ($\xi \sim 2.567468375$).

To conclude we need to prove that for $\beta = \beta(d) = \xi\frac{d}{a}$ the quadratic terms in the variables β, d appearing in Equations (8), (3), and (2) satisfy the inequality

$$abd^2 + \left(-4a^2b + 3a + \frac{(a-2)}{2}\right)\beta d + \left(2a^3 b - \frac{3a^2}{2}\right)\beta^2 + 2d^2 > \frac{3}{2}d^2, \tag{11}$$

or equivalently,

$$\left(ab+\frac{1}{2}\right)d^2 + \left(-4a^2b + \frac{7}{2}a - 1\right)\beta d + a^2\left(2ab - \frac{3}{2}\right)\beta^2 > 0. \tag{12}$$

By making the substitution $\beta = \beta(d) = \xi\frac{d}{a}$ in the left-side term of Equation (12), since $b \geq 1$, we have

$$\begin{aligned}
&\left(ab+\frac{1}{2}\right)d^2 + \left(-4a^2b + \frac{7}{2}a - 1\right)\xi\frac{d^2}{a} + a^2\left(2ab - \frac{3}{2}\right)\xi^2\frac{d^2}{a^2} \qquad (13)\\
&= \frac{d^2}{a}\left[a\left(ab+\frac{1}{2}\right) + \left(-4a^2b + \frac{7}{2}a - 1\right)\xi + a\left(2ab - \frac{3}{2}\right)\xi^2\right]\\
&= \frac{d^2}{a}\left[a^2b\left(1-4\xi+2\xi^2\right) + a\left(\frac{1}{2}+\frac{7}{2}\xi-\frac{3}{2}\xi^2\right) - \xi\right]\\
&\geq \frac{d^2}{a}\left[a^2\left(1-4\xi+2\xi^2\right) + a\left(\frac{1}{2}+\frac{7}{2}\xi-\frac{3}{2}\xi^2\right) - \xi\right].
\end{aligned}$$

In particular

$$a^2\left(1-4\xi+2\xi^2\right) + a\left(\frac{1}{2}+\frac{7}{2}\xi-\frac{3}{2}\xi^2\right) - \xi > 0 \Longleftrightarrow a < a_1 \vee a > a_2 \tag{14}$$

where

$$a_1 = \frac{-\left(\frac{1}{2}+\frac{7}{2}\xi-\frac{3}{2}\xi^2\right)-\sqrt[2]{\left(\frac{1}{2}+\frac{7}{2}\xi-\frac{3}{2}\xi^2\right)^2+4\xi\left(1-4\xi+2\xi^2\right)}}{2\left(1-4\xi+2\xi^2\right)} < 0$$

and

$$a_2 = \frac{-\left(\frac{1}{2}+\frac{7}{2}\xi-\frac{3}{2}\xi^2\right)+\sqrt[2]{\left(\frac{1}{2}+\frac{7}{2}\xi-\frac{3}{2}\xi^2\right)^2+4\xi\left(1-4\xi+2\xi^2\right)}}{2\left(1-4\xi+2\xi^2\right)}$$
$$\sim 0,8628701083.$$

Thus $\beta(d)$ satisfies Equation (11) for all $a, b \in \mathbb{N}$ with $a \geq 1$ and $b \geq 1$.
Let $h = \frac{d}{a}$, so that $\beta(d) = \beta(h) = h\xi$. The fractional part $\{h\xi\}$ of $h\xi$ is uniformly dense in $(0,1)$. Hence for all $\epsilon \in (0,1)$ there exists $h \gg 0$ and sufficiently divisible such that $h\xi - \epsilon \in \mathbb{N}$. Equivalently set

$$\epsilon' = 1 - \epsilon,$$

then $\beta = \beta(h) + \epsilon' = h\xi + \epsilon' \in \mathbb{N}$, for $h \gg 0$ and sufficiently divisible.
To conclude the Lemma it is enough to show that there exists $\epsilon' \in (0,1)$ such that

$$\beta = \lceil h\xi \rceil = h\xi + \epsilon' \in \mathbb{N},$$

and fulfill Equation (7), for some $h \gg 0$.
As before we start with the cubic terms and Equation (9):

$$\frac{ad^2}{2}\beta - a^2 d\beta^2 + \frac{a^3}{3}\beta^3 - \frac{d^3}{3} = \frac{1}{6}(2a^3\beta^3 - 6a^2 d\beta^2 + 3ad^2\beta - 2d^3)$$
$$= \frac{1}{6}(2a^3\beta^3 - 6a^3h\beta^2 + 3a^3h^2\beta - 2a^3h^3) = \frac{a^3}{6}(2\beta^3 - 6h\beta^2 + 3h^2\beta - 2h^3)$$
$$= \frac{a^3}{6}[2(\beta(h)^3 + 3\beta(h)^2\epsilon' + 3\beta(h)\epsilon'^2 + \epsilon'^3) - 6h(\beta(h)^2 + 2\beta(h)\epsilon' + \epsilon'^2)$$
$$+3h^2(\beta(h)+\epsilon') - 2h^3].$$

Since $\beta(h) = h\xi$ solves (9), the previous expression becomes

$$\frac{a^3}{6}[2(3\beta(h)^2\epsilon' + 3\beta(h)\epsilon'^2 + \epsilon'^3) - 6h(2\beta(h)\epsilon' + \epsilon'^2) + 3h^2\epsilon']$$
$$= \frac{a^3\epsilon'}{6}[2(3h^2\xi^2 + 3h\xi\epsilon' + \epsilon'^2) - 6h(2h\xi + \epsilon') + 3h^2]$$
$$= \frac{a^3\epsilon'}{6}[3h^2(2\xi^2 - 4\xi + 1) + 6h\epsilon'(\xi - 1) + 2\epsilon'^2] > 0. \tag{15}$$

The presence of a term h^2 forces us to study also the quadratic terms and Equation (12):

$$\left(ab+\frac{1}{2}\right)d^2 + \left(-4a^2b + \frac{7}{2}a - 1\right)\beta d + a^2\left(2ab - \frac{3}{2}\right)\beta^2$$
$$= \left(ab+\frac{1}{2}\right)a^2h^2 + \left(-4a^2b + \frac{7}{2}a - 1\right)\beta ah + a^2\left(2ab - \frac{3}{2}\right)\beta^2 = \left(a^3b + \frac{a^2}{2}\right)h^2$$

$$+\left(-4a^3b+\frac{7}{2}a^2-a\right)h(h\xi+\epsilon')+\left(2a^3b-\frac{3}{2}a^2\right)(h^2\xi^2+2h\xi\epsilon'+\epsilon'^2)$$

$$=\left[a^3b+\frac{a^2}{2}+\xi\left(-4a^3b+\frac{7}{2}a^2-a\right)+\xi^2\left(2a^3b-\frac{3}{2}a^2\right)\right]h^2 \tag{16}$$

$$+\left[\epsilon'\left(-4a^3b+\frac{7}{2}a^2-a\right)+2\xi\epsilon'\left(2a^3b-\frac{3}{2}a^2\right)\right]h+\epsilon'^2\left(2a^3b-\frac{3}{2}a^2\right).$$

The number $\beta(h)$ satisfies Equation (12) thus Equation (16) is positive for any $\epsilon' \ll 1$ when $a \geq 1$ and $b \geq 1$. Therefore for any $h \gg 0$ and sufficiently divisible, the equation (7) is satisfied. □

We are now able, via Lemma 2.9, to establish the existence of the Cremona equivalences we are interested in.

Corollary 3.3. *Let $S \subset \mathbb{P}^3$ be a uniruled surface, $\mathcal{L}$ and $\Lambda_M \subset |aC_0 + \pi^*(B+B_1)|$ be as above. Assume that* $\operatorname{mult}_{[1,0,0,0]} S < \deg S - 1$*. Then there are positive integers β and d such that*

$$M_d(p_0, S_M)^{\not{C}} \cap M_d(p_4, S_M)^{\not{C}} \neq \emptyset.$$

Proof. Lemma 3.2 produces elements in $M_d(p_0, S_M)^{\not{C}}$ with vertex also in p_4. If Ξ contains $C_{p_4}(S_M)$ then $\Xi = C_{p_4}(S_M) + R$, for some residual divisor R of degree $d - \deg C_{p_4}(S_M) = d - \deg S$. By hypothesis

$$\operatorname{mult}_{[1,0,0,0]} S = \operatorname{mult}_{p_0} C_{p_4}(S_M) < \deg S - 1.$$

Since $\operatorname{mult}_{p_0} \Xi \geq d-1$ this gives the contradiction

$$\deg R = d - \deg S < d - 1 - \operatorname{mult}_{p_0} C_{p_4}(S_M) \leq \operatorname{mult}_{p_0} R.$$

□

We are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Set $X = \mathbb{P}^1 \times C$ and let $\mathcal{L} := \{L_0, L_1, L_2, L_3\}$ be a linear system that realizes the uniruled surface $S \subset \mathbb{P}^3$ with a ruling of curves of degree a, that is $S = \varphi_{\mathcal{L}}(X)$ and $\mathcal{L} \subseteq |aC_0 + \pi^*B|$. Up to a linear automorphism we may assume that $\operatorname{mult}_{[1,0,0,0]} S < \deg S - 1$. Fix sufficiently ample divisors $B_i \in \operatorname{Pic}(C)$, with $\deg B_i = \beta_i$, for $i \in \{0, \ldots, 4\}$, and a general section $\Gamma \in |C_0 + \pi^*G|$. Let $M_0 \in |(a-1)C_0 + \pi^*(B + B_0 - G)|$. Let

$$\Lambda_{M_0} := \left\{L_0 + \sum_{i=1}^{\beta_0}\overline{F_i}, L_1 + \sum_{i=1}^{\beta_0}\overline{F_i}, L_2 + \sum_{i=1}^{\beta_0}\overline{F_i}, L_3 + \sum_{i=1}^{\beta_0}\overline{F_i}, \Gamma + M_0\right\},$$

and $S_{M_0} = \varphi_{\Lambda_{M_0}}(X) \subset \mathbb{P}^4$.

By Corollary 3.3 we may choose β_0, d such that

$$M_d(p_0, S_{M_0})^{\not{C}} \cap M_d(p_4, S_{M_0})^{\not{C}} \neq \emptyset.$$

By construction $\varphi_{i|S_{M_0}}$ is birational, and the surface S_{M_0} is non degenerate therefore by Lemma 2.9 $S=\varphi_4(S_{M_0})$ is Cremona equivalent to $S_1:=\varphi_0(S_{M_0})$.

The surface S_1 is realized by the linear system

$$\mathcal{L}_1 := \left\{ L_1 + \sum_{i=1}^{\beta_0} \overline{F_i}, L_2 + \sum_{i=1}^{\beta_0} \overline{F_i}, L_3 + \sum_{i=1}^{\beta_0} \overline{F_i}, \Gamma + M_0 \right\}.$$

For M_0 general we may assume that $\operatorname{mult}_{[1,0,0,0]} S_1 < \deg S_1 - 1$. This allows to iterate the construction via general divisors $M_i \in |(a-1)C_0 + \pi^*(B + \sum_{j=0}^{i} B_j - G)|$, for $i \in \{1,2,3\}$, to produce surfaces S_{i+1} Cremona equivalent to S and such that S_4 is associated to the linear system

$$\mathcal{L}_4 := \left\{ \Gamma + M_0 + \sum_{i=1}^{\beta_1+\beta_2+\beta_3} \overline{F}_i, \Gamma + M_1 + \sum_{i=1}^{\beta_2+\beta_3} \overline{F}_i, \Gamma + M_2 + \sum_{i=1}^{\beta_3} \overline{F}_i, \Gamma + M_3 \right\}.$$

Since Γ is a fixed component we may remove it from the linear system. Therefore S_4 is realized by the linear system

$$\left\{ M_0 + \sum_{i=1}^{\beta_1+\beta_2+\beta_3} \overline{F}_i, M_1 + \sum_{i=1}^{\beta_2+\beta_3} \overline{F}_i, M_2 + \sum_{i=1}^{\beta_3} \overline{F}_i, M_3 \right\},$$

and it is ruled by curves of degree $(a-1)$. Then by induction on a the surface S is Cremona equivalent to a scroll. □

Remark 3.4. *Let $S \subset \mathbb{P}^3$ be a uniruled surface. The inductive procedure that produces the Cremona equivalence of S to a scroll increases its degree and produces points with high multiplicity. In particular the resulting scroll is a very special projection into $\mathbb{P}^3$ of the general scroll $\mathbb{P}^1 \times C$ embedded in $\mathbb{P}^N$ by a linear system $|C_0 + \pi^* B|$, for some very ample $B \in \operatorname{Pic}(C)$. See also Example 4.3 for a more precise statement.*

As already observed in the introduction Theorem 1.1 is equivalent to the following.

Corollary 3.5. *Let $cr_3 : Cr_3 \times \operatorname{Hilb}(\mathbb{P}^3) \dashrightarrow \operatorname{Hilb}(\mathbb{P}^3)$ be the natural action induced by the group of Cremona transformations. Then any one dimensional family of rational curves is rectifiable by cr_3, that is it can be mapped into the Klein quadric (the irreducible variety representing lines of $\mathbb{P}^3$) by cr_3.*

4. The Cremona action on scrolls

In this section we collect examples and remarks about the problem of determining Cremona equivalent scrolls. Let us stress from the beginning that the picture is very far from being unveiled.

Example 4.1. *Let C be a curve and consider the ruled surface $\mathbb{P}^1 \times C$, with ruled structure $\pi : \mathbb{P}^1 \times C \to C$ and $q : \mathbb{P}^1 \times C \to \mathbb{P}^1$. Then for any* *birational linear system* $\mathcal{L}\subset|\pi^*D+q^*\mathcal{O}_{\mathbb{P}^1}(1)|$, *with* $D\in\operatorname{Pic}(C)$, *of projective dimension 3, we produce a scroll* $S_\mathcal{L}\subset\mathbb{P}^3$. *Moreover if* $g(C)>0$ *any scroll is obtained in this way. Let* $\chi:\mathbb{P}^3\dashrightarrow\mathbb{P}^3$ *be a general Cremona transformation of type* (d_1,d_2). *Then* $\chi(S_\mathcal{L})\subset\mathbb{P}^3$ *is ruled by rational curves of degree* d_1 *and has degree greater then* $S_\mathcal{L}$. *Via Theorem 1.1* $\chi(S_\mathcal{L})$ *is Cremona equivalent to a scroll of even higher degree. This shows that the subset of scrolls Cremona equivalent to a given surface is not only infinite but contains a dense subset of the Cremona group of* $\mathbb{P}^3$.

The following is a slight modification of [MP2, Lemma 2.2]

Lemma 4.2. *Let* X^{n-1} *be an irreducible and reduced projective variety. Let* $\mathcal{L}$ *and* $\mathcal{G}$ *be birational embeddings of* X *into* $\mathbb{P}^n$, *of degree respectively* l *and* g. *Assume that* $l\geq g$ *and* $\varphi_\mathcal{L}(X)$ *is Cremona equivalent, but not projectively equivalent, to* $\varphi_\mathcal{G}(X)$. *Then the pair* $(\mathbb{P}^n,\frac{n+1}{l}\varphi_\mathcal{L}(X))$ *has not terminal singularities. In particular if* $X,X_1\subset\mathbb{P}^n$ *are two non projectively equivalent irreducible divisors and* $(\mathbb{P}^n,\frac{n+1}{\deg X}X)$ *and* $(\mathbb{P}^n,\frac{n+1}{\deg X_1}X_1)$ *are terminal pairs then* X *is not Cremona equivalent to* X_1.

Proof. Let $\Phi:\mathbb{P}^n\dashrightarrow\mathbb{P}^n$ be a Cremona equivalence between $\varphi_\mathcal{L}(X)$ and $\varphi_\mathcal{G}(X)$. Fix a resolution of Φ

$$\begin{array}{ccc} & Z & \\ {}^{p}\swarrow & & \searrow^{q} \\ \mathbb{P}^n & \overset{\Phi}{\dashrightarrow} & \mathbb{P}^n \end{array}$$

Then, for any ϵ we have

$$p^*(\mathcal{O}(K_{\mathbb{P}^n}+(1+\epsilon)(\frac{n+1}{l}\varphi_\mathcal{L}(X)))=K_Z+(1+\epsilon)\frac{n+1}{l}X_Z-\sum a_iE_i$$

and

$$q^*(\mathcal{O}(K_{\mathbb{P}^n}+(1+\epsilon)(\frac{n+1}{l}\varphi_\mathcal{G}(X)))=K_Z+(1+\epsilon)\frac{n+1}{l}X_Z-\sum b_iF_i$$

where E_i, respectively F_i, are p, respectively q, exceptional divisors. Let $r\subset\mathbb{P}^n$ be a general line in the right hand side $\mathbb{P}^n$ and Γ its strict transform on the left hand side, with $\deg\Gamma=\gamma>1$. The hypothesis $l\geq g$ yields

$$\begin{aligned}\epsilon\frac{n+1}{l}g&=q^{-1}r\cdot(q^*(\mathcal{O}(K_{\mathbb{P}^n}+(1+\epsilon)\frac{n+1}{l}\varphi_\mathcal{G}(X))\\ &=q^{-1}r\cdot(p^*(\epsilon\frac{n+1}{l}\varphi_\mathcal{L}(X))+\sum a_iE_i)\\ =\epsilon\frac{n+1}{l}\gamma l+(\sum a_iE_i)\cdot q^{-1}r&>\epsilon\frac{n+1}{l}g+(\sum a_iE_i)\cdot q^{-1}r.\end{aligned}$$

This proves that at least one $a_i<0$. Therefore for any $\epsilon>0$ the pair $(\mathbb{P}^n,(1+\epsilon)(\frac{n+1}{l}\varphi_L(X)))$ has not canonical singularities. Hence the pair $(\mathbb{P}^n,\frac{n+1}{l}\varphi_L(X))$ has not terminal singularities. □

As we already observed our procedure to produce a Cremona equivalence to a scroll increases the degree and produces quite bad singularities. Here we want to go a bit deeper in this and show that this is not a lack of our construction. Fix a surface $\mathbb{P}^1 \times C$ and a birational linear system of dimension 3 $\mathcal{L} \subset |q^*\mathcal{O}_{\mathbb{P}^1}(1) + \pi^*B|$, for some divisor $B \in \mathrm{Pic}(C)$. This is equivalent to give a linear projection to $\mathbb{P}^3$. It is classically known that a general projection of a surface to $\mathbb{P}^3$ has mild singularities, a curve of double points and isolated 3-ple points. This suggests that we do not expect that a uniruled surface is Cremona equivalent to a general projection. The next example aims to formalize this feeling.

Example 4.3. *Let C be a curve of positive genus. Consider the embedding of $\mathbb{P}^1 \times C \subset \mathbb{P}^N$ given by $|q^*\mathcal{O}_{\mathbb{P}^1}(a) + \pi^*D|$, for $a \geq 2$ and $D \in \mathrm{Pic}(C)$ divisor of degree $d \geq 4$. Let S_π be a general projection in $\mathbb{P}^3$ of degree $2ad \geq 16$. Then the singularities of S_π are an irreducible curve of double points and isolated triple points. In particular for any point $x \in S_\pi$ we have*

$$\mathrm{mult}_x S_\pi \frac{4}{2ad} \leq \frac{12}{2ad} < 1,$$

and the pair $(\mathbb{P}^3, \frac{4}{2ad}S_\pi)$ has terminal singularities.

Let $Y_D \subset \mathbb{P}^N$ be the embedding of $\mathbb{P}^1 \times C$ with a linear system $|q^*\mathcal{O}_{\mathbb{P}^1}(1) + \pi^*B|$, for some $B \in \mathrm{Pic}(C)$. Then, exactly as above, for any B and any general projection, say S^B, the pair $(\mathbb{P}^3, \frac{4}{\deg S^B}S^B)$ has terminal singularities as soon as $\deg B \geq 7$. Then by Lemma 4.2 S_π is not Cremona equivalent to S^B if $\deg B \geq 7$. If $\deg B \leq 6$ then $\deg S^B \leq 12$ and again by Lemma 4.2, it is not Cremona equivalent to S_π.

References

[AM] Abhyankar, S.S., Moh, T.T.: *Embeddings of the line in the plane*. J. Reine Angew. Math. **276**, 148–166 (1975)

[BB] Bogomolov, F., Böhning, C.: *On uniformly rational varieties*, accepted for publication in S.P. Novikov's 75th anniversary volume, to be published by the AMS, arXiv:1307.0102[math.AG]

[CC1] Calabri, A., Ciliberto C.: *Birational classification of curves on rational surfaces* Nagoya Math. J. **199** (2010), 4393.

[CC2] ______: *On Cremona contractibility of unions of lines in the plane*, arXiv:1503.05850 [math.AG]

[CCMRZ] Ciliberto, C., Cueto, M.A., Mella, M., Ranestad, K., Zwiernik, P.: *Cremona linearizations of some classical varieties*, arXiv:1403.1814v2 [math.AG] (2014)

[Coo] Coolidge, J. L.: *A treatise on algebraic plane curves*, Dover Publ., New York, 1959.

[Con] Conforto, F.: *Le superficie razionali*, Zanichelli Ed., Bologna, 1939 (Federigo Enriques had been author of this book as well, but his name had been omitted because of Italian laws against Hebrews, issued in 1938).

[Isk87] Iskovskikh, V. A.: *On the rationality problem for conic bundles*, Duke Math. J. **54** (1987), no. 2, 271–294.

[Je] Jelonek, Z.: *The extension of regular and rational embeddings*. Math. Ann. **277**, 113–120 (1987)

[Ka] Kantor, S.: *Die Typen der linearen Complexe rationaler Curve in R_r* Amer. J. of Math. **23** (1901), 1-28.

[Ma] Maruyama, M.: *On classification of ruled surfaces*, Lectures in Mathematics, Department of Mathematics, Kyoto University (**3**), Kinokuniya Book-Store Co., Tokyo (1970)

[Me] Mella, M.: *Equivalent birational embeddings III: cones*, arXiv:1407.8075v1 [math.AG] to appear on Rendiconti del Seminario Matematico di Torino, special volume in honour of Alberto Conte's 70$^{\text{th}}$ birthday (2014)

[MP1] Mella, M., Polastri, E.: *Equivalent birational embeddings*, Bull. Lond. Math. Soc., 41 (**1**), 89-93 (2009)

[MP2] ———: *Equivalent birational embeddings II: divisors*, Math. Zeit., **270**, 1141-1161 (2012)

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