

# ENERGY AND INVARIANT MEASURES FOR BIRATIONAL SURFACE MAPS

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## Abstract

*Given a birational self-map of a compact complex surface, it is useful to find an invariant measure that relates the dynamics of the map to its action on cohomology. Under a very weak hypothesis on the map, we show how to construct such a measure. The main point in the construction is to make sense of the wedge product of two positive, closed  $(1, 1)$ -currents. We are able to do this in our case because local potentials for each current have “finite energy” with respect to the other. Our methods also suffice to show that the resulting measure is mixing, does not charge curves, and has nonzero Lyapunov exponents.*

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## 0. Introduction

Let  $X$  be a compact Kähler surface, and let  $f : X \rightarrow X$  be a bimeromorphic mapping. We consider  $(f, X)$  as a dynamical system, which means that we consider the behavior of the iterates  $f^n = f \circ \cdots \circ f$  as  $n$  tends to infinity. Since  $f$  is invertible, we may consider both forward and backward dynamics, that is,  $f^n$  as  $n \rightarrow +\infty$  and  $n \rightarrow -\infty$ . A meromorphic map of a surface is holomorphic outside a finite set,  $I(f)$ , of points that are blown up to curves. Thus  $f$  is not in general a continuous map, so

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it is not clear to what extent there is a standard category of dynamical systems into which such an object falls.

We consider two bimeromorphic maps to be equivalent if they are bimeromorphically conjugate. Two complex surfaces can be bimeromorphically equivalent, however, without being homeomorphic. One approach that has proved fruitful in complex dynamics is to start with the induced action  $f^*$  on the cohomology group  $H^{1,1}(X)$ . A question that arises when  $f$  has points of indeterminacy is whether the passage to cohomology is natural for the dynamics, that is, whether  $(f^n)^* = (f^*)^n$ . This happens exactly when the condition

$$\bigcup_{n \geq 0} f^{-n} I(f) \cap \bigcup_{n \geq 0} f^n I(f^{-1}) = \emptyset \quad (1)$$

holds. This condition may be viewed as a separation between the obstructions to forward and backward dynamics. Diller and Favre [DF] showed that any bimeromorphic surface map  $f : X \rightarrow X$  is bimeromorphically equivalent to a map  $\hat{f} : \hat{X} \rightarrow \hat{X}$  for which (1) holds. In general, the spectral radius  $\rho$  of  $\hat{f}^*$  on  $H^{1,1}(X)$  is greater than or equal to one, and it was shown in [DF] that, if  $\rho = 1$ , then either  $f$  is a dynamically trivial automorphism or  $f$  preserves a rational or elliptic fibration and exhibits a dynamic that is essentially one-dimensional.

We assume in this paper that  $\rho > 1$ . In this case (see [DF, Section 6] or Theorem 2.6) there are stable/unstable currents  $\mu^\pm$  whose cohomology classes generate the  $f^*$ - and  $f_*$ -eigenspaces for  $\rho$ , and, in fact,  $f^* \mu^+ = \rho \mu^+$  and  $f_* \mu^- = \rho \mu^-$ . The currents  $\mu^\pm$  carry geometric information of (complex) dimension one and are useful in analyzing the dynamics of  $f$ .

A natural hope is that the wedge product  $\mu := \mu^+ \wedge \mu^-$  might define an invariant measure that serves as a bridge between the action of  $f^*$  on  $H^{1,1}$  and the ergodic properties of  $f$  on  $X$ . This was shown to happen for polynomial automorphisms of  $\mathbf{C}^2$  in [BS] and [FS1]; for automorphisms of K3 surfaces in [C]; and for certain birational maps in [FG], [Di3], and [G]. There has also been some work in dimensions larger than two (see, for instance, [BP] and [GH] for the case of polynomial automorphisms and [DS] for automorphisms of compact complex manifolds). Typically, one considers the positive, closed currents  $\mu^\pm = dd^c g^\pm$  in terms of local potentials. The operation of the wedge product is then interpreted in terms of the so-called complex Monge-Ampère operator  $dd^c g^+ \wedge dd^c g^-$ . As is well known, this operation is possible if at least one of the potentials  $g^+$  or  $g^-$  is locally bounded. And this is what happens in all of the papers cited above. On the other hand, it is possible for both local potentials  $g^+$  and  $g^-$  to be locally unbounded at a point, as is the case for the “golden mean” family, which was analyzed in detail in [BDu].

The condition

$$\overline{\bigcup_{n \geq 0} f^{-n} I(f)} \cap \overline{\bigcup_{n \geq 0} f^n I(f^{-1})} = \emptyset \tag{2}$$

was introduced in [Di3], and it was shown to be equivalent to the condition that for each point there be a neighborhood on which one of the local potentials  $g^+$  or  $g^-$  is continuous. In this paper we employ a quantitative condition stronger than (1) and weaker than (2):

$$\sum_{n \geq 0} \rho^{-n} \log \text{dist}(f^n I(f^{-1}), I(f)) > -\infty. \tag{3}$$

By Theorem 4.3, this is equivalent to  $g^+(x) > -\infty$  for all  $x \in I(f)$ .

**THEOREM**

*If (3) holds, then  $\mu := \mu^+ \wedge \mu^-$  is a probability measure that puts no mass on any algebraic set;  $\mu$  is invariant and mixing for  $f$ . Further,*

$$\int |\log \|Df\|| \mu < \infty, \tag{4}$$

*and thus the Lyapunov exponents of  $f$  with respect to  $\mu$  are well defined and finite. Finally, the Lyapunov exponents satisfy*

$$\lambda^- \leq -\frac{\log \rho}{8} < 0 < \frac{\log \rho}{8} \leq \lambda^+,$$

*and thus  $\mu$  is a hyperbolic measure of saddle type.*

The finiteness of the integral in (4), and thus of the Lyapunov exponents, seems to be closely linked with condition (3). On the other hand, Favre [F1] has constructed a mapping that does not satisfy (3). Favre’s example depends on the existence of an invariant complex line whose rotation number satisfies a delicate number-theoretic property. In any case, condition (3) seems to be generic (cf. Proposition 4.5).

We define  $\mu^+ \wedge \mu^-$  using an “energy” approach to interpret the complex Monge-Ampère operator, as was done in [BT] and [B]. Specifically, if  $T$  is a positive, closed  $(1, 1)$ -current, then we define the energy of a function  $\varphi$  to be

$$\mathcal{E}_T(\varphi) := \int d\varphi \wedge d^c \varphi \wedge T.$$

The approach from [BT] is that if  $\varphi$  is essentially plurisubharmonic (psh), and if  $\mathcal{E}_T(\varphi) < \infty$ , then  $dd^c \varphi \wedge T$  defines a measure and  $\varphi$  is integrable with respect to this measure. In the situation at hand, we show that  $\mathcal{E}_T(\varphi) < \infty$  for  $T = \mu^-$  and  $\varphi = g^+$ .

The currents  $\mu^\pm$  are obtained dynamically by starting with Kähler forms  $\beta_1$  and  $\beta_2$  and taking normalized limits of pullbacks:

$$\mu^+ = c_1 \cdot \lim_{n \rightarrow \infty} \rho^{-n} f^{*n} \beta_1, \quad \mu^- = c_2 \cdot \lim_{n \rightarrow \infty} \rho^{-n} f_*^n \beta_2. \tag{5}$$

We show in Corollary 4.9 that the measure  $\mu$  is also obtained as

$$\mu = c \cdot \lim_{n,m \rightarrow \infty} \rho^{-n-m} f^{*n} \beta_1 \wedge f_*^m \beta_2. \tag{6}$$

The contents of the paper are as follows. In Section 1 we discuss the pullbacks of currents and the associated (local) potential functions. The fact of convergence in (5) was established in [DF]. However, in order to pass from (5) to (6), we need to know how the intermediate pullbacks depend on  $\beta_1$  and  $\beta_2$ . This dependence is clarified in Section 2. In Section 3 we discuss properties of the energy integral. In Section 4 we discuss condition (3); we show that when (3) holds, the gradients of the local potentials of  $\mu^\pm$  belong to  $L^2$ . Thus  $\mu := \mu^+ \wedge \mu^-$  is well defined. We show in Theorem 4.11 that  $\mu$  is invariant. Section 5 is dedicated to showing that  $\mu$  is mixing, and Section 6 gives the estimates on the Lyapunov exponents.

**1. Pullbacks under birational maps**

Throughout this paper we let  $X$  denote a compact Kähler surface endowed with the Hermitian metric associated to a fixed Kähler form  $\beta$ . Let  $f : X \rightarrow X$  be a bimeromorphic self-map. That is, there is a compact surface  $\Gamma$  (the desingularized graph of  $f$ ) with proper modifications (i.e., generically injective holomorphic maps)  $\pi_1, \pi_2 : \Gamma \rightarrow X$  such that  $f = \pi_2 \circ \pi_1^{-1}$ . The set

$$\mathcal{C}(\pi_j) := \{x \in \Gamma : \#(\pi_j^{-1} \pi_j(x)) > 1\} = \{x \in \Gamma : \dim(\pi_j^{-1} \pi_j(x)) = 1\}$$

is the critical set for  $\pi_j$ . The images  $I(f) := \pi_1(\mathcal{C}(\pi_1))$  and  $\mathcal{C}(f) := \pi_1(\mathcal{C}(\pi_2))$  are the *indeterminacy* and *critical* sets, respectively, for  $f$ . Note that in this case the critical set is actually an exceptional set since the irreducible components are mapped to points. It is shown in [DF, Section 2] that after a finite number of blowups we may assume that (1) holds. In this case,  $I(f^n) = \bigcup_{j=0}^{n-1} f^{-j} I(f)$ .

Since  $f$  is ill defined at points of indeterminacy, it is useful to adopt some conventions concerning images of points and curves under  $f$ . Given any  $x \in X$ , we set  $f(x) = \pi_2(\pi_1^{-1}(x))$ , with the effect that  $f(x)$  is a point if  $x \notin I(f)$  and a component of  $\mathcal{C}(f^{-1})$  otherwise. Given any curve  $V \subset X$ , we set  $f(V) = \overline{f(V \setminus I(f))}$ . For irreducible  $V$ , it follows that  $f(V)$  is a point if  $V \subset \mathcal{C}(f)$  and an irreducible curve if not.

PROPOSITION 1.1

There exist constants  $A, B > 0$  such that

$$\|D_x f\| \leq A \operatorname{dist}(x, I)^{-B}$$

for all  $x \in X$ . Further, given a second point  $y \in X$ , one has

$$\text{dist}(f(x), f(y)) \leq A \text{dist}(\{x, y\}, I)^{-B} \text{dist}(x, y).$$

*Proof*

Choose any Hermitian metric on the graph  $\Gamma$  of  $f$ . Then  $\|D\pi_2\|$  is uniformly bounded on  $\Gamma$ , so it suffices to prove the first inequality for  $\pi_1^{-1}$  in place of  $f$ .

In local coordinates, the entries of  $D\pi_1$  are holomorphic functions, so the entries of  $(D\pi_1)^{-1}$  are meromorphic functions with poles in  $\mathcal{C}(\pi_1)$ . Since  $\Gamma$  is compact, there are constants  $A, B > 0$  such that

$$\|(D_y\pi_1)^{-1}\| \leq A \text{dist}(y, \mathcal{C}(\pi_1))^{-B}$$

for all  $y \in \Gamma$ . But  $\|D\pi_1\|$  is uniformly bounded on  $\Gamma$ , so this implies

$$\|D_{\pi_1(y)}(\pi_1^{-1})\| = \|(D_y\pi_1)^{-1}\| \leq A \text{dist}(\pi_1(y), I)^{-B}.$$

The first inequality now follows because  $\pi_1$  is surjective.

The second inequality follows from the first by integrating along a path from  $x$  to  $y$ . □

We consider the Hermitian inner product on the set of smooth  $(1, 1)$ -forms given by

$$\langle a, a' \rangle := \int_X a \wedge \bar{a}'.$$

It follows that any smooth  $(1, 1)$ -form defines an element of the dual space of  $(1, 1)$ -forms and thus defines a  $(1, 1)$ -current. The  $(1, 1)$  cohomology group  $H^{1,1}(X)$  may be given as the smooth, closed  $(1, 1)$ -forms modulo the exact ones. It follows from Stokes's theorem that the Hermitian pairing on  $(1, 1)$ -forms induces a pairing on  $H^{1,1}(X)$ . In fact, this pairing is a nondegenerate duality.

If  $T$  is a closed  $(1, 1)$ -current, then  $T(d\xi) = 0$ , which means that  $T$  annihilates all  $d$ -exact  $(1, 1)$ -forms. Thus the restriction of  $T$  to the closed forms defines an element of  $H^{1,1}(X)^*$ , and there is a cohomology class  $\{T\} \in H^{1,1}(X)$  that represents this restriction in the sense that  $T = \langle \cdot, \{T\} \rangle$ .

The  $\partial\bar{\partial}$ -lemma from Kähler geometry (see [GH, page 149]) asserts that if  $T_1$  and  $T_2$  are closed  $(1, 1)$ -currents that define the same cohomology class, then there is a current  $S$  of degree zero such that

$$T_1 = T_2 + dd^c S.$$

In particular, if  $T$  is a closed  $(1, 1)$ -current on  $X$ , there is a smooth  $(1, 1)$ -form  $\alpha$  defining the cohomology class  $\{T\}$ , and by the  $\partial\bar{\partial}$ -lemma there is a current  $h$  such that  $T = \alpha + dd^c h$ .

Next we define the pullback of a smooth form. If  $\alpha$  is a smooth  $(1, 1)$ -form on  $X$ , then  $\pi_2^*\alpha$  is a smooth  $(1, 1)$ -form on  $\Gamma$ . By duality,  $\pi_2^*\alpha$  defines a current on  $\Gamma$  of bidegree  $(1, 1)$ . Thus

$$f^*\alpha := \pi_{1*}(\pi_2^*\alpha)$$

is a current on  $X$ . The pullback  $f^*$  commutes with  $d$  and with the complex structure, so closed (resp., exact) forms are pulled back to closed (resp., exact) currents of the same bidegree. This gives a well-defined map  $f^*$  on  $H^{1,1}(X)$ . Similarly, we set  $f_*\eta := (f^{-1})^*\eta = \pi_{2*}\pi_1^*\eta$ . In other words, we set  $f_* = (f^{-1})^*$ . Note that  $f^*$  and  $f_*$  are adjoint with respect to the intersection form  $\langle \cdot, \cdot \rangle$  on cohomology classes, which is to say,

$$\langle f^*\alpha, \alpha' \rangle = \langle \pi_2^*\alpha, \pi_1^*\alpha' \rangle = \langle \alpha, f_*\alpha' \rangle.$$

We can also define the pullback  $f^*T$  if  $T$  is a positive, closed  $(1, 1)$ -current on  $X$ . By pulling back local potentials of  $T$ , we may define  $f^*T$  on  $X - I(f)$ . Now, for any  $x \in I(f)$ , we may choose a pseudoconvex neighborhood  $U$  of  $x$  with  $H^2(U - \{x\}) = 0$ . Thus there is a potential  $p$  on  $U - \{x\}$  such that  $f^*T = dd^c p$  on  $U - \{x\}$ . Since  $p$  is psh on  $U - \{x\}$ , it follows that  $p$  has a psh extension  $\tilde{p}$  to  $U$ . We define  $f^*T := dd^c p$  on  $U$ .

In order to discuss the singularities caused by pulling back forms and currents, let us recall that the Lelong number of a positive, closed  $(1, 1)$ -current  $T$  at a point  $x \in X$  is the nonnegative number

$$\nu(T, x) := \lim_{r \rightarrow 0} \frac{C}{r^2} \int_{B_x(r)} \beta \wedge T.$$

If  $u$  is a local potential for  $T$  in a neighborhood of  $x$ , that is, if  $T = dd^c u$ , then

$$\nu(T, x) := \sup \{t \geq 0 : u(y) < t \log \text{dist}(x, y) + O(1)\}$$

(see [D2, Equation 5.5(e)]).

We use Proposition 1.1 to gain control over the singularities of pullbacks of smooth  $(1, 1)$ -forms.

PROPOSITION 1.2

Let  $\omega$  be a Kähler form on  $X$ , and let  $\omega'$  be a smooth form cohomologous to  $f^*\omega$ . Then we can write

$$f^*\omega = \omega' + dd^c u, \tag{7}$$

where  $u$  is smooth and negative on  $X \setminus I(f)$  and satisfies

$$A \log \text{dist}(x, I) - B \leq u(x) \leq A' \log \text{dist}(x, I) + B'$$

for some constants  $A, B, A', B' > 0$  and every  $x \in X$ .

*Proof*

If  $\omega'$  and  $f^*\omega$  represent the same element of  $H^{1,1}(X)$ , then there exists a  $u$  satisfying (7). The current  $f^*\omega$  is positive, so  $u$  is given locally as the sum of a smooth function  $u_1$  and a plurisubharmonic function  $u_2$ . In particular, we can assume that  $u$  is negative. The remaining assertion in the proposition only concerns some (any) choice of  $u_2$  in the neighborhood of a point  $y \in I(f)$ .

For each component  $V'$  of  $\pi_1^{-1}(y)$  and its image  $V = \pi_2(V')$ , we have

$$\int_{V'} \pi_2^* \omega = \int_V \omega > 0.$$

The intersection form on  $\pi_1^{-1}(y)$  is negative definite, so we can choose a nontrivial effective divisor  $V'$  supported on  $\pi_1^{-1}(y)$  such that  $\pi_2^* \omega + [V']$  is cohomologically trivial near  $V$ . In particular, we can write  $\pi_2^* \omega + [V'] = dd^c v$  for some function  $v$  defined in a neighborhood  $U'$  of  $\pi_1^{-1}(y)$  and smooth off  $\pi_1^{-1}(y)$ . Therefore  $v \circ \pi_1^{-1}$  is a local potential for

$$\pi_{1*}(\pi_2^* \omega + [V']) = \pi_{1*} \pi_2^* \omega$$

on the neighborhood  $U = \pi_1(U')$  of  $y$ . The singularities of  $v$  come entirely from local potentials for  $[V']$ . Hence we can arrange

$$v(x') \geq A \log \text{dist}(p', \pi_1^{-1}(y))$$

for some  $A > 0$  and all  $x' \in U'$ . Finally, since  $\pi_1$  is uniformly Lipschitz, we obtain, after adjusting  $A$ ,

$$u_2 := v \circ \pi_1^{-1}(x) \geq A \log \text{dist}(x, y),$$

which finishes the proof of the lower bound for  $u_2$ .

To obtain the upper bound for  $u_2$ , we rely on the push-pull formula [DF, Theorem 3.3] applied to  $\pi_1$ . This gives

$$\pi_1^* f^* \omega = \pi_1^* \pi_{1*}(\pi_2^* \omega) = \pi_2^* \omega + [V'],$$

where  $V'$  is an effective divisor such that  $\pi(\text{supp } V') = I(f)$  whose support contains at least one component of  $\mathcal{C}(\pi_1) \cap \pi_2^{-1}(V)$  for every irreducible  $V \subset \mathcal{C}(f^{-1})$ . In particular, the Lelong number of the positive current  $\pi_1^* f^* \omega$  is positive at some point in  $\pi_1^{-1}(p)$  for every  $p \in I(f) = \pi_2 \mathcal{C}(f^{-1})$ . It follows from [F2, Theorem 2] that  $f^* \omega$  has a positive Lelong number at each point in  $I(f) = \pi_1(\mathcal{C}(\pi_1))$ . We conclude that any local potential  $u_2$  for  $f^* \omega$  near  $y \in I(f)$  must satisfy

$$u_2(x) \leq A' \log \text{dist}(x, y) + B'$$

for some  $A', B' > 0$ . □

Let  $P(X)$  denote the set of upper semicontinuous functions  $u$  on  $X$  such that  $dd^c u \geq -c\beta$  for some  $c \in \mathbf{R}$ . Such functions are locally the sum of a psh function and a smooth function. (Since  $X$  is compact, there are no global psh functions.) Given a finite set  $S \subset X$ , let  $\tilde{P}(X, S)$  denote those functions  $u \in P(X) \cap C^\infty(X \setminus S)$  such that

$$u(x) \geq A \log \text{dist}(x, S) - B$$

for some  $A, B > 0$  and all  $x \in X$ .

PROPOSITION 1.3

Suppose that  $S \subset X$  is finite and disjoint from  $I(f^{-1})$ . Then  $\tilde{u} \in \tilde{P}(X, S)$  implies that  $u \circ f$  is a difference of functions in  $\tilde{P}(X, f^{-1}(S) \cup I(f))$ .

*Proof*

Because  $u \in \tilde{P}(X, S)$ , we get

$$0 \leq dd^c u \circ f + cf^*\beta = dd^c(u \circ f + v) + \beta' \leq dd^c(u \circ f + v) + c\beta,$$

where  $\beta'$  is a smooth  $(1, 1)$ -form cohomologous to  $f^*\beta$ ,  $v \in \tilde{P}(X, I(f))$ , and  $c > 0$  is chosen large enough that  $c\beta \geq \beta'$ . Moreover, since  $u \in \tilde{P}(X, S)$ , we see from Proposition 1.1 that for  $f(x)$  near  $S$ , and therefore uniformly far from  $I(f^{-1})$ ,

$$\begin{aligned} u \circ f(x) &\geq A \log \text{dist}(f(x), S) - B \\ &\geq A \log \text{dist}(x, f^{-1}(S)) - B + C \log \text{dist}(f(x), I(f^{-1})) \\ &\geq A \log \text{dist}(x, f^{-1}(S)) - B. \end{aligned}$$

Combining the two displayed inequalities, we see that

$$u \circ f = (u \circ f + v) - v,$$

where  $v \in \tilde{P}(X, I(f))$  and  $u \circ f + v \in \tilde{P}(X, f^{-1}(S) \cup I(f))$ . □

**2. Invariant cohomology classes and currents**

Condition (1) implies that  $(f^n)^* = (f^*)^n$  on  $H^{1,1}(X)$  for every  $n \in \mathbf{Z}$  (see [FS1] and [DF, Theorem 1.14]). In this case the bimeromorphically invariant quantity

$$\rho := \lim_{n \rightarrow \infty} \|f^{n*}|_{H^{1,1}}\|^{1/n} \geq 1$$

is the modulus of the largest eigenvalue of  $f^*$  on  $H^{1,1}(X)$ . In this paper, we assume that

$$\rho > 1. \tag{8}$$

An element of  $H^{1,1}$  is a Kähler class if it contains a Kähler form. We say that a cohomology class is numerically effective (nef) if it is in the closure of the Kähler classes. Alternatively, a class is nef if and only if (see [L] or [Bu]) its intersections with the fundamental classes of curves and with the class of the Kähler form  $\beta$  are all nonnegative.

The following is [DF, Theorem 5.1].

**THEOREM 2.1**

*If (8) holds, then  $\rho$  is the unique (counting multiplicity) eigenvalue of  $f^*$  of modulus larger than one; and the associated eigenspace is generated by a nef class  $\theta^+$ . If  $\theta^-$  is a nef class generating the corresponding eigenspace for  $f_*$ , then  $\langle \theta^+, \theta^- \rangle > 0$ .*

We note that it is also shown in [DF] that if  $\langle \theta^+, \theta^+ \rangle = 0$ , then  $f$  is birationally conjugate to an automorphism of some complex surface. The results from this paper for that case have already been obtained by Cantat in his thesis [C]. So while it does not much simplify the exposition, there is no harm in assuming in what follows that  $\langle \theta^+, \theta^+ \rangle > 0$ , and similarly for  $\theta^-$ . Indeed, under this assumption, [DF, Theorem 7.2] states that the surface  $X$  must be rational.

For convenience, we scale  $\theta^\pm$  and  $\beta$  so that

$$\langle \theta^+, \theta^- \rangle = \langle \theta^+, \beta \rangle = \langle \theta^-, \beta \rangle = 1. \tag{9}$$

This completely determines  $\theta^+$  and  $\theta^-$ .

We fix Kähler forms  $\omega_1, \dots, \omega_N$  whose cohomology classes form a basis for  $H^{1,1}(X)$ , and we let  $\Omega$  denote the linear span of these forms. We also assume for convenience that  $\Omega$  contains the Kähler form  $\beta$  corresponding to the metric on  $X$ . We endow  $\Omega$  with the norm  $\|\omega\| = (\sum |c_j|^2)^{1/2}$ , where  $\omega = \sum c_j \omega_j$ . Let  $\omega^+, \omega^- \in \Omega$  denote the unique elements representing the classes  $\theta^+$  and  $\theta^-$ , respectively. By Theorem 2.1, an element  $\eta \in \Omega$  has a decomposition

$$\eta = \eta^\perp + c\omega^+, \tag{10}$$

where  $\eta^\perp$  belongs to the span of the eigenspaces corresponding to eigenvalues other than  $\rho$ . The fact that  $f^*$  and  $(f^{-1})^*$  are adjoint gives  $c = \langle \omega, \theta^- \rangle$ .

If  $\eta$  is a closed  $(1, 1)$ -current, then we let  $\omega(\eta)$  denote the element of  $\Omega$  which corresponds to the cohomology class  $\{\eta\}$  defined by  $\eta$ . Thus  $\omega^\pm = \omega(\theta^\pm)$ . It is evident that, as a mapping from currents to  $\Omega$ ,  $\omega$  is a projection; that is,  $\omega \circ \omega = \omega$ . There is a current  $p(\eta)$  such that

$$\eta = \omega(\eta) + dd^c p(\eta). \tag{11}$$

Since  $\omega(\eta)$  is smooth, it follows that  $p(\eta)$  is smooth wherever  $\eta$  is. The potential  $p(\eta)$  is uniquely defined modulo an additive constant, and we specify it uniquely by

the condition

$$\langle p(\eta), \beta \wedge \beta \rangle = 0.$$

Now we investigate the interplay between the decomposition (11) and  $f^*$ . If  $\eta$  is positive, then  $p(\eta) \in P(X)$ , and we may apply  $f^*$  to (11) to obtain  $f^*\eta = f^*\omega(\eta) + dd^c f^*p(\eta)$ . Then we set

$$\gamma^+ = pf^*\omega$$

and apply the decomposition (11) to obtain

$$f^*\eta = \omega f^*\omega(\eta) + dd^c[\gamma^+(\eta) + f^*p(\eta)].$$

The operators  $\eta \mapsto \omega\eta$  and  $\eta \mapsto \gamma^+\eta$  are linear in  $\eta$  and depend only on the cohomology class  $\{\eta\}$ . The map  $\omega$  induces an isomorphism  $\omega : H^{1,1}(X) \rightarrow \Omega$ . This provides a conjugacy between the action of  $f^*$  on  $H^{1,1}$  and the action of  $\omega f^*$  on  $\Omega$ . Thus we have  $\omega f^* = \omega f^*\omega$ , and we may iterate the previous equation to obtain

$$f^{n*}\eta = \omega f^{n*}\eta + \rho^n dd^c g_n^+, \tag{12}$$

where we define

$$g_n^+ \eta = \frac{1}{\rho^n} \left( f^{*n} p(\eta) + \sum_{j=0}^{n-1} f^{(n-j-1)*} \gamma^+(f^{j*}\eta) \right). \tag{13}$$

The cone  $H_{\text{nef}}^{1,1}$  of nef classes is closed convex and “strict.” The last condition, which follows from the fact that nef classes are represented by positive, closed currents, means that if  $\theta$  and  $-\theta$  are nef, then  $\theta = 0$ . It follows that there is an affine hyperplane  $H \subset H^{1,1}$  such that  $H \cap H_{\text{nef}}^{1,1}$  is compact, convex, and generates  $H_{\text{nef}}^{1,1}$  as a real cone. If we set

$$K = \{ \eta \in \Omega : \langle \eta, V \rangle \geq 0 \text{ for every irreducible } V \subset \mathcal{C}(f^{-1}) \},$$

then  $K$  is a cone defined by a finite number of linear inequalities, and it follows from [L] that  $H_{\text{nef}}^{1,1} \subset K$ . (Here, as in other places, we identify  $K$  with the corresponding set of cohomology classes.)

LEMMA 2.2

Let  $K$  be the subset defined above. Then the function

$$M(\eta) := \sup_{x \in X-I(f)} \gamma^+(\eta)_x$$

is finite for  $\eta \in K \cap H \cap \Omega$ .

*Proof*

It is enough to show that for each point  $x \in I(f)$  there are a neighborhood  $U$  and a local potential for  $f^*\eta$  which is bounded above. We have  $f^*\eta = \pi_{1*}\pi_2^*\eta$ , so if  $U \cap I(f) = \{x\}$ , we may argue as in Proposition 1.3 to conclude that on  $\pi_1^{-1}U$  we have

$$\pi_1^*f^*\eta = \pi_2^*\eta + [V],$$

where  $V$  is a (possibly trivial) effective divisor supported on a fiber  $\pi_1^{-1}(x)$ . Hence  $\pi_1^*f^*\eta = dd^c v$  for some function  $v$  on  $\pi_1^{-1}(U)$  whose singularities come entirely from local potentials for  $[V]$ . Thus  $v$  is bounded above. Now the pushforward  $v \circ \pi_1^{-1}$  is a local potential for  $f^*\eta$  on  $U$  and is bounded above, as desired.  $\square$

**THEOREM 2.3**

*There are positive constants  $A, B$  such that for any  $\eta \in \Omega$ ,*

$$|\gamma^+\eta(x)| \leq \|\eta\| (A + B |\log \text{dist}(x, I(f))|) \tag{14}$$

*holds for all  $x \in X$ . Further, there exists a constant  $C$  such that if the cohomology class of  $\eta$  is nef, then  $\gamma^+\eta(x), g_n^+\eta(x) \leq C \|\eta\|$  for all  $n \in \mathbf{N}$  and  $x \in X$ .*

*Proof*

The first assertion follows from writing  $\eta$  as a linear combination of the basis elements  $\omega_1, \dots, \omega_N$  and applying Propositions 1.2 and 1.3.

By definition,  $K$  is a convex cone defined by finitely many linear inequalities, and since  $H \cap H_{\text{nef}}^{1,1}$  is compact, we can choose finitely many elements  $\eta_1, \dots, \eta_m \in K$  whose convex hull contains  $H \cap H_{\text{nef}}^{1,1}$ . The expression  $M(\eta)$  from Lemma 2.2 is a convex function of  $\eta$ , so we conclude that

$$C := \max_{1 \leq j \leq m} M(\eta_j) \geq \sup_{\eta \in H \cap H_{\text{nef}}^{1,1}} M(\eta)$$

gives the upper bound for  $\gamma^+\eta(x)$  when  $\eta$  is nef. Together with (13), the fact that  $\|f^{j*}\eta\| \leq C'\rho^j \|\eta\|$  allows us to extend the bound on  $\gamma^+\eta$  to an upper bound for  $g_n^+\eta$  which does not depend on  $n$ .  $\square$

**PROPOSITION 2.4**

*Given  $t > 1$ , there exists a constant  $C$  such that for any form  $\omega \in \Omega$  and any  $n \in \mathbf{N}$ ,*

$$\int |\gamma^+(\omega)| \circ f^j dV \leq Ct^j \|\omega\|.$$

*Proof*

From [DF, Section 6], we have that for any  $t > 1$ , there exist constants  $C_1, C_2 > 0$

such that

$$\text{Vol } f^{-n}(B_{I(f)}(r)) \leq C_1 r^{C_2/t^n}$$

for all  $n \in \mathbf{N}$  and all  $r > 0$ .

From Theorem 2.3, we have

$$\int |\gamma^+ \omega| \circ f^j dV \leq \|\omega\| \left( A + B \int |\log \text{dist}(f^j(x), I(f))| dV(x) \right).$$

Now the volume estimate above gives

$$\begin{aligned} \int |\log \text{dist}(f^j(x), I(f))| dV(x) &\leq A + \int_0^\infty \text{Vol } f^{-j}(B_{I(f)}(e^{-s})) ds \\ &\leq A + \int_0^\infty C_1 e^{-C_2 s/t^j} ds \leq Ct^j, \end{aligned}$$

which combines with the first estimate to finish the proof. □

Let us define  $\gamma^+ := \gamma^+ \omega^+ = \gamma^+ \theta^+$  so that we have

$$dd^c \gamma^+ = f^* \omega^+ - \rho \omega^+.$$

This form is smooth away from  $I(f)$ , so  $\gamma^+$  is smooth away from  $I(f)$ . And since the class of  $\omega^+$  is nef, Theorem 2.3 tells us that  $\gamma^+$  is bounded above. Adjusting the value of  $\gamma^+$  at points in  $I(f)$  if necessary, we may therefore assume that  $\gamma^+$  is upper semicontinuous. Thus the infinite sum

$$g^+ := \sum_{j=0}^\infty \frac{\gamma^+ \circ f^j}{\rho^j} \tag{15}$$

is essentially decreasing and defines an upper semicontinuous function (which is possibly  $-\infty$  at some points).

**THEOREM 2.5**

*The function  $g^+$  in (15) belongs to  $L^1(X)$ . Further, for any smooth, closed  $(1, 1)$ -form  $\eta$ , we have*

$$\lim_{n \rightarrow \infty} g_n^+ \eta = c \cdot g^+,$$

where  $c = \langle \eta, \theta^- \rangle$ , and the convergence takes place in  $L^1(X)$ .

*Proof*

Let us first consider the case  $\eta = \omega^+$ . Recall that  $\{f^{n*} \omega^+\} = \rho^n \theta^+$  and that  $\gamma^+$

depends only on the cohomology class. If we set  $\gamma^+ := \gamma^+ \omega^+ = \gamma^+ \theta^+$ , then  $\gamma^+ f^{j*} \omega^+ = \rho^j \gamma^+$ . Further, since  $\omega(\omega^+) = \omega^+$ , we have  $p(\omega^+) = 0$ . Thus

$$g_n^+ \omega^+ = \sum_{j=0}^{n-1} \frac{\gamma^+ \circ f^j}{\rho^j}.$$

If we take  $1 < t < \rho$ , then by Proposition 2.4 we have

$$\int |\gamma^+ \circ f^j| dV \leq C' t^j.$$

Thus the sequence  $\{g_n^+\}$  converges in  $L^1(X)$  to

$$g^+ := \sum_{j=0}^{\infty} \frac{\gamma^+ \circ f^j}{\rho^j}.$$

Since  $\gamma^+ \eta$  depends only on the cohomology class  $\{\eta\}$ , we may assume  $\eta \in \Omega$ . We use the decomposition (10):  $\eta = c\omega^+ + \eta^\perp$ . Thus

$$g_n^+ \eta^\perp = \rho^{-n} f^{n*} p \eta^\perp + \rho^{-n} \sum_{j=0}^{n-1} f^{(n-j-1)*} \gamma^+ (f^{j*} \eta^\perp).$$

Since  $\eta^\perp$  is smooth, so is  $p \eta^\perp$ , and so we have  $|f^{n*} p \eta^\perp| \leq C$  on  $X$ . Thus  $\rho^{-n} f^{n*} p \eta^\perp$  converges uniformly to zero. By Proposition 2.4 again and from the fact that  $\gamma^+ = \gamma^+ \circ \omega$ , we have

$$\int |g_n^+ \eta^\perp| dV \leq C \frac{\text{Vol}(X)}{\rho^n} + \frac{1}{\rho^n} \sum_{j=0}^{n-1} C t^{n-j-1} \|\omega f^{j*} \eta^\perp\|.$$

By Theorem 2.1, there is a constant  $C'$  such that  $\|\omega f^{j*} \eta^\perp\| \leq C' t^j$ . Thus

$$\int |g_n^+ \eta^\perp| dV \leq C \frac{\text{Vol}(X)}{\rho^n} + C C' n \frac{t^{n-1}}{\rho^n}.$$

This tends to zero as  $n \rightarrow \infty$ , and  $g_n^+ \eta$  is linear in  $\eta$ , so the theorem follows. □

**THEOREM 2.6**

*The current  $\mu^+ := \omega^+ + dd^c g^+$  has the following properties:*

- *for every smooth, closed (1, 1)-form  $\eta$  on  $X$ , we have*

$$\lim_{n \rightarrow \infty} \frac{f^{n*} \eta}{\rho^n} = \mu^+ \cdot \langle \eta, \theta^- \rangle;$$

- $\mu^+$  is positive;
- $f^*\mu^+ = \rho\mu^+$ .

*Proof*

By Theorem 2.5,

$$\lim_{n \rightarrow \infty} \frac{f^{n*}\eta}{\rho^n} = \lim_{n \rightarrow \infty} \frac{\omega f^{n*}\eta}{\rho^n} + dd^c \lim_{n \rightarrow \infty} g_n^+ \eta = \langle \eta, \theta^- \rangle (\omega^+ + dd^c g^+).$$

Taking  $\eta = \beta$  to be the Kähler form on  $X$ , we have  $\langle \beta, \theta^- \rangle = 1$ . Thus  $\mu^+ = \lim_{n \rightarrow \infty} \rho^{-n} f^{n*}\beta$  is a limit of positive currents and therefore positive. Since  $f^*$  acts continuously on positive, closed  $(1, 1)$ -currents, we also get

$$f^*\mu^+ = \lim_{n \rightarrow \infty} \rho^{-n} f^{(n+1)*}\beta = \rho\mu^+. \quad \square$$

The following observation of Favre (see [F2, Theorem 1]) is useful for us later.

COROLLARY 2.7

*The Lelong number  $v(\mu^+, x)$  vanishes for  $x \in X - \bigcup_{n \geq 1} I(f^n)$ .*

We close this section by discussing the extent to which  $\mu^+$  is invariant under bimeromorphic conjugacy.

PROPOSITION 2.8

*Let  $h : Y \rightarrow X$  be a proper modification of  $X$ . Suppose that both  $f : X \dashrightarrow X$  and  $f_Y := h^{-1} \circ f \circ h : Y \dashrightarrow Y$  satisfy (1) and (8). Let  $\mu^+$  and  $\mu_Y^+$  denote the invariant currents associated to  $f$  and  $f_Y$ . Then  $h_*\mu_Y^+$  is a positive multiple of  $\mu^+$ .*

*Proof*

By hypothesis,  $I(h) = \mathcal{C}(h) = \emptyset$ . Thus  $h^*\beta$  is a smooth, positive, and closed  $(1, 1)$ -form on  $X$ , and we compute

$$ch_*\mu_Y^+ = h_* \lim_{n \rightarrow \infty} \frac{f_Y^{n*}(h^*\beta)}{\rho^n} = \lim_{n \rightarrow \infty} \frac{h_* f_Y^{n*} h^*\beta}{\rho^n} = \lim_{n \rightarrow \infty} \frac{f^{n*}\beta}{\rho^n} = \mu^+.$$

The first and last equalities follow from Theorem 2.6. The second inequality follows from the continuity of  $h_*$  acting on positive, closed  $(1, 1)$ -currents, and the third equality is a consequence of the proof of [DF, Proposition 1.13]. Since  $\mu^+$  and  $h_*\mu_Y^+$  are positive and  $\mu^+$  is nontrivial, it follows that  $c > 0$ . □

PROPOSITION 2.9

*Let  $h : Y \rightarrow X$  be a bimeromorphic map. Suppose that both  $f$  and  $f_Y := h^{-1} \circ f \circ h :$*

$Y \circlearrowleft$  satisfy (1) and (8), and let  $\mu^+$  and  $\mu_Y^+$  be the associated invariant currents. Then  $h^*\mu^+ = c\mu_Y^+ + [V]$ , where  $c > 0$  and  $V$  is an effective divisor supported on  $\mathcal{C}(h)$ .

*Proof*

Let  $G$  be the desingularized graph of  $h$ , and let  $\pi_Y : G \rightarrow Y$ ,  $\pi_X : G \rightarrow X$  be the projections onto the first and second factors. That is,  $h = \pi_X \circ \pi_Y^{-1}$ . After blowing up points in  $G$  if necessary, we can assume that the common lift  $F : G \rightarrow G$  of  $f$  and  $f_Y$  to  $G$  satisfies (1). Let  $\nu^+$  denote the invariant current associated to  $F$ . Then by the previous proposition, we see that

$$\pi_{X*}\nu^+ = c_1\mu^+, \quad \pi_{Y*}\nu^+ = c_2\mu_Y^+$$

for constants  $c_1, c_2 > 0$ . Hence  $\pi_X^*\mu^+ - [V'] = c\pi_Y^*\mu_Y^+ - [V'']$ , where  $c > 0$  and  $V'$  and  $V''$  are effective divisors (with possibly noninteger coefficients) supported on  $\mathcal{C}(\pi_X)$  and  $\mathcal{C}(\pi_Y)$ , respectively. We apply the push-pull formula [DF, Theorem 3.3] to  $\pi_Y$  and conclude that

$$c\mu_Y^+ = c\pi_{Y*}\pi_Y^*\mu_Y^+ = \pi_{Y*}(\pi_X^*\mu^+ + [V''] - [V']) = h^*\mu^+ - [\pi_{Y*}V'].$$

Since  $V := \pi_{Y*}V'$  is an effective divisor supported on  $\mathcal{C}(h)$ , we are done. □

### 3. Energy

Let  $T$  be a positive, closed  $(1, 1)$ -current on  $X$ . Then  $T$  defines an inner product on the space of smooth, real functions on  $X$  via the formula

$$\mathcal{E}_T(\varphi, \psi) := \int d\varphi \wedge d^c\psi \wedge T.$$

We denote the seminorm associated with this inner product by

$$|\varphi|_T = (\mathcal{E}(\varphi, \varphi))^{1/2} = \left( \int d\varphi \wedge d^c\varphi \wedge T \right)^{1/2}.$$

We say that functions  $u_j \in C^\infty(X)$ ,  $j \geq 0$ , form a *regularizing sequence* for a function  $u$  if  $u_j$  decreases pointwise to  $u$  and  $dd^c u_j \geq -c\beta$  for some  $c > 0$  and all  $j$ . The limit  $u$  necessarily belongs to  $P$ , and, indeed, any  $u \in P$  admits a regularizing sequence (see [D1, Theorem 1.1]). We use the following property of a function  $u \in P$ :

$$\text{Every regularizing sequence } \{u_j\} \text{ for } u \text{ is Cauchy in } |\cdot|_T. \tag{16}$$

The union of two regularizing sequences is (essentially) a regularizing sequence. Thus if  $u$  satisfies (16), then all regularizing sequences define the same element of the completion with respect to  $|\cdot|_T$ . In particular, if  $u$  and  $v$  satisfy (16), then we may define  $\mathcal{E}_T(u, v)$  by taking the limit along any regularizing sequences.

The special case  $T = \beta$  is classical: condition (16) for  $T = \beta$  is equivalent to the condition  $\nabla u \in L^2$ .

Our principal use of condition (16) is to define  $(dd^c u) \wedge T$ . If  $u$  satisfies (16), then we may define  $(dd^c u) \wedge T$  as a distribution via the pairing

$$\psi \mapsto \langle dd^c u \wedge T, \psi \rangle := -\mathcal{E}_T(\psi, u).$$

It is evident that  $dd^c u \wedge T + c\beta \wedge T \geq 0$ , so  $dd^c u \wedge T$  is represented by a (signed) Borel measure. Further, since  $dd^c u_j \geq -c\beta$ , it also follows from (16) that  $dd^c u_j \wedge T$  converges to  $dd^c u \wedge T$  in the weak\* topology on the space of measures.

PROPOSITION 3.1

If  $u \in L^1(T \wedge \beta)$ , and if  $u$  satisfies (16), then  $(dd^c u) \wedge T = dd^c(uT)$ .

*Proof*

Let us remark first that if  $u \in L^1(T \wedge \beta)$ , then  $uT$  is a well-defined  $(1, 1)$ -current and thus  $dd^c(uT)$  is a well-defined current. If  $\{u_j\}$  is a sequence satisfying (16), then  $u_j T$  converges to  $uT$  weakly as currents. Thus  $dd^c(u_j T)$  converges to  $dd^c(uT)$ . Finally,  $(dd^c u_j) \wedge T = dd^c(u_j T)$  when  $u_j$  is smooth, and we have observed above that  $\lim_{j \rightarrow \infty} (dd^c u_j) \wedge T = (dd^c u) \wedge T$ . □

PROPOSITION 3.2

If  $u, v \in P$  both satisfy (16), and if  $v \in L^1(T \wedge \beta)$ , then  $v \in L^1(dd^c u \wedge T)$ .

*Proof*

Let  $\{u_j\}$  and  $\{v_k\}$  denote regularizing sequences for  $u$  and  $v$ . For fixed  $j$  and  $k$ , an integration by parts gives  $\int v_k dd^c u_j \wedge T = -\int dv_k \wedge d^c u_j \wedge T$ . Now  $dd^c u_j + c\beta \geq 0$ , so  $(dd^c u + c\beta) \wedge T$  defines (positive) Borel measure. Letting  $j \rightarrow \infty$ , we have

$$\int |v_k| (dd^c u + c\beta) \wedge T = -\mathcal{E}_T(v_k, u) + c \int |v_k| \beta \wedge T.$$

If we let  $k \rightarrow \infty$ , then the right-hand side stays bounded since  $v \in L^1(T \wedge \beta)$ , and thus  $v \in L^1(dd^c u \wedge T)$  by monotone convergence. □

The motivation for our work in the following sections is as follows. We show that  $g^+ \in L^1(T \wedge \beta)$  and that  $g^+$  satisfies (16) for the current  $T = \mu^- + \beta$ . It then follows that  $dd^c g^+ \wedge \mu^-$  is well defined, so the wedge product defines a (signed) measure

$$\mu = \mu^+ \wedge \mu^- = \omega^+ \wedge \mu^- + dd^c g^+ \wedge \mu^-, \tag{17}$$

and  $g^+ \in L^1(\mu)$ . Since  $\mu^\pm \geq 0$ , it follows that  $\mu$  is positive. The total mass of  $\mu$  is  $\int \omega^+ \wedge \omega^- = \langle \theta^+, \theta^- \rangle = 1$ , so  $\mu$  is a probability measure.

LEMMA 3.3

Let  $u, v \in C^\infty(X)$  satisfy  $dd^c u, dd^c v \geq -c\beta$ , and  $v \geq u$ . Then for any positive, closed  $(1, 1)$ -current  $T$ ,

$$\begin{aligned}\mathcal{E}_T(u, v) - \mathcal{E}_T(v, v) &\geq -c \int (v - u) \beta \wedge T, \\ \mathcal{E}_T(u, u) - \mathcal{E}_T(u, v) &\geq -c \int (v - u) \beta \wedge T.\end{aligned}$$

*Proof*

It is sufficient to prove the first inequality:

$$\begin{aligned}\mathcal{E}_T(u, v) - \mathcal{E}_T(v, v) &= \int d(u - v) \wedge d^c v \wedge T \\ &= \int (v - u) dd^c v \wedge T \geq -c \int (v - u) \beta \wedge T.\end{aligned}$$

Here we used Stokes's theorem to pass from the first line to the second line, and the inequality is obtained because  $v - u \geq 0$  and  $dd^c v \geq -c\beta$ .  $\square$

THEOREM 3.4

Suppose that  $T$  is a positive, closed  $(1, 1)$ -current and that  $u \in L^1(\beta \wedge T)$ . If there exists a regularizing sequence  $\{u_j\}$  for which  $\{|u_j|_T\}$  is bounded, then  $u$  satisfies (16).

*Proof*

By hypothesis, there exists  $c > 0$  such that  $dd^c u_j + c\beta \geq 0$  for all  $j$ . Now  $\beta \wedge T$  is a positive, finite Borel measure, and by the monotonicity of the sequence  $\{u_j\}$ , we have

$$\lim_{j \rightarrow \infty} \int |u_j - u| \beta \wedge T = \lim_{j, k \rightarrow \infty} \int |u_j - u_k| \beta \wedge T = 0.$$

It follows from Lemma 3.3 that for  $k \geq j$ , we have

$$\begin{aligned}\mathcal{E}_T(u_k, u_k) - \mathcal{E}_T(u_j, u_j) &\geq \mathcal{E}_T(u_k, u_k) - \mathcal{E}_T(u_k, u_j) + \mathcal{E}_T(u_k, u_j) - \mathcal{E}_T(u_j, u_j) \\ &\geq -2c \int |u_k - u_j| \beta \wedge T.\end{aligned}$$

Thus the sequence  $|u_k|_T = \mathcal{E}_T(u_k, u_k)^{1/2}$  is essentially increasing. Since we have assumed that it is also bounded, we conclude that  $\lim_{k \rightarrow \infty} |u_k|_T$  exists and is finite.

Now we observe that

$$|u_j - u_k|_T^2 = \mathcal{E}_T(u_k, u_k) - 2\mathcal{E}_T(u_j, u_k) + \mathcal{E}_T(u_j, u_j).$$

If  $k \geq j$ , then  $u_k \leq u_j$ , so by Lemma 3.3, we have  $\mathcal{E}_T(u_j, u_j) \leq \mathcal{E}_T(u_j, u_k) \leq \mathcal{E}_T(u_k, u_k)$ , modulo an error of size  $2c \int |u_k - u_j| \beta \wedge T$ . Thus

$$\lim_{j \rightarrow \infty} \mathcal{E}_T(u_j, u_j) = \lim_{j,k \rightarrow \infty} \mathcal{E}_T(u_j, u_k) = \lim_{k \rightarrow \infty} \mathcal{E}_T(u_k, u_k),$$

and so  $\lim_{j,k \rightarrow \infty} |u_j - u_k|_T = 0$ .

Now we show that (16) holds. Let  $\{v_j\}$  be any regularizing sequence for  $u$ . Since  $v_j$  is smooth, there exists  $k = k_j$  such that  $u_{k_j} \leq v_j$ . Thus by Lemma 3.3,  $|v_j|_T$  is essentially bounded by  $|u_{k_j}|_T$ . From the first part of the proof, it then follows that  $\{v_j\}$  is Cauchy.  $\square$

PROPOSITION 3.5

Suppose that  $u \in \tilde{P}(X, S)$  and that  $T$  is a positive, closed  $(1, 1)$ -current. Then  $u \in L^1(T \wedge \beta)$ .

*Proof*

First, recall from Jensen’s inequality that  $[\beta \wedge T](B_S(r)) \leq Cr^2$  for all  $r \geq 0$ . Since  $u(x) \geq A \log \text{dist}(x, S) + B$ , we have

$$[\beta \wedge T]\{|u| \geq t\} \leq [\beta \wedge T](B_S(e^{-(t+B)/A})) \leq C'e^{-2t/A}$$

for constants  $C, C' > 0$  and all  $t \geq 0$ . Therefore

$$\int |u| \beta \wedge T = \int_0^\infty [\beta \wedge T]\{|u| \geq t\} dt \leq C' \int_0^\infty r^{-2t/A} dt < \infty. \quad \square$$

THEOREM 3.6

Suppose that  $u \in \tilde{P}(X, S)$  and that  $T$  is a positive, closed  $(1, 1)$ -current with local potentials that are finite at each point in  $S$ . Then  $u$  satisfies (16), and so  $|u|_T := \lim_{j \rightarrow \infty} |u_j|_T$  is well defined and finite, and

$$|u|_T^2 = \int_{X-S} du \wedge d^c u \wedge T.$$

The expression  $du \wedge T$  defines a current on  $X - S$  which has finite mass. The (trivial) extension of  $du \wedge T$  to  $X$  is equal to the current  $d(uT)$ ; that is, for all smooth 1-forms  $\eta$  we have

$$\langle d(uT), \eta \rangle = \int_{X \setminus S} \eta \wedge du \wedge T.$$

*Proof*

First we show that  $u$  satisfies (16). Choose a function  $m \in C^\infty(\mathbf{R})$  which is convex, increasing, and equal to  $\max\{0, x\}$  outside a small neighborhood of zero. For all  $j \geq$

0, let  $u_j(p) = m(u(p) + j) - j \approx \max\{u, -j\}$ . Clearly,  $u_j$  is smooth and decreases to  $u$  pointwise on  $X$ . If  $dd^c u \geq -c\beta$ , then a quick computation using the fact that  $m' \leq 1$  verifies that  $dd^c u_j \geq -c\beta$  as well, with the same constant  $c$ . That is,  $u_j$  regularizes  $u$ . By Theorem 3.4, it suffices to show that  $|u_j|_T$  is bounded.

Let  $L \in C^\infty(X \setminus S)$  be a function satisfying  $L \leq u$  and  $L(z) = C(q) \log \|z\|$  with respect to local coordinates centered at each point  $q \in S$ . For each  $j \in \mathbb{N}$ , let  $L_j(z) = m(L(p) + j) - j$ . Then the above argument applied to  $L, L_j$  instead of  $u, u_j$  shows that  $L \in L^1(\beta \wedge T)$  and that  $L_j$  regularizes  $L$ . Since  $u_j \geq L_j$ , we have from Lemma 3.3 that

$$|u_j|_T \leq |L_j|_T + C \int (u_j - L_j) \beta \wedge T \leq |L_j|_T + C \int (\|u\|_\infty - L) \beta \wedge T \leq |L_j|_T + C$$

for every  $j \in \mathbb{N}$ . Hence our problem reduces to showing that  $\{|L_j|_T\}$  is bounded.

To do this, we can restrict attention to a coordinate neighborhood  $B_0(2)$  centered at  $q \in X$  and assume that  $L(z) = C \log \|z\|$  in these coordinates. We choose a smooth, radially symmetric, and compactly supported function  $\chi : B_0(2) \rightarrow [0, 1]$  such that  $\chi \equiv 1$  on  $B_0(1)$ . We choose a local potential  $v$  for  $T$  on  $B_0(2)$  and estimate

$$\int_{B_0(1)} dL_j(z) \wedge d^c L_j(z) \wedge T \leq \int \chi dL_j(z) \wedge d^c L_j(z) \wedge dd^c v.$$

Integrating by parts, we see that the right side is dominated by

$$\begin{aligned} & \left| \int v dd^c \chi \wedge dL_j \wedge d^c L_j \right| \\ & + 2 \left| \int v d\chi \wedge d^c L_j \wedge dd^c L_j \right| + \left| \int \chi v dd^c L_j \wedge dd^c L_j \right| \\ & = \left| \int v dd^c \chi \wedge dL \wedge d^c L \right| + 2 \left| \int v d\chi \wedge d^c L \wedge dd^c L \right| \\ & + \left| \int \chi v dd^c L_j \wedge dd^c L_j \right| = C + \int (-v) \chi dd^c L_j \wedge dd^c L_j \end{aligned}$$

for  $j$  large enough that  $L_j = L$  on  $\text{supp } d\chi$ . The measures  $\chi(dd^c L_j)^2$  are radially symmetric and convergent to a point mass at the origin as  $j \rightarrow \infty$ . Since  $v$  is subharmonic, we obtain

$$-\lim_{j \rightarrow \infty} \int \chi v dd^c L_j \wedge dd^c L_j = -v(0) < \infty.$$

Thus  $\{|L_j|_T\}$  is bounded, and (16) holds by Theorem 3.4.

The sequence  $\{du_j\}$  is bounded in  $L^2_T(X - S)$  and thus has a weak limit in  $L^2_T(X - S)$  which must be  $du$ . Further, we must have  $\int_{X - S} du \wedge d^c u \wedge T \leq \lim_{j \rightarrow \infty} |u_j|_T = |u|_T$ . It follows that  $du \wedge T$  defines a current on  $X - S$  of finite total mass, and the

trivial extension of this current to  $X$  is  $d(uT)$ . This proves the second equation in the statement of the theorem.

Finally, in order to prove the first equation in the statement of the theorem, it suffices to show that for  $x \in S$  we may choose  $r$  small enough that the integral  $\int_{B_x(r)} du_j \wedge d^c u_j \wedge T$  is arbitrarily small, uniformly in  $j$ . Arguing as above, we may assume that  $x = 0$  and replace  $u_j$  by the logarithm  $L(z)$ , so it suffices to show that  $\int_{B_x(r)} du_j \wedge d^c u_j \wedge T$  is uniformly small in  $j$ . In local coordinates near  $x = 0$ , we have  $dL \wedge d^c L \leq c\|z\|^{-2}\beta$ . We may identify the Laplacian of the local potential  $v$  for  $T$  as the measure  $\Delta v = T \wedge \beta$ . Since  $-\|z\|^{-2}$  is the Newtonian potential on  $\mathbf{R}^4$ , we may assume that  $v$  is given by convolution:  $v(0) = -v \star \|z\|^{-2}(0) = -\int \|z\|^{-2} \Delta v$ . Since  $v(0)$  is finite, it follows that  $\|z\|^{-2}$  is integrable with respect to the measure  $\Delta v$ . Thus the integral over  $B_x(r)$  may be taken uniformly small in  $j$ . □

**COROLLARY 3.7**

Let  $u \in \tilde{P}(X, S)$ , where  $S \cap I(f^{-1}) = \emptyset$ , and let  $T$  be a positive, closed  $(1, 1)$ -current with local potentials that are finite at each point in  $I(f) \cup f^{-1}(S)$ . Then  $|f^*u|_T$  is well defined, and

- $\int_{X-S} du \wedge d^c u \wedge f_*T < \infty$ ;
- $|f^*u|_T^2 = \int_{X-(I(f) \cup f^{-1}S)} d(f^*u) \wedge d^c(f^*u) \wedge T < \infty$ ;
- $|u|_{f_*T} = |f^*u|_T$ .

*Proof*

The hypotheses imply that  $f_*T$  does not charge  $\mathcal{C}(f^{-1}) = f(I(f))$  and that  $f_*T$  has local potentials that are finite at each point  $p \in S \subset f(f^{-1}(S))$ . The function  $f^*u$  is a difference of elements of  $\tilde{P}(X, f^{-1}(S) \cup I(f))$  by Proposition 1.3. So by Theorem 3.6, the integrals defining  $|u|_{f_*T}$  and  $|f^*u|_T$  are finite. We compute

$$\begin{aligned} |u|_{f_*T}^2 &= \int_{X-I(f^{-1})-\mathcal{C}(f^{-1})} du \wedge d^c u \wedge f_*T \\ &= \int_{X-I(f)-\mathcal{C}(f)} d(u \circ f) \wedge d^c(u \circ f) \wedge T \\ &= \int_{X-I(f)} d(u \circ f) \wedge d^c(u \circ f) \wedge T = |f^*u|_T^2. \end{aligned}$$

The first equality holds because  $f_*T$  charges neither points nor  $\mathcal{C}(f^{-1})$ . The second equality follows by the change of variables formula because  $f : X - I(f) - \mathcal{C}(f) \rightarrow X - I(f^{-1}) - \mathcal{C}(f^{-1})$  is a biholomorphism. The third equality is a consequence of the fact that  $u \circ f$  is constant on  $\mathcal{C}(f) - I(f)$ . Finally, the fourth equality holds because  $T$  does not charge points. □

**4. Invariant measure**

Up to this point, we have required that conditions (1) and (8) hold. We now impose two further conditions. The first of these is

$$\langle \theta^+, f(x) \rangle > 0 \quad \text{for every } x \in I(f)$$

and

$$\langle \theta^-, f^{-1}(y) \rangle > 0 \quad \text{for every } y \in I(f^{-1}). \tag{18}$$

Like condition (1), condition (18) may be thought of as a property of the underlying space  $X$  used to represent the map  $f$ ; it is shown in Proposition 4.1 that (18) may always be assumed to hold. Next, we consider condition (3) more carefully (see Theorems 4.3 and 4.6). After this, we assume for the rest of the paper that (3) holds, by which we mean implicitly that (1), (3), (8), and (18) all hold. The main results of this section are that if (3) holds, then the expression  $\mu$  in (17) is well defined (see Theorem 4.7) and invariant (see Theorem 4.11).

PROPOSITION 4.1

*If  $f : X \rightarrow X$  satisfies (1), we may blow down curves in  $X$  if necessary so that both (1) and (18) hold.*

*Proof*

Suppose to the contrary that  $\langle \theta^+, f(x) \rangle \leq 0$  for some  $x \in I$ . Then since  $\theta^+$  is nef, we have  $\langle \theta^+, V \rangle = 0$  for every component  $V \subset f(x)$ . From this and the Hodge index theorem on surfaces, we see that either  $\langle \theta^+, \theta^+ \rangle = 0$  or the intersection form is negative definite on  $f(x)$ . In the first case, [DF, Theorem 0.4] guarantees that after blowing down an appropriate curve,  $f$  conjugates to an automorphism and satisfies the conclusion of the proposition vacuously.

In the second case, we note that  $\pi_1^{-1}(x)$  constitutes a single connected component of  $\mathcal{C}(\pi_1)$ . We can therefore apply the argument of [DF, Proposition 1.7] to obtain a smooth rational curve of self-intersection  $-1$  in  $f(x)$ . After blowing this curve down, (1) still holds. However, the dimension of  $H^{1,1}(X)$  drops by one. If on the new surface we still have  $\langle \theta^\pm, f(x) \rangle = 0$  for some  $x \in I(f^{\pm 1})$ , then we can repeat this process. This cannot happen more than  $\dim H^{1,1}(X)$  times, so eventually we descend to a surface on which  $f$  satisfies both (1) and (18). □

The following is a companion to Corollary 2.7.

COROLLARY 4.2

*If  $f$  satisfies (18), then  $v(\mu^+, x) > 0$  for  $x \in I(f^n)$ ,  $n \geq 1$ .*

*Proof*

The proof uses the fact that  $\mu^+ = \rho^{-n} f^{n*} \mu^+$ , but it is otherwise identical to the proof of the second conclusion in Proposition 1.2. □

**THEOREM 4.3**

Suppose that (1), (8), and (18) hold. Then (3) holds if and only if the function  $g^+$  defined in (15) is finite at each point of  $I(f^{-1})$ . More generally, (3) implies that

$$\lim_{n \rightarrow \infty} (g_n^+ \eta)(p) = \langle \eta, \theta^- \rangle g^+(p) \tag{19}$$

for every smooth (1, 1)-form  $\eta$  and every  $p \in I(f^{-1})$ .

*Proof*

Remarking that  $f^* \omega^+ = \rho \omega^+ + dd^c \gamma^+$ , we see from Proposition 1.3 and Corollary 4.2 that

$$A \log \text{dist}(x, I(f)) - B \leq \gamma^+(x) \leq A' \log \text{dist}(x, I(f)) + B'$$

Replacing  $x$  by  $f^j x$  and summing, we see that  $g^+$  is bounded above and below by infinite sums of the form  $S := A \sum \rho^{-n} \log \text{dist}(f^n x, I(f)) + B$ . Thus  $g^+(x) > -\infty$  if and only if  $S$  is finite. Since  $I(f^{-1})$  is a finite set, we have

$$\text{dist}(f^n I(f^{-1}), I(f)) = \min_{x \in I(f^{-1})} \text{dist}(f^j x, I(f)),$$

and we see that (3) holds if and only if  $S$  is finite for all  $x \in I(f^{-1})$ .

Equation (19) is established using the same argument together with Theorem 2.3 and equations (10) and (11). □

The following theorem says that condition (3) is symmetric in  $f$  and  $f^{-1}$ ; we refer the reader to [Di1, Theorem 5.2] for a proof.

**THEOREM 4.4**

Suppose that (1), (8), and (18) hold. Then (3) holds if and only if

$$\sum_{j=0}^{\infty} \frac{\log \text{dist}(f^{-j}(I(f)), I(f^{-1}))}{\rho^j} < \infty. \tag{20}$$

Another important thing about condition (3) is that it tends to be satisfied by “most” birational maps. The referee pointed out the following result in this direction.

PROPOSITION 4.5

Let  $f : \mathbf{P}^2 \rightarrow \mathbf{P}^2$  be a birational map satisfying (1) and (3). Then for all  $A \in \text{PGL}(3, \mathbf{C})$  outside a pluripolar set, the map  $A \circ f$  satisfies (1) and (3).

*Proof*

$H^{1,1}(\mathbf{P}^2)$  is one-dimensional, so the spectral radius  $\rho$  of  $f^*$  must be a positive integer wherever (1) is satisfied, and we may suppose that  $\omega^+ = \beta$  is the Fubini-Study Kähler form. For each  $n \in \mathbf{N}$ , the condition  $f^n I(f^{-1}) \cap f^{-n} I(f) \neq \emptyset$  is algebraic, so (1) is satisfied outside a countable union of subvarieties of  $\text{PGL}(3, \mathbf{C})$ . Since the complement of these subvarieties is connected and  $\rho$  varies continuously, it follows that  $\rho$  is constant wherever (1) holds.

Let  $\pi : \mathbf{C}^3 - \{0\} \rightarrow \mathbf{P}^2$  be the holomorphic map sending complex lines through the origin onto points in  $\mathbf{P}^2$ . The map  $f$  lifts to a homogeneous map  $F : \mathbf{C}^3 \rightarrow \mathbf{C}^3$ , and  $\pi^* \omega^+ = dd^c \log \|z\|$ . Hence

$$\pi^*(A \circ f)^{n*} \omega^+ = dd^c \log \|(A \circ F)^n(z)\|$$

for all  $n \geq 0$ . The functions  $\log \|(A \circ F)^n(z)\|$  are plurisubharmonic in both  $A$  and  $z$ , and on each compact subset of  $\text{PGL}(3, \mathbf{C})$ , they are essentially decreasing in  $n$ . We let  $G$  denote their limit. For each fixed  $A$  such that  $A \circ f$  satisfies (1), we have that  $G(\cdot, A)$  is a potential for the lift  $\pi^* \mu^+$  of the invariant current  $\mu^+$  corresponding to  $A \circ f$ . In particular,  $G = \lim G_n$  is not identically  $-\infty$  on  $\mathbf{C}^3 \times \text{PGL}(3, \mathbf{C})$  and must instead be plurisubharmonic.

The map  $A \circ f$  satisfies condition (3) if and only if  $G(z) \neq -\infty$  for  $\pi(z) \in I(f^{-1})$ . On the other hand,

$$\{(z, A) \in \mathbf{C}^3 \times \text{PGL}(3, \mathbf{C}) : \pi(z) \in I((A \circ f)^{-1})\}$$

is an analytic variety, and on each of its irreducible components,  $G$  must be finite outside a pluripolar set or identically  $-\infty$ . Each component contains a point of the form  $(z, \text{id})$ , where  $z \neq 0$ , so our hypothesis on  $f$  implies that the latter possibility does not occur. That is, (3) is satisfied by  $A \circ f$  for all  $A$  outside a pluripolar set.  $\square$

PROPOSITION 4.6

If (3) holds, then there exists a constant  $C > 0$  such that for each  $\eta \in \Omega$  and every  $n \in \mathbf{N}$ ,

$$|f^{n*} \gamma^+ \eta|_{\beta + \mu^-} \leq C \rho^{n/2} \|\eta\|.$$

*Proof*

The function  $\gamma^+ \eta \circ f^n$  is a difference of functions in  $\tilde{P}(X, I(f^n))$  by Theorem 2.3, and  $f_*^n \beta$  is smooth away from  $I(f^{-n})$ , so by Corollary 3.7,  $f^{n*} \gamma^+ \eta$  is a difference

of functions that satisfy condition (16) for  $T = f_*^m \beta$ . Condition (3) together with Proposition 4.3 implies that  $f^{n*} \gamma^+ \eta$  is also a difference of functions satisfying (16) for  $T = \mu^-$ .

We have

$$|f^{n*} \gamma^+ \eta|_{\mu^-} = |\gamma^+ \eta|_{f_*^n \mu^-} = \rho^{n/2} |\gamma^+ \eta|_{\mu^-} \leq C \rho^{n/2} \|\eta\|.$$

The first equality uses Corollary 3.7, the second proceeds from the invariance of  $\mu$ , and the inequality follows from writing  $\eta$  as a linear combination of the basis elements  $\omega_1, \dots, \omega_n \in \Omega$  and noting that  $|\gamma^+ \omega_j|_{\mu^-} < \infty$  for each  $j$ .

Corollary 3.7 also gives

$$|f^{n*} \gamma^+ \eta|_{\beta} = \rho^{n/2} |\gamma^+ \eta|_{\rho^{-n} f_*^n \beta} \leq \rho^{n/2} \|\eta\| \max_k |\gamma^+ \omega_k|_{\rho^{-n} f_*^n \beta}.$$

It suffices to show that  $|\gamma^+ \omega_k|_{\rho^{-n} f_*^n \beta}$  is bounded. Using the notation from Section 2, we write

$$\rho^{-n} f_*^n \beta = \rho^{-n} \omega f_*^n \beta + dd^c g_n^- \beta.$$

Hence

$$|\gamma^+ \omega_k|_{\rho^{-n} f_*^n \beta}^2 \leq |\gamma^+ \omega_k|_{\rho^{-n} \omega f_*^n \beta}^2 + |\gamma^+ \omega_k|_{dd^c g_n^- \beta}^2. \tag{21}$$

The sequence  $\{\rho^{-n} \omega f_*^n \beta : n = 1, 2, 3, \dots\} \subset \Omega$  converges uniformly to  $c\omega^-$ , so the first term on the right-hand side is bounded.

Before we analyze the second term, we make some observations. Since  $\gamma^+ \omega_k \in P$  is smooth except for logarithmic singularities at  $I(f)$ , the wedge product yields a well-defined measure (see [D2])

$$(dd^c \gamma^+ \omega_k)^2 = \sum_{x \in I(f)} c_x \delta_x + (dd^c \gamma^+ \omega_k)^2|_{X-I(f)}.$$

We claim that the integral  $\int (g_n^- \beta) (dd^c \gamma^+ \omega_k)^2$  is uniformly bounded in  $n$  if (3) holds. To see this we use the projections  $\pi_1, \pi_2 : \Gamma \rightarrow X$  and the formula  $f^* \omega_k = \pi_{1*} \pi_2^* \omega_k$ . Since  $\pi_1 : \Gamma - \mathcal{C}(\pi_1) \rightarrow X - I(f)$  is biholomorphic, we may pull back by  $\pi_1$  to  $\Gamma$ , obtaining

$$\begin{aligned} \int_{X-I(f)} (g_n^- \beta) (dd^c \gamma^+ \omega_k)^2 &= \int_{X-I(f)} (g_n^- \beta) (f^* \omega - \omega f^* \omega_k)^2 \\ &= \int_{\Gamma} \pi_1^* (g_n^- \beta) (\pi_2^* \omega_k - \pi_1^* \omega f^* \omega_k)^2. \end{aligned}$$

Now  $\pi_2^* \omega_k - \pi_1^* \omega f^* \omega_k$  is a smooth  $(1, 1)$ -form on  $\Gamma$ . By Theorems 2.3 and 2.5,  $\{g_n^- \beta\}_{n \in \mathbb{N}}$  is uniformly bounded above and converges in  $L^1$  to  $g^-$ . In particular, no subsequence of  $\{\pi_1^* (g_n^- \beta)\}$  converges uniformly to  $-\infty$ . Also, since  $\beta$  is positive,

$dd^c \pi^*(g_n^- \beta) \geq -C \pi^* \beta$  for some  $C > 0$  and all  $n \in \mathbf{N}$ . Therefore a standard result (see [H, Theorem 4.1.9]) concerning sequences of subharmonic functions shows that the integral on the right side of the previous display is bounded independent of  $n$ . On the indeterminacy set itself, we have that

$$\int_{I(f)} (g_n^- \beta)(dd^c \gamma^+ \omega_k)^2 = \sum_{x \in I(f)} c_x (g_n^- \beta)(x)$$

is uniformly bounded by Theorems 4.3 and 4.4.

We now claim that

$$|\gamma^+ \omega_k|_{dd^c g_n^- \beta}^2 = \int d(\gamma^+ \omega_k) \wedge d^c(\gamma^+ \omega_k) \wedge dd^c g_n^- \beta = - \int (g_n^- \beta)(dd^c \gamma^+ \omega_k)^2.$$

In light of the previous paragraph, this implies that the second term in (21) is bounded. To see that the claim is true, replace  $\gamma^+ \omega_k$  by a regularizing sequence  $\gamma_j$ , which coincides with  $\gamma^+ \omega_k$  outside a neighborhood of  $I(f)$ . Performing two integrations by parts, we have

$$|\gamma_j|_{dd^c g_n^- \beta} = - \int (g_n^- \beta)(dd^c \gamma_j)^2.$$

As  $j \rightarrow \infty$ , the measures  $(dd^c \gamma_j)^2$  converge weakly to  $(dd^c \gamma^+ \omega_k)^2$ . Further, since  $g_n^- \beta$  is continuous in a neighborhood of  $I(f)$ , we may assume that it is continuous on the set where  $\gamma_j \neq \gamma^+ \omega_k$ . Letting  $j \rightarrow \infty$  therefore establishes our claim.  $\square$

Now we sharpen Theorem 2.5.

**THEOREM 4.7**

*If (3) holds, then  $g^+ \in L^1(\mu^- \wedge \beta)$ , and  $g^+$  satisfies (16) for  $T = \beta + \mu^-$ . In fact, if  $\eta$  is any smooth, closed (1, 1)-form on  $X$ , then*

$$\lim_{n \rightarrow \infty} |g_n^+ \eta - \langle \eta, \theta^- \rangle g^+|_{\beta + \mu^-} = 0.$$

*Proof*

First we show that for every smooth, closed (1, 1)-form  $\eta$ , the sequence  $\{g_n^+ \eta\}$  is Cauchy with respect to  $|\cdot|_{\beta + \mu^-}$ . We consider increasingly general forms  $\eta$ . If  $\eta = \omega^+$ , it follows from (15) and Proposition 4.5 that  $\{g_n^+ \omega^+\}$  is Cauchy. Now for  $\eta \in \Omega$ , we write  $\eta = c\omega^+ + \eta^\perp$  as in (10), where  $c = \langle \eta, \theta^- \rangle$ . By Theorem 2.1, we have  $\|\omega f^{j*} \eta^\perp\| \leq C_t t^j \|\eta^\perp\|$  for any  $t$  greater than 1. By Proposition 4.5 again, we have

$$|\rho^{-n} f^{(n-j-1)*} \gamma^+ f^{j*} \eta^\perp|_{\beta + \mu^-} \leq C \rho^{-n} \rho^{(n-j-1)/2} t^j.$$

Choosing  $1 < t < \sqrt{\rho}$ , we see from this estimate and (13) that  $|g_n^+ \eta^\perp|_{\beta + \mu^-}$  converges to zero. Since  $g_n^+ \eta$  is linear in  $\eta$ , we see that  $g_n^+ \eta$  satisfies (16) with  $T = \beta + \mu^-$ .

Finally, for a general, smooth, closed (1, 1)-form  $\eta$ , we see that  $g_n^+ \eta$  differs from  $g_n^+ \omega \eta$  by  $\rho^{-n} f^{n*} p \eta$ . The seminorm of this additional term is

$$|\rho^{-n} f^{n*} p \eta|_T^2 = \rho^{-2n} |p \eta|_{f_*^n T}^2 = \rho^{-n} |p \eta|_{\rho^{-n} f_*^n T}^2.$$

Since  $p \eta$  is smooth, and since  $\rho^{-n} f_*^n T$  converges to  $c \mu^-$ , it follows that  $|p \eta|_{\rho^{-n} f_*^n T}$  converges to  $|p \eta|_{c \mu^-}$ . Thus  $\rho^{-n} f^{n*} p \eta$  converges to zero, and we conclude that  $\{g_n^+ \eta\}$  is Cauchy.

Next, we show that  $g^+$  satisfies (16) with respect to the current  $T = \beta + \mu^-$ . We have from Theorems 2.4 and 2.6 that  $g_n^+ \rightarrow g^+$  in  $L^1(X)$  and  $g^+ \in P$ . The sequence  $\{g_n^+ \beta\}$  is uniformly bounded above everywhere on  $X$  by Theorem 2.6. Since  $dd^c g_n^+ \beta = \rho^{-n} f^{n*} \beta - \rho^{-n} \omega f^{n*} \beta$  and  $\rho^{-n} \omega f^{n*} \beta$  converges uniformly to  $\omega^+$ , we also have  $dd^c g_n^+ \beta \geq -c \beta$  for some constant  $c$  and all  $n \in \mathbf{N}$ . It follows (see [H, Theorem 4.1.9]) that  $\overline{\lim} g_n^+ \beta \leq g^+$  everywhere on  $X$ . Indeed, if  $\{u_n\}$  is a regularizing sequence for  $g^+$ , then we can pass to a subsequence and assume that  $u_n \geq g_n^+$  for all  $n \in \mathbf{N}$  large enough. As in the proof of Theorem 3.4, we can conclude that

$$|u_n|_{\beta + \mu^-} \leq |g_n^+|_{\beta + \mu^-} + C$$

for some constant  $C$  independent of  $n$ . By the first part of the proof, the sequence of energies  $\{|g_n^+|_{\beta + \mu^-}\}$  converges, so we conclude that  $\{|u_n|_{\beta + \mu^-}\}$  is bounded. Theorem 3.4 therefore implies that the limiting function  $g^+$  satisfies (16).

Theorem 4.4 and the paragraph above tell us that  $g^-$  satisfies (16) with  $T = \mu^+ + \beta$ . So if we set  $v = g^+$ ,  $u = g^-$ , and  $T = \beta$  in Proposition 3.2, then it follows that  $g^+ \in L^1(\mu^- \wedge \beta)$ . □

COROLLARY 4.8

If (3) holds, then expression (17) defines a probability measure  $\mu$  such that  $g^\pm \in L^1(\mu)$ .

*Proof*

This follows from Proposition 3.2 with  $v = u = g^+$  and  $T = \mu^-$ . □

Together with Theorem 4.4, the next corollary shows among other things that the roles of  $g^-$  and  $g^+$  can be reversed without changing the measure  $\mu$ . It is also the principal place in which we use the fact that  $g^\pm$  satisfies (16) with  $T = \beta$ .

COROLLARY 4.9

Suppose that (3) holds. Then for any smooth, closed (1, 1)-forms  $\eta_1, \eta_2$  on  $X$ , we have

$$\lim_{n,m \rightarrow \infty} \frac{f^{n*} \eta_1}{\rho^n} \wedge \frac{f_*^m \eta_2}{\rho^m} = \mu \cdot \langle \eta_1, \theta^- \rangle \langle \eta_2, \theta^+ \rangle$$

in the weak\* topology on Borel measures.

*Proof*

Without loss of generality, we may assume that  $\langle \eta_1, \theta^- \rangle = \langle \eta_2, \theta^+ \rangle = 1$ . We write

$$\mu_n^+ := \rho^{-n} f^{n*} \eta_1 = \omega_n^+ + dd^c \tilde{g}_n^+,$$

where  $\omega_n^+ = \omega \mu_n^+$  and  $dd^c \tilde{g}_n^+ = \mu_n^+ - \omega_n^+$ . We define similarly  $\mu_m^-, \tilde{g}_m^-$ , and  $\omega_m^-$ . Let  $\psi : X \rightarrow \mathbf{R}$  be a smooth test function. We show that

$$\int \psi \mu_n^+ \wedge \mu_m^- - \mu = \int \psi \mu_n^+ \wedge (\mu_m^- - \mu^-) + \int \psi (\mu_n^+ - \mu^+) \wedge \mu^-$$

is small for  $n, m$  large by dealing with each term on the right side separately.

The second term is easier because it depends only on  $n$ . Since  $\langle \eta_1, \theta^- \rangle = 1$ , it follows that  $\omega_n^+$  converges uniformly to  $\omega^+$ . Theorem 2.5 tells us, moreover, that  $\tilde{g}_n^+$  converges in  $L^1$  to  $g^+$ , and Theorem 4.7 tells us that  $\tilde{g}_n^+ \rightarrow g^+$  with respect to the energy seminorm  $|\cdot|_{\beta+\mu^-}$ . Thus

$$\int \psi (\mu_n^+ - \mu^+) \wedge \mu^- = \int \psi (\omega_n^+ - \omega^+) \wedge \mu^- - \int d\psi \wedge d^c (\tilde{g}_n^+ - g^+) \wedge \mu^- \rightarrow 0$$

as  $n \rightarrow \infty$ .

Turning to the first term, we have

$$\begin{aligned} \left| \int \psi \mu_n^+ \wedge (\mu_m^- - \mu^-) \right| &\leq \left| \int \psi \mu_n^+ \wedge (\omega_m^- - \omega^-) \right| \\ &\quad + \left| \int \psi \omega_n^+ \wedge (\mu_m^- - \mu^-) \right| \\ &\quad + \left| \int \psi dd^c \tilde{g}_n^+ \wedge dd^c (\tilde{g}_m^- - g^-) \right|. \end{aligned}$$

The first two terms on the right side of the inequality tend to zero as  $n$  and  $m$  increase because one factor in each integrand converges uniformly whereas the other tends weakly to zero. As for the third term on the right, we have

$$\begin{aligned} \left| \int \psi dd^c \tilde{g}_n^+ \wedge dd^c (\tilde{g}_m^- - g^-) \right| &= \left| \int dd^c \psi \wedge d\tilde{g}_n^+ \wedge d^c (\tilde{g}_m^- - g^-) \right| \\ &\leq |\psi|_{C^2} |\tilde{g}_n^+|_{\beta} |\tilde{g}_m^- - g^-|_{\beta} \rightarrow 0 \end{aligned}$$

as  $m \rightarrow \infty$  because  $|\tilde{g}_n^+|_{\beta}$  is bounded uniformly in  $n$  and  $|\tilde{g}_m^- - g^-|_{\beta} \rightarrow 0$ . □

**THEOREM 4.10**

*The measure  $\mu$  does not charge points.*

*Proof*

By Corollary 2.7, there is no point in  $X$  at which both  $\mu^+$  and  $\mu^-$  have positive Lelong number. So given  $x \in X$ , we may assume that  $\nu(\mu^-, x) = 0$ . Choose a local coordinate system such that  $x = 0$  and  $U$  is the unit ball, and choose a smooth, compactly supported function  $\chi : U \rightarrow [0, 1]$  such that  $\chi(z) = 1$  for  $\|z\|$  small enough. Set  $\chi_r(z) = \chi(r^{-1}z)$  for all  $0 < r < 1$ . We have

$$\begin{aligned} \mu(x) &\leq \int \chi_r \mu \leq \left| \int \chi_r \omega^+ \wedge \mu^- \right| + |\langle d\chi_r, d^c(g^+ \wedge \mu^-) \rangle| \\ &\leq C_1 r^2 + C_2 |\chi_r|_{\mu^-} \leq C_1 r^2 + \frac{C_2}{r^2} \int_{B_x(r)} \beta \wedge \mu^-. \end{aligned}$$

The last expression on the right converges to a multiple of  $\nu(\mu^-, x) = 0$  as  $r \rightarrow 0$ .  $\square$

Since  $\mu$  does not charge  $I(f)$ , its pushforward  $f_*\mu$  is well defined.

**THEOREM 4.11**

*The measure  $\mu$  is  $f$ -invariant.*

*Proof*

Let  $\psi : X \rightarrow \mathbf{R}$  be a continuous function. Then by definition and by Corollary 4.9,

$$\int \psi f_*\mu = \int (\psi \circ f) \mu = \lim_{n \rightarrow \infty} \int (\psi \circ f) \frac{f^{n*}\beta}{\rho^n} \wedge \frac{f_*\beta}{\rho^n}.$$

The measure  $f^{n*}\beta \wedge f_*\beta$  does not charge points because there is no point in  $X$  at which both factors are singular. Both factors are smooth at points outside a finite set, so this measure does not charge curves either. Thus

$$\begin{aligned} \int_X (\psi \circ f) \frac{f^{n*}\beta}{\rho^n} \wedge \frac{f_*\beta}{\rho^n} &= \int_{X - (\mathcal{C}(f) \cup I(f))} \psi \circ f \frac{f^{n*}\beta}{\rho^n} \wedge \frac{f_*\beta}{\rho^n} \\ &= \int_{X - (\mathcal{C}(f^{-1}) \cup I(f^{-1}))} \psi \frac{f^{(n-1)*}\beta}{\rho^{n-1}} \wedge \frac{f_*^{n+1}\beta}{\rho^{n+1}} \\ &= \int_X \psi \frac{f^{(n-1)*}\beta}{\rho^{n-1}} \wedge \frac{f_*^{n+1}\beta}{\rho^{n+1}}, \end{aligned}$$

where the second equality holds because  $f : X - \mathcal{C}(f) - I(f) \rightarrow X - \mathcal{C}(f^{-1}) - I(f^{-1})$  is a biholomorphism. If we let  $n \rightarrow \infty$ , then the last expression on the right converges to  $\int \psi \mu$ . We conclude that  $f_*\mu = \mu$ .  $\square$

**COROLLARY 4.12**

*The measure  $\mu$  does not charge  $\mathcal{C}(f)$  or  $\mathcal{C}(f^{-1})$ .*

*Proof*

We already know that  $\mu(I(f)) = 0$ . So by invariance,

$$\mu(\mathcal{C}(f)) = \mu(\mathcal{C}(f) - I(f)) = \mu(I(f^{-1})) = 0.$$

Similarly,  $\mu(\mathcal{C}(f^{-1})) = 0$ . □

### 5. Mixing

Let  $T$  be a positive, closed  $(1, 1)$ -current, and let  $J$  denote the operator on 1-forms induced by the complex structure. For a smooth 1-form  $\eta$ ,  $\eta \wedge J\eta$  is a positive  $(1, 1)$ -form, and we define the  $L^2(T)$ -seminorm

$$\|\eta\|_{L^2(T)}^2 := \int \eta \wedge J\eta \wedge T.$$

If  $\varphi \in L^1(T \wedge \beta)$ , then we may define the quantity

$$|\varphi|_T^{\natural} := \sup \left\{ \int \varphi \wedge d\eta \wedge T : \eta \text{ a smooth 1-form with } \|\eta\|_{L^2(T)} \leq 1 \right\};$$

$|\cdot|_T^{\natural}$  is a seminorm on the space  $\{\varphi \in L^1(T \wedge \beta) : |\varphi|_T^{\natural} < \infty\}$ . If  $\varphi$  is smooth, then we may take  $\eta = d^c \varphi$ , integrate by parts in the integral defining  $|\varphi|_T^{\natural}$ , and apply Cauchy's inequality to see that  $|\varphi|_T^{\natural} = |\varphi|_T$ . Thus  $|\cdot|_T^{\natural}$  extends  $|\cdot|_T$  from the space of smooth functions to a larger space.

LEMMA 5.1

Suppose that (3) holds. If  $\psi : X \rightarrow \mathbf{R}$  is a smooth function, then

$$|\psi \circ f^n|_{\mu^+}^{\natural} \leq \rho^{-n/2} |\psi|_{\mu^+}.$$

*Proof*

Recall that  $f^n : X - (\mathcal{C}(f^n) \cup I(f^n)) \rightarrow X - (\mathcal{C}(f^{-n}) \cup I(f^{-n}))$  is biholomorphic. The function  $f^{n*}\psi$  is smooth and bounded on  $X - I(f^n)$ , and  $\beta \wedge \mu^+$  puts no mass on  $I(f^n)$ , so  $f^{n*}\psi \in L^1(\beta \wedge \mu^+)$ . Since  $\beta \wedge \mu^+$  also puts no mass on  $\mathcal{C}(f^n)$ , it follows that if  $\eta$  is a smooth 1-form, then

$$\begin{aligned} - \int (f^{n*}\psi) d\eta \wedge \mu^+ &= - \int_{X - (\mathcal{C}(f^n) \cup I(f^n))} (f^{n*}\psi) d\eta \wedge \mu^+ \\ &= - \int_{X - (\mathcal{C}(f^n) \cup I(f^n))} f^{n*} \left( \psi \frac{\mu^+}{\rho^n} \right) \wedge d\eta. \end{aligned}$$

Now let  $\Gamma$  denote the (desingularized) graph of  $f^n$ , and let  $\pi_1, \pi_2 : \Gamma \rightarrow X$  be the associated projections. Thus  $f^n = \pi_2 \pi_1^{-1}$  and  $f^{n*} = \pi_{1*} \pi_2^*$ . Now  $\pi_1 : \Gamma - \mathcal{C}(\pi_1) \rightarrow$

$X - I(f^n)$  is a biholomorphism, so we may pull this last integral back under  $\pi_1^*$  to obtain an integral over  $\Gamma$ :

$$\begin{aligned} &= - \int_{\Gamma - \mathcal{C}(\pi_1)} \pi_2^*(\psi \wedge \rho^{-n} \mu^+) \wedge \pi_1^*(d\eta) \\ &= - \int_{\Gamma - \mathcal{C}(\pi_1)} \pi_2^*(\psi) \wedge \pi_1^*(d\eta) \wedge \pi_2^*(\rho^{-n} \mu^+). \end{aligned}$$

Now  $\pi_2^*\psi$  and  $\pi_1^*(d\eta)$  are smooth on  $\Gamma$ , and  $\pi_2^*\psi$  and  $\pi_1^*(d\eta)$  are smooth on  $\Gamma$ , and  $\pi_2^*(\rho^{-n} \mu^+)$  puts no mass on  $\mathcal{C}(\pi_1) = \pi_1^{-1}(I(f^n))$ , so we obtain

$$\begin{aligned} &= - \int_{\Gamma} \pi_2^*(\psi) \wedge \pi_1^*(d\eta) \wedge \pi_2^*(\rho^{-n} \mu^+) \\ &= \int_{\Gamma} \pi_2^*(d\psi) \wedge \pi_1^*(\eta) \wedge \pi_2^*(\rho^{-n} \mu^+). \end{aligned}$$

Applying the Schwarz inequality to this last term, we have

$$\begin{aligned} \left| \int (f^{n*} \psi) \wedge d\eta \wedge \mu^+ \right| &\leq \left( \int_{\Gamma} \pi_2^*(d\psi) \wedge \pi_2^*(d^c \psi) \wedge \pi_2^*(\rho^{-n} \mu^+) \right)^{1/2} \\ &\quad \times \left( \int_{\Gamma} \pi_1^*(\eta) \wedge J\pi_1^*(\eta) \wedge \pi_2^*(\rho^{-n} \mu^+) \right)^{1/2}. \end{aligned}$$

Again, since  $\pi_2^*(d\psi)$  and  $\pi_1^*(\eta)$  are smooth, there is no change if we integrate over just  $\Gamma - \mathcal{C}(\pi_1)$  or  $\Gamma - \mathcal{C}(\pi_2)$ . Pushing the last integral forward under  $\pi_{1*}$  and the first integral forward under  $\pi_{2*}$ , we obtain

$$\begin{aligned} \left| \int (f^{n*} \psi) \wedge d\eta \wedge \mu^+ \right| &\leq \left( \int_{X-I(f^n)} d\psi \wedge d^c \psi \wedge \rho^{-n} \mu^+ \right)^{1/2} \\ &\quad \times \left( \int \eta \wedge J\eta \wedge f^{n*}(\rho^{-n} \mu^+) \right)^{1/2} \\ &= \rho^{-n/2} |\psi|_{\mu^+} \|\eta\|_{L^2(\mu^+)}. \end{aligned}$$

Now we take the supremum over  $\eta$  with  $\|\eta\|_{L^2(\mu^+)} \leq 1$ , and the lemma follows.  $\square$

**THEOREM 5.2**

Let  $\psi : X \rightarrow \mathbf{R}$  be a smooth function. Then

$$\lim_{n \rightarrow \infty} (\psi \circ f^n) \mu^+ = \mu^+ \cdot \int \psi \mu.$$

*Proof*

We assume without loss of generality that  $0 \leq \psi \leq 1$ , so that  $0 \leq (\psi \circ f^n) \mu^+ \leq \mu^+$ . Thus every subsequence of  $\{(\psi \circ f^n) \mu^+\}$  has a subsubsequence converging to

a positive  $(1, 1)$ -current  $S$  dominated by  $\mu^+$ . To see that  $S$  is closed, let  $\eta$  be any smooth 1-form

$$|\langle \eta, dS \rangle| = \lim_{n_j \rightarrow \infty} |\langle -d\eta, (\psi \circ f^{n_j})\mu^+ \rangle| \leq \|\eta\|_{L^2(\mu^+)} |\psi \circ f^n|_{\mu^+}^{\frac{1}{2}}.$$

The right-hand side tends to zero by Lemma 5.1. The current  $\mu^+$  is extremal in its cohomology class (see [G, Theorem 5.1]), so  $S = c\mu^+$ . We conclude the proof by showing that  $c = \int \psi \mu$ .

Let  $\pi_1, \pi_2 : \Gamma \rightarrow X$  be as in the proof of Lemma 5.1; so  $f^{n*} = \pi_{1*}\pi_2^*$ . Since  $\beta$  is smooth and  $\mu^+$  puts no mass on  $\mathcal{C}(f^{-1}) \cup \mathcal{C}(f)$ , we may pull integrals up from  $X$  to  $\Gamma$  and push them back down as before:

$$\begin{aligned} \int_X (\psi \circ f^n) \mu^+ \wedge \beta &= \int_X (f^{n*} \psi) \wedge (\rho^{-n} f^{n*} \mu^+) \wedge \beta \\ &= \int_{\Gamma} (\pi_1^* \psi) \wedge (\rho^{-n} \pi_1^* \mu^+) \wedge \pi_2^* \beta \\ &= \int_X \psi \wedge \pi_{1*} \left( \pi_2^* \beta \wedge \frac{\pi_1^* \mu^+}{\rho^n} \right) = \int_X \psi \frac{f_*^n \beta}{\rho^n} \wedge \mu^+. \end{aligned}$$

Note that the second and fourth equalities depend on the fact that none of the measures involved charge curves. Finally,

$$\begin{aligned} c \int \mu^+ \wedge \beta &= \int S \wedge \beta = \lim_{n \rightarrow \infty} \int (\psi \circ f^n) \mu^+ \wedge \beta \\ &= \left( \int \psi \mu \right) \langle \beta, \theta^+ \rangle = \left( \int \psi \mu \right) \int \mu^+ \wedge \beta. \end{aligned}$$

The third equality follows from the previous display and Corollary 4.9; the fourth follows from the fact that  $\theta^+$  is the cohomology class of  $\mu^+$ . □

**THEOREM 5.3**

*The measure  $\mu$  is mixing for  $f$ .*

*Proof*

We show that for any smooth functions  $\varphi$  and  $\psi$ , we have

$$\lim_{n \rightarrow \infty} \int (\psi \circ f^n) \cdot \varphi \mu = \int \psi \mu \cdot \int \varphi \mu.$$

Let  $\{g_j^-\}$  be a regularizing sequence for  $g^-$ . By Theorem 5.2, we have

$$\lim_{n \rightarrow \infty} \int (\psi \circ f^n) \mu^+ \wedge \varphi \wedge (\omega^- + dd^c g_j^-) = \int \varphi \mu^+ \wedge (\omega^- + dd^c g_j^-) \cdot \int \psi \mu. \tag{22}$$

By Theorem 4.7,  $\mu^+ \wedge (\omega^- + dd^c g_j^-)$  converges to  $\mu$  as  $j \rightarrow \infty$ . Thus it suffices to show that we may interchange the limits  $n \rightarrow \infty$  and  $j \rightarrow \infty$  in the left-hand integral in (22). For this, it suffices to show that

$$\lim_{j,k \rightarrow \infty} \int (\psi \circ f^n) \varphi \mu^+ \wedge dd^c (g_j^- - g_k^-) = 0 \tag{23}$$

holds uniformly in  $n$ . Integration by parts gives

$$\begin{aligned} & \int (\psi \circ f^n) \varphi \mu^+ \wedge dd^c (g_j^- - g_k^-) \\ &= - \int \varphi \wedge d(\psi \circ f^n) \wedge d^c (g_j^- - g_k^-) \wedge \mu^+ \\ & \quad - \int (\psi \circ f^n) d\varphi \wedge d^c (g_j^- - g_k^-) \wedge \mu^+ = I + II. \end{aligned}$$

The first integral is estimated by

$$|I| \leq |\psi \circ f^n|_{\mu^+}^{\sharp} \|\varphi d^c (g_j^- - g_k^-)\|_{L^2(\mu^+)} \leq |\psi \circ f^n|_{\mu^+}^{\sharp} \|\varphi\|_{L^\infty} |g_j^- - g_k^-|_{\mu^+}.$$

This term converges to zero as  $j, k \rightarrow \infty$  because  $|\psi \circ f^n|_{\mu^+}^{\sharp}$  tends to zero by Lemma 5.1 and because  $\{g_j^-\}$  is Cauchy for  $|\cdot|_{\mu^+}$  by Theorem 4.7. The second term is estimated by

$$|II| \leq \left( \int (\psi \circ f^n)^2 d\varphi \wedge d^c \varphi \wedge \mu^+ \right)^{1/2} |g_j^- - g_k^-|_{\mu^+} \leq \|\psi\|_{L^\infty} |\varphi|_{\mu^+} |g_j^- - g_k^-|_{\mu^+},$$

which converges to zero because  $\{g_j^-\}$  is Cauchy. □

We can now strengthen Corollary 4.12.

**COROLLARY 5.4**

*Every compact curve in  $X$  has zero  $\mu$ -measure.*

*Proof*

Let  $V$  be a compact irreducible curve. By Corollary 4.12, we can assume that  $V$  is not critical for any iterate of  $f$ . If  $V$  is not fixed by any iterate of  $f$ , then  $f^n(V) \cap f^m(V)$  is finite for any  $n \neq m$ . Hence

$$\infty > \mu(X) \geq \mu\left(\bigcup_{n=0}^{\infty} f^n(V)\right) = \sum_{n=0}^{\infty} \mu(f^n(V)) = \sum_{n=0}^{\infty} \mu(V),$$

which can happen only if  $\mu(V) = 0$ .

Now suppose that  $V = f^n(V)$  is fixed by an iterate of  $f$ . Since  $\mu$  is mixing, we see that  $V$  has either full or zero  $\mu$ -measure. In the former case, it would follow that  $f^n|_V$  lifts to an automorphism of a compact Riemann surface (the desingularization of  $V$ ) with a nontrivial mixing invariant measure. No such automorphism exists in dimension one, so we must have  $\mu(V) = 0$ . □

**THEOREM 5.5**

*Any bimeromorphic map  $h : X \rightarrow Y$  is defined at  $\mu$  a.e. point. The pushforward  $h_*\mu$  is a probability measure, does not charge compact curves in  $Y$ , and is invariant and mixing for  $g := h^{-1} \circ f \circ h : Y \rightarrow Y$ . If  $g$  satisfies conditions (3), and  $\mu_g^+$  and  $\mu_g^-$  denote the associated currents, then  $h_*\mu = \mu_g^+ \wedge \mu_g^-$ .*

*Proof*

Let  $\tilde{h}$  denote the restriction of  $h$  to  $X - (I(h) \cup \mathcal{C}(h))$ . All but the last conclusion are immediate from Corollary 5.4 and the fact that  $\tilde{h} : X - (I(h) \cup \mathcal{C}(h)) \rightarrow Y - (I(h^{-1}) \cup \mathcal{C}(h^{-1}))$  is a biholomorphism. Let us use the notation  $\tilde{\cdot}$  to denote the restriction of a current or measure on  $Y$  to  $Y - (I(h^{-1}) \cup \mathcal{C}(h^{-1}))$ . With this notation, it follows from Corollary 2.9 that  $\tilde{h}^* \tilde{\mu}_g^+$  is equal to the restriction of  $\mu^+$  to  $X - (I(h) \cup \mathcal{C}(h))$ . Thus, since  $h_* = (\tilde{h}^*)^{-1}$  on  $X - (I(h) \cup \mathcal{C}(h))$ , we have  $h_*\mu = \tilde{h}_*\mu = \tilde{h}_*\mu^+ \wedge \tilde{h}_*\mu^- = \mu_g^+ \wedge \mu_g^-$ . □

**6. Lyapunov exponents**

In this section we show that  $\mu$  is hyperbolic (see Theorem 6.4). In order to do this, we first show that the Lyapunov exponents are finite  $\mu$  a.e. Next, we give an estimate on  $\mu$  of a ball with a certain mapping property (see Proposition 6.3). Then, using the machinery of Pesin theory, we convert this estimate into a proof of Theorem 6.4.

**PROPOSITION 6.1**

*The quantities  $\log^+ \|Df\|$ ,  $\log^+ \|(Df)^{-1}\|$ , and  $\log^+ \|D^2f\|$  are  $\mu$ -integrable.*

*Proof*

It suffices to consider a neighborhood of a point  $x \in I(f)$ ; let  $z$  be a local coordinate system such that  $x$  corresponds to  $z = 0$ . By Proposition 1.1, we have  $\log^+ \|Df\| \leq C |\log \|z\||$ . Now if  $v = \log \|z\|$ ,  $u = g^+$ , and  $T = \mu^-$ , we may apply Theorems 4.7 and 3.6 and Proposition 3.2 to conclude that  $v$ , and thus  $\log^+ \|Df\|$ , is  $\mu$ -integrable. Likewise,  $\log^+ \|D(f^{-1})\|$  is integrable with respect to  $\mu$ . Since  $\mu$  is  $f$ -invariant, this gives the  $\mu$ -integrability of  $\log^+ \|(Df)^{-1}\|$ .

By the Cauchy estimates applied to the entries of  $Df$ , we have  $\log \|D^2f\| \leq C' |\log \|z\||$ , so  $\log \|D^2f\|$  is  $\mu$ -integrable. □

Proposition 6.1 allows us to invoke Oseledec’s theorem (see [KH, Theorem S.2.9]) to conclude that the following holds.

PROPOSITION 6.2

*The limits*

$$\chi^+ = \lim_{n \rightarrow \infty} \frac{1}{n} \log \|Df_x^n\|, \quad \chi^- = - \lim_{n \rightarrow \infty} \frac{1}{n} \log \|Df_x^{-n}\|$$

exist and are finite at  $\mu$  a.e. point  $x \in X$ . Since  $\mu$  is ergodic, these limits are a.e. independent of  $x$ .

For  $v \in T_x X - \{0\}$ , the Lyapunov exponent of  $f$  at  $x$  in the direction  $v$  is defined as the limit

$$\chi(x, v) = \lim_{n \rightarrow \infty} \frac{1}{n} \log |Df_x^n v|,$$

provided that this limit exists. If  $\chi^+ \neq \chi^-$ , it is a consequence of the Oseledec theorem that there is an  $f$ -invariant splitting  $T_x X = E_x^s \oplus E_x^u$  for  $\mu$  a.e.  $x$ , and

$$\chi^\pm = \chi(x, v) = \lim_{n \rightarrow \infty} \frac{1}{n} \log |Df_x^n v|$$

for  $v \in E_x^{u/s} - \{0\}$ .

PROPOSITION 6.3

There exist  $C < \infty$  and  $R > 0$  such that if  $f^n(B_x(2r)) \subset B_y(R)$  for some  $n \geq 0$ , then

$$\mu(B_x(r)) \leq Cp^{-n/2}.$$

*Proof*

From the invariance of  $\mu$  and Lemma 5.1, we obtain

$$\begin{aligned} \left| \int \varphi \mu \right| &= \left| \int (\varphi \circ f^{-n}) \mu \right| \\ &\leq \left| \int (\varphi \circ f^{-n}) \omega^+ \wedge \mu^- \right| + \left| \int d(\varphi \circ f^{-n}) \wedge d^c g^+ \wedge \mu^- \right| \\ &\leq \frac{1}{\rho^n} \left| \int \varphi \wedge (f^{n*} \omega^+) \wedge \mu^- \right| + |g^+|_{\mu^-} |\varphi \circ f^{-n}|_{\mu^-} \\ &\leq \frac{1}{\rho^n} \left| \int \varphi \wedge (f^{n*} \omega^+) \wedge \mu^- \right| + \frac{C|\varphi|_{\mu^-}}{\rho^{n/2}} \end{aligned}$$

for any  $\varphi \in C^\infty(X)$ . Now let us take  $\varphi$  to be bounded above and below by the characteristic functions for  $B_x(2r)$  and  $B_x(r)$ , respectively. We may choose  $\varphi$  such that  $\|\varphi\|_{C^1}^2, \|\varphi\|_{C^2} \leq Cr^{-2}$  for some constant  $C$  independent of  $x$  and  $r$ .

Let us choose  $R > 0$  small enough that there is a local potential  $u$  for  $\omega^+$  on  $B_y(R)$ . Since  $\omega^+$  is smooth, we may assume that the  $L^\infty$ -norm of  $u$  is bounded above independently of  $y$ . We use the local potential  $dd^c u = \omega^+$  on the first right-hand term in the inequality above and integrate by parts twice to find

$$\begin{aligned} \left| \int \varphi \mu \right| &\leq \frac{1}{\rho^n} \left| \int_{B_\rho(2r)} (u \circ f^n) dd^c \varphi \wedge \mu^- \right| + \frac{C}{\rho^{n/2}} \left( \int_{B_\rho(2r)} d\varphi \wedge d^c \varphi \wedge \mu^- \right)^{1/2} \\ &\leq \frac{C_1}{r^2 \rho^n} \int_{B_\rho(2r)} \beta \wedge \mu^- + \frac{C_2}{r \rho^{n/2}} \left( \int_{B_\rho(2r)} \beta \wedge \mu^- \right)^{1/2} \\ &\leq \frac{C_1}{\rho^n} + \frac{C_2}{\rho^{n/2}} \leq \frac{C}{\rho^{n/2}}, \end{aligned}$$

where  $C$  does not depend on  $x, y$ , or  $r$ . □

**THEOREM 6.4**

*The Lyapunov exponents satisfy*

$$\chi^- \leq -\frac{\log \rho}{8} < 0 < \frac{\log \rho}{8} \leq \chi^+.$$

*In particular,  $\mu$  is hyperbolic of saddle type.*

*Proof*

Pesin theory provides us with the following setup (see [KH, Theorem S.3.1]). For any  $\epsilon > 0$ , there exist measurable radius and distortion functions  $r, A : X \rightarrow \mathbf{R}^+$ , and a constant  $c > 0$  with the following properties. For  $\mu$  a.e.  $x \in X$ , there is an embedding  $\psi_x : B_0(r(x)) \rightarrow X$  such that

- $\psi_x(\mathbf{0}) = x$ ;
- both  $r$  and  $A$  are “ $\epsilon$ -slowly varying”; that is,  $-\epsilon < \log A(fx)/A(x), \log r(fx)/r(x) < \epsilon$ ;
- $c \operatorname{dist}(\psi_x(a), \psi_x(b)) \leq \operatorname{dist}(a, b) \leq A(x) \operatorname{dist}(\psi_x(a), \psi_x(b))$  for  $a, b \in B_x(r(x))$ ;
- if  $f_x = \psi_{f(x)}^{-1} \circ f \circ \psi_x$ , then  $D_0 f_x$  is a diagonal matrix with diagonal entries  $e^{\chi_1}, e^{\chi_2}$  satisfying

$$|\operatorname{Re} \chi_1 - \chi^+|, |\operatorname{Re} \chi_2 - \chi^-| < \epsilon;$$

- $\|f_x - D_0 f_x\|_{C^1} \leq \epsilon$  on the domain of  $f_x$ .
- With this notation, we set  $r_n(x) = e^{-n(\chi^+ + 3\epsilon)} cr(x)/A(x)$ . By the properties above, it follows that for  $1 \leq j \leq n$  we have

$$f^j|_{B_x(r_n(x))} = \psi_{f^j(x)} \circ f_{f^{j-1}(x)} \circ \cdots \circ f_x \circ \psi_x^{-1}.$$

That is, by keeping track of the diameters of the successive images of  $B_x(r_n(x))$ , one sees that each stage of the composition on the right side is well defined and that, moreover,

$$f^n B_x(r_n(x)) \subset B_{f^n(x)}(r(f^n(x))).$$

Therefore Lemma 6.3 gives us the bound

$$\mu(B_x(r_n(x))) \leq C\rho^{-n/2}.$$

Now Lusin's theorem provides us with a compact subset  $K \subset \text{supp } \mu$  such that  $\mu(K) > 1/2$  and on which  $r$  and  $A$  vary continuously. Thus, for all  $x \in K$ , the radius of  $B_n(x)$  is bounded below by  $Ce^{-n(\chi^+ + 3\epsilon)}$ . We can therefore choose  $m \leq Ce^{4n(\chi^+ + 3\epsilon)}$  points  $x_1, \dots, x_m \in K$  such that  $K \subset \bigcup_{j=1}^m B_{x_j}(r_n(x_j))$ . Using this cover, we estimate

$$\frac{1}{2} < \mu(K) \leq \sum_{j=1}^m \mu(B_{x_j}(r_n(x_j))) \leq Ce^{4n(\chi^+ + 3\epsilon)} \rho^{-n/2}.$$

Letting  $n$  tend to infinity and then letting  $\epsilon$  tend to zero yields

$$1 < e^{4\chi^+} \rho^{-1/2},$$

which yields the estimate of Theorem 6.4. □

Using Theorem 6.4 and the fact that  $\mu$  is mixing, one can apply a contraction mapping argument (see, e.g., [Di3, Section 8]) to obtain the following.

#### COROLLARY 6.5

*The support of  $\mu$  is contained in the closure of the saddle periodic points of  $\mu$ .*

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