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DYNAMICS OF RATIONAL SURFACE AUTOMORPHISMS: ROTATION DOMAINS

By ERIC BEDFORD and KYOUNGHEE KIM

Abstract. A Fatou component is said to be a rotation domain if the automorphism induces a torus action on it. We construct a rational surface automorphism with positive entropy and a rotation domain which contains both a curve of fixed points and isolated fixed points. This Fatou component cannot be imbedded into complex Euclidean space, so we introduce a global linear model space and show that it can be globally linearized in this model.

Introduction. Let X denote a compact complex surface, and let f be a (biholomorphic) automorphism of X . The regular part of the dynamics of f occurs on the Fatou set $\mathcal{F}(f) \subset X$, where the forward iterates are equicontinuous. As in [BS, U], we call a connected component $U \subset \mathcal{F}(f)$ a *rotation domain of rank d* if $f^p(U) = U$ for some $p \geq 1$, and $f^p|_U$ generates a (real torus) \mathbf{T}^d -action on U . In dimension 1, rotation domains correspond to Siegel disks or Herman rings, which have a (circle) \mathbf{T}^1 -action. Here we consider surface automorphisms with the property that the induced map f^* on $H^2(X, \mathbf{R})$ has an eigenvalue greater than one. This is equivalent to the condition that f have positive entropy (see [C2]).

Let us consider generally the possibilities for Fatou sets of surface automorphisms. If X is a complex 2-torus, then an automorphism is affine. Positive entropy implies that the eigenvalues of the linear part are $|\lambda_1| < 1 < |\lambda_2|$, and in this case the Fatou set is empty. A second possibility is given by $K3$ surfaces (or their unramified quotients called Enriques surfaces). Since there is an invariant volume form, the only possible Fatou components are rotation domains (see Section 1). McMullen [M1] has shown the existence of non-algebraic $K3$ surfaces with rotation domains of rank 2 (see also [O]).

By Cantat [C1], the only other possibilities for compact surfaces with automorphisms of positive entropy are rational surfaces. In fact, by [BK2] there exist arbitrarily high dimensional families of rational surfaces which carry automorphisms with positive entropy. By definition, a rational surface is birationally (or bimeromorphically) equivalent to \mathbf{P}^2 , and by Nagata [N, Theorem 5] we may assume that it is obtained by iterated blowups of \mathbf{P}^2 . Rotation domains of rank 1 and 2, as

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well as attracting/repelling basins are known to occur for rational surface automorphisms (see [M2, BK1]).

In this paper we show that a positive entropy automorphism can have a rotation domain which is large in the sense that it contains a curve of fixed points as well as isolated fixed points. To be more precise, we will use the birational maps of the plane which are defined in affine coordinates (x, y) by

$$(0.1) \quad f(x, y) = (y, -\delta x + cy + y^{-1}),$$

with c and δ to be specified. We choose δ to be a root of the polynomial

$$(0.2) \quad \chi_{n,m}(t) = \frac{t(t^{nm} - 1)(t^n - 2t^{n-1} + 1)}{(t^n - 1)(t - 1)} + 1.$$

For $1 \leq j \leq n - 1$, $(j, n) = 1$, we set $c = 2\sqrt{\delta} \cos(j\pi/n)$. Although there is apparent ambiguity in the choice of $\pm\sqrt{\delta}$, the set of values obtained as we range over all possible j will be the same. Let Σ_0 denote the line at infinity in \mathbf{P}^2 . With this choice of c , the restriction $f|_{\Sigma_0}$ has period n , so the n th iterate f^n fixes the line at infinity Σ_0 pointwise. Let $p_0 = \Sigma_0 \cap \{x = 0\}$, and let $p_s := f^s(p_0)$, $0 \leq s \leq n - 1$ denote its orbit. We construct a complex algebraic surface $\pi : X \rightarrow \mathbf{P}^2$ by performing iterated blowups to level 3. First we let \mathcal{F}_s^1 denote the exceptional divisor obtained by blowing up $p_s \in \Sigma_0$, $0 \leq s \leq n - 1$, and we let \mathcal{F}_s^2 denote the exceptional divisor obtained by blowing up a point $q_s = \mathcal{F}_s^1 \cap L_s$ where L_s is the strict transform of the line in \mathbf{P}^2 joining the origin and p_s . In Theorem 2.3, we show that with this choice of δ and c , we may blow up points $r_{s,\ell} \in \mathcal{F}_s^2$, $0 \leq s \leq n - 1$, $1 \leq \ell \leq m$, so that the induced map $f_X := \pi^{-1} \circ f \circ \pi$ is an automorphism of X . We use Σ_0 to denote both the line at infinity in \mathbf{P}^2 and its strict transform in X .

THEOREM A. *Let $n \geq 4$, $m \geq 1$, or if $n = 3$, $m \geq 2$, and let δ , c , f , and X be as above. Then f is a positive entropy automorphism of the rational surface X . If $|\delta| = 1$, $h := f^n$ has a rotation domain U which contains a curve of fixed points Σ_0 as well as invariant curves $\mathcal{F}_0^1, \dots, \mathcal{F}_{n-1}^1$. The domain U is a union of invariant (Siegel) disks on each of which h acts as an irrational rotation.*

The geometry of this rotation domain is illustrated in Figure 1. Here r indicates a general point of Σ_0 , which is the center of a Siegel disk S_r for h . By \mathcal{F}_s^j we denote the exceptional divisor at level j over a point p_s , $0 \leq s \leq n - 1$. Theorem 3.3 shows that, in addition, the two fixed points of f in \mathbf{C}^2 can be centers of rank 2 rotation domains, in which case U is not dense in X .

Linearization is a useful technique to give the existence of rotation domains, but it is a local technique. In order to understand the global nature of the Fatou component U , we introduce a global model. We start with the linear map $L = \lambda^{-1}I$ on \mathbf{C}^2 , which is scalar times the identity transformation, with λ chosen as in (3.3). L defines a holomorphic map of \mathbf{P}^2 which fixes the line at infinity Σ_0 , and the

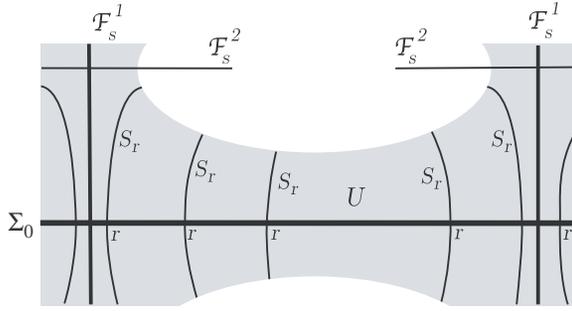


Figure 1. Rotation domain: one parameter family of Siegel disks S_r .

multiplier in the normal direction is λ . We define a new manifold $\pi : X_L \rightarrow \mathbf{P}^2$ by blowing up the points $p_s \in \Sigma_0$ and then the points $q_s \in \mathcal{F}_s^1$, the p_s and q_s being the same points that are blown up to make X . After this, however, the procedure differs from the construction of X : we blow up the second fixed point of $L|_{\mathcal{F}_s^2}$ (the other fixed point is $\mathcal{F}_s^1 \cap \mathcal{F}_s^2$). At each stage, the centers of blowup are fixed by L , so L extends to an automorphism of X_L , and (L, X_L) is our linear model space. This rotation domain can be linearized globally on this model space:

THEOREM B. *There is a domain $\Omega \subset X_L$ and a biholomorphic conjugacy $\Phi : U \rightarrow \Omega$ taking (h, U) to (L, Ω) . In particular, h has no periodic points in $U - \pi^{-1}\Sigma_0$. Further, $\pi(\Omega) - \Sigma_0$ is a pseudoconvex, circled domain in \mathbf{C}^2 which is complete at infinity.*

X_L is well suited to the example h since it “contains” the whole rotation domain. On the other hand, we may replace the curve of fixed points, Σ_0 , by a single point O . This is because Σ_0 has self-intersection $1 - n < 0$, so we may blow it down to obtain a singular space \check{X}_L . We may represent the quotient space \check{X}_L locally in a neighborhood of O as a neighborhood of 0 in the variety $\{x = (x_0, \dots, x_{n-1}) \in \mathbf{C}^n : x_j x_k = x_\ell x_m, \forall j + k = \ell + m\}$ (see [L, p. 54]). In the x -coordinates near 0 , the induced map \check{L} becomes multiplication by λ .

By Proposition 2.4, we may also blow down the variety $\Gamma = \Sigma_0 \cup \bigcup_s (\mathcal{F}_s^1 \cup \mathcal{F}_s^2)$ to a point, so that $\check{X} := X/\Gamma$ is a complex space. There is an induced map \check{f} on \check{X} , and the variety Γ becomes a fixed point \check{O} for \check{f} . \check{O} will be a normal singular point, so \check{f} will be holomorphic at \check{O} . However, the 2-form $dx \wedge dy$ is holomorphic on $X - \Gamma$ but does not extend to be holomorphic on X , so this singularity is not a quotient singularity (see [Du]).

If \mathbf{C}^2/σ is a quotient singularity, then a linear map commuting with σ can be used to give a rotation domain with 0 as a fixed point. If we blow up 0 and resolve the quotient singularity, then we obtain a rotation domain with an invariant curve. The purpose of the preceding paragraph is to show that the rotation domains constructed below are not, even locally, of this form.

This paper is organized as follows: Section 1 discusses rotation domains generally and global linear models. Section 2 develops a number of the properties of the maps (0.1). Section 3 treats linearization at the fixed points of h in \mathbb{C}^2 , which are non-resonant, and Σ_0 is shown to be in the Fatou set. In Section 4 the resonant fixed points $\mathcal{F}_s^1 \cap \mathcal{F}_s^2$ are linearized and shown to belong to the same Fatou component as Σ_0 ; Theorem A is a consequence of Theorem 4.9. Section 5 gives the global linearization, and Theorem 5.1 yields Theorem B.

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1. Rotation domains. In this section we consider an automorphism f of a general compact, complex manifold \mathcal{M} of arbitrary dimension. We assume throughout that f has infinite order, which is to say that f^n is not the identity map if $n \neq 0$. Recall that the (forward) Fatou set consists of all points which have neighborhoods U' such that the restrictions of the forward iterates $\{f^n|_{U'}, n \geq 0\}$ form a normal family. Let U denote an f -invariant, connected component of the Fatou set. We define the set of all normal limits of subsequences

$$\mathcal{G} = \mathcal{G}(U) := \{g : U \rightarrow \overline{U}, g = \lim f^{n_j}, n_j \rightarrow \infty\}$$

and we consider the condition

(*) $\text{For each } g \in \mathcal{G}(U), g(U) \text{ contains an open set.}$

PROPOSITION 1.1. *If f preserves a smooth volume form, then every Fatou component satisfies (*). If (*) holds, then \mathcal{G} is a subgroup of $\text{Aut}(U)$.*

Proof. Suppose that g is a normal limit of f^{n_j} . The jacobian determinant of f^{n_j} has modulus one, and so this holds for g . Thus g is an open mapping, and $g(U) \subset U$, and g satisfies (*). Next, suppose that $\varphi = \lim_{j \rightarrow \infty} f^{n_j} \in \mathcal{G}$. Passing to a subsequence, we may assume that both $m_j = n_{j+1} - n_j$ and $p_j = n_{j+1} - 2n_j$ converge to $+\infty$ as $j \rightarrow \infty$. Passing to further subsequences, we may suppose that there is convergence: $f^{m_j} \rightarrow g$ and $f^{p_j} \rightarrow h$. Since $f^{m_j} \circ f^{n_j} = f^{n_{j+1}}$, we may use (*) to pass to the limit and conclude that $g \circ \varphi = \varphi$ on an open subset of U , so that g is the identity element. Similarly, $f^{p_j} \circ f^{n_j} = f^{m_j}$, which by (*) converges to $h \circ \varphi = g$ on an open subset of U . Thus h is the inverse of φ . We conclude that $\varphi \in \text{Aut}(U)$. Now it is evident that \mathcal{G} is closed under composition, so \mathcal{G} is a subgroup of $\text{Aut}(U)$. □

Thus if (*) holds, then U is a Siegel domain in the terminology of Fornaess and Sibony [FS]. By Proposition 1.2, we have a group action $\mathcal{G} \times U \rightarrow U$. Since the iterates are a normal family, it follows that \mathcal{G} is a compact group in the compact-open

topology. Now we may apply the proof of a Theorem of H. Cartan (as presented, for instance, in [N, Chapter IV]) to conclude:

THEOREM 1.2. *If (*) holds, then \mathcal{G} is a compact, abelian, Lie group, and the action of \mathcal{G} on U is real analytic.*

We let \mathcal{G}_0 denote the connected component of the identity in \mathcal{G} . Since \mathcal{G} is a compact, infinite abelian Lie group, \mathcal{G}_0 is a torus of positive dimension d . Thus if (*) holds, then there is a real torus acting on U , and if there is such a torus action, we call U a *rotation domain of rank d* .

THEOREM 1.3. *If U is a rotation domain, then the rank satisfies $d \leq 2 \dim_{\mathbb{C}} \mathcal{M}$. If $d = 2 \dim_{\mathbb{C}} \mathcal{M}$, then $U = \mathcal{M}$, and \mathcal{M} is a torus.*

Proof. Let T denote the torus acting on U , and let $z_0 \in U$ be a point where the orbit $T \cdot z_0$ has maximal dimension. If the dimension d of T is greater than the real dimension of \mathcal{M} , then the stabilizer subgroup $S_{z_0} = \{g \in T : gz_0 = z_0\}$ will have positive dimension. We note that S_{z_0} is closed, and closed subgroups are rigid in T . Since S_z depends continuously on z for z near z_0 , we conclude that $S = S_z$ is independent of z . It follows that for $g \in S$ we have $gz = z$ for z near z_0 , and thus S acts trivially on U . This is a contradiction since we have taken T as a subgroup of $\text{Aut}(U)$.

Now suppose $d = 2 \dim_{\mathbb{C}} \mathcal{M}$. Since the Lie algebra of \mathcal{G}_0 is abelian, we see that a generic orbit $T \cdot z_0$ will be a d -torus. Thus we must have $T \cdot z_0 = \mathcal{M}$, and so $U = \mathcal{M}$. □

THEOREM 1.4. *If U is a rotation domain, then it is pseudoconvex.*

Proof. Pseudoconvexity is a local property of the boundary. The Lie algebra of \mathcal{G} is generated by holomorphic vector fields. For a boundary point $p \in \partial U$, we may write a vector field locally in terms of analytic functions $\sum a_j \partial_{z_j}$. If U is not pseudoconvex, then there will be a coordinate neighborhood on which the a_j have analytic continuations to a larger set. So the vector field, and thus the torus action, extends to a larger open set $\tilde{U} \supset U$. The larger set \tilde{U} , however, belongs to the Fatou set, which contradicts the fact that U is a Fatou component. □

We may regard \mathbb{C}^2 as both an \mathbf{R} -linear and a \mathbf{C} -linear vector space. Every 2-dimensional \mathbf{R} -linear subspace $S \subset \mathbb{C}^2$ is either complex, or it contains no nonzero complex-linear subspace. In the second case it is said to be totally real.

THEOREM 1.5. *Let X denote a compact, Kähler surface, and let f be an automorphism of X with positive entropy. Then we have $d \leq 2$. If $d = 2$, then the generic orbit of \mathcal{G}_0 is a totally real 2-torus, which means that the tangent space to the orbit is totally real at each point. If $d = 1$, then there is a holomorphic vector field \mathcal{V} , and each orbit of \mathcal{V} is invariant under f .*

Proof. Since f has positive entropy, $U \neq X$. Thus by Theorem 1.3, $d \leq 3$. Now suppose that $d = 3$. We may suppose that the Lie algebra of \mathcal{G}_0 is generated by $\partial/\partial\theta_j$, $j = 1, 2, 3$. Let \mathcal{V}_j be the vector field on U induced by $\partial/\partial\theta_j$. Now choose a generic point $z_0 \in U$, so that the orbit $\mathbf{T}^3 \cdot z_0$ is a 3-torus. If T denotes the tangent space of the orbit at z_0 , then $H_{z_0} = T \cap JT$ is the unique complex subspace of T . Let $Z \neq 0$ denote a $(1, 0)$ vector in H_{z_0} . Since \mathbf{T}^3 acts on U by biholomorphic maps, it follows that Z generates a $(1, 0)$ vector field on $\mathbf{T}^3 \cdot z_0$. Let ν denote the \mathbf{T}^3 -invariant probability measure on $\mathbf{T}^3 \cdot z_0$. Now, since the vectors $\partial/\partial\theta_j$ commute, it follows that the current $S := iZ \wedge \bar{Z}\nu$ is a positive, closed $(1, 1)$ -current with support on $\mathbf{T}^3 \cdot z_0$. We may move z_0 to a nearby point z'_0 so that $\mathbf{T}^3 \cdot z_0$ is disjoint from $\mathbf{T}^3 \cdot z'_0$. We have an invariant $(1, 0)$ -form Z' on this orbit and a corresponding positive, closed $(1, 1)$ -current S' . Let $\{S\} = \{S'\} \in H^{1,1}$ represent the associated cohomology class. Since the supports of S and S' are disjoint, we have $\{S\}^2 = 0$.

On the other hand, since f has positive entropy, there are a cohomology class $\theta_+ \in H^{1,1}$ and a $\lambda > 1$ such that $f^*\theta_+ = \lambda\theta_+$. By [DF] we must have $\theta_+ \cdot \theta_+ = 0$. Let ω_+ be a smooth $(1, 1)$ -form representing this cohomology class. We may take the limit $T_+ = \lim_{n \rightarrow \infty} \lambda^{-n} f^{*n}\omega_+$ and obtain a current which represents the cohomology class θ_+ . However, since the $\{f^n, n \geq 0\}$ are a normal family on U , it follows that $T_+ = 0$ on U . We conclude that $\theta_+ \cdot S = 0$. However, this makes a 2-dimensional linear subspace of $\{v : v \cdot v = 0\}$, which contradicts the Hodge Index Theorem. Thus $d \leq 2$.

Now suppose that $d = 2$, and let \mathcal{V}_1 and \mathcal{V}_2 be vector fields which generate the Lie algebra of \mathcal{G}_0 . For generic $p \in X$, the span of these vector fields will have real dimension 2. If the span of these vector fields at a point p is not a complex 1-dimensional subspace of the tangent space TX , then the \mathcal{G}_0 -orbit will be a totally real 2-torus. Otherwise, if there is an open set where it is complex, we may repeat the argument above. (Actually, the argument above shows that an orbit of a complex tangency is isolated.) Thus the generic \mathcal{G}_0 -orbit of a point of U must be a totally real 2-torus.

Finally, if $d = 1$, then there is a (holomorphic) vector field \mathcal{V} which generates the Lie algebra. That is, \mathcal{V} generates a foliation of U by Riemann surfaces, and the real part of \mathcal{V} generates the action of \mathcal{G}_0 . In particular, each leaf is invariant under \mathcal{G}_0 . □

Let us remark that there is always a semi-global model of a torus action since we may linearize in the neighborhood of a totally real orbit. More precisely, suppose that a 2-torus T acts on a complex surface, and z_0 is a point whose orbit $T \cdot z_0$ is a totally real 2-torus. Then there is a neighborhood Ω of T and an imbedding Φ of Ω into \mathbf{C}^2 such that $\Phi(\Omega)$ is a Reinhardt domain, then the T -action is taken to the standard \mathbf{T}^2 -action on \mathbf{C}^2 (see [BBD]). This does not require the T -action to have a fixed point.

In Sections 3 and 4 we will use local linearization to show that certain fixed points belong to the Fatou set. The converse is easier: linearization is always

possible at a fixed point inside the Fatou set. We will recall the statement of an easily proved result about domains in \mathbf{C}^k (see [H]), in which Φ is defined on all of U .

PROPOSITION 1.6. *Suppose that $U \subset \mathbf{C}^k$ is bounded and invariant under a holomorphic map h . Suppose that $z^0 = 0$ is a fixed point for h ; let A denote the differential of h at z^0 , and suppose A is unitary. Then $\Phi = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} A^{-n} h^n$ defines a holomorphic map $\Phi : U \rightarrow \mathbf{C}^k$ such that $\Phi \circ h = A \circ \Phi$.*

Global Linear Model. If U is not contained in \mathbf{C}^2 , however, we must define a global model if we want to have a global linearization. Let us give some examples of (zero entropy) maps which illustrate some possibilities for global rotation. We start with the linear map on \mathbf{P}^2 which is given as $M[t : x : y] = [t : \mu_1 x : \mu_2 y]$. M has three fixed points on \mathbf{P}^2 : $[1 : 0 : 0]$, $[0 : 1 : 0]$, and $[0 : 0 : 1]$. The multipliers at $[1 : 0 : 0] = (0, 0) \in \mathbf{C}^2$ are $\{\mu_1, \mu_2\}$, the multipliers at $[0 : 1 : 0]$ (the point where the x -axis intersects the line at infinity) are $\{\mu_1^{-1}, \mu_2/\mu_1\}$, and at $[0 : 0 : 1]$, where the y -axis intersects the line at infinity, the multipliers are $\{\mu_2^{-1}, \mu_1/\mu_2\}$. This is shown on the left hand side of Figure 2. The fixed points are marked; the axes are X and Y , and the multipliers at the fixed points are indicated.

Rank 2. We suppose now that $|\mu_1| = |\mu_2| = 1$, so M generates a torus action if μ_1 and μ_2 are not both roots of unity. If $\mu_1^{j_1} \mu_2^{j_2} \neq 1$ for all $j_1, j_2 \in \mathbf{Z}$, $(j_1, j_2) \neq (0, 0)$, then μ_1 and μ_2 are said to be multiplicatively independent, and in this case M generates a \mathbf{T}^2 action on \mathbf{P}^2 .

Rank 1. In the case of multiplicative dependence, we have a \mathbf{T}^1 action. We may suppose that $\mu_1 = t^p$, $\mu_2 = t^q$, where t is not a root of unity, and $(p, q) = 1$. Thus $\mu_1^q \mu_2^{-p} = 1$. The standard (p, q) -action acts on a point $(x, y) \in \mathbf{C}^2$ according to $\mathbf{T}^1 \ni \theta \mapsto (e^{ip\theta} x, e^{iq\theta} y)$. M preserves the curves $\{x^q = cy^p\}$ for any fixed $c \in \mathbf{C}$. We say that $\{\mu_1, \mu_2\}$ are *resonant* if $\mu_1^{k_1} \mu_2^{k_2} = \mu_s$ with $s = 1$ or 2 , and $k_1, k_2 \geq 0$ with $k_1 + k_2 \geq 2$. There is a special case where the multipliers are $\{1, t\}$, but otherwise in the resonant case, we have a (p, q) -action with $pq < 0$, which means that only two of the invariant curves pass through the origin. In the non-resonant case, all of the invariant curves pass through the origin.

If $p > q > 0$, then the fixed point $[0 : 1 : 0]$ will be non-resonant, while the fixed point $[0 : 0 : 1]$ will be resonant. If $p = q = 1$, then $(0, 0)$ is non-resonant, but the whole line at infinity Σ_0 is fixed, with multipliers $\{1, t^{-1}\}$, so all points of the fixed line are resonant.

Blow up. Now let $\pi : Z \rightarrow \mathbf{P}^2$ be \mathbf{P}^2 blown up at a fixed point p with multipliers $\{\nu_1, \nu_2\}$, corresponding to directions X and Y , respectively. This will induce the map \tilde{M} on Z . We denote the resulting exceptional divisor by P . We suppose that $\nu_1 \neq \nu_2$, so $\tilde{M}|_P$ will have two fixed points. Since ν_1 is the multiplier in the direction X , the multipliers at the new fixed point $X \cap P$ will be ν_1 in the direction X and ν_2/ν_1 in the direction P . Similarly, the multipliers at $Y \cap P$ will be

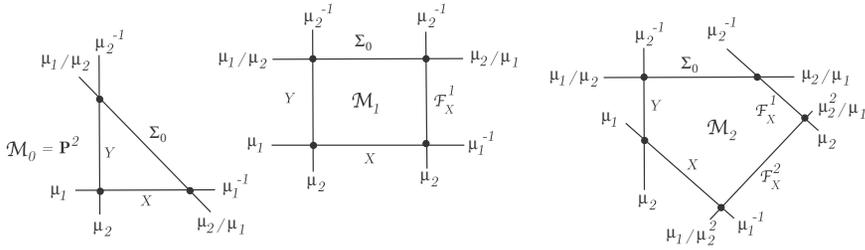


Figure 2. Global linear models \mathcal{M}_j at stage j of blowup; fixed points and multipliers.

$\{\nu_2, \nu_1/\nu_2\}$. The diagram in the middle of Figure 2 shows the invariant curves and their multipliers after the intersection point $\Sigma_0 \cap X$ is blown up; the exceptional divisor is denoted \mathcal{F}_X^1 . The right hand side of Figure 2 shows the space obtained after the further blow up of the point $X \cap \mathcal{F}_X^1$. The multipliers at the fixed points are determined by the reasoning described above. We may repeat this process of blowing up fixed points of \tilde{M} and obtain a map with an arbitrary number of fixed points. In the case of a rank 1 rotation (that is, $\mu_1 = \mu_2$) this produces both resonant and non-resonant fixed points.

2. Rational automorphisms. Here we present a family of automorphisms which are different from the ones given in [BK1, BK2, M2, D]. They are determined by a choice of m, n, δ , and j , and for different choices of these parameters, we get maps that are not birationally equivalent. (This is because if we change (m, n) , we change the dynamical degree; the number δ is the rotation number around Σ_0 ; and j/n is the rotation number within Σ_0 .) Let us imbed \mathbf{C}^2 into \mathbf{P}^2 via the map $(x, y) \mapsto [1 : x : y]$, and let us use coordinates $t = x_0, x = x_1, y = x_2$. Let δ be a root of the polynomial $\chi_{n,m}$ in (0.2) with $n \geq 4, m \geq 1$ or $n \geq 3, m \geq 2$ and $\delta^3 \neq -1$, and let $f(x, y)$ be a map of the form (0.1). In homogeneous coordinates on \mathbf{P}^2 , f takes the form

$$(2.1) \quad f[t : x : y] = [ty : y^2 : -\delta xy + cy^2 + t^2].$$

The exceptional curve for f is $\Sigma_2 = \{y = 0\}$, and $\Sigma_1 = \{x = 0\}$ is the exceptional curve for f^{-1} .

$$f : \Sigma_2 \mapsto e_2 = [0 : 0 : 1], \quad f^{-1} : \Sigma_1 \mapsto e_1 = [0 : 1 : 0]$$

Since $f[0 : 1 : w] = [0 : 1 : c - \delta/w]$, the line at infinity $\Sigma_0 = \{t = 0\}$ is invariant and $f|_{\Sigma_0}$ is equivalent to the linear fractional transformation $g(w) := c - \delta/w$. Let us set

$$C_n(\delta) := \{2\sqrt{\delta} \cos(j\pi/n) : 0 < j < n, (j, n) = 1\}.$$

Note that since $2\sqrt{\delta} \cos(j\pi/n) = 2(-\sqrt{\delta}) \cos((n-j)\pi/n)$, $C_n(\delta)$ does not depend on the choice of $\pm\sqrt{\delta}$.

LEMMA 2.1. *If $c \in C_n(\delta)$ then $f|_{\Sigma_0}$ is periodic with period n .*

Proof. Let $c = 2\sqrt{\delta} \cos(j\pi/n)$ for some j relatively prime to n . The fixed points of g , $w_{\text{fix}} = (c \pm \sqrt{c^2 - 4\delta})/2$. Since $g'(w_{\text{fix}}) = \delta/w_{\text{fix}}^2$ it follows from our choices of c and δ that $(g'(w_{\text{fix}}))^n = 1$, which means that the n th iterate of g is the identity. \square

Thus $c \in C_n(\delta)$ if and only if $g^{n-1}(c) = g^{-1}(c) = \infty$. Let us use the notation $\omega_s = g^{s-1}(c)$ for $1 \leq s \leq n-1$, that is $f^s e_2 = [0 : 1 : \omega_s]$, $1 \leq s \leq n-1$.

LEMMA 2.2. *Suppose $c \in C_n(\delta)$. For $1 \leq j \leq n-2$, $\omega_j \omega_{n-1-j} = \delta$. If n is even, then $\omega_1 \cdots \omega_{n-2} = \delta^{(n-2)/2}$. If n is odd, then we let $\omega_* = \omega_{(n-1)/2}$. In this case, we have $\omega_1 \cdots \omega_{n-2} = \delta^{(n-3)/2} \omega_*$ and $\omega_*^2 = \delta$.*

Proof. Note that $g^{-1}(w) = \delta/(c-w)$. Since $\omega_{n-1} = 0$, we have $\omega_{n-2} = \delta/c$. It follows that $\omega_1 \omega_{n-2} = c \cdot \delta/c = \delta$. If $\omega_j \omega_{n-1-j} = \delta$ then $\omega_{j+1} = c - \delta/\omega_j$, $\omega_{n-1-(j+1)} = g^{-1}(\delta/\omega_j) = \delta/(c - \delta/\omega_j)$, and thus $\omega_{j+1} \omega_{n-1-(j+1)} = \delta$. The Lemma follows by induction on j . \square

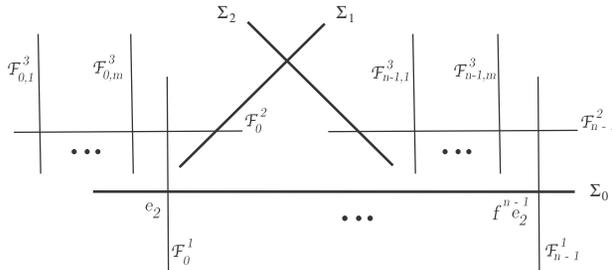


Figure 3. Construction of X .

For any δ and any $c \in C_n(\delta)$, we construct the manifold $\pi_1 : X^1 \rightarrow \mathbf{P}^2$ by blowing up the set of n points in the line at infinity $f^s e_2, 0 \leq s \leq n-1$. Let $\mathcal{F}_s^1 := \pi_1^{-1}(f^s e_2)$ denote the exceptional divisor. For \mathcal{F}_0^1 we will use $\pi_1(s_1, \eta_1)_0 = [s_1 : s_1 \eta_1 : 1]$ and for $\mathcal{F}_s^1, 1 \leq s \leq n-1$, we use the coordinate chart $\pi_1(s_1, \eta_1)_s = [s_1 : 1 : s_1 \eta_1 + \omega_s]$. The induced map f_{X^1} maps Σ_2 to a point in the exceptional divisor \mathcal{F}_0^1 :

$$f_{X^1}[1 : x : y] = (s_1, \eta_1)_0 = \left(\frac{y}{1 - \delta xy + cy^2}, y \right)_0.$$

Letting $y \rightarrow 0$, we see that

$$f_{X^1}(\Sigma_2) = (0, 0)_0 = \mathcal{F}_0^1 \cap \{x = 0\}.$$

Similarly we see that

$$f_{X^1}^{-1}(\Sigma_1) = (0, 0)_{n-1} = \mathcal{F}_{n-1}^1 \cap \{y = 0\}.$$

If we map forward by f_{X^1} from \mathcal{F}_s^1 to \mathcal{F}_{s+1}^1 , we have

$$(2.2) \quad f_{X^1} : \begin{cases} \mathcal{F}_0^1 \ni (0, \eta_1)_0 \mapsto (0, -\delta\eta_1)_1 \in \mathcal{F}_1^1 \\ \mathcal{F}_s^1 \ni (0, \eta_1)_s \mapsto (0, \delta\eta_1/\omega_s)_{s+1} \in \mathcal{F}_{s+1}^1, & 1 \leq s \leq n-2 \\ \mathcal{F}_{n-1}^1 \ni (0, \eta_1)_{n-1} \mapsto (0, \eta_1)_0 \in \mathcal{F}_0^1. \end{cases}$$

Thus the orbit of the exceptional line Σ_2 lands at the point of indeterminacy:

$$(2.3) \quad f_{X^1} : \mathcal{F}_s^1 \cap L_s \ni (0, 0)_s \mapsto (0, 0)_{s+1} \in \mathcal{F}_{s+1}^1 \cap L_{s+1}$$

where L_0 is the line $\{x = 0\}$ and L_s is the line $\{y = \omega_s x\}$, $1 \leq s \leq n-1$.

Next, construct $\pi_2 : X^2 \rightarrow X^1$ by blowing up the points $(0, 0)_s = \mathcal{F}_s^1 \cap L_s$, for all $0 \leq s \leq n-1$. Denote the new exceptional divisor by \mathcal{F}_s^2 . For \mathcal{F}_s^2 we use local coordinates $\pi_2(\xi_2, x_2)_s = (\xi_2 x_2, x_2)_s = (s_1, \eta_1)_s$ and we have

$$(2.4) \quad f_{X^2} : \begin{cases} (\xi_2, x_2)_0 \mapsto \left(\frac{\xi_2}{\xi_2 - \delta}, x_2(-\delta + \xi_2) \right)_1 \\ (\xi_2, x_2)_s \mapsto \left(\frac{\omega_s \xi_2}{\delta x_2^2 \xi_2 + \omega_s(\xi_2 + \delta)}, \frac{x_2(\delta x_2^2 \xi_2 + \omega_s(\xi_2 + \delta))}{\omega_s(\omega_s + x_2^2 \xi_2)} \right)_{s+1}, & 1 \leq s \leq n-2 \\ (\xi_2, x_2)_{n-1} \mapsto \left(\frac{\xi_2}{\xi_2 - \delta + cx_2^2 \xi_2}, x_2 \right)_0. \end{cases}$$

Thus the induced map f_{X^2} maps exceptional divisors to other exceptional divisors:

$$(2.5) \quad f_{X^2} : \begin{cases} \mathcal{F}_0^2 \ni (\xi_2, 0)_0 \mapsto (\xi_2/(\xi_2 - \delta), 0)_1 \in \mathcal{F}_1^2 \\ \mathcal{F}_s^2 \ni (\xi_2, 0)_s \mapsto (\xi_2/(\xi_2 + \delta), 0)_{s+1} \in \mathcal{F}_{s+1}^2, & 1 \leq s \leq n-2 \\ \mathcal{F}_{n-1}^2 \ni (\xi_2, 0)_{n-1} \mapsto (\xi_2/(\xi_2 - \delta), 0)_0 \in \mathcal{F}_0^2. \end{cases}$$

Near Σ_2 we have

$$f_{X^2}[1 : x : y] = (\xi_2, x_2)_0 = \left(\frac{1}{1 - \delta xy + cy^2}, y \right)_0.$$

The inverse map is $f^{-1}(x, y) = ((cx - y + \frac{1}{x})/\delta, x)$, which lifts to

$$f_{X^2}^{-1}(x, y) = (\xi_2, x_2)_{n-1} = \left(\frac{\delta}{1 + cx - cyx}, x \right)_{n-1}.$$

Thus $f_{X^2}^{-1}(\Sigma_1) = (\delta, 0)_{n-1} \in \mathcal{F}_{n-1}^2$. So in order to have the exceptional curve Σ_2 land on the point of indeterminacy after n steps, we must have

$$(2.6) \quad (f_{X^2})^n \Sigma_2 = f_{X^2}^{-1} \Sigma_1 = (\delta, 0)_{n-1} \in \mathcal{F}_{n-1}^2.$$

Now we use the first line of (2.5) to see that $f_{X^2}^2 \Sigma_2 = f_{X^2}(1, 0)_0 = (1/(1 - \delta), 0)_1$. We map this point forward by iterating the second part of (2.5) $n - 2$ times. Since

$\mathcal{F}_{n-1}^2 \cong \mathbf{P}^1$, in homogeneous coordinates we can identify $(\delta, 0)_{n-1}$ with $[\delta : 1] = [1 : 1/\delta]$. So we may write equation (2.6) as:

$$\begin{pmatrix} 1 & 0 \\ 1 & \delta \end{pmatrix}^{n-2} \begin{pmatrix} 1 \\ 1-\delta \end{pmatrix} = \begin{pmatrix} 1 \\ \delta^{-1} \end{pmatrix}.$$

From this we see that (2.6) holds if and only if δ is a root of $\chi_{n,1}$, as defined in (0.2).

In case (2.6) does not hold, then $f_{X^2}^n(\Sigma_2)$ is not indeterminate, and f_{X^2} will map it to \mathcal{F}_0^2 , and we may map it through the sequence $\mathcal{F}_0^2 \rightarrow \dots \rightarrow \mathcal{F}_{n-1}^2$ again. Since $\begin{pmatrix} 1 & 0 \\ 1 & -\delta \end{pmatrix}^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, we derive an alternative to (2.6): the condition that $f_{X^2}\Sigma_2$ ends up at the point of indeterminacy after m times through this cycle is given (projectively) by

$$\left(\begin{pmatrix} 1 & 0 \\ 1 & \delta \end{pmatrix}^{n-2} \begin{pmatrix} 1 & 0 \\ 1 & -\delta \end{pmatrix}^2 \right)^m \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ \delta^{-1} \end{pmatrix}.$$

This happens exactly when δ is a root of $\chi_{n,m}$.

We now make the space $\pi_3 : X^3 \rightarrow X^2$ by blowing up at the centers $f_{X^2}^{j+1}\Sigma_2 \in \mathcal{F}_*^2$ for $0 \leq j \leq nm - 1$, and we denote the blowup fiber by $\mathcal{F}_{s,\ell}^3$ as in Figure 3. We set $X := X^3$. Using a similar computation as above we see that the induced map f_X maps Σ_2 to the third exceptional divisor:

$$f_X : \Sigma_2 \ni [t : x : 0] \mapsto \left(\frac{x\delta}{t}, 0 \right)_0 \in \mathcal{F}_{0,1}^3$$

and the mapping from $\mathcal{F}_{n-1,m}^3$ to Σ_1 is a local diffeomorphism. From our construction we conclude:

THEOREM 2.3. *Let $n \geq 3, m \geq 1$ such that $(n, m) \neq (3, 1)$, and let δ be a root of polynomial $\chi_{n,m}$ in (0.2) such that $\delta^3 \neq -1$. Let f be a map of the form (2.1). If $c \in C_n(\delta)$, then the induced map $f_X : X \rightarrow X$ is an automorphism. The exceptional divisors are mapped according to:*

$$\begin{aligned} \Sigma_0 &\longrightarrow \Sigma_0, & \mathcal{F}_0^j &\longrightarrow \mathcal{F}_1^j \longrightarrow \dots \longrightarrow \mathcal{F}_{n-1}^j \longrightarrow \mathcal{F}_0^j, & j = 1, 2 \\ \Sigma_2 &\longrightarrow \mathcal{F}_{0,1}^3 \longrightarrow \dots \longrightarrow \mathcal{F}_{n-1,1}^3 \longrightarrow \mathcal{F}_{0,2}^3 \longrightarrow \dots \longrightarrow \mathcal{F}_{n-1,2}^3 \\ &\longrightarrow \dots \longrightarrow \mathcal{F}_{0,m}^3 \longrightarrow \dots \longrightarrow \mathcal{F}_{n-1,m}^3 \longrightarrow \Sigma_1. \end{aligned}$$

We let $\text{Pic}(X)$ denote the Picard group of integral divisors on X . The classes of Σ_0 , and $\mathcal{F}_s^j, 0 \leq s \leq n - 1, j = 1, 2$ and $\mathcal{F}_{s,\ell}^3, 0 \leq s \leq n - 1, 1 \leq \ell \leq m$ form a basis of $\text{Pic}(X)$. Let $Z_{s,\ell} \in \text{Pic}(X)$ be defined as

$$Z_{s,\ell} = \Sigma_0 - \mathcal{F}_s^2 - 2\mathcal{F}_{s,\ell}^3 - \sum_{i \neq \ell} \mathcal{F}_{s,i}^3 + \sum_{t \neq s} \left[\mathcal{F}_t^1 + \mathcal{F}_t^2 + \sum_{i=1}^m \mathcal{F}_{t,i}^3 \right].$$

It follows that $Z_{s,\ell} \cdot \mathcal{F}_{s,\ell}^3 = 1$ and the intersection product with every other element of the basis of $\text{Pic}(X)$ is zero. With the notation $\mathcal{F}^j = \sum_s \mathcal{F}_s^j$ for $j = 1, 2$ and $\mathcal{F}^3 = \sum_{s,\ell} \mathcal{F}_{s,\ell}^3$, we have

$$Z := \sum_{s,\ell} Z_{s,\ell} = mn\Sigma_0 + m(n-1)\mathcal{F}^1 + m(n-2)\mathcal{F}^2 + (m(n-2)-1)\mathcal{F}^3.$$

Let S denote the span in $\text{Pic}(X)$ of Σ_0 and \mathcal{F}_s^j , $j = 1, 2$, $0 \leq s \leq n-1$. By Theorem 2.3. we see that S is f^* -invariant.

PROPOSITION 2.4. *The intersection form on S is negative definite.*

Proof. Since each $Z_{s,\ell}$ is orthogonal to S , it follows that $Z \perp S$. Further, we observe that $Z \cdot Z = mn(mn - 2m - 1) > 0$, so by the Hodge Index Theorem, there is no $s \in S$ with $s \cdot s > 0$. □

PROPOSITION 2.5. *Let C be an effective divisor, and let $\{C\}$ denote its class in $\text{Pic}(X)$. If $\{C\} \in S$, then there are nonnegative integers n_0 and $n_{j,s}$ such that $C = n_0\Sigma_0 + \sum_s (n_{1,s}\mathcal{F}_s^1 + n_{2,s}\mathcal{F}_s^2)$.*

Proof. C is linearly equivalent to $A - B$, where A and B are transverse effective divisors supported on the curves defining S . A must be nontrivial since C is effective and cannot be linearly equivalent to the negative of an effective divisor. Now suppose that $C = \sum n_j V_j$, $n_j > 0$, where V_j are transverse to the curves defining S . Then we have $A \cdot (B + C) = A \cdot A < 0$, but on the other hand $A \cdot (B + C) \geq 0$ because A is transverse to B and C . □

For the next Proposition, we use rational coefficients on $\text{Pic}(X)$. Let $T := S^\perp$ be the orthogonal complement of S . By Proposition 2.4, we have $S \cap T = 0$. Thus $\text{Pic}(X) = S \oplus T$. Let $\gamma_{s,\ell}$ denote the projection to T of the class $\mathcal{F}_{s,\ell}^3 \in \text{Pic}(X)$, and let λ_s denote the projection of the strict transform in X of the line $\Lambda_s := \overline{0p_s} \subset \mathbf{P}^2$.

PROPOSITION 2.6. $\lambda_s = \sum_\ell (-\gamma_{s,\ell} + \sum_{t \neq s} \gamma_{t,\ell})$. Thus we may represent the restriction $f_{X*}|_T$ as

$$\begin{aligned} \lambda_{n-1} &\longrightarrow \gamma_{0,1} \longrightarrow \gamma_{1,1} \longrightarrow \cdots \longrightarrow \gamma_{n-1,1} \longrightarrow \gamma_{0,2} \\ &\longrightarrow \cdots \longrightarrow \gamma_{n-1,m} \longrightarrow \lambda_0 = \sum_\ell \left(-\gamma_{0,\ell} + \sum_{s \neq 0} \gamma_{s,\ell} \right). \end{aligned}$$

The characteristic polynomial of $f_{X*}|_T$ is $\chi_{n,m}$ in (0.2). The spectral radius of f_{X*} is the largest zero of the polynomial $\chi_{n,m}$.

Proof. We may assume that $s = 0$, that is $L_0 = \Sigma_1$. Since $\Sigma_0 = \Sigma_1 \in \text{Pic}(\mathbf{P}^2)$, we pull back by π_1 and have $\Sigma_1 + \mathcal{F}_0^1 = \Sigma_0 + \sum_s \mathcal{F}_s^1 \in \text{Pic}(X^1)$. From (2.3) we see that the center of the blowup for \mathcal{F}_0^2 is $\mathcal{F}_0^1 \cap L_0$ and there are no centers of blowup

of the second blowup fibers in Σ_0 . Thus we have $\Sigma_1 + \mathcal{F}_0^1 + 2\mathcal{F}_0^2 = \Sigma_0 + \sum_s (\mathcal{F}_s^1 + \mathcal{F}_s^2) \in \text{Pic}(X^2)$. Pulling back by π_3 gives

$$\Sigma_1 + \mathcal{F}_0^1 + 2\mathcal{F}_0^2 + 2 \sum_{\ell} \mathcal{F}_{0,\ell}^3 = \Sigma_0 + \sum_s \left(\mathcal{F}_s^1 + \mathcal{F}_s^2 + \sum_{\ell} \mathcal{F}_{s,\ell}^3 \right) \in \text{Pic}(X).$$

When we project everything to $T = S^\perp$, we have $\lambda_0 = \sum_{\ell} (-\gamma_{0,\ell} + \sum_{s \neq 0} \gamma_{s,\ell})$. By Proposition 2.3 we obtain our representation of the restriction $f_{X^*}|_T$. The eigenvalues of $f_{X^*}|_S$ lie in the unit circle since the intersection form is negative definite on S . Thus the spectral radius is given by the restriction $f_{X^*}|_T$, and a direct computation shows that (0.2) is the characteristic polynomial of the transformation defined by the restriction $f_{X^*}|_T$. \square

Remark. Let $\lambda_{n,m}$ denote the largest root of $\chi_{n,m}$, so by Proposition 2.6, $\lambda_{n,m} = \delta(f)$ is the dynamical degree. Since f_X is an automorphism, it follows that if $\lambda_{n,m} > 1$, then $\chi_{n,m}$ is a Salem polynomial, which means a polynomial of degree ≥ 3 such that there are real roots $\lambda = \lambda_{n,m} > 1 > \lambda^{-1}$, and all other roots have modulus one. In fact, $\chi_{n,m}$ has at least one cyclotomic factor and is thus reducible. Note that

$$\chi_{n,m+1}(t) = (t^{n(m+1)} - 1)(t^{nm} - 1)\chi_{n,m}(t) - \frac{t^{nm}(t^n - 1)}{t^{nm} - 1},$$

and

$$\begin{aligned} \chi_{n+1,m}(t) &= \frac{(t^{nm} - 1)(t^n - 2t^{n-1} + 1)(t^{n+1} - 1)}{(t^{(n+1)m} - 1)(t^{n+1} - 2t^n + 1)(t^n - 1)} \chi_{n,m}(t) \\ &\quad - \frac{2(t-1)^2 t^{n-1} (t^n - 1)}{(t^{(n+1)m} - 1)(t^{n+1} - 2t^n + 1)(t^n - 1)}. \end{aligned}$$

It follows that, if $t_* > 1$ is the largest real root of $\chi_{n,m}(t)$ then $\chi_{n,m+1}(t_*) < 0$ and $\chi_{n+1,m}(t_*) < 0$ and therefore both the largest real root of $\chi_{n,m+1}(t)$ and the largest real root of $\chi_{n+1,m}(t)$ are bigger than t_* . Thus we have if $n \geq 4, m \geq 1$, or if $n = 3, m \geq 2$, then the dynamical degree of $\lambda_{n,m} \geq \lambda_{3,2} \approx 1.55603 > 1$. For fixed n , $\lambda_{n,m}$ increases to the largest real root of $t^{n+1} - 2t^n + 1$ and for each $m \geq 1$, $\lambda_{n,m}$ increases to 2 as $n \rightarrow \infty$.

COROLLARY 2.7. *The only invariant curves are unions of Σ_0 and \mathcal{F}_s^j with $0 \leq s \leq n - 1$ and $1 \leq j \leq 2$.*

Proof. If C is an invariant curve, then $\{C\} \in \text{Pic}(X)$ is an eigenvector with eigenvalue 1. The number 1 is not a zero of $\chi_{n,m}$, so by Proposition 2.6, 1 is not an eigenvalue of $f_*|_T$, so the projection of $\{C\}$ to T is not fixed by $f_*|_T$. Thus $\{C\} \in S$, and so the corollary follows from Proposition 2.5. \square

As in [BK2, Theorem 5], we can use Corollary 2.7 to show that (f, X) is a minimal dynamical system if $n \neq 2$.

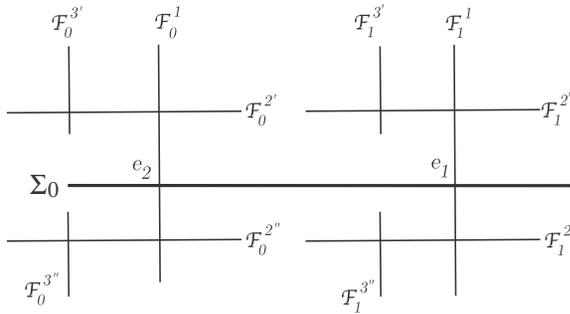


Figure 4. Manifold for Example 2.8.

Example 2.8. We consider the family of maps given by

$$(2.7) \quad k(x, y) = \left(y, -x + 1 + \frac{a}{y} \right).$$

The restriction $k|_{\Sigma_0}$ interchanges $e_1 \leftrightarrow e_2$. As before, we find that $k : \Sigma_2 \rightarrow e_2$, and $k^{-1} : \Sigma_1 \rightarrow e_1$, so we blow up the point e_2 (resp. e_1) and denote the resulting exceptional divisor as \mathcal{F}_0^1 (resp. \mathcal{F}_1^1). On the new manifold, we have $k : \Sigma_2 \rightarrow 0 \in \mathcal{F}_0^1$. The exceptional divisors map according to

$$\mathcal{F}_0^1 \ni \xi \mapsto 1 - \xi \in \mathcal{F}_1^1, \quad \mathcal{F}_1^1 \ni \xi \mapsto \xi \in \mathcal{F}_0^1.$$

The orbit of $0 \in \mathcal{F}_0^1$ ends up at the point of indeterminacy after going twice around:

$$0 \in \mathcal{F}_0^1 \longrightarrow 1 \in \mathcal{F}_1^1 \longrightarrow 1 \in \mathcal{F}_0^1 \longrightarrow 0 \in \mathcal{F}_1^1.$$

We blow up this orbit, and label the new exceptional divisors so that $\mathcal{F}_0^{2'} \rightarrow \mathcal{F}_1^{2'} \rightarrow \mathcal{F}_0^{2''} \rightarrow \mathcal{F}_1^{2''}$. We find that Σ_2 is still exceptional for k ; it is mapped to $a \in \mathcal{F}_0^{2'}$. Finally, we blow up this orbit and obtain an automorphism.

As in the previous case, we consider the invariant subspace $S \subset \text{Pic}$, which is generated by Σ_0 and the blowup fibers up to level 2. The intersection product restricted to S is negative semidefinite but is not negative definite because it has a zero eigenvalue. Thus $T = S^\perp$ intersects S in a one-dimensional subspace. When we compute $k_*|_T$, we find a 3×3 Jordan block with eigenvalues of modulus one, so k_* has quadratic growth.

3. First linearizations. The next three sections will be devoted to linearizations in various contexts; here we consider the more standard cases. First we will show that in many cases the fixed points of f are centers of rank 2 rotation domains. By itself, this result is not new because the possibility of such domains was

shown already for $K3$ surfaces in [M1] and for rational surfaces in [M2] (see also [BK1]), but the relevance for our map f is that the rotation domain U we have constructed is not dense. At the end of this section, we note that Σ_0 is the center of a one parameter family of Siegel disks.

The map f has two isolated fixed points in \mathbf{C}^2 . Choose one of them and let $\lambda_i, i = 1, 2$ be the multipliers of the differential at that point. Thus the λ_i are roots of

$$(3.1) \quad P(t) = t^2 + (1 + \delta - 2c)t + \delta.$$

We will show that these multipliers are multiplicatively independent in certain cases.

LEMMA 3.1. *The multipliers have modulus 1 if and only if $|\operatorname{Re} \sqrt{\delta} - 2\cos(j\pi/n)| \leq 1$.*

Proof. By (3.1), $\lambda_i = -((1 + \delta)/2 - c) \pm \sqrt{-\delta + ((1 + \delta)/2 - c)^2}$, $i = 1, 2$. Since $|\delta| = 1$, we may set $\delta = e^{i\theta}$. Using the expression $c = 2\sqrt{\delta}\cos(j\pi/n)$ and trigonometric identities we can see that

$$\lambda_i = \sqrt{\delta} \left[2\cos(j\pi/n) - \cos(\theta/2) \pm \sqrt{(\cos(\theta/2) - 2\cos(j\pi/n))^2 - 1} \right].$$

Since $|\delta| = 1$, the two multipliers will have modulus 1 if and only if $(\cos(\theta/2) - 2\cos(j\pi/n))^2 - 1 \leq 0$. □

LEMMA 3.2. *The multipliers λ_1 and λ_2 are multiplicatively independent.*

Proof. Suppose $\lambda_1^{p_1}\lambda_2^{p_2} = 1$ for some integers $p_1, p_2 \in \mathbf{Z}$. By (3.1), $\lambda_1\lambda_2 = \delta$ and δ is not a root of unity, we may suppose that $p_1 < p_2 \in \mathbf{Z}$ and $\lambda_1 = \delta^{p_2/(p_2-p_1)}$ and $\lambda_2 = (1/\delta)^{p_1/(p_2-p_1)}$. Choose integers m, k such that $(m, k) = 1$ and $p_2/(p_2 - p_1) = m/k$ and set $\mu = \delta^{1/k}$. It follows that μ^m and μ^{k-m} satisfy (3.1), so if we evaluate $P(\mu^m)$ and use the definition of $c \in C_n(\delta)$, we have

$$(3.2) \quad \frac{1}{4}(\mu^{m-k/2} + \mu^{k/2-m} + \mu^{k/2} + \mu^{-k/2}) = \cos(j\pi/n).$$

It is known (see [R, pages 6–7]) that the n th Chebyshev Polynomial T_n of the first kind takes on its extrema ± 1 at the points $\cos(j\pi/n)$ for $1 \leq j \leq n - 1$. Thus we have

$$T_n(t) + 1 = \tau_n^{(1)}(t) \prod_{j:\text{odd}} (t - \cos(j\pi/n))^2$$

and

$$T_n(t) - 1 = \tau_n^{(2)}(t) \prod_{j:\text{even}} (t - \cos(j\pi/n))^2$$

where $\tau_n^{(i)}(t), i = 1, 2$ are polynomials with no real root. Let us define $\zeta(t) = (t^{2m-k} + t^{k-2m} + t^k + t^{-k})/4$ so that $\zeta(\sqrt{\mu})$ is the left hand side of equation (3.2), and let us set

$$Q(t) = 4^n t^{kn} (T_n(\zeta(t)) \pm 1).$$

Since T_n is an integer polynomial of degree n , we see that $Q(t)$ is a polynomial with integer coefficients, and $Q(\sqrt{\mu}) = 0$. Since $\mu^k = \delta$ and δ is not a root of unity, we conclude that the minimal polynomial of $\sqrt{\mu}$ contains exactly one real root, t_* outside the unit circle. Since t_* is positive real and strictly greater than 1, equation (3.2) gives us

$$\zeta(t_*) = \frac{1}{4} \left(t_*^{2m-k} + \frac{1}{t_*^{2m-k}} + t_*^k + \frac{1}{t_*^k} \right) > 1 \geq \cos(j\pi/n),$$

which gives a contradiction. □

THEOREM 3.3. *If $|\operatorname{Re} \sqrt{\delta} - 2 \cos(j\pi/n)| \leq 1$, then each of the two fixed points is the center of a rotation domain of rank 2.*

Proof. By Lemma 3.1 the multipliers λ_1, λ_2 both have modulus 1, and by Lemma 3.2 the two multipliers are multiplicatively independent. Thus there is a formal power series solution to the linearization equation. Multiplicative independence means that $T(m_1, m_2) := m_1 \log \lambda_1 + m_2 \log \lambda_2 \neq 0$ for all $m_1, m_2 \in \mathbf{Z}, (m_1, m_2) \neq (0, 0)$. Since $\cos(j\pi/n)$ and δ are algebraic, λ_1 and λ_2 are algebraic. It follows from [Ba, Theorem 3.1] that there are $\epsilon > 0$ and $\mu < \infty$ such that $|T(m_1, m_2)| \geq \epsilon(|m_1| + |m_2|)^{-\mu}$ for all $m_1, m_2 \in \mathbf{Z}, (m_1, m_2) \neq (0, 0)$. This condition is sufficient (see, for instance [P] or [Z]) to show that the formal power series converges in a neighborhood of the origin. □

Remark. The condition in Theorem 3.3 can be restated as $\operatorname{Re}(\sqrt{\delta} - 1)/2 \leq \cos(j\pi/n) \leq \operatorname{Re} \sqrt{\delta}/2 + 1/2$. Since $|\delta| = 1$, the length of the interval $[\operatorname{Re}(\sqrt{\delta} - 1)/2, \operatorname{Re}(\sqrt{\delta} + 1)/2] \cap [-1, 1]$ is equal to 1. Using a computer, we see that for many choices of n, m, δ , more than half of values of $0 < j < n$ satisfy this condition.

Now let us turn our attention to the question of linearizing the map $f_{\mathcal{X}}^n$ in a neighborhood of a point of Σ_0 . To simplify the notation we set $f := f_{\mathcal{X}}$ and $h := f^n$. Let us set

$$(3.3) \quad \begin{aligned} \lambda &:= -\delta^{-n/2} && \text{if } n : \text{even} \\ \lambda &:= -1/(\delta^{(n-1)/2} \omega_*) && \text{if } n : \text{odd} \end{aligned}$$

where ω_* is defined in Lemma 2.2.

LEMMA 3.4. *Every point on Σ_0 is fixed under h and the multipliers of h at each point of Σ_0 are 1 and λ .*

Proof. From Lemma 2.1 we see that Σ_0 is a curve of fixed points for $h = f^n$. We have $f(x, y) = M\begin{pmatrix} x \\ y \end{pmatrix} + O(1/y)$ with $M := \begin{pmatrix} 0 & -\delta \\ 1 & c \end{pmatrix}$. We chose c so that the restriction of $h = f^n$ to Σ_0 will be the identity. Since h looks like M^n at Σ_0 , we know that M^n should induce the identity map on Σ_0 . Thus $M^n = \begin{pmatrix} \nu & 0 \\ 0 & \nu \end{pmatrix}$ is a multiple of the identity matrix, and this means that the multipliers at any point of Σ_0 will be $\{1, \nu^{-1}\}$. Since the determinant of M is δ , we conclude that $\nu^2 = \delta^n$, or $\nu = \pm\delta^{n/2}$. It remains to show that the correct sign is the one given in (3.3).

To get the multipliers at Σ_0 we use $\tilde{\pi}_1(\xi_1, t_1)_0 = [t_1\xi_1 : t_1 : 1]$ for \mathcal{F}_0^1 . In these local coordinates, $\{t_1 = 0\} = \mathcal{F}_0^1$ and $\{\xi_1 = 0\} = \Sigma_0$. For \mathcal{F}_s^1 , $1 \leq s \leq n-1$, we use the coordinate chart $\tilde{\pi}_1(\xi_1, t_1)_s = [t_1\xi_1 : 1 : t_1 + \omega_s]$. We also set $h_s = \tilde{\pi}_1^{-1} \circ f \circ \tilde{\pi}_1 : (\xi_1, t_1)_s \mapsto (\xi'_1, t'_1)_{s+1}$ for $0 \leq s \leq n-2$ and $h_{n-1} : (\xi_1, t_1)_{n-1} \mapsto (\xi'_1, t'_1)_0$. It follows that $f^n = h_{n-1} \circ h_{n-2} \circ \dots \circ h_0$ in $\Sigma_0 \setminus \{e_1\}$. Direct computation shows that we have

$$Dh_0(t_1, 0)_0 = \begin{pmatrix} -1/\delta & 0 \\ 0 & \star \end{pmatrix}, \quad Dh_{n-1}(t_1, 0) = \begin{pmatrix} 1 & 0 \\ 0 & \star \end{pmatrix},$$

$$Dh_s(t_1, 0) = \begin{pmatrix} \omega_s/\delta & 0 \\ 0 & \star \end{pmatrix}$$

for $1 \leq s \leq n-2$. It follows that

$$Dh((t_1, 0)_0) = \begin{pmatrix} -(\omega_1 \cdots \omega_{n-2})/\delta^{n-1} & 0 \\ 0 & \star \end{pmatrix} \quad \text{for all } (t_1, 0)_0 \in \Sigma_0.$$

Since every point in Σ_0 is fixed by h , we conclude that the entry \star in this matrix is equal to 1. From Lemma 2.2. we see that $-(\omega_1 \cdots \omega_{n-2})/\delta^{n-1} = \lambda$. □

Now we will linearize h semi-globally, i.e., in a neighborhood of Σ_0 . For this, let λ be as in (3.3), and let $L = \lambda^{-1}I$ be the linear map which is defined by multiplication by λ^{-1} on \mathbf{C}^2 . This map extends to \mathbf{P}^2 and fixes the line at infinity Σ_0 , where the local multipliers are 1 and λ . Now let \tilde{X} denote the space \mathbf{P}^2 , blown up at the points p_0, \dots, p_{n-1} . Then L lifts to an automorphism of \tilde{X} . Let $\iota : X \dashrightarrow \tilde{X}$ be the birational map induced by the identity map on \mathbf{P}^2 . It is evident that we may consider ι to be the identity map in a neighborhood of Σ_0 . Let us consider a local coordinate system (t, ξ) , as in Lemma 3.4, in which $\{t = 0\} \subset \Sigma_0$. h fixes Σ_0 , and by Lemma 3.4, the multiplier normal to Σ_0 at each point is λ . Thus if we write h in the (t, ξ) coordinates, the second coordinate can have no term which is linear in t , which gives us $h(t, \xi) = (\lambda t + t^2\star, \xi + t^2\star)$, where $t^2\star$ denotes terms divisible by t^2 . It is now not hard to solve for the higher order terms in the function $\Phi(t, \xi) = (t, \xi) + (t^2\star, t^2\star) = \iota + O(t^2)$ to obtain a formal solution of the equation $\Phi \circ h = L \circ \Phi$. Similarly, we may work in the coordinate system (t, η) , with $\eta\xi = 1$, to obtain this result at all points of Σ_0 . Since λ is algebraic, it satisfies the correct Diophantine condition, and so the series defining Φ is in fact locally convergent (see [P, Ro] or [Ra]):

PROPOSITION 3.5. *For each $p \in \Sigma_0$, there is a local holomorphic conjugacy Φ_p at p taking h to a linear map. The condition of being tangent to the identity at p defines the local germ Φ_p uniquely. Thus there is a neighborhood U_0 of Σ_0 in X and a map $\Phi : U_0 \rightarrow \tilde{X}$ which is tangent to the identity at each point of Σ_0 such that $L \circ \Phi = \Phi \circ h$.*

4. Linearization at isolated resonant points. Suppose h is a local self-map of \mathbb{C}^2 fixing the origin. Let η_1, η_2 be two resonant multipliers of modulus 1, that is, $|\eta_i| = 1$ and there exist a non-negative integer pair $(a, b) \in \mathbb{N} \times \mathbb{N} \setminus \{(0, 0)\}$ such that $\eta_1^a \eta_2^b = 1$. It follows that there are infinitely many resonant monomials.

Let us define the following spaces spanned by non-resonant monomials, where ‘‘Span’’ means that we take finite or infinite sums over $j_1, j_2 \geq 0$ in the space of polynomials, convergent power series, or formal power series, depending on the situation:

$$\begin{aligned}
 \check{S}_1 &= \text{Span} \{x^{j_1} y^{j_2} : j_1 = (a/b)j_2 + 1, j_2 \geq 1\} \\
 \check{S}_2 &= \text{Span} \{x^{j_1} y^{j_2} : j_2 = (b/a)j_1 + 1, j_1 \geq 1\} \\
 \mathcal{S}_1 &= \text{Span} \{x^{j_1} y^{j_2} : j_1 > (a/b)j_2 + 1\} \\
 \hat{S}_1 &= \text{Span} \{x^{j_1} y^{j_2} : j_1 \geq (a/b)(j_2 - 1)\}
 \end{aligned}
 \tag{4.1}$$

with a similar definition for \mathcal{S}_2 and \hat{S}_2 . Note that $\hat{S}_1 \supset \mathcal{S}_1 \cup \check{S}_1 \cup \check{S}_2$.

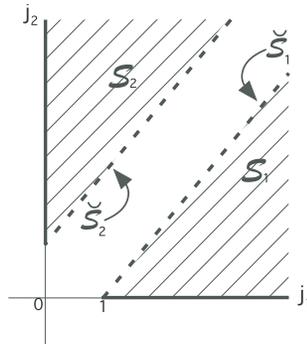


Figure 5. Regions of non-vanishing monomials.

LEMMA 4.1. \mathcal{S}_k and \hat{S}_k are closed under multiplication for $k = 1, 2$. Let us set $\mu_1 = (a/b), \mu_2 = (b/a)$. Then for $\{k, \ell\} = 1, 2$ we have:

- (a) For $n \geq 1$, $\mathcal{S}_k^n = \text{Span}\{x^{j_1} y^{j_2} : j_k > \mu_k j_\ell + n\}$.
- (b) For $n \geq 1$, $\hat{S}_k^n = \text{Span}\{x^{j_1} y^{j_2} : j_k \geq \mu_k(j_\ell - n)\}$.
- (c) If $j_1 > (a/b)j_2 + 1$ then $(x + \mathcal{S}_1)^{j_1} (y + \hat{S}_1)^{j_2} \in \mathcal{S}_1$.
- (d) If $j_1 \geq (a/b)(j_2 - 1)$ then $(x + \mathcal{S}_1)^{j_1} (y + \hat{S}_1)^{j_2} \in \hat{S}_1$.
- (e) If $j_2 > (b/a)j_1 + 1$ then $(x + \hat{S}_2)^{j_1} (y + \mathcal{S}_2)^{j_2} \in \mathcal{S}_2$.
- (f) If $j_2 \geq (b/a)(j_1 - 1)$ then $(x + \hat{S}_2)^{j_1} (y + \mathcal{S}_2)^{j_2} \in \hat{S}_2$.

Proof. Let us suppose $k = 1$. An element in \mathcal{S}_1 has a form $s = \sum_{q=1}^m x^{j_1,q} y^{j_2,q} \in \mathcal{S}_1$, $m \geq 1$. Thus for $n \geq 1$, an element in \mathcal{S}_1^n is a sum of monomials $x^{j_1} y^{j_2}$ where $j_1 = j_{1,q_1} + \dots + j_{1,q_n} \geq (a/b)(j_{2,q_1} + \dots + j_{2,q_n}) + n = (a/b)j_2 + n$. This gives part (a). A similar argument applies for the case (b). Using (a) and (b) it is clear that \mathcal{S}_k and $\hat{\mathcal{S}}_k$ are closed under multiplication and we have

$$(x + \mathcal{S}_1)^{j_1} (y + \hat{\mathcal{S}}_1)^{j_2} = \left(\sum_{i_1} x^{i_1} \mathcal{S}_1^{j_1-i_1} \right) \cdot \left(\sum_{i_2} y^{i_2} \hat{\mathcal{S}}_1^{j_2-i_2} \right).$$

Thus to prove (c) and (d), it suffices to check a monomial in $x^{i_1} \mathcal{S}_1^{j_1-i_1} \cdot y^{i_2} \hat{\mathcal{S}}_1^{j_2-i_2}$ for each $i_1 \leq j_1, i_2 \leq j_2$. Consider $x^{i_1+\alpha_1+\beta_1} y^{i_2+\alpha_2+\beta_2}$ where $x^{\alpha_1} y^{\alpha_2} \in \mathcal{S}_1^{j_1-i_1}$ and $x^{\beta_1} y^{\beta_2} \in \hat{\mathcal{S}}_1^{j_2-i_2}$. Using (a) and (b), we see that $\alpha_1 > (a/b)\alpha_2 + j_1 - i_1$ and $\beta_1 \geq (a/b)(\beta_2 - j_2 + i_2)$. It follows that if $j_1 > (a/b)j_2 + 1$, we have

$$i_1 + \alpha_1 + \beta_1 \geq (a/b)(\alpha_2 + \beta_2 + i_2) + j_1 - (a/b)j_2 > (a/b)(\alpha_2 + \beta_2 + i_2) + 1$$

and therefore $x^{i_1+\alpha_1+\beta_1} y^{i_2+\alpha_2+\beta_2} \in \mathcal{S}_1$. This gives part (c). Similarly if $j_1 \geq (a/b)(j_2 - 1)$ we have

$$i_1 + \alpha_1 + \beta_1 \geq (a/b)(\alpha_2 + \beta_2 + i_2) + j_1 - (a/b)j_2 > (a/b)(\alpha_2 + \beta_2 + i_2 - 1),$$

which gives us that $x^{i_1+\alpha_1+\beta_1} y^{i_2+\alpha_2+\beta_2} \in \hat{\mathcal{S}}_1$. The proof for the case $k = 2$ is similar. □

PROPOSITION 4.2. *Suppose both $f_i : \mathcal{M} \rightarrow \mathcal{M}$, $i = 1, 2$, fix the origin and*

$$f_i(x, y) \in (\alpha_1^{(i)} x, \alpha_2^{(i)} y) + \mathcal{S}_1 \times \hat{\mathcal{S}}_1$$

where $\alpha_1^{(i)}, \alpha_2^{(i)} \in \mathbf{C} \setminus \{0\}$. Then we have

$$f_1 \circ f_2 : (x, y) \mapsto (\alpha_1^{(1)} \alpha_1^{(2)} x, \alpha_2^{(1)} \alpha_2^{(2)} y) + \mathcal{S}_1 \times \hat{\mathcal{S}}_1.$$

with a similar result for $\hat{\mathcal{S}}_2 \times \mathcal{S}_2$. Further $f_1^{-1}(x, y) \in (x/\alpha_1^{(i)}, y/\alpha_2^{(i)}) + \mathcal{S}_1 \times \hat{\mathcal{S}}_1$.

Proof. The statement about $f_1 \circ f_2$ is a direct application of Lemma 4.1. Now let us find f_1^{-1} as an infinite composition $\lim_{n \rightarrow \infty} g_n \circ \dots \circ g_1$, where $g_1(x, y) = (x/\alpha_1^{(i)}, y/\alpha_2^{(i)})$. We may proceed by induction, supposing that we have g_1, \dots, g_{n-1} such that $g_{n-1} \circ \dots \circ g_1 \circ f_1(x, y) = (x, y) + \tilde{h}_n(x, y) \in (x, y) + \mathcal{S}_1 \times \hat{\mathcal{S}}_1$ with $\tilde{h}_n = O_n$. Now we set $g_n(x, y) = (x, y) - \tilde{h}_n(x, y)$, so by the first part of this proposition, we have $g_n \circ \dots \circ g_1 \circ f_1 = (x, y) + \tilde{h}_{n+1} \in (x, y) + \mathcal{S}_1 \times \hat{\mathcal{S}}_1$ with $\tilde{h}_{n+1} \in O_{n+1}$. It follows that $\lim_{n \rightarrow \infty} g_n \circ \dots \circ g_1$ converges as formal power series. Since f_1 is locally invertible, the power series for its inverse is convergent and belongs to $(x/\alpha_1^{(i)}, y/\alpha_2^{(i)}) + \mathcal{S}_1 \times \hat{\mathcal{S}}_1$. □

THEOREM 4.3. *If h has a local expansion of the form $h(x, y) \in (\eta_1 x, \eta_2 y) + \mathcal{S}_1 \times \hat{\mathcal{S}}_1$ or $h(x, y) \in (\eta_1 x, \eta_2 y) + \hat{\mathcal{S}}_2 \times \mathcal{S}_2$, then there is a formal power series expansion Φ such that $\Phi \circ h = L \circ \Phi$ where $L(x, y) = (\eta_1 x, \eta_2 y)$. Furthermore if η_1, η_2 are algebraic, then the formal series Φ is convergent, so h is analytically linearizable.*

Proof. It suffices to assume that the higher order terms are in $\mathcal{S}_1 \times \hat{\mathcal{S}}_1$. For $j \geq 2$, we will define a sequence of maps $\Phi_j : (x, y) \mapsto (x, y) +$ homogeneous polynomials of degree j . Since h has no resonant monomials, we may follow the Poincaré Linearization Theorem (see, for instance, [A, Section 25]) and find Φ_2 such that

$$\Phi_2^{-1} \circ h \circ \Phi_2(x, y) = L(x, y) + \tilde{h}_3$$

where order of $\tilde{h}_3 \geq 3$. Since $h(x, y) \in (\eta_1 x, \eta_2 y) + \mathcal{S}_1 \times \hat{\mathcal{S}}_1$, we may assume that $\Phi_2(x, y) \in (x, y) + \mathcal{S}_1 \times \hat{\mathcal{S}}_1$ by setting coefficient of all resonant terms equal to zero. From Proposition 4.2 we see that $\tilde{h}_3 \in \mathcal{S}_1 \times \hat{\mathcal{S}}_1$ and thus it has no resonant monomials. We proceed with an induction on j and find Φ_j such that

$$\Phi_j^{-1} \circ \dots \circ \Phi_2^{-1} \circ h \circ \Phi_2 \circ \dots \circ \Phi_j = L + \tilde{h}_{j+1}$$

where order of $\tilde{h}_{j+1} \geq j + 1$ and $\tilde{h}_{j+1} \in \mathcal{S}_1 \times \hat{\mathcal{S}}_1$. Thus we obtain a (formal) linearization $\Phi = \lim_{n \rightarrow \infty} \Phi_2 \circ \Phi_3 \circ \dots \circ \Phi_n$.

We define $T(m_1, m_2) = m_1 \log \eta_1 + m_2 \log \eta_2$. Because of resonances in the multipliers, $T(m_1, m_2)$ can vanish, but by our construction there are no nonvanishing resonant monomials, which means that the coefficient of $x^{m_1} y^{m_2}$ will also vanish when $T(m_1, m_2)$ vanishes. Since η_1 and η_2 are algebraic, by [Ba, Theorem 3.1] we will have $|T(m_1, m_2)| \geq \epsilon(|m_1| + |m_2|)^{-\mu}$ for all values for which $T(m_1, m_2)$ does not vanish. Thus this estimate holds whenever the coefficient of $x^{m_1} y^{m_2}$, does not vanish. By [P, Z] it follows that the power series of Φ actually converges, and thus h is linearizable. □

Remark. A related linearization, at resonant points with invariant manifolds, is given by Raissy [Ra].

Example. Let λ be a number of modulus 1 which is not a root of unity, and consider the map

$$f(x, y) = (\lambda x + x^5 y^2, \lambda^{-1} y + x^2 + x^3 y).$$

The multipliers at the origin exhibit the resonance corresponding to $\lambda^a (\lambda^{-1})^b = 1$ with $a = b = 1$. Thus the monomials in f belong to \mathcal{S}_1 , so by Theorem 4.3, f can be formally linearized at the origin. If λ is algebraic, then the formal conjugacy actually converges and gives a holomorphic linearization of f .

We may reformulate Theorem 4.3 to give non-linearizability.

COROLLARY 4.4. *Suppose the local expansion is $h = (\eta_1x, \eta_2y) + (h_1, h_2)$. Suppose that for $k = 1$ or $k = 2$, $h_1, h_2 \in \hat{S}_k$ and $h_k \notin S_k$ then h is not linearizable.*

Proof. Suppose n is the smallest integer such that h_k has a monomial of order n in \hat{S}_k . Using Theorem 4.3. we see that there are $\Phi_2, \dots, \Phi_{n-1}$ such that

$$\Phi_{n-1}^{-1} \circ \dots \circ \Phi_2^{-1} \circ h \circ \Phi_2 \circ \dots \circ \Phi_{n-1} = L + \tilde{h}_{n+1}$$

where \tilde{h}_n has resonant monomials. It follows that there is no formal power series expansion Φ such that $\Phi \circ h = L \circ \Phi$ and thus h is not linearizable. □

Let us apply this discussion to the map $h := f^n$. By Theorem 2.3. we see that the exceptional fibers $\mathcal{F}_s^j, 0 \leq s \leq n - 1, j = 1, 2$ are invariant under h . Thus $\mathcal{F}_s^1 \cap \mathcal{F}_s^2$ is fixed by h .

LEMMA 4.5. *The multipliers of h at $\mathcal{F}_s^1 \cap \mathcal{F}_s^2$ are λ^2 and $1/\lambda$, where λ is defined in (3.3).*

Proof. Let us rewrite f near $\mathcal{F}_s^1 \cap \mathcal{F}_s^2$ for $0 \leq s \leq n - 1$ using the local coordinates $(\xi_2, x_2)_s$ defined in Section 2, so $\{\xi_2 = 0\} = \mathcal{F}_s^1$ and $\{x_2 = 0\} = \mathcal{F}_s^2$. Using the expression in (2.4) we see that differential at the origin of each mapping is diagonal and thus we have

$$\begin{aligned} dh|(0,0)_s &= \begin{pmatrix} -1/\delta & 0 \\ 0 & -\delta \end{pmatrix} \begin{pmatrix} 1/\delta & 0 \\ 0 & \delta/\omega_1 \end{pmatrix} \cdots \begin{pmatrix} 1/\delta & 0 \\ 0 & \delta/\omega_{n-2} \end{pmatrix} \begin{pmatrix} -1/\delta & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1/\delta^n & 0 \\ 0 & -\delta^{n-1}/(\omega_1 \cdots \omega_{n-2}) \end{pmatrix} = \begin{pmatrix} \lambda^2 & 0 \\ 0 & 1/\lambda \end{pmatrix}. \end{aligned}$$

The last equality in the second line comes from Lemma 2.2. □

LEMMA 4.6. *For $c, \delta \in \mathbf{C}$ and $g(w) = c - \delta/w$, let $g_n = g^n(c)$ for $n \geq 0$. Then we have*

$$g_n = c - \frac{\delta}{c} - \frac{\delta^2}{c^2 g_1} - \frac{\delta^3}{c^2 g_1^2 g_2} - \cdots - \frac{\delta^n}{c^2 g_1^2 \cdots g_{n-2}^2 g_{n-1}}$$

Proof. Note that $c = g_0$. The conclusion is equivalent to

$$g_n - g_{n-1} = -\delta^n / (g_0^2 g_1^2 \cdots g_{n-2}^2 g_{n-1}).$$

Since $g_1 = c - \delta/c$, it is easy to see that $g_1 - g_0 = -\delta/g_0$. We proceed by induction on n :

$$g_{n+1} - g_n = g(g_n) - g(g_{n-1}) = -\frac{\delta(g_{n-1} - g_n)}{g_{n-1}g_n}$$

Replacing $g_{n-1} - g_n$ by $\delta^n / (g_0^2 g_1^2 \cdots g_{n-2}^2 g_{n-1})$ we have the conclusion. □

LEMMA 4.7. *The local expansion of h at the fixed point $\mathcal{F}_s^1 \cap \mathcal{F}_s^2$ is given by*

$$h(\xi, x) \in \left(\lambda^2 \xi, \frac{1}{\lambda} x \right) + \mathcal{S}_1 \times \hat{\mathcal{S}}_1.$$

Proof. Using the expression (2.4), we can rewrite the mappings near fixed points $\mathcal{F}_s^1 \cap \mathcal{F}_s^2$ as following:

$$f_{X^2} : \begin{cases} (\xi_2, x_2)_0 \mapsto \left(-\frac{\xi_2}{\delta} + \mathcal{S}_1, -\delta x_2 + \hat{\mathcal{S}}_1 \right)_1 \\ (\xi_2, x_2)_s \mapsto \left(\sum_{m=0}^{\infty} \frac{(-1)^m}{\delta \omega_s^m} x_2^{2m} \xi_2^{m+1} + \mathcal{S}_1, \frac{\delta}{\omega_s} x_2 + \hat{\mathcal{S}}_1 \right)_{s+1}, \quad 1 \leq s \leq n-2 \\ (\xi_2, x_2)_{n-1} \mapsto \left(-\sum_{m=0}^{\infty} \frac{c^m}{\delta^{m+1}} x_2^{2m} \xi_2^{m+1} + \mathcal{S}_1, x_2 \right)_0. \end{cases}$$

Using Lemma 4.1. it suffices to show that the first component of h is in \mathcal{S}_1 . First note that if $j_1 = (a/b)j_2 + 1$ then for $\xi_2^{\alpha_1} x_2^{\alpha_2} \in \mathcal{S}_1^{j_1}$ and $\xi_2^{\beta_1} x_2^{\beta_2} \in \hat{\mathcal{S}}_1^{j_2}$ we have $\alpha_1 + \beta_1 > (1/2)\alpha_2 + j_1 + (1/2)(\beta_2 - j_2) = (1/2)(\alpha_2 + \beta_2) + 1$. It follows that

$$(4.2) \quad \mathcal{S}_1^{j_1} \hat{\mathcal{S}}_1^{j_2} \in \mathcal{S}_1 \quad \text{for } j_1 = (a/b)j_2 + 1.$$

Using (4.2) we see that

$$f^2(\xi_2, x_2)_0 = \left(-\sum_{m=0}^{\infty} \frac{\delta^{m-2}}{\omega_1^m} x_2^{2m} \xi_2^{m+1} + \mathcal{S}_1, -\frac{\delta^2}{\omega_1} x_2 + \hat{\mathcal{S}}_1 \right)_2.$$

Using the binomial expansion we can keep track of the coefficient of $x_2^{2n} \xi_2^{n+1}$ in the first coordinate:

$$f^3(\xi_2, x_2)_0 = \left(-\sum_{m=0}^{\infty} \delta^{m-3} \left(\frac{\omega_1 \omega_2 + \delta}{\omega_1^2 \omega_2} \right)^m x_2^{2m} \xi_2^{m+1} + \mathcal{S}_1, -\frac{\delta^3}{\omega_1 \omega_2} x_2 + \hat{\mathcal{S}}_1 \right)_3.$$

Again by the binomial expansion we see that the first coordinate of $f^n(\xi_2, x_2)_0$ is given by

$$-\sum_{m=0}^{\infty} \delta^{m+1-n} \left(-c + \frac{\delta}{\omega_1} + \frac{\delta^2}{\omega_1^2 \omega_2} + \dots + \frac{\delta^n}{\omega_1^2 \dots \omega_{n-3}^2 \omega_{n-2}} \right)^m x_2^{2m} \xi_2^{m+1} + \mathcal{S}_1.$$

Since $c = \omega_1$ and $\omega_j = g^{j-1}(c)$, from Lemma 4.6 the coefficient of $x_2^{2m} \xi_2^{m+1}$ vanishes for all $m \geq 1$ if and only if $g^{n-2}(c) = 0$. Recall that $c \in C_n(\delta)$ if and only if $g^{n-2}(c) = 0$. Since $c \in C_n(\delta)$ we have the desired conclusion. \square

PROPOSITION 4.8. *There is a holomorphic conjugacy Φ defined in a neighborhood of $\mathcal{F}_s^1 \cap \mathcal{F}_s^2$ taking h to a linear map.*

Proof. Using Lemma 4.6 and Theorem 4.3, we see that there is a formal expansion of Φ such that $\Phi \circ h = L \circ \Phi$. The multipliers are powers of a root of the Salem polynomial $\chi_{n,m}$, and are not roots of unity. It follows that Φ is holomorphic. \square

THEOREM 4.9. *If $|\delta| = 1$, there is a Fatou component U which is a rotation domain of rank 1 and which contains $\Sigma_0 \cup \mathcal{F}_0^1 \cup \dots \cup \mathcal{F}_{n-1}^1$. In particular U contains a curve Σ_0 of fixed points as well as isolated fixed points $\{q_0, \dots, q_{n-1}\}$.*

Proof. We will show that there is a neighborhood U_0 containing $\Sigma_0 \cup \bigcup_s \mathcal{F}_s^1$ and a unique conjugacy which is tangent to the identity along $\Sigma_0 \cup \bigcup_s \{q_s\}$ taking (h, U_0) to $(L, \Phi(U_0))$. Let Φ' denote the local conjugacy from Proposition 4.8, which is defined in a neighborhood of $q_s = \mathcal{F}_s^1 \cap \mathcal{F}_s^2$. Let Φ'' denote the local conjugacy from Proposition 3.5, which is defined in a neighborhood U_0 containing $p_s = \Sigma_0 \cap \mathcal{F}_s^1$. It suffices to show that these two conjugacies may be analytically continued together to one conjugacy which is defined in a neighborhood of \mathcal{F}_s^1 .

We use coordinates (ξ, x) such that $q_s = (0, 0)$, and $\mathcal{F}_s^1 = \{\xi = 0\}$. The series expressing Φ' has the form $\sum_k \sum_{j \leq 2k+1} a_{j,k} x^j \xi^k$, and we may assume that it converges for $\{|x|, |\xi| < 1\}$. Thus if $R < \infty$ and we set $\epsilon = R^{-2}$, and it follows that the series for Φ' converges in V' which contains $\{|\xi| < \epsilon, |x| < R\}$.

Now let us use coordinates (s, η) so that $p_s = (0, 0)$, and $\{s = 0\} = \mathcal{F}_s^1$. We may assume that Φ'' is defined in $V'' := \{|s| < \epsilon, |\eta| < 1\}$. Choosing R sufficiently large, we may assume that $V' \supset V_0 := \{|s| < \epsilon, \frac{1}{2} < |\eta| < 1\}$. Now with an appropriate coordinate change both Φ' and Φ'' conjugate the map $h|_{V_0}$ to the linear map $L(x, y) = (x, \lambda y)$. It follows that $\tilde{\phi}(x, y) := \Phi' \circ \Phi''^{-1}$ commutes with L . In other words, if we write $\tilde{\phi}$ as a Laurent series on $\{|x| < \epsilon, r_1 < |y| < r_2\}$, then the second coordinate of $\tilde{\phi}$ satisfies $\lambda \tilde{\phi}_2(x, y) = \tilde{\phi}_2(x, \lambda y)$. Since λ is not a root of unity, we conclude that there is a $c(x)$ so that $\tilde{\phi}(x, y) = (x, c(x)y)$. Thus $\tilde{\phi}$ extends holomorphically to V'' . Since we have $\Phi' = \tilde{\phi} \circ \Phi''$, it follows that Φ' extends analytically to $V' \cup V''$, which is a neighborhood of \mathcal{F}_s^1 .

Finally, since Φ'' and the extended map Φ' are both tangent to the identity at p_s , they agree in a neighborhood of p_s , so they combine to give a conjugacy in a neighborhood of $\Sigma_0 \cup \bigcup_s \mathcal{F}_s^1$. \square

5. Global linearization. Our global model (L, X_L) is defined as follows. Let L be the linear map of \mathbf{C}^2 given by the diagonal matrix $L = \text{diag}(\lambda^{-1}, \lambda^{-1})$ with λ as in (3.3), and extend L to an automorphism of \mathbf{P}^2 . To construct X_L , we blow up the points $p_0, \dots, p_{n-1} \in \Sigma_0$ as in the construction of the manifold \tilde{X} for Proposition 3.5. The fixed points of $L|_{\mathcal{F}_s^1}$ are $\{p_s, q_s\}$, so we continue as with the construction of X and blow up the points $q_s \in \mathcal{F}_s^1, 0 \leq s \leq n-1$. So far, we have completed the first two steps of the construction of X . The restriction of L to \mathcal{F}_s^2 fixes a second point $r_s \in \mathcal{F}_s^2$, and we construct X_L by blowing up $r_s, 0 \leq s \leq n-1$. The exceptional divisor is written $\tilde{\mathcal{F}}_s^3$. This is shown in Figure 6, where we put three hollow dots in \mathcal{F}_s^2 to denote the m points which would have been blown up if we

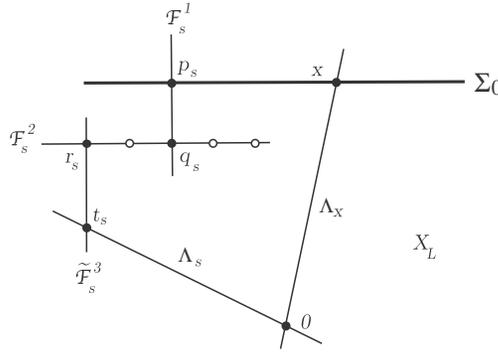


Figure 6. Global linear model X_L .

wanted to make X . For $x \in \Sigma_0$ we let Λ_x denote the strict transform in X_L of the line $\overline{0x} \subset \mathbf{P}^2$. We denote Λ_{p_s} by Λ_s ; Figure 6 shows the difference between Λ_x and Λ_s . We note that X may be considered to coincide with X_L in a neighborhood of $\Sigma_0 \cup \mathcal{F}_0^1 \cup \dots \cup \mathcal{F}_{n-1}^1$.

THEOREM 5.1. *There is a domain $\Omega \subset X_L$ and a holomorphic conjugacy $\Phi : U \rightarrow \Omega$ taking (h, U) to (L, Ω) . In particular, h has no periodic points in $U \cap \pi^{-1}\mathbf{C}^2$.*

One consequence is the following:

COROLLARY 5.2. *The domains $\Omega - \bigcup_s (\mathcal{F}_s^1 \cup \mathcal{F}_s^2 \cup \bigcup_\ell \mathcal{F}_{s,\ell}^3)$ and $\Omega' := \pi(\Omega) - \Sigma_0 \subset \mathbf{C}^2$ are pseudoconvex. Further, Ω' has the complete-circular property that if $(x, y) \in \Omega'$, and if $\zeta \in \mathbf{C}$, $|\zeta| \geq 1$, then $(\zeta x, \zeta y) \in \Omega'$.*

The rest of this section will be devoted to proving Theorem 5.1. First we note that since the conjugacies obtained in Proposition 3.5 and Theorem 4.9 are uniquely determined by being tangent to the identity at the fixed points, we have the following:

LEMMA 5.3. *There is a neighborhood U_0 containing $\Sigma_0 \cup \bigcup_s \mathcal{F}_s^1$ and a unique conjugacy $\Phi : U_0 \rightarrow \Phi(U_0)$ which is tangent to the identity along $\Sigma_0 \cup \bigcup_s \{q_s\}$ taking (h, U_0) to $(L, \Phi(U_0))$.*

For $x \in \Sigma_0$, the curve $\Lambda_x \subset X_L$ is invariant under L . The restriction $\Phi^{-1}|_{\Phi(U_0) \cap \Lambda_x}$ is analytic in a neighborhood of x , and we let $\omega_x \subset \Lambda_x$ denote a maximal domain such that Φ^{-1} has an analytic continuation to a map $\psi_x : \omega_x \rightarrow U$. Since ψ_x preserves the circle action, $\omega_x \subset \Lambda_x$ is a disk centered at x . The sets ω_{q_s} and ω_{r_s} are defined analogously.

LEMMA 5.4. *For $x \in \Sigma'_0$, ω_x is a relatively compact sub-disk of $\Lambda_x - \{0\}$.*

Proof. First we observe that $\psi_x(\omega_x)$ cannot be contained in an algebraic curve. For, since x is fixed, and ψ_x commutes with the group action, it follows that this

algebraic curve is invariant. But by Corollary 2.7, the only invariant curves are Σ_0 , \mathcal{F}_s^1 and \mathcal{F}_s^2 , which do not contain $\psi_x(\omega_x)$.

Now if $\omega_x = \Lambda_x - \{0\}$, we define

$$A_r := \frac{1}{\#_r} \int_0^r \frac{dt}{t} [\psi_x(D_t)]$$

where $D_t = \Lambda_x \cap \{|(x, y)| > 1/t\}$, and $\#_r$ is chosen so that A_r has mass 1. Since $\psi_x(\omega_x)$ is not contained in an algebraic curve, we may pass to a subsequence $r_j \rightarrow \infty$ which converges to a closed (Ahlfors) current A . Since x is fixed, and ψ_x commutes with the circle action, we have a current satisfying $h_*(A) = A$. The corresponding class $\{A\} \in \text{Pic}(X)$ is fixed under h_* . By Section 2, then, we have $\{A\} \in S$, and the intersection form is negative definite on S . On the other hand, a property of the Ahlfors current is that it has nonnegative self-intersection $\{A\}^2 \geq 0$ (see [Br, Lemme 1, where the current Φ is the same as A but in different notation]). This contradiction shows that ω_x must be a proper sub-disk of Λ_x . \square

Recall that by Section 1, there is a holomorphic vector field V on U whose real part gives the \mathbf{T}^1 action on U . We let \mathcal{V} denote the foliation of U by complex manifolds which are the complexifications of the orbits of the \mathbf{T}^1 action. The leaves of \mathcal{V} are the same as the complex orbits of V . For $x \in U$, we let \mathcal{V}_x denote the leaf of \mathcal{V} containing x . By the maximality of ω_x we have $\psi_x(\omega_x) = \mathcal{V}_x$ for $x \in \Sigma'_0$.

LEMMA 5.5. *For $x \in \Sigma'_0$, the map $\psi_x : \omega_x \rightarrow U$ is proper. Thus \mathcal{V}_x is a properly embedded disk, and ψ_x is a bijection between ω_x and \mathcal{V}_x .*

Proof. If $K \subset U$ is compact, there exists $\eta > 0$ such that for each $y_0 \in K$, the leaf of \mathcal{V} passing through y_0 has inner radius at least η . We may assume that η is less than the distance from K to ∂U and let \tilde{K} denote the closure of an η -neighborhood of K . Thus \tilde{K} is a compact subset of U . The circle action on Λ_x is generated by the vector field $i\zeta \frac{\partial}{\partial \zeta}$, and ψ_x maps this to a constant multiple of V . Since V is bounded on \tilde{K} , it follows that there is a constant M such that the differential of ψ_x at ζ_0 is bounded by M for all $\zeta_0 \in \omega_x$ such that $\psi_x(\zeta_0) \in \tilde{K}$. It follows that if $\zeta_0 \in \psi_x^{-1}K$, then ψ_x extends to the disk of radius η/M centered at ζ_0 . Thus the distance of $\psi_x^{-1}K$ to $\partial\omega_x$ is at least η/M , so ψ_x is proper. \square

Following the recipe for determining multipliers in Figure 2, we have:

LEMMA 5.6. *The local multipliers of (L, X_L) at q_s are $\{\lambda^{-1}, \lambda^2\}$, and at r_s they are $\{\lambda^3, \lambda^{-2}\}$.*

Proof of Theorem 5.1. First we assume that $\mathcal{F}_s^2 \not\subset U$, and define

$$\Omega = \bigcup_{x \in \Sigma'_0} \omega_x \cup \bigcup_{s=0}^{n-1} (\mathcal{F}_s^1 \cup \omega_{q_s}).$$

Let us set $U' := \bigcup_{x \in \Sigma'_0} \mathcal{V}_x$ and $\Omega' := \bigcup_{x \in \Sigma'_0} \omega_x$. By Lemma 5.5, we may extend the definition of Φ from $\mathcal{V}_x \cap U_0$, $x \in \Sigma'_0$, to \mathcal{V}_x by setting $\Phi = \psi_x^{-1}$. Now by equivariance, Φ satisfies the holomorphic differential equation $\Phi_* V = ci\zeta\partial_{\bar{z}}$ for some real constant c . Thus holomorphicity propagates along the leaves of \mathcal{V} , so Φ is holomorphic on U' . By Lemma 5.3, Φ is holomorphic on all of Ω . Since the sets ω_x are disjoint, Φ is an injection, so it follows that $\Phi : U \rightarrow \Omega$ is a biholomorphism.

Now if $\mathcal{F}_s^2 \subset U$, then we define

$$\Omega = \bigcup_{x \in \Sigma'_0} \omega_x \cup \bigcup_{s=0}^{n-1} (\mathcal{F}_s^1 \cup \mathcal{F}_s^2 \cup \omega_{r_s}).$$

All of the previous arguments apply in this case, except that we need to show that Φ is holomorphic in a neighborhood of r_s . This is similar to the proof of Theorem 4.9. We may choose coordinates so that $\mathcal{F}_s^2 = \{x = 0\}$, and $r_s = (0, 0)$. We may suppose that the map Φ is holomorphic on the set $\{|x| > 1, |y| < 1\}$. Further, since r_s is in the Fatou set, we know that H can be linearized in a neighborhood of r_s . Thus we have another map Φ' conjugating H to its linear part, which by Lemma 5.6 is $diag(\lambda^{-2}, \lambda^3)$. We may assume that Φ' is analytic on $\{|x| < 2, |y| < 1\}$, and that $\Xi := \Phi' \circ \Phi^{-1} = \sum a_{i,j} x^i y^j$ commutes with this linear map on the set $\{1 < |x| < 2, |y| < 1\}$. We then have that the first coordinate is $\lambda^{-2}\Xi^{(1)}(x, y) = \Xi^{(1)}(\lambda^{-2}x, \lambda^3y)$, from which we conclude that $\lambda^{-2} = \lambda^{-2i+3j}$ for all nonvanishing coefficients $a_{i,j}$. Since we have $j \geq 0$, and λ is not a root of unity, it follows that we must have $i \geq 1$. Looking at the second coordinate, we get $\lambda^3 = \lambda^{-2i+3j}$, so in this case, we cannot have $i < 0$. It follows that all exponents i, j in Ξ are positive, so Ξ is analytic in $\{|x| < 2, |y| < 1\}$. Thus we conclude that Ξ , and thus Φ extends holomorphically through r_s . □

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