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Linear recurrences in the degree sequences of monomial mappings

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Abstract. Let A be an integer matrix, and let f_A be the associated monomial map. We give a connection between the eigenvalues of A and the existence of a linear recurrence relation in the sequence of degrees.

1. Introduction

Let A denote a $k \times k$ matrix of rank k which has integer entries. The monomial map $f_A : \mathbb{C}^k \rightarrow \mathbb{C}^k$ defined by

$$f_A(x) = x^A = \left(\prod_j x_j^{a_{1,j}}, \dots, \prod_j x_j^{a_{n,j}} \right) \tag{1.1}$$

is a dominant rational map. The iterates are given by $f_A^n = f_A \circ \dots \circ f_A = f_{A^n}$. If $A \in GL(n, \mathbb{Z})$, then f_A is birational, and $f_A^{-1} = f_{A^{-1}}$. A rational map f on projective space \mathbb{P}^k induces a linear map f^* on $H^{p,p}(\mathbb{P}^k; \mathbb{Z}) \cong \mathbb{Z}$. We define the degree of f^n in codimension p to be $d_p^{(n)} := (f^n)^* | H^{p,p}(\mathbb{P}^k)$; or, equivalently (see [RS]), if ω is a Kähler form on \mathbb{P}^k normalized so that $\int_{\mathbb{P}^k} \omega^k = 1$, then

$$d_p^{(n)} = \int_{\mathbb{P}^k} \omega^{k-p} \wedge (f^n)^* \omega^p.$$

If $f : \mathbb{P}^k \rightarrow \mathbb{P}^k$ is rational, X is a compact Kähler manifold, and $\pi : X \rightarrow \mathbb{P}^k$ is holomorphic and bimeromorphic, then we have a map $\tilde{f} = \pi^{-1} \circ f \circ \pi : X \rightarrow X$. There will be an induced linear map

$$\tilde{f}^* : H^{p,p}(X) \rightarrow H^{p,p}(X).$$

Let $\chi(x) = x^m + \alpha_{m-1}x^{m-1} + \dots + \alpha_0 \in \mathbb{Z}[x]$ be the characteristic polynomial of $\tilde{f}^* | H^{p,p}$. If we have

$$(\tilde{f}^n)^* = (\tilde{f}^*)^n \quad \text{on } H^{p,p}(X), \tag{1.2}$$

then by [DF, Corollary 2.2] the sequence $\{d_p^{(n)}\}_{n \in \mathbb{N}}$ satisfies the linear recurrence

$$d_p^{(n+m)} + \alpha_{m-1}d_p^{(n+m-1)} + \dots + \alpha_0d_p^{(n)} = 0 \tag{1.3}$$

for all $n \in \mathbb{N}$.

In dimension $k = 2$, there is only the case $p = 1$ to consider. Favre [F] has given necessary and sufficient conditions for a monomial map in dimension two to have a regularization $\pi : X \rightarrow \mathbb{P}^2$ satisfying (1.2). Diller and Favre [DF] showed that for every bimeromorphic surface map, there is such a regularization. Favre and Jonsson [FJ] have shown that the degree sequence of a polynomial map of \mathbb{C}^2 always satisfies (1.3).

In dimension $k = 3$, Hasselblatt and Propp [HP] showed that there is a matrix $A \in GL(3, \mathbb{Z})$ such that the degree sequence $\{d_1^{(n)}\} = \{\deg(f_A^n)\}$ does not satisfy any linear recurrence.

In homogeneous coordinates we have

$$f_A[x_0 : x_1 : \dots : x_k] = \left[1 : \prod_j (x_j/x_0)^{a_{1,j}} : \dots : \prod_j (x_j/x_0)^{a_{n,j}} \right], \tag{1.4}$$

and if we rewrite f_A so that the coordinates are homogeneous polynomials, then their degree is $d_1^{(1)} = D(A)$, with

$$D(A) := \max \left(0, \sum_{j=1}^k a_{1,j}, \dots, \sum_{j=1}^k a_{n,j} \right) + \sum_{j=1}^k \max(0, -a_{1,j}, \dots, -a_{n,j}). \tag{1.5}$$

It is evident that there is a set \mathcal{C} consisting of $(k + 1)^{k+1}$ linear functionals $L_C : \mathbb{M}_k \rightarrow \mathbb{R}$ on the space of real $k \times k$ matrices such that

$$D(A) = \max\{L_C(A) \mid C \in \mathcal{C}\}. \tag{1.6}$$

For any linear functional L on the set of $k \times k$ real matrices, the sequence $\{L(A^n)\}_{n \in \mathbb{N}}$ satisfies the linear recurrence (1.3) (see Lemma 2.1), where $\chi(x)$ is the characteristic polynomial of A . It follows that the degree sequence $\{d_1^{(n)}\} = \{D(A^n)\}$ is ‘almost’ the solution to a linear recurrence: it is the finite maximum over $C \in \mathcal{C}$ of the sequences $\{L_C(A^n)\}_{n \in \mathbb{N}}$, each of which satisfies (1.3) but which may have different initial conditions. Another way of describing this phenomenon is that the space of matrices is divided into different cells defined by

$$S_C := \{M \in \mathbb{M}_k \mid D(M) = L_C(M)\}. \tag{1.7}$$

For each n , A^n belongs to one of the cells $S_{C(n)}$, so $d_1^{(n)} = L_{C(n)}(A^n)$. Although there are only finitely many cells, the cell $C(n)$ might change in a sufficiently irregular way that $\{d_1^{(n)}\}_{n \in \mathbb{N}}$ does not satisfy any linear recurrence at all. This is the approach taken in [HP] and also used in §3 of this paper.

Here we extend the results of [HP] to obtain the following.

THEOREM 1.1. *Let A be a $k \times k$ integer matrix with rank k . Suppose that for every eigenvalue λ of A with $|\lambda| \geq 1$, $\bar{\lambda}/\lambda$ is a root of unity; then $\{d_1^{(n)}\}_{n \in \mathbb{N}}$ satisfies a linear recurrence. On the other hand, if the eigenvalues of largest modulus consist of a conjugate pair λ and $\bar{\lambda}$ of simple eigenvalues, where $\bar{\lambda}/\lambda$ is not a root of unity, then $\{d_1^{(n)}\}_{n \in \mathbb{N}}$ does not satisfy any linear recurrence.*

By the duality between $H^{1,1}$ and $H^{k-1,k-1}$, we see that $f_A^*|H^{k-1,k-1}$ is dual to $(f_A)_*|H^{1,1} = f_{A^{-1}}^*|H^{1,1}$. Thus, in dimension three, $\{d_2(f_A^n)\} = \{d_1(f_{A^{-1}}^n)\}$. By this duality, Theorem 1.1 gives a rather complete treatment of the cases in dimension three.

THEOREM 1.2. *Suppose that the matrix $A \in GL(3, \mathbb{Z})$ has no eigenvalues of modulus one. In the case where all three eigenvalues are real, both $\{d_1^{(n)}\}_{n \in \mathbb{N}}$ and $\{d_2^{(n)}\}_{n \in \mathbb{N}}$ satisfy linear recurrences. In the case where there is a non-real eigenvalue, we may write the eigenvalues as $\lambda, \bar{\lambda}$ and $\pm|\lambda|^{-2}$. If $|\lambda| > 1$, then $\{d_1^{(n)}\}_{n \in \mathbb{N}}$ does not satisfy a linear recurrence, but $\{d_2^{(n)}\}_{n \in \mathbb{N}}$ does; and vice versa if $|\lambda| < 1$.*

In dimension $k = 4$, Theorem 1.1 and duality, applied to the map $f_A : \mathbb{P}^4 \rightarrow \mathbb{P}^4$, give the following result.

THEOREM 1.3. *Suppose that the eigenvalues of $A \in GL(4, \mathbb{Z})$ are two conjugate pairs $\lambda_j, \bar{\lambda}_j, j = 1, 2$, such that neither $\bar{\lambda}_1/\lambda_1$ nor $\bar{\lambda}_2/\lambda_2$ is a root of unity, and $|\lambda_1| < 1 < |\lambda_2|$. Then neither $\{d_1^{(n)}\}_{n \in \mathbb{N}}$ nor $\{d_3^{(n)}\}_{n \in \mathbb{N}}$ is given by a linear recurrence. In particular, there is no map \tilde{f}_A satisfying (1.2) for $p = 1$ or $p = 3$.*

In §2 we prove the first part of Theorem 1.1, which involves matrices with essentially real eigenvalues; the second part is proved in §3.

2. Monomial mappings: linear recurrence for $d_1^{(n)}$

Recall the linear functions L_C in (1.6) and the cells S_C in (1.7). From the form of $D(A)$ in (1.5), we see that the boundary ∂S_C is contained in a union of hyperplanes in the set $\{A = (a_{i,j})\} \subset \mathbb{M}_k$ of $k \times k$ matrices. These hyperplanes are defined, for fixed i, j and m , by $\{a_{i,j} = 0\}, \{a_{i,j} = a_{m,j}\}, \{\sum_{\sigma} a_{i,\sigma} = 0\}$ and $\{\sum_{\sigma} a_{i,\sigma} = \sum_{\sigma} a_{m,\sigma}\}$. We now make two observations about linear recurrences.

LEMMA 2.1. *Let a $k \times k$ matrix A be given. For fixed $1 \leq i, j \leq k$, the sequence of i, j elements $\{(A^n)_{i,j}\}_{n \in \mathbb{N}}$ satisfies (1.3), and thus, for fixed $C \in \mathcal{C}$, the sequence $\{L_C(A^n)\}_{n \in \mathbb{N}}$ satisfies (1.3).*

Proof. By the Cayley–Hamilton theorem, the characteristic polynomial satisfies $\chi(A) = 0$. Thus each entry $(A^n)_{i,j}$ satisfies (1.3), and so the lemma follows. □

COROLLARY 2.2. *Suppose A is a $k \times k$ integer matrix and $N < \infty$ is such that $A^n \in S_C$ for some fixed $C \in \mathcal{C}$ and for all $n \geq N$. Then the degree sequence $\{d_1^{(n)}\}$ for f_A satisfies a linear recurrence relation with constant coefficients.*

Let $\lambda_1, \dots, \lambda_t$ denote the distinct non-zero eigenvalues of A , and let $\mu_j \geq 1$ denote the size of the largest Jordan block corresponding to the eigenvalue λ_j . Then there exist constants $\alpha_{i,j}(s, \ell)$, not depending on n , such that

$$(A^n)_{i,j} = \sum_{\ell=1}^t \left(\sum_{s=0}^{\mu_{\ell}-1} \alpha_{i,j}(s, \ell) \binom{n}{s} \lambda_{\ell}^{n-s} \right). \tag{2.1}$$

Case 1. All eigenvalues are positive: $\lambda_1 > \dots > \lambda_t > 0$. For a linear functional $L : \mathbb{M}_k \rightarrow \mathbb{R}$, we define $Q_L(n) := L(A^n)$. With reference to (2.1), we set

$$Q_{L,\ell}(n) = \sum_{s=0}^{\mu_\ell-1} L(\alpha_{i,j}(s, \ell)) \binom{n}{s} \lambda_\ell^{n-s},$$

and so we have

$$Q_L(n) = \sum_{\ell=1}^t Q_{L,\ell}(n). \tag{2.2}$$

LEMMA 2.3. *In Case 1, there is an integer N such that either $Q_L(n) \geq 0$ for all $n \geq N$ or $Q_L(n) \leq 0$ for all $n \geq N$.*

Proof. Let us look at the form of $Q_{L,\ell}(n)$. It is a sum involving terms $\binom{n}{s}$, which are polynomial in n , and terms λ_ℓ^{n-s} , which are exponential in n . Thus $Q_{L,\ell}(n)$ may be written as a polynomial $P_\ell(n)$ in n multiplied by λ_ℓ^n . Let ℓ_0 denote the first value of ℓ for which P_{ℓ_0} is not the zero polynomial. It follows that $|P_{\ell_0}(n)|$ grows like a power of n , multiplied by $\lambda_{\ell_0}^n$. Summing over the remaining ℓ , we see that $Q_L(n)$ exhibits the same kind of growth. Thus $Q_L(n)$ must ultimately be either non-negative or non-positive. \square

LEMMA 2.4. *In Case 1, there exists a positive integer N such that for all $n \geq N$, A^n belongs to one particular cell.*

Proof. For $C \in \mathcal{C}$, we consider a linear functional that defines one of the sides of a cell S_C . By Lemma 2.3, we know that the sequence $L(A^n)$ is ultimately either non-negative or non-positive. By applying this knowledge to all of the linear functionals defining S_C , we find that for all n sufficiently large, either $A^n \in S_C$ or $A^n \notin S_C$. On the other hand, the set of all $S_C, C \in \mathcal{C}$, exhausts \mathbb{M}_k ; therefore A^n must belong to one particular cell. \square

Case 2. *Eigenvalues whose modulus is greater than or equal to 1 are positive.* Let us choose t_0 such that the eigenvalues of A are given by $\lambda_1 > \lambda_2 > \dots > \lambda_{t_0} \geq 1 > |\lambda_{t_0+1}| \geq \dots \geq |\lambda_{t_1}| > 0$. As before, we let $L : \mathbb{M}_k \rightarrow \mathbb{R}$ be a linear functional, but now we suppose that it has integer coefficients. Then, with Q_L and $Q_{L,\ell}$ as above, we define $R_L(n)$ by

$$Q_L(n) = \sum_{\ell=1}^{t_0} Q_{L,\ell}(n) + R_L(n).$$

LEMMA 2.5. *In Case 2, there is an integer N such that either $Q_L(n) \geq 0$ for all $n \geq N$ or $Q_L(n) \leq 0$ for all $n \geq N$.*

Proof. As in the proof of Lemma 2.3, each $Q_{L,\ell}(n)$ is either identically zero or grows like a power of n times λ_ℓ^n . The conclusion of the lemma must hold, then, unless $Q_{L,\ell}(n) = 0$ for ℓ equal to $1, \dots, t_0$. This means that $Q_L(n) = R_L(n)$. But now we recall that L has integer coefficients, so that $Q_L(n) \in \mathbb{Z}$. On the other hand, $R_L(n)$ is made up of exponentials with modulus less than 1, so we have $R_L(n) \rightarrow 0$ as $n \rightarrow \infty$. This can happen only if $Q_L(n) = 0$ for all n . \square

We observe that the hyperplanes bounding S_C are defined by functions with integer coefficients; so, as in Lemma 2.4, we deduce the following.

LEMMA 2.6. *In Case 2, there exists a positive integer N such that for all $n \geq N$, A^n belongs to one particular cell.*

THEOREM 2.7. *Let A be a $k \times k$ integer matrix with rank k . Suppose that for every eigenvalue λ of A with $|\lambda| \geq 1$, $\bar{\lambda}/\lambda$ is a root of unity; then $\{d_1^{(n)}\}_{n \in \mathbb{N}}$ satisfies a linear recurrence. If the eigenvalues with $|\lambda| \geq 1$ are all positive, then the degree sequence satisfies the linear recurrence relation given by the characteristic polynomial of A .*

Proof. For each such eigenvalue λ with $|\lambda| \geq 1$, we choose $\tau > 1$ such that $(\lambda/|\lambda|)^\tau = 1$. Let $\tilde{\tau}$ be the least common multiple of all such τ . Now the eigenvalues of A^τ are in Case 2. By Lemma 2.6, we find that for fixed ν with $0 \leq \nu \leq \tilde{\tau} - 1$, $\{d_1^{\tilde{\tau}n+\nu}\}$ satisfies a linear recurrence whose coefficients are given by the characteristic polynomial of A^τ . Thus the full sequence also satisfies a linear recurrence. \square

3. *Monomial mappings: no linear recurrence for $d_1^{(n)}$*

We will use a fact from combinatorics (see [S, Ch. 4]): if $\{c_k\}$ and $\{d_k\}$ both satisfy linear recurrence relations, then the sequence of indices k for which $c_k - d_k = 0$ is eventually periodic.

PROPOSITION 3.1. *Suppose that $A = (a_{i,j})$ is an integer matrix of rank k , and suppose A has exactly two eigenvalues $\delta, \bar{\delta}$ of maximum modulus, and $\bar{\delta}/\delta$ is not a root of unity. Then the degree sequence $d_1^{(n)}$ for f_A does not satisfy a linear recurrence.*

Proof. Let m denote the size of the largest Jordan block with eigenvalue δ or $\bar{\delta}$. We choose $C \in \mathcal{C}$ such that $A^n \in C$ for infinitely many n . Writing $\lambda_\ell, 1 \leq \ell \leq t$, for the other eigenvalues of A , we have

$$c_n := L_C(A^n) = \sum_{s=0}^{m-1} \binom{n}{s} \operatorname{Re}(\beta(s)\delta^{n-s}) + \sum_{\ell=1}^t \left(\sum_{s=0}^{\mu_\ell-1} \alpha_{i,j}(s, \ell) \binom{n}{s} \lambda_\ell^{n-s} \right).$$

For the values of n such that $A^n \in S_C$ we have $d_1^{(n)} = L_C(A^n)$. Since $d_1^{(n)}$ grows like $|\delta|^n$, and since $|\lambda_j| < |\delta|$, we see that not all the coefficients $\beta(s)$ can be equal to zero.

By Lemma 2.1, $\{c_n\}$ satisfies a linear recurrence. If $\{d_1^{(n)}\}$ also satisfies a (possibly different) linear recurrence, then by the combinatorial fact above, the indices for which $d_1^{(n)} = c_n$ are eventually periodic. This means that they agree for n belonging to an arithmetic progression $An + B$. Now we write $\eta = (\delta/|\delta|)^A$. Since η is not a root of unity, the numbers $\operatorname{Re}(\eta^n)$ are dense in the unit circle. Restricting to this arithmetic sequence, we have

$$d_1^{(An+B)} = c_{An+B} = |\delta|^{An} \sum_{s=0}^{m-1} \binom{An+B}{s} \operatorname{Re}(\delta^{B-sA} \beta(s)\eta^n) + O(|\lambda|^{An}).$$

We let s_0 denote the largest value of s such that $\beta(s_0) \neq 0$, and for large n this will give the dominant term in the summation. However, there are arbitrarily large values of n for which

$$\operatorname{Re}(\dots) \leq -\frac{1}{2} |\delta^{B-s_0A} \beta(s_0)|.$$

So, for such a value of n which is sufficiently large, $d_1^{(n)}$ will be negative, which is a contradiction. \square

Proof of Theorem 1.1. This follows directly from Theorems 2.7 and 3.1. □

Proof of Theorem 1.2. The characteristic polynomial of A is an irreducible cubic, so $\bar{\lambda}/\lambda$ cannot be a root of unity. Theorem 1.2 then follows from Theorem 1.1. □

Example of Hasselblatt and Propp. The example given in [HP] is

$$f_A(x_1, x_2, x_3) := \left(\frac{x_2}{x_1}, \frac{x_3}{x_1}, x_1 \right).$$

The eigenvalues of A are a conjugate pair $\lambda, \bar{\lambda}$ with $|\lambda| > 1$, and $\bar{\lambda}/\lambda$ is not a root of unity. By [HP] (or by Proposition 3.1), the degree sequence $d_1^{(n)} = \deg(f^n)$ does not satisfy any linear recurrence relation with constant coefficients. On the other hand, the inverse map is a polynomial map

$$g := f_A^{-1}(x_1, x_2, x_3) = (x_3, x_1x_3, x_2x_3).$$

By Theorem 2.7, the degree sequence $d_k^- := \deg(f_A^{-k})$ satisfies $d_k^- = d_{k-3}^- + d_{k-2}^- + d_{k-1}^-$.

Finally, we show that in fact g can be made 1-regular in the sense of (1.2). We start with the induced map on \mathbb{P}^3 ,

$$g : [x_0 : x_1 : x_2 : x_3] \mapsto [x_0^2 : x_0x_3 : x_1x_3 : x_2x_3].$$

The indeterminacy locus is $I(g) = \{x_0 = x_3 = 0\} \cup \{e_3 := [0 : 0 : 0 : 1]\}$. The orbits of the exceptional hypersurfaces are

$$\begin{aligned} g : \{x_0 = 0\} &\mapsto \{x_0 = x_1 = 0\} \mapsto e_3 \in I(g) \\ \{x_3 = 0\} &\mapsto [1 : 0 : 0 : 0] \mapsto [1 : 0 : 0 : 0]. \end{aligned}$$

We see that g does not satisfy (1.2) by the criterion of [FS]: an exceptional hypersurface is mapped, after two iterates, completely inside the indeterminacy locus. Consider the complex manifold $\pi : X \rightarrow \mathbb{P}^3$ obtained by blowing up the point e_3 and then the line $\Sigma_{01} = \{x_0 = x_1 = 0\}$. We denote by E_3 and S_{01} the exceptional fibers over e_3 and Σ_{01} , respectively. Under the induced map g_X , we have

$$g_X : \Sigma_0 := \{x_0 = 0\} \mapsto S_{01} \cap \Sigma_0 \mapsto E_3 \cap \Sigma_{01} \mapsto E_3 \cap \Sigma_0 \mapsto S_{01} \cap \Sigma_0.$$

It follows from [BK, Theorem 1.4] that the induced map g_X is 1-regular. And, by duality, $g_X^{-1} = \tilde{f}_A$ is 2-regular.

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