A TRANSCENDENTAL DYNAMICAL DEGREE

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ABSTRACT. We give an example of a dominant rational selfmap of the projective plane whose dynamical degree is a transcendental number.

INTRODUCTION

The most fundamental dynamical invariant of a dominant rational selfmap $f\colon X \dashrightarrow X$ of a smooth projective variety is, arguably, the (first) *dynamical degree* $\lambda(f)$. It can be defined as the limit $\lim_{n\to\infty}(f^{n*}H \cdot H^{\dim X-1})^{1/n}$ for any ample divisor H on X. Its value does not depend on the choice of H, and it is also invariant under birational conjugacy: if $h\colon X' \dashrightarrow X$ is a birational map, then $f' := h \circ f \circ h^{-1}\colon X' \dashrightarrow X'$ is a dominant rational map with $\lambda(f') = \lambda(f)$.

The dynamical degree is often difficult to compute. When f is *algebraically stable* in the sense that $f^{n*} = f^{*n}$ on the Picard group $\mathrm{Pic}(X)$ of X [Sib99], $\lambda(f)$ is equal to the spectral radius of the $\mathbb{Z}$-linear operator $f^*\colon \mathrm{Pic}(X)\otimes_{\mathbb{Z}}\mathbb{R} \to \mathrm{Pic}(X)\otimes_{\mathbb{Z}}\mathbb{R}$, which implies that $\lambda(f)$ is an algebraic integer. For certain classes of maps, such as birational maps of $\mathbb{P}^2$ [DF01] or polynomial maps of $\mathbb{A}^2$ [FJ07, FJ11], we can achieve algebraic stability after birational conjugation; hence the dynamical degree is an algebraic integer in these cases. It has been shown, moreover, that there are only countably many different dynamical degrees among all rational maps, algebraically stable or not [BF00, Ure18].

All of this leads naturally to the question [Via08, p.1379]: is the dynamical degree always an algebraic integer, or at least an algebraic number? The answer is negative.

Main Theorem. *There exists a dominant rational map $f\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ whose dynamical degree is a transcendental number.*

Our examples are completely explicit, of the form $f = \tau\circ\sigma$, where

$$\sigma(y_1, y_2) = \left(-y_1\frac{1-y_1+y_2}{1-y_1-y_2}, -y_2\frac{1+y_1-y_2}{1-y_1-y_2}\right)$$

is a fixed birational involution and $\tau(y_1, y_2) = (y_1^a y_2^b, y_1^{-b} y_2^a)$ is monomial. We show that if $(a+bi)^n \notin \mathbb{R}$ for all integers $n > 0$, then $\lambda(f)$ is transcendental. Favre [Fav03] showed that under the same condition on $a + bi$, the monomial map τ cannot be birationally conjugated to an algebraically stable map, though $\lambda(f) = |a + bi|$ is in fact a quadratic integer. A large class of 'volume preserving' maps, including f, were

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considered by the second author and J.-L. Lin in [DL16] where it was shown that failure of stabilizability for τ implies the same for f.

Our first step toward showing $\lambda(f)$ is transcendental is to relate degrees for iterates of τ to those of f. Writing $D_j := \deg \tau^j := (\tau^{j*}H \cdot H)$ for $j \geq 0$, with $H \subset \mathbb{P}^2$ a line, we show that the dynamical degree $\lambda = \lambda(f)$ is the unique positive solution to the equation

$$\sum_{j=1}^{\infty} D_j \lambda^{-j} = 1. \qquad (\star)$$

In order to derive $(\star)$, we study the action of σ and τ on the space of valuations on the function field of $\mathbb{P}^2$, and on the group of (Cartier) b-divisor classes on $\mathbb{P}^2$. These actions are functorial and well-suited to study the degree growth of rational maps, see [FJ07, BFJ08, Can11, FJ11]. In our study, we also use the fact that τ and σ interact well with the toric structure of $\mathbb{P}^2$. This is of course quite clear for the monomial map τ, but less evident for the involution σ, which is not monomial, but preserves a torus-invariant rational 2-form, see [DL16].

One computes by elementary means that $D_j = \mathrm{Re}(\gamma(j)\zeta^j)$, where $\zeta = a + bi$ and $\gamma(j) \in \{-2, \pm 2i, 1 \pm 2i\}$ is chosen to be whichever element maximizes the right side. The condition $\zeta^n \notin \mathbb{R}$ means that the argument of ζ is $2\pi\theta$, where θ is irrational. Were θ rational, the Gaussian integer $\gamma(j)$ would be periodic in j, the analytic function

$$D(z) := \sum_{j \geq 1} D_j z^j$$

rational, and λ algebraic. However, as Hasselblatt–Propp [HP07] observed, when θ is irrational, the sequence $(D_j)_{j \geq 1}$ does not satisfy any finite linear recurrence relation.

One suspects under these circumstances that $D(z)$ should be transcendental for *any* algebraic number $z \neq 0$. There are many results of this type in the literature, see e.g. [Nis96, FM97, AC03, AC06, Beu06, AB07a, AB07b, BBC15], but we were not able to locate any that applies in our setting. Instead, we develop a tailor-made method based on results by Evertse and others on S-unit equations, see [EG]; these in turn rely on the p-adic Subspace Theorem by Schlickewei [Sch77]. Our method draws inspiration from earlier work of Corvaja and Zannier [CZ02] and Adamczewski and Bugeaud [AB07a, AB07b], who used the subspace theorem to establish transcendence of special values of certain classes of power series.

The idea is that if m/n is a continued fraction approximant of θ, then ζ^n is nearly real, the Gaussian integers $\gamma(j)$ are nearly n-periodic in j, and $D(z)$ is well-approximated by the rational function $D^{(n)}(z) = (1 - z^n)^{-1} \sum_{j=1}^{n} D_j z^j$ obtained by assuming the $\gamma(j)$ are precisely n-periodic. If the approximations improve sufficiently quickly with n and z is algebraic, then $D_n(z)$ approximates $D(z)$ too well for the latter to also be algebraic. Unfortunately this seems a little too much to hope for, or at least more than is actually known about θ.

To deal with the possibility that θ is badly approximable by rational numbers, we need a more subtle argument, which uses another result on unit equations, this time by Evertse, Schlickewei and Schmidt [ESS02]. In addition, Evertse's theorem on S-unit equations does not apply to the rational functions $D_n(z)$, and instead we work with related but slightly more complicated functions, see §3 for details.

Dynamical degrees play a key role in algebraic, complex and arithmetic dynamics. To any dominant rational map $f\colon X \dashrightarrow X$ of a projective variety X over a field K is in fact associated a sequence $(\lambda_p(f))_{p=1}^{\dim X}$ of dynamical degrees, see [DS05, Tru18, Dan17]; the dynamical degree above corresponds to $p = 1$.

Naturally defined in the context of algebraic dynamics, dynamical degrees were first introduced in *complex* dynamics by Friedland [Fri91], who showed that when $K = \mathbb{C}$ and f is a holomorphic endomorphism, the topological entropy of f is given by $\log \max_p \lambda_p(f)$; this generalized earlier work by Gromov, see [Gro03], and was later extended (as an inequality) by Dinh and Sibony [DS05] to the case of dominant rational maps.[1] Dynamical degrees are furthermore essential for defining and analyzing natural invariant currents and measures, see for example [RS97, Gue10, DS17] and the references therein.

In complex dimension two, the only relevant degrees are the first $\lambda_1 = \lambda$ and last λ_2 (the 'topological degree', equal to the number of preimages of a typical point). Their relationship determines which of two types of dynamical behavior (saddle or repelling) predominates (see [DDG1-3] and [Gue05]). The class of examples we consider here includes both types. If, for instance, $\zeta = 1 + 2i$, then we obtain a map f of *small topological degree*

$$\lambda_2(f) = \lambda_2(\tau) = |\zeta|^2 = 5 < \lambda_1(f) = 6.8575574092\dots.$$

Squaring to get $\zeta = -3 + 4i$, gives a map with *large topological degree* $\lambda_2(f) = 25 > \lambda_1(f) = 13.4496076817\dots$.

In *arithmetic* dynamics, K is a global field, and the (first) dynamical degree serves as an upper bound for the asymptotics of the growth of heights along orbits [Sil12, KS16, Mat16]; the question of when equality holds is part of the *Kawaguchi–Silverman conjecture*, which recently has attracted a lot of attention.

As already mentioned, the set of all possible dynamical degrees is countable, and our Main Theorem shows that it contains transcendental numbers. It would obviously be interesting to say more about it. Note that the set of dynamical degrees of birational surface maps is much better understood, see e.g. [BK06, McM07, Ueh16, BC16].

It would also be interesting to study the complex and arithmetic dynamics of the rational map $f = f_\zeta$ considered here. For example, does f admit a unique measure of maximal entropy, and is the topological entropy equal to $\log \lambda(f)$? The fact that f is defined over $\mathbb{Q}$ may be useful, see [JR18]. On the arithmetic side, one may ask whether the Kawaguchi–Silverman conjecture holds: does every point with Zariski dense orbit have arithmetic degree equal to $\lambda(f)$? Note that what we call the Kawaguchi–Silverman conjecture is part (d) of [KS16, Conjecture 6]. Given our Main Theorem, the existence of a point as above would in fact contradict part (b); see also [LS19].

[1] Motivated by these results, the logarithm of the first dynamical degree is sometimes called the "algebraic entropy" [BV98], but this ignores the role of the higher dynamical degrees.

The paper is organized as follows. In §§1-2 we establish the formula ($\star$), using the action of rational maps on valuations and b-divisor classes, largely following [BFJ08]. Then, in §3 we prove that the solution to ($\star$) is transcendental.

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1. Dynamics of dominant rational maps of the projective plane

In this section we study dominant rational selfmaps of $\mathbb{P}^2$ using the induced action on valuations and b-divisor classes. The exposition largely follows [BFJ08] but with additional attention paid to the structure of $\mathbb{P}^2$ as a toric variety.

1.1. **Setup.** We work over a field K of characteristic zero, which we for convenience assume is algebraically closed.[2] Fix homogeneous coordinates $[x_0 : x_1 : x_2]$ on $\mathbb{P}^2$ and use affine coordinates $(y_1, y_2) = (x_1/x_0, x_2/x_0)$ on the affine chart $\{x_0 \neq 0\} \simeq \mathbb{A}^2$.

1.2. **Rational maps and their degrees.** A dominant rational selfmap of $\mathbb{P}^2$ is given in homogeneous coordinates by

$$f\colon [x_0 : x_1 : x_2] \mapsto [P(x_0, x_1, x_2) : Q(x_0, x_1, x_2) : R(x_0, x_1, x_2)],$$

where P, Q and R are homogeneous polynomials of the same degree $d \geq 1$, and with no factor in common. The integer $\deg f := d$ is called the *degree* of f. The sequence $(\deg f^n)_{n\geq 1}$ is submultiplicative; hence the limit

$$\lambda(f) := \lim_{n\to\infty} (\deg f^n)^{1/n} \in [1, \infty)$$

exists and is equal to the dynamical degree of f as defined in the introduction.

1.3. **Monomial maps.** Any nondegenerate 2×2 matrix $A = (a_{ij})_{i,j}$ with integer coefficients defines a dominant rational self map $\tau_A\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, which in affine coordinates is given by $\tau_A\colon (y_1, y_2) \mapsto (y_1^{a_{11}} y_2^{a_{21}}, y_1^{a_{12}} y_2^{a_{22}})$. Such a rational map is called a *monomial map*. Note that $\tau_A^n = \tau_{A^n}$. The degree of τ_A is given by

$$\deg \tau_A = \max\{0, a_{11} + a_{21}, a_{12} + a_{22}\} + \max\{0, -a_{11}, -a_{12}\} + \max\{0, -a_{21}, -a_{22}\},$$

[2] For the purposes of this section, assuming K algebraically closed is not a restriction, since degrees of rational maps are invariant under ground field extension.

see [HP07, BK08]. We can view A as a linear selfmap of $\mathbb{R}^2$. Now identify $\mathbb{R}^2$ with $\mathbb{C}$, and assume that A is given by multiplication with a Gaussian integer ζ, so that

$$A = \begin{pmatrix} \operatorname{Re}\zeta & -\operatorname{Im}\zeta \\ \operatorname{Im}\zeta & \operatorname{Re}\zeta \end{pmatrix}$$

In this case, we write τ_ζ instead of τ_A. We have

$$\begin{aligned} \deg \tau_\zeta &= \max\{0, \operatorname{Re}\zeta + \operatorname{Im}\zeta, \operatorname{Re}\zeta - \operatorname{Im}\zeta\} \\ &\quad + \max\{0, -\operatorname{Re}\zeta, -\operatorname{Im}\zeta\} + \max\{0, \operatorname{Im}\zeta, -\operatorname{Re}\zeta\} \\ &= \max\{\operatorname{Re}\zeta + 2\operatorname{Im}\zeta, 2\operatorname{Im}\zeta, -2\operatorname{Re}\zeta, -2\operatorname{Im}\zeta, \operatorname{Re}\zeta - 2\operatorname{Im}\zeta\}. \end{aligned} \tag{1.1}$$

Note that $(\tau_\zeta)^n = \tau_{\zeta^n}$ for $n \geq 1$. It follows easily that the dynamical degree of τ_ζ is $|\zeta|$, but Hasselblatt and Propp [HP07] (see also [Fav03, BK08]) proved that the sequence $\deg(\tau_\zeta^n)_{n\geq 1}$ does not satisfy a linear recursion when ζ has the property that $\zeta^n \notin \mathbb{R}$ for $n \geq 1$. As a concrete example, we can take $\zeta = 1 + 2i$, in which case the monomial map is given by $\tau_\zeta\colon (y_1, y_2) \mapsto (y_1 y_2^2, y_2 y_1^{-2})$.

1.4. **Valuations and tropicalization.** The function field of $\mathbb{P}^2$ is $K(\mathbb{P}^2) = K(y_1, y_2)$. Let V be the set of valuations $v\colon K(\mathbb{P}^2)^* \to \mathbb{R}$ that are trivial on the ground field K. We extend any such valuation to $K(\mathbb{P}^2)$ by $v(0) := +\infty$.

The choice of affine coordinates (y_1, y_2) defines a *tropicalization map*

$$\operatorname{trop}\colon V \to \mathbb{R}^2$$

by evaluation: $\operatorname{trop}(v) := (v(y_1), v(y_2))$. The tropicalization map has a canonical section $\mathbb{R}^2 \to V$ whose image is the set V^{mon} of all *monomial* (or *toric*) valuations. More precisely, the image of $t = (t_1, t_2) \in \mathbb{R}^2$ is the valuation v_t on $K(y_1, y_2)$ whose restriction to $K[y_1, y_2]$ is given by

$$v_t\Big(\sum_{i_1, i_2 \geq 0} a_{i_1 i_2} y_1^{i_1} y_2^{i_2}\Big) = \min\{t_1 i + t_2 i_2 \mid a_{i_1 i_2} \neq 0\}.$$

1.5. **Blowups and centers.** By a *blowup* of $\mathbb{P}^2$ we mean a birational morphism $\pi\colon X_\pi \to \mathbb{P}^2$, where X_π is a smooth surface. Up to isomorphism, π is then a finite composition of point blowups. If π and π' are blowups of $\mathbb{P}^2$, we say that π' dominates π, written $\pi' \geq \pi$, if there exists a birational morphism $\mu\colon X_{\pi'} \to X_\pi$ such that $\pi' = \pi \circ \mu$. Any two blowups, can be dominated by a third, so the set $\mathfrak{B}$ of isomorphism classes of blowups is a directed set.

If $\pi\colon X_\pi \to \mathbb{P}^2$ is a blowup, then π induces an isomorphism between function fields, so we can view any $v \in V$ as a valuation on $K(X_\pi)$. Since X_π is proper, any $v \in V$ admits a *center* on X_π; this is the unique scheme-theoretic point $\xi = c_{X_\pi}(v) \in X_\pi$ such that $v \geq 0$ on the local ring $\mathcal{O}_{X_\pi,\xi}$ and $v > 0$ on its maximal ideal. If $\pi' \geq \pi$, then the associated map $X_{\pi'} \to X_\pi$ sends $c_{X_{\pi'}}(v)$ to $c_{X_\pi}(v)$.

The center of v is the generic point of X_π iff v is the trivial valuation on $K(\mathbb{P}^2)$. A valuation is *divisorial* iff there exists a blowup $\pi\colon X_\pi \to \mathbb{P}^2$ such that $c_{X_\pi}(v)$ is the generic point of a prime divisor E on X_π; then $v = c \operatorname{ord}_E$, where $c > 0$ and $\operatorname{ord}_E\colon K(\mathbb{P}^2)^* \to \mathbb{Z}$ is the order of vanishing along E. Further, π admits a factorization $\pi' = \pi \circ \mu$, where μ is a birational morphism and $\pi\colon X_\pi \to \mathbb{P}^2$ is the composition of the blowup $\pi_j \colon X_j \to X_{j-1}$, $1 \le j \le n$ at the center of v on X_{j-1}. Here $X_0 = \mathbb{P}^2$ and $X_n = X_\pi$.

1.6. **Centers and tropicalization.** If $\pi \colon X_\pi \to \mathbb{P}^2$ is a blowup, write $Z_{\pi,i}$ for the pullback of the divisor $\{x_i = 0\}$, $i = 0, 1, 2$ on $\mathbb{P}^2$. Then $Z_\pi := \sum_{i=0}^2 Z_{\pi,i}$ is an effective divisor with simple normal crossings support. The following technical result will be used later on.

Lemma 1.1. *Let $E \subset X_\pi$ be a π-exceptional prime divisor such that the associated divisorial valuation ord_E is non-monomial and $t := \mathrm{trop}(\mathrm{ord}_E) \neq 0$. Then:*

(a) *the center of the monomial valuation v_t on X_π is a prime divisor;*
(b) *the divisors $Z_{\pi,i}$, $i = 0, 1, 2$ are proportional in a neighborhood of E;*
(c) *for any valuation $v \in V$ with $c_{X_\pi}(v) \in E$, we have $\mathrm{trop}(v) \in \mathbb{R}_+^* t$.*

Proof. By assumption, $\eta := \pi(E) \in \mathbb{P}^2$ is a closed point, which furthermore lies on the curve $\{x_0x_1x_2 = 0\}$, since otherwise $\mathrm{trop}(\mathrm{ord}_E) \neq 0$. There are two cases:

(i) η is a regular point on $\{x_0x_1x_2 = 0\}$. For definiteness, we assume η lies on $\{x_2 = 0\}$ but not on $\{x_i = 0\}$, $i = 0, 1$.
(ii) η is a singular point on $\{x_0x_1x_2 = 0\}$, say η lies on $\{x_1 = 0\}$ and $\{x_2 = 0\}$ but not on $\{x_0 = 0\}$.

In case (i), we have $t = (0, t_2)$, where $t_2 > 0$, so the center of the monomial valuation v_t on X_π is equal to the strict transform of the curve $\{x_2 = 0\}$, proving (a). Further, E is disjoint from the support of $Z_{\pi,0}$ and $Z_{\pi,1}$, but contained in the support of $Z_{\pi,2}$. Thus (b) is clear, and (c) follows since $v(y_1) = 0 < v(y_2)$.

Now consider case (ii). Then E is disjoint from $Z_{\pi,0}$ but contained in the supports of $Z_{\pi,1}$ and $Z_{\pi,2}$. Let us first prove that (b) implies (c). Thus assume there exist integers $a_1, a_2 > 0$ such that $a_1 Z_{\pi,x_2} = a_2 Z_{\pi,x_1}$ in a neighborhood of E. This implies that the rational function $y_2^{a_1}/y_1^{a_2}$ is regular and nonvanishing in a neighborhood of E. As a consequence, we have $a_1 v(y_2) = a_2 v(y_1) > 0$ for all v centered on E, so $\mathrm{trop}(v)$ is proportional to the vector $(a_1, a_2) \in \mathbb{R}^2$.

It remains to prove (a) and (b). Up to isomorphism, we have $\pi = \pi' \circ \mu$, where $\pi' \colon X_{\pi'} \to \mathbb{P}^2$ is defined by successively blowing up the center of ord_E, and μ is a further composition of point blowups. By an easy induction on the number of blowups in μ, we reduce to the case $\pi = \pi'$. Thus $\pi = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_n$, where $\pi_j \colon X_j \to X_{j-1}$ is the blowup at η_{j-1}, the center of v on X_{j-1}, with exceptional divisor $E_{j-1} \subset X_j$. Here $X_0 = \mathbb{P}^2$, $\eta_0 = \eta$, and $X_n = X_\pi$. Let $Z_{j,i}$ be the pullback to X_j of the line $\{x_i = 0\}$, $i = 0, 1, 2$, and set $Z_j := \sum_{i=0}^2 Z_{j,i}$. Then η_j lies in the support of Z_j for $0 \le j < n$.

Let $m \in \{0, 1, \dots, n-1\}$ be the largest integer such that η_j is a singular point of the support of Z_j for $0 \le j \le m$. The valuation ord_{E_j} is then monomial for $0 \le j \le m$. Thus $m < n - 1$, since ord_E is not monomial, so η_{m+1} is a regular point of E_m. Let a_1 and a_2 be the coefficients of E_m of $Z_{m+1,1}$ and $Z_{m+1,2}$, respectively. Then the rational function $y_1^{a_2}/y_2^{a_1}$ on X_{m+1} is regular and nonvanishing at η_{m+1}. It follows that the same function on $X_\pi = X_n$ is regular and nonvanishing in a neighborhood of E. This implies that $a_1 Z_{\pi,2} = a_2 Z_{\pi,1}$ in a neighborhood of E, proving (b). Further, the center of the monomial valuation v_t on X_π is the strict transform of E_m, so (a) holds as well. $\square$

1.7. **Induced map on valuations.** Any dominant rational map $f\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ induces an endomorphism $f^*\colon K(\mathbb{P}^2) \to K(\mathbb{P}^2)$, and in turn a selfmap of V that we also denote by f and is given by $f(v)(\phi) := v(f^*\phi)$ for $\phi \in K(\mathbb{P}^2)^*$.

Any monomial map $\tau = \tau_A$ preserves the set V^{mon} of monomial valuations: we have $\tau_A(v_t) = v_{A^*t}$ for $t \in \mathbb{R}^2$, where A^* is the transpose of A. It further commutes with the tropicalization map in the sense that

$$\begin{array}{ccc} V & \xrightarrow{\tau_A} & V \\ \downarrow{\scriptstyle \mathrm{trop}} & & \downarrow{\scriptstyle \mathrm{trop}} \\ \mathbb{R}^2 & \xrightarrow{A^*} & \mathbb{R}^2 \end{array}$$

If we identify $\mathbb{R}^2 \simeq \mathbb{C}$ and A is given by multiplication with $\zeta \in \mathbb{Z} + \mathbb{Z}i$, then A^* is given by multiplication by $\overline{\zeta}$.

1.8. **b-divisor classes.** For any blowup π, denote by $\mathrm{Pic}(X_\pi)$ the Picard group on X_π, i.e. the set of linear equivalence classes of divisors on X_π. When $\pi' \geq \pi$, the birational morphism $\mu\colon X_{\pi'} \to X_\pi$ induces an injective homomorphism $\mu^*\colon \mathrm{Pic}(X_{\pi'}) \to \mathrm{Pic}(X_\pi)$. The group of *b-divisor classes* on $\mathbb{P}^2$ is defined as the direct limit

$$\mathcal{C} := \varinjlim_{\pi \in \mathfrak{B}} \mathrm{Pic}(X_\pi).$$ [3]

Concretely, an element of $\mathcal{C}$ is an element of $\mathrm{Pic}(X_\pi)$ for some blowup π, where two elements $\alpha' \in \mathrm{Pic}(X_\pi)$, $\alpha' \in \mathrm{Pic}(X_{\pi'})$ are identified iff they pull back to the same class on some blowup dominating both π and π'. A class in the image of $\mathrm{Pic}(X_\pi) \to \mathcal{C}$ is said to be *determined* on X_π. We write ω for the class determined by $\mathcal{O}_{\mathbb{P}^2}(1)$ on $\mathbb{P}^2$.

The projection formula implies that there is a natural intersection pairing $\mathcal{C} \times \mathcal{C} \to \mathbb{Z}$, denoted $(\alpha \cdot \beta)$ for $\alpha, \beta \in \mathcal{C}$. This is defined as the intersection pairing on X_π for any π on which α and β are determined.

A *toric blowup* of $\mathbb{P}^2$ is a blowup $\pi\colon X_\pi \to \mathbb{P}^2$, with X_π a (smooth, projective) toric surface and π equivariant under the torus action. The set of *toric b-divisor classes* $\mathcal{C}^{\mathrm{tor}} \subset \mathcal{C}$ is the direct limit $\varinjlim_\pi \mathrm{Pic}(X_\pi)$, where π runs over all toric blowups of $\mathbb{P}^2$.

1.9. **Action by rational maps on b-divisor classes.** Now consider a dominant rational map $f\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. For any blowups π, π' of $\mathbb{P}^2$ we have an induced rational map $f_{\pi\pi'}\colon X_{\pi'} \dashrightarrow X_\pi$. Given π, we can always choose π' such that $f_{\pi\pi'}$ is a morphism. Using this fact, we can define a group homomorphism

$$f^*\colon \mathcal{C} \to \mathcal{C}$$

as follows: if $\alpha \in \mathcal{C}$ is determined on X_π, pick a blowup π' such that $f_{\pi\pi'}\colon X_{\pi'} \to X_\pi$ is a morphism, and declare $f^*\alpha \in \mathcal{C}$ to be the class determined on $X_{\pi'}$ determined by $f^*_{\pi\pi'}\alpha$. This action is functorial: if f and g are dominant rational maps of $\mathbb{P}^2$, then

[3]Here b stands for birational, following Shokurov. In [BFJ08], the elements of $\mathcal{C}$ were referred to as Cartier classes on the Riemann–Zariski space of $\mathbb{P}^2$.

$(f\circ g)^* = g^*f^*$ on $\mathcal{C}$. When $\tau\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is monomial, we have $\tau^*\mathcal{C}^{\mathrm{tor}} \subset \mathcal{C}^{\mathrm{tor}}$. The degree of a rational map can be computed as follows:

$$\deg f = (f^*\omega\cdot\omega).$$

2. The degree sequence of certain rational maps

We now specialize the considerations above to a particular class of maps that will later be shown to have transcendental dynamical degrees.

2.1. **A volume preserving involution.** As in [DL16] we consider the involution $\sigma\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ defined in homogeneous coordinates by

$$\sigma\colon [x_0 : x_1 : x_2] \to [x_0(x_1+x_2-x_0) : x_1(x_2+x_0-x_1) : x_2(x_0+x_1-x_2)]$$

The automorphism $[x_0 : x_1 : x_2] \mapsto [x_1+x_2-x_0 : x_2+x_0-x_1 : x_0+x_1-x_2]$ conjugates σ to the Cremona involution $[x_0 : x_1 : x_2] \mapsto [x_1x_2 : x_2x_0 : x_0x_1]$. As a consequence, we have the following geometric description. Consider the three points $P_0 = [0 : 1 : 1]$, $P_1 = [1 : 0 : 1]$, $P_2 = [1 : 1 : 0]$ and the three lines $L_0 = \{x_0 = x_1+x_2\}$, $L_1 = \{x_1 = x_2+x_0\}$, $L_2 = \{x_2 = x_0+x_1\}$ on $\mathbb{P}^2$. Let X^0 be the blowup of $\mathbb{P}^2$ at $\{P_0, P_1, P_2\}$, with exceptional divisors E_0, E_1, E_2. Then σ induces an automorphism of X^0 of order two that sends E_i to the strict transform of L_i for $i = 0, 1, 2$.

The salient underlying feature of σ is that while it does not directly respect the torus action on $\mathbf{P}^2$, it does preserves the torus invariant form $\frac{dy_1}{y_1}\wedge\frac{dy_2}{y_2}$, and this implies that σ interacts well with torus invariant valuations and blowups.

First, σ preserves most (but not all) of the fibers of the tropicalization map.

Lemma 2.1. *If $t \notin \mathbb{R}_+\{(1,0),(0,1),(-1,-1)\}$, then $\sigma(\mathrm{trop}^{-1}(t)) = \mathrm{trop}^{-1}(t)$.*

Proof. Since σ is an involution, it suffices to prove the inclusion $\sigma(\mathrm{trop}^{-1}(t)) \subset \mathrm{trop}^{-1}(t)$. The condition on t means that if $v \in \mathrm{trop}^{-1}(t)$, then either $v(y_1), v(y_2) > 0$, $v(1/y_1), v(y_2/y_1) > 0$ or $v(1/y_2), v(y_1/y_2) > 0$. In all three cases we must show that $\sigma(v)(y_i) = v(y_i)$, $i = 1, 2$. We treat the first case and leave the other two to the reader. In affine coordinates (y_1, y_2), σ is given by

$$(y_1, y_2) \mapsto \left(-y_1\frac{1-y_1+y_2}{1-y_1-y_2}, -y_2\frac{1+y_1-y_2}{1-y_1-y_2}\right),$$

so since $v(y_i) > 0$, $i = 1, 2$, it is evident that $\sigma(v)(y_i) = v(y_i)$, $i = 1, 2$. □

Second, σ preserves the space of monomial valuations:

Lemma 2.2. *The induced map $\sigma\colon V \to V$ is the identity on the set V^{mon} of monomial valuations: we have $\sigma(v_t) = v_t$ for all $t \in \mathbb{R}^2$.*

Proof. This can be verified by direct calculation. For an argument that relies more explicitly on invariance of $\frac{dy_1}{y_1}\wedge\frac{dy_2}{y_2}$, see [DL16, p.322]. □

Now consider a monomial map $\tau = \tau_\zeta$ associated to a Gaussian integer ζ. We will construct a subset $V' \subset V \setminus V^{\mathrm{mon}}$ that is backward invariant under both σ and τ. As before, we identify $\mathbb{R}^2$ with $\mathbb{C}$. Define $N' \subset \mathbb{C}$ by

$$N' := \bigcup_{j\geq 1} \overline{\zeta}^{-j}\mathbb{R}_+^*\{1, i, -1-i\} = \bigcup_{j\geq 1} \zeta^j\mathbb{R}_+^*\{1, i, -1-i\}.$$

Assuming that $\zeta^j \notin \mathbb{R}$ for all $j \geq 1$, this is a dense set of rays in $\mathbb{C}$, but the complement is also dense. Now define a subset $V' \subset V$ by

$$V' := \bigcup_{t \in N'} \operatorname{trop}^{-1}(t) \setminus \{v_t\}.$$

Corollary 2.3. *We have $\sigma^{-1}(V') \subset V'$ and $\tau^{-1}(V') \subset V'$.*

Proof. The second inclusion follows from the construction of N', the fact that τ commutes with the tropicalization map, and that $\tau^{-1}(V^{\mathrm{mon}}) = V^{\mathrm{mon}}$. The first inclusion follows from Lemmas 2.1 and 2.2. □

Next we study the action of σ and τ on the group $\mathcal{C}$ of b-divisor classes. Define $\mathcal{C}'$ as the set of classes $\alpha \in \mathcal{C}$ that can be represented by a divisor $\sum_j c_j E_j$ on some blowup of $\mathbb{P}^2$, with $\operatorname{ord}_{E_j} \in V'$ for all j. Note that this implies $(\alpha \cdot \omega) = 0$. By convention, $0 \in \mathcal{C}'$, so $\mathcal{C}'$ is a subgroup of $\mathcal{C}$.

Corollary 2.4. *We have $\sigma^* \mathcal{C}' \subset \mathcal{C}'$ and $\tau^* \mathcal{C}' \subset \mathcal{C}'$.*

Proof. Pick $\varphi \in \{\sigma, \tau\}$ and $\alpha \in \mathcal{C}'$. We must prove that $\varphi^* \alpha \in \mathcal{C}'$. By linearity it suffices to consider the case when α is represented by a prime divisor E on X_π for some blowup π, such that $\operatorname{ord}_E \in V'$. Set $t = \operatorname{trop}(\operatorname{ord}_E) \in \mathbb{R}^2$. Then $t \in N'$, but $\operatorname{ord}_E \neq v_t$. In particular, $t \neq 0$, so Lemma 1.1 implies that the center of v_t on X_π is a prime divisor F.

Pick a blowup π' such that the rational map $\varphi_{\pi\pi'} \colon X_{\pi'} \to X_\pi$ induced by φ is a morphism. Then $\varphi^* \alpha$ is represented by the divisor $\varphi^*_{\pi\pi'} E$ on $X_{\pi'}$. Write $\varphi^*_{\pi\pi'} E = \sum_j a_j E'_j$, where $a_j > 0$ and $E'_j \subset X_{\pi'}$ is a prime divisor. We must prove that $v'_j := \operatorname{ord}_{E'_j} \in V'$ for all j. By Corollary 2.3 it suffices to show that $v_j := \varphi(v'_j) \in V'$ for all j. Now the center of v_j on X_π is contained in E, since $\varphi_{\pi\pi'}(E'_j) \subset E$, so Lemma 1.1 shows that $\operatorname{trop}(v_j) = c_j \operatorname{trop}(\operatorname{ord}_E) = c_j t$ for some $c_j > 0$. In particular, $\operatorname{trop}(v_j) \in N'$. Further, v_j is not a monomial valuation, or else the center of v_j on X_π would be equal to F, a contradiction. It follows that $v_j \in V'$. □

Next we study the action of σ on toric Cartier classes. Given a toric blowup $\pi \colon X_\pi \to \mathbb{P}^2$, let X^0_π be the blowup of X_π along the strict transform of the set $\{P_0, P_1, P_2\}$ defined above.

Lemma 2.5. *Given any toric blowup $\pi \colon X_\pi \to \mathbb{P}^2$, the induced birational map $\sigma^0_\pi \colon X^0_\pi \dashrightarrow X^0_\pi$ is a morphism.*

Proof. Suppose σ^0_π is not a morphism. As σ is an involution, we can then find a prime divisor E on X^0_π such that $\sigma^0_\pi(E)$ is a point. If the image of E on X^0 is a curve, then this curve would be contracted to a point by the involution $\sigma^0 \colon X^0 \to X^0$, a contradiction. Thus E must be exceptional for $X^0_\pi \to X^0$. Since π is a monomial blowup, this means that ord_E is a monomial divisorial valuation. But then $\sigma(\operatorname{ord}_E) = \operatorname{ord}_E$, by Lemma 2.2, so $\sigma^0_\pi(E) = E$, a contradiction. □

Lemma 2.6. *We have $\sigma^* \omega = 2\omega + \beta$, where $\beta \in \mathcal{C}$ satisfies $\tau^* \beta \in \mathcal{C}'$.*

Proof. We use the notation introduced earlier in the subsection. On X^0, ω is represented by the divisor $\frac{1}{3}\sum_{i=0}^{2}(L_i+2E_i)$, so $\sigma^*\omega$ is represented by $\frac{1}{3}\sum_{i=0}^{2}(2L_i+E_i) = 2\omega+\beta$, where $\beta := -\sum_{i=0}^{2} E_i$. It only remains to see that $\tau^*\beta \in \mathcal{C}'$. Pick a blowup $\pi\colon X_\pi \to \mathbb{P}^2$ such that τ induces a morphism $\tau_\pi\colon X_\pi \to X^0$. Then $\tau^*\beta$ is represented by the divisor $\sum_{i=0}^{2}\tau_\pi^* E_i$ on X_π. Let us prove that $\tau^*E_0 \in \mathcal{C}'$; the argument for the other two terms is similar. It suffices to prove that if E is a prime divisor on X_π such that $\tau_\pi(E) \subset E_0$, then $\mathrm{ord}_E \in V'$. Now $\tau_\pi(E)\subset E_0$ implies that $\tau(\mathrm{ord}_E)(y_1) > 0$ and $\tau(\mathrm{ord}_E)(y_2) = 0$, so $s := \mathrm{trop}(\tau(\mathrm{ord}_E)) \in \mathbb{R}_+^*(1,0)$. Further, $\tau(\mathrm{ord}_E)$ is not monomial, since otherwise $\tau_\pi(E)$ would be equal to the strict transform of the line $\{x_0 = 0\}$ on $\mathbb{P}^2$. Thus $\tau(\mathrm{ord}_E) \in \mathrm{trop}^{-1}(s)\setminus\{v_s\}$. This implies $\mathrm{ord}_E \in \mathrm{trop}^{-1}(t)\setminus\{v_t\}$, where $t = \overline{\zeta}^{-1}s \in N'$, and hence $\mathrm{ord}_E \in V'$. □

Lemma 2.7. *If $\alpha \in \mathcal{C}^{\mathrm{tor}}$ and $(\alpha\cdot\omega) = 0$, then $\sigma^*\alpha = \alpha$.*

Proof. There exists a toric blowup $\pi\colon X_\pi \to \mathbb{P}^2$ such that α is represented by a torus invariant π-exceptional divisor on X_π. By Lemma 2.5, the induced birational map $\sigma_\pi^0\colon X_\pi^0 \dashrightarrow X_\pi^0$ is a morphism. If $E \subset X_\pi^0$ is a π-exceptional prime divisor, then $\mathrm{ord}_E \in V^{\mathrm{mon}}$ and hence $\sigma(\mathrm{ord}_E) = \mathrm{ord}_E$ by Lemma 2.2. As σ_π^0 is an automorphism, this gives $(\sigma_\pi^0)^*E = E$. Thus $\sigma^*\alpha = \alpha$. □

2.2. **Degree sequence.** Let σ be the involution above, and $\tau = \tau_\zeta$ the monomial map associated to a Gaussian integer ζ such that $\zeta^n \notin \mathbb{R}$ for all $n \geq 1$. Set $f = \tau\circ\sigma$. Write

$$D_n = \deg(\tau^n) = (\tau^{n*}\omega\cdot\omega) \quad\text{and}\quad E_n = \deg(f^n) = (f^{n*}\omega\cdot\omega)$$

for $n \geq 1$. It will also be convenient to set $D_0 = 1$ and $E_0 = 2$. Our aim is to prove the following recursion formula.

Proposition 2.8. *We have $E_n = \sum_{j=0}^{n-1} E_j D_{n-j}$ for $n \geq 1$.*

Proof. We will prove the following more precise result by induction on n.

$$f^{n*}\omega = E_0\tau^{n*}\omega + \cdots + E_{n-1}\tau^*\omega \mod \mathcal{C}' \qquad (A_n)$$

$$\sigma^* f^{n*}\omega = E_0\tau^{n*}\omega + \cdots + E_{n-1}\tau^*\omega + E_n\omega + E_n\beta \mod \mathcal{C}' \qquad (B_n)$$

for $n \geq 1$. Pairing (A_n) with ω implies the desired result since any class in $\mathcal{C}'$ is orthogonal to ω.

Now (A_1) amounts to $\tau^*\sigma^*\omega = 2\tau^*\omega \bmod \mathcal{C}'$ and follows from Lemma 2.6. It therefore suffices to prove that $(A_n) \implies (B_n)$ and $(B_n) \implies (A_{n+1})$ for $n \geq 1$. The latter implication follows immediately by applying τ^* and using that $\tau^*\beta \in \mathcal{C}'$. For the former implication, note that (A_n) gives

$$f^{n*}\omega = E_n\omega + (E_0\tau^{n*}\omega + \cdots + E_{n-1}\tau^*\omega - E_n\omega) \bmod \mathcal{C}'$$

The expression in brackets lies in $\mathcal{C}^{\mathrm{tor}}$ and is orthogonal to ω. Lemmas 2.6 and 2.7 therefore give

$$\begin{aligned}\sigma^* f^{n*}\omega &= 2E_n\omega + E_n\beta + (E_0\tau^{n*}\omega + \cdots + E_{n-1}\tau^*\omega - E_n\omega) \mod \mathcal{C}' \\ &= E_0\tau^{n*}\omega + \cdots + E_{n-1}\tau^*\omega + E_n\omega + E_n\beta \mod \mathcal{C}',\end{aligned}$$

which completes the proof. □

2.3. **Dynamical degree.** Set $D(z) := \sum_{j=1}^{\infty} D_j z^j$ and $E(z) := \sum_{j=0}^{\infty} E_j z^j$. These are power series with radius of convergence equal to $|\zeta|^{-1}$ and λ^{-1}, respectively, where λ is the dynamical degree of f. Proposition 2.8 shows that

$$E(z)(1 - D(z)) = 2,$$

and since $D(z)$ has positive coefficients, this implies

Proposition 2.9. *The dynamical degree $\lambda = \lambda(f)$ satisfies $\lambda > |\zeta|$, and λ is the unique positive solution to the equation $\sum_{j=1}^{\infty} D_j \lambda^{-j} = 1$, where $D_j = \deg \tau^j$.*

Using this proposition together with the formula

$$D_j = \max\{\operatorname{Re}\zeta^j + 2\operatorname{Im}\zeta^j, 2\operatorname{Im}\zeta^j, -2\operatorname{Re}\zeta^j, -2\operatorname{Im}\zeta^j, \operatorname{Re}\zeta^j - 2\operatorname{Im}\zeta^j\}, \tag{2.1}$$

see (1.1), we will prove that λ is transcendental.

3. Proof of transcendence

In what follow, we will use Vinogradov notation: if $(a_n)_n$ and $(b_n)_n$ are sequences of positive numbers indexed by an infinite subset $A \subset \mathbb{N}$, then we write $a_n \ll b_n$ and $b_n \gg a_n$ iff there is a positive constant C such that $a_n \leq C b_n$ for large $n \in A$. We will also use various standard results about continued fraction approximants of irrational numbers, see [HW, §10] (especially Theorems 167 and 171).

3.1. **Setup.** Pick any Gaussian integer ζ such that $\zeta^n \notin \mathbb{R}$ for $n \neq 0$, and write

$$\zeta = |\zeta| e^{2\pi i\theta},$$

where $\theta \in (0,1)$ is irrational. Consider the rational map $f = f_\zeta = \tau_\zeta \circ \sigma$ as in §2. By Proposition 2.9 and (2.1), the dynamical degree $\lambda = \lambda(f_\zeta)$ is the reciprocal of the unique real and positive solution x of the equation $\sum_{j=1}^{\infty} \psi(\zeta^j) x^j = 1$, where

$$\psi(z) := \max_{\gamma \in \{-2, \pm 2i, 1 \pm 2i\}} \operatorname{Re}(\gamma z).$$

The function ψ is convex, nonnegative and piecewise $\mathbb{R}$-linear; see Figure 1.

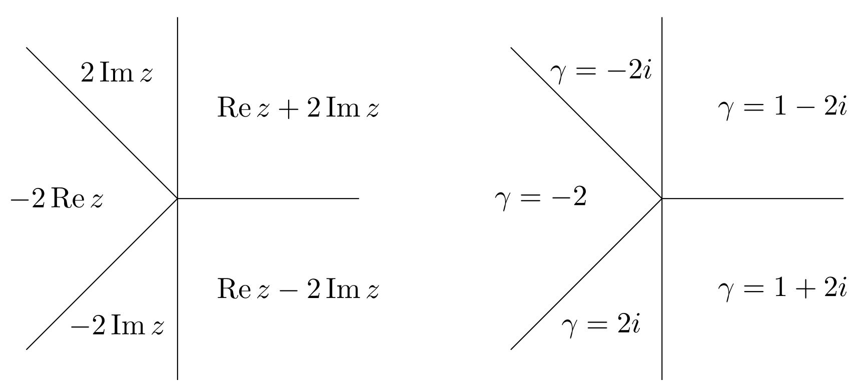

Figure 1. The picture on the left show the piecewise $\mathbb{R}$-linear function ψ on $\mathbb{C}$. On the right is shown the $\gamma \in \{-2, \pm 2i, 1 \pm 2i\}$ that realizes the maximum in the definition of ψ.

Set $\alpha = \lambda^{-1}\zeta$, so that $|\alpha| < 1$. We can then rewrite the equation defining λ as

$$1 = \sum_{j=1}^{\infty} \psi((\lambda^{-1}\zeta)^j) = \operatorname{Re} F(\alpha),$$

where F is a complex analytic function given on the unit disk by

$$F(z) := \sum_{j=1}^{\infty} \gamma(j) z^j,$$

and where the coefficient $\gamma(j)$ is the element in $\gamma \in \{-2, \pm 2i, 1 \pm 2i\}$ for which $\operatorname{Re}(\gamma\alpha^j)$, or equivalent $\operatorname{Re}(\gamma\zeta^j)$, is maximized, see Figure 1. It follows that $\gamma(j)$ only depends on the image of $j\theta$ in $\mathbb{R} \bmod \mathbb{Z}$, and more specifically which interval $(k/8, (k+1)/8)$ contains $j\theta \bmod 1$.

Our aim is to show by contradiction that α, and therefore λ, is a transcendental number. We assume henceforth that α is algebraic and will arrive eventually at a contradiction.

Lemma 3.1. *The numbers α and $\bar{\alpha}$ generate a free multiplicative subgroup of $\mathbb{C}^*$.*

Proof. We have $\arg(\alpha) = 2\pi\theta$, so if $\alpha^i\bar{\alpha}^j = 1$, then $i = j$, as θ is irrational. But then $\alpha^i\bar{\alpha}^j = |\alpha|^{2i}$, and hence $i = j = 0$ since $|\alpha| < 1$. □

Since θ is irrational, the sequence $(\gamma(j))_{j\geq 1}$ is not n-periodic, or even eventually n-periodic, for any $n \in \mathbb{N}$. Nevertheless, as we will make precise below, it comes close to being n-periodic when n is chosen to be the denominator in some continued fraction approximant m/n of θ. For such n, it will be illuminating to compare the analytic function $F(z)$ with approximations by rational functions of the form

$$F_n(z) := (1 - z^n)^{-1} \sum_{1\leq j\leq n} \gamma(j) z^j = \sum_{j\geq 1} \gamma_n(j) z^j,$$

where $\gamma_n(j)$ denotes the n-periodic extension of the initial sequence $\gamma(1), \dots, \gamma(n)$.

Lemma 3.2. *For any sufficiently large n, we have $0 < \operatorname{Re} F_n(\alpha) < 1$.*

Proof. By definition, we have

$$1 - \operatorname{Re} F_n(\alpha) = \operatorname{Re}(F(\alpha) - F_n(\alpha)) = \sum_{j>n} \operatorname{Re}((\gamma(j) - \gamma_n(j))\alpha^j).$$

Since $|\alpha| < 1$, the right hand side tends to zero as $n \to \infty$; in particular, $\operatorname{Re} F_n(\alpha) > 0$ for large n. Now, for each j, $\gamma(j)$ maximizes $\operatorname{Re}(\gamma\alpha^j)$ over $\gamma \in \{-2, \pm 2i, 1 \pm 2i\}$, so $\operatorname{Re}((\gamma(j) - \gamma_n(j)\alpha^j) \geq 0$. Thus $\operatorname{Re} F_n(\alpha) \leq 1$, and to see that the inequality is strict, it suffices to find j such that $\operatorname{Re}((\gamma(j) - \gamma_n(j)\alpha^j) > 0$. Since θ is irrational, we can find $p \geq 1$ such that $p\theta \in (7/8, 1) \bmod 1$. Assume $n > p$, and pick $m \geq 1$ such that if $j = mn + p$, then $j\theta \in (0, 1/8) \bmod 1$. Then $\gamma(j) - \gamma(n - j) = -4i$, see Figure 1, so since $\arg(\alpha^j) \in (0, \pi/4)$, it follows that $\operatorname{Re}(-4i\alpha^j) > 0$. □

In order to apply results about S-unit equations, we also consider the following rescaled function:

$$\begin{aligned} E_n(z) &:= 2|1-z^n|^2 \operatorname{Re}(F(z) - F_n(z)) \\ &= 2\mathrm{Re}\left((1-\bar{z}^n) \sum_{j>n} (\gamma(j) - \gamma(j-n)) z^j \right). \end{aligned} \tag{3.1}$$

Again, $E_n(\alpha) > 0$ for all n. The following notion will be convenient.

Definition 3.3. *We say that an index $j > n$ is* n-regular *if $\gamma(j) = \gamma(j-n)$, and* n-irregular *otherwise.*

Since θ is irrational, there are infinitely many n-irregular indices, but they nevertheless form a rather sparse subset of $\mathbb{N}$, as will be explored below.

3.2. **A theorem of Evertse.** Let K be a number field, and set $d := [K : \mathbb{Q}]$. We let $M(K)$ denote the set of places of K. There are two types of places: finite places, which correspond to prime ideals of the ring of integers $\mathcal{O}_K$ of K; and infinite places, which correspond to embeddings (respectively, distinct conjugate pairs of embeddings) of K into $\mathbb{R}$ (respectively, into $\mathbb{C}$); the former places are called *real infinite* places and the latter are called *complex infinite* places. We let $M(K)_{\mathrm{fin}}$ and $M(K)_{\mathrm{inf}}$ denote, respectively, the set of finite places and the set of infinite places. The degree d equals the number of real infinite places plus twice the number of complex infinite places.

Given a place $v \in M(K)$, we normalize the corresponding absolute value $|\cdot|_v$ on K as follows. If v is a finite place, corresponding to a prime ideal P of $\mathcal{O}_K$, and $x \in K$, then we set $|x|_v = 0$ if $x = 0$ and $|x|_v = \mathrm{N}(P)^{-\mathrm{ord}_P(x)}$ if $x \neq 0$, where $\mathrm{N}(P)$ is the cardinality of the finite field $\mathcal{O}_K/P$. If v is an infinite place corresponding to a real embedding $\tau : K \to \mathbb{R}$, then we take $|x|_v = |\tau(x)|$, where $|\cdot|$ is the ordinary absolute value on $\mathbb{R}$. Finally, if v corresponds to a distinct pair $\tau, \bar{\tau}$ of complex embeddings of K into $\mathbb{C}$ then we take $|x|_v = |\tau(x)|^2 = |\bar{\tau}(x)|^2$.

A nonzero element $x \in K$ has the property that $|x|_v = 1$ for all but finitely many places. With the above normalizations, the following *product formula* holds:

$$\prod_{v \in M(K)} |x|_v = 1 \quad \text{for } x \in K^*,$$

and we have $\prod_{v \in M(K)_{\mathrm{inf}}} |b|_v = b^d$ for every positive integer b.

We make use of the following general result of Evertse [Eve84] on unit equations, see [EG, Proposition 6.2.1]. If $S \subset M(K)$ is a finite set of places containing all infinite places, then the set of *S-integers* in K is defined by $\mathcal{O}_S := \{a \in K\colon |a|_v \leq 1 \text{ for all } v \in M(K) \setminus S\}$. Given a vector $\mathbf{x} = (x_1, \ldots, x_m) \in \mathcal{O}_S^m$ we set

$$H_S(\mathbf{x}) = \prod_{v \in S} \max\{|x_1|_v, \ldots, |x_m|_v\}.$$

Theorem 3.4. *Let $S \subset M(K)$ be a finite set of places of K containing all infinite places, T a subset of S, $m \geq 2$ an integer, and $\epsilon > 0$. Then there is a constant*

$C = C(K, S, m, \epsilon) > 0$ *such that if* $\mathbf{x} = (x_1, \dots, x_m) \in \mathcal{O}_S^m$ *and* $\sum_{k\in I} x_k \neq 0$ *for every nonempty subset* $I \subset \{1, 2, \dots, m\}$*, then*

$$\prod_{v\in T} |x_1 + \cdots + x_m|_v \geq C \frac{\prod_{v\in T} \max\{|x_1|_v, \dots, |x_m|_v\}}{H_S(\mathbf{x})^\epsilon \prod_{v\in S} \prod_{k=1}^m |x_k|_v}$$

Note that the assumption on $\mathbf{x}$ implies that $x_k \neq 0$ for all k.

3.3. **Initial choices and estimates.** Now we specialize the considerations in the previous section. In what follows, K is a fixed Galois extension of $\mathbb{Q}$ containing α, $\bar{\alpha}$ and i (and therefore all Gaussian integers). Write v_0 for the complex infinite place given by our fixed (complex) embedding $K \hookrightarrow \mathbb{C}$, so that $|a|_{v_0} := |a|^2$. We set

$$\Gamma := \{0, \pm 2, \pm 2i, \pm 1 \pm 2i, \pm 4, \pm 4i, \pm 2 \pm 4i, \pm 3 \pm 2i, \pm 1 \pm 4i\};$$

in particular $\gamma(j) \in \Gamma$ for all j, and $\pm\gamma(i) \pm \gamma(j) \in \Gamma$ for all i, j. Fix $T = \{v_0\}$ and let $S \subset M(K)$ be the set of all infinite places of K together with all finite places v such that $|x|_v \neq 1$ for some $x \in \{\alpha, \bar{\alpha}\} \cup \Gamma$.

We will apply Theorem 3.4 to vectors $(x_1, \dots, x_m)$ whose components are polynomials in α and $\bar{\alpha}$ with coefficients in Γ. Such polynomials naturally occur when approximating $E_n(\alpha)$, see (3.1), and they are all S-integers. Let us establish some basic estimates for them, in order to apply Theorem 3.4.

Lemma 3.5. *There exists a positive constant* R *such that for all polynomials*

$$x = \sum_{1\leq i+j\leq n} \gamma_{ij} \alpha^i \bar{\alpha}^j$$

of degree $\leq n$ *with coefficients* $\gamma_{ij} \in \Gamma$*, we have*

$$\prod_{v\in M(K)} \max\{|x|_v, 1\} \ll R^n.$$

Proof. Pick a positive integer b such that $b\Gamma$, $b\alpha$, and $b\bar{\alpha}$ are all contained in $\mathcal{O}_K$. Then

$$b^{n+1} x = \sum_{1\leq i+j\leq n} (b\gamma_{ij})(b\alpha)^i (b\bar{\alpha})^j b^{n-i-j} \in \mathcal{O}_K.$$

Thus $|b^{n+1}|_v \leq 1$ and $|b^{n+1} x|_v \leq 1$ for every finite place $v \in M(K)$, so

$$\begin{aligned}\prod_{v\in M(K)_{\text{fin}}} \max\{1, |x|_v\} &\leq \prod_{v\in M(K)_{\text{fin}}} \max\{1, |b^{-(n+1)}|_v\} \\ &= \prod_{v\in M(K)_{\text{fin}}} |b^{-(n+1)}|_v = \prod_{v\in M(K)_{\text{inf}}} |b^{(n+1)}|_v = (b^{n+1})^d,\end{aligned}$$

where we used the product formula and the fact that the degree d of K is the number of real infinite places plus twice the number of complex infinite places. Let R_0 be the maximum of 1 and the quantities $|\sigma(\alpha)|, |\sigma(\bar{\alpha})|$ as σ ranges over elements of the Galois group of K. As the coefficient set Γ is finite, we obtain $|x|_v \ll n^2 R_0^n$ for any real infinite place v, and $|x|_v \ll n^4 R_0^{2n}$ for any complex infinite place. Thus

$$\prod_{v\in M(K)_{\text{inf}}} \max\{1, |x|_v\} \ll n^{2d} R_0^{dn},$$

and the lemma follows on taking R to be strictly larger than $(bR_0)^d$. □

Corollary 3.6. *If x is a polynomial of degree $\leq n$ as in Lemma 3.5, then*

$$\prod_{v\in S} |x|_v \ll R^n.$$

Proof. Immediate. □

Corollary 3.7. *If $x_1, \ldots, x_m$ are polynomials as in Lemma 3.5 and $\sum_{k=1}^m \deg x_k \leq n$, then*

$$H_S(x_1, \ldots, x_m) \ll R^n.$$

Proof. Let $n_k = \deg x_k$. Then, by Lemma 3.5,

$$H_S(x_1, \ldots, x_m) = \prod_{v\in S} \max\{|x_1|_v, \ldots, |x_m|_v\} \leq \prod_{v\in S}\prod_{k=1}^m \max\{1, |x_k|_v\} \ll \prod_{k=1}^m R^{n_k} \leq R^n,$$

completing the proof. □

3.4. **The well approximable case.** Since θ is irrational, it admits continued fraction approximants m_i/n_i with $(n_i)_{i\in\mathbb{N}}$ strictly increasing and $|n_i\theta - m_i| < \frac{1}{n_i}$. It is standard to call θ *badly approximable*[4] if there exists $\kappa > 0$ such that $|n\theta - m| \geq \frac{\kappa}{n}$ for all integers m, n (or equivalently, only for all m_i, n_i). We will call θ *well approximable* otherwise. In fact, we do not know whether θ is well or badly approximable, so our argument will need to take account of both possibilities. In this section we deal with the (easier) well approximable case.

Proposition 3.8. *Suppose that θ is well approximable. Then, for any $C \geq 1$, there are arbitrarily large $n \in \mathbb{N}$ such that all indices $j \in (n, Cn]$ are n-regular.*

Proof. Let $\epsilon = \frac{1}{16(C+1)}$. By assumption, there exist infinitely many n such that $|n\theta - m| < \frac{\epsilon}{n}$ for some $m \in \mathbb{N}$. We claim that any such n will do.

To see this, pick any $j \in (n, Cn]$, and let us show that $\gamma(j) = \gamma(j-n)$. We may assume $|j\theta - \frac{k}{8}| < \epsilon/n$ for some integer k, or else the statement is clear. Thus

$$|8mj - kn| \leq |8jn\theta - 8mj| + |8jn\theta - kn| < 8\epsilon\left(\frac{j}{n} + 1\right) \leq 8\epsilon(C+1) = \frac{1}{2}.$$

Hence $8mj = kn$, and since $\gcd(m, n) = 1$, it follows that $j = \frac{k'n}{8}$ where $k' = k/m \in (8, 8C]$ is an integer.

Choose $p \in [0, 8)$ congruent to mk' mod 8. We claim that both $j\theta$ and $(j-n)\theta$ are equivalent mod 1 to elements of $[\frac{p}{8}, \frac{p+1}{8})$; in particular $\gamma(j) = \gamma(j-n)$. To see this, we suppose for definiteness' sake that $m/n < \theta$, i.e. $0 < n\theta - m < \epsilon/n$. Thus

$$0 < j\theta - \frac{mk'}{8} = \frac{k'}{8}(n\theta - m) < \frac{\epsilon k'}{8n} \leq \frac{C\epsilon}{n} < \frac{1}{16},$$

[4] Badly approximable numbers are also known as irrational numbers of *bounded type*; they are characterized by the coefficients of the continued fractions expansion being bounded.

and by the same token

$$0 < (j-n)\theta - \frac{m(k'-8)}{8} = \frac{(k'-8)}{8}(n\theta - m) < \frac{\epsilon(k'-8)}{8n} < \frac{C\epsilon}{n} < \frac{1}{16}.$$

Since both $mk'/8$ and $m(k'-8)/8$ equate to $p/8$ mod 1, the claim is proved. □

We now apply Theorem 3.4, with $T = \{v_0\}$ and S as in the beginning of §3.3, to the vector $(x_1, x_2) = (-2|1-\alpha^n|^2, 2|1-\alpha^n|^2 \operatorname{Re} F_n(\alpha))$. Thus

$$x_1 = -2(1-\alpha^n)(1-\bar{\alpha}^n) \quad \text{and} \quad x_2 = (1-\bar{\alpha}^n)\sum_{j=1}^{n} \gamma(j)\alpha^j + (1-\alpha^n)\sum_{j=1}^{n} \bar{\gamma}(j)\bar{\alpha}^j,$$

x_1 and x_2 are both polynomials of degree $2n$ in α and $\bar{\alpha}$, with coefficients in Γ. Further, no subsum of $x_1 + x_2$ vanishes. Indeed, it is clear that $x_1 \neq 0$, and Lemma 3.2 implies that $x_2 \neq 0$ and $x_1 + x_2 \neq 0$ for large n.

Hence Theorem 3.4, together with Corollaries 3.6 and 3.7, says for any $\epsilon > 0$ that

$$|x_1 + x_2|^2 \gg \frac{\max\{|x_1|^2, |x_2|^2\}}{H_S(x_1, x_2)^\epsilon \left(\prod_{v\in S} |x_1|_v |x_2|_v\right)} \gg \frac{1}{R^{4\epsilon n} \cdot R^{4n}} = R^{-4n(1+\epsilon)}.$$

On the other hand, for any fixed $C \geq 1$, Proposition 3.8 and (3.1) tell us that

$$|x_1 + x_2| = 2|1-\alpha^n|^2 |1 - \operatorname{Re} F_n(\alpha)| \ll |\alpha|^{Cn}.$$

Taking C large enough so that $|\alpha|^C < R^{-2(1+\epsilon)}$, we find the previous two estimates contradict each other for large n. So if θ is well approximable, α cannot be algebraic.

3.5. **Unit equations.** We need a little extra machinery from the theory of unit equations to deal with the possibility that θ is badly approximable. Specifically, we need the following result by Evertse, Schlickewei and Schmidt, see [ESS02, Theorem 1.1] and also [EG, Theorem 6.1.3].

Theorem 3.9. *Let L be a field of characteristic zero, $a_1, \ldots, a_l \in L^*$, and $H \subset (L^*)^l$ a subgroup of finite rank, where $l \geq 1$. Then there are only finitely many non-degenerate solutions to the equation $a_1 y_1 + \cdots + a_l y_l = 1$ with $(y_1, \ldots, y_l) \in H$.*

Here a solution is *non-degenerate* if $\sum_{k\in I} a_k y_k \neq 0$ for every nonempty subset $I \subset \{1, \ldots, l\}$, and H is said to be of finite rank r if there exists a free subgroup $H' \subset H$ of rank r such that every element of H/H' has finite order. While we won't need it, the number of non-degenerate solutions can in fact be explicitly bounded as a function of l and r. Note that Theorem 6.1.3 in [EG] is only stated for $l \geq 2$, but the case $l = 1$ is trivial.

We now apply this theorem. Let α, K and Γ be as in §3.3.

Corollary 3.10. *For any integer $m \geq 1$ there exists $N = N(m) \in \mathbb{N}$ such that*

$$\gamma_1 \alpha^{i_1} \bar{\alpha}^{j_1} + \cdots + \gamma_m \alpha^{i_m} \bar{\alpha}^{j_m} \neq 0$$

whenever $\gamma_1, \ldots, \gamma_m \in \Gamma$ are not all zero, and $|i_k - i_\ell| + |j_k - j_\ell| \geq N$ for all $k \neq \ell$.

Proof. We may of course assume $m \geq 2$. Since Γ is a finite set, it suffices to consider a fixed vector $(\gamma_1, \dots, \gamma_m)$, and we may further assume that $\gamma_k \neq 0$ for all k. By Lemma 3.1, α and $\bar{\alpha}$ generate a free abelian subgroup $G \subset K^*$. It therefore suffices to prove that for any $(\gamma_1, \dots, \gamma_m) \in (\Gamma \setminus \{0\})^m$ there are only finitely many non-degenerate solutions to the equation

$$\gamma_1 \alpha^{i_1} \bar{\alpha}^{j_1} + \cdots + \gamma_{m-1} \alpha^{i_{m-1}} \bar{\alpha}^{j_{m-1}} + \gamma_m = 0,$$

which follows from Theorem 3.9 with $L = K$, $l = m-1$, $a_k = -\gamma_k/\gamma_m$, $H = (G^*)^l$. □

Theorem 3.4 now allows us to render Corollary 3.10 effective:

Corollary 3.11. *Given $\delta, \rho > 0$, $C \geq 1$ and an integer $m \geq 1$, the following is true for n large enough. Suppose $i_1, j_1, \dots, i_m, j_m \geq 0$ are integers satisfying*

- *$i_k + j_k \leq Cn$ for all k;*
- *$|i_k - i_\ell| + |j_k - j_\ell| \geq \delta n$ for all $k \neq \ell$;*

and suppose $\gamma_1, \dots, \gamma_m \in \Gamma$ do not all vanish. Then

$$|\gamma_1 \alpha^{i_1} \bar{\alpha}^{j_1} + \cdots + \gamma_m \alpha^{i_m} \bar{\alpha}^{j_m}| \gg |\alpha|^{\min\{i_k + j_k | \gamma_k \neq 0\} + \rho n} \geq |\alpha|^{(C+\rho)n}.$$

Proof. Suppose without loss of generality that *no* γ_k vanishes. Corollary 3.10 tells us that no subset of the sum on the left adds to zero. Let $S \subset M(K)$ and $T = \{v_0\}$ be as in the beginning of §3.3. Setting $x_k = \gamma_k \alpha^{i_k} \bar{\alpha}^{j_k}$ for $1 \leq k \leq m$, we have $|x_k|_v = 1$ for all $v \notin S$, so the product formula gives $\prod_{v \in S} \prod_{k=1}^m |x_k|_v = 1$. Further,

$$\max\{|x_1|^2, \dots, |x_m|^2\} = \max_k |\gamma_k \alpha^{i_k} \bar{\alpha}^{j_k}|^2 \gg |\alpha|^{2 \min\{i_k + j_k | \gamma_k \neq 0\}},$$

whereas Corollary 3.7 shows that

$$H_S(x_1, \dots, x_m) \ll R^{Cmn},$$

where $R > 0$ only depends on Γ. Thus Theorem 3.4 gives

$$|x_1 + \cdots + x_m|^2 \gg \frac{\max\{|x_1|^2, \dots, |x_m|^2\}}{H_S(x_1, \dots, x_m)^\epsilon \prod_{v \in S} \prod_{k=1}^m |x_k|_v} \gg \frac{|\alpha|^{2 \min\{i_k + j_k | \gamma_k \neq 0\}}}{R^{Cmn\epsilon}}.$$

Shrinking ϵ then gives us the lower bound we seek. □

3.6. **The badly approximable case.** From now on we treat the case when θ is badly approximable. The next result is the analogue of Proposition 3.8.

Proposition 3.12. *Suppose θ is badly approximable. Then there exist $B = B(\theta) > 0$, $\delta = \delta(\theta) > 0$ and arbitrarily large $n \in \mathbb{N}$ such that*

(i) *$j - n \geq \delta n$ for any n-irregular index $j > n$;*
(ii) *$|j - j'| \geq \delta n$ for any distinct n-irregular indices $j, j' > n$;*
(iii) *$|j - j' - n| \geq \delta n$ for any n-irregular indices $j, j' > n$ such that $j \neq j' + n$;*
(iv) *for any $C \geq 1$, there are at most C/δ n-irregular indices in the interval $(n, Cn]$, and at least one n-irregular index in the interval $(Cn, BCn]$.*

Proof. Since θ is badly approximable, there exists $\kappa > 0$ such that $|n\theta - m| \geq \kappa/n$ for any rational number m/n. On the other hand, if m_i/n_i is the ith continued fraction approximation of θ, then $|n_i\theta - m_i| < 1/n_i$. Further, if i is odd, then $m_{i+1}/n_{i+1} < \theta < m_i/n_i$. In what follows we pick $n = n_i$, with i odd.

If $j > n$ is n-irregular, then there is $k \in \mathbb{N}$ such that $|j\theta - k/8| < \frac{1}{n}$. Hence

$$\frac{\kappa}{8(j-n)} \leq |8(j-n)\theta - (k-8m)| < \frac{16}{n}.$$

So $j - n > \frac{\kappa n}{128}$. And if $j' > j$ is another n-irregular index, then $|j'\theta - k'/8| < \frac{1}{n}$ for some $k' \in \mathbb{N}$. Hence

$$\frac{\kappa}{8(j'-j)} \leq |8(j'-j)\theta - (k'-k)| < \frac{16}{n},$$

so again $j' - j > \frac{\kappa n}{128}$. One shows similarly that if $j' - n \neq j$, then $|j - j' - n| > \frac{\kappa n}{192}$. All told, (i)–(iii) hold with $\delta = \frac{\kappa}{192}$.

The first part of (iv) follows immediately from (ii). To prove the second part, pick m'/n' equal to the first even-index continued fraction approximant for θ such that $n' > Cn$. Then $m'/n' < \theta$, and since θ is badly approximable, there exists $A = A(\theta) > 0$ (independent of i) such that $n' \leq ACn$. We now have

$$0 < m' - n'\theta < n\theta - m < \frac{1}{n}.$$

Assuming $n \geq 8$, we find that $n'\theta \bmod 1$ lies in $(\frac{7}{8}, 1)$. From the same inequalities we infer

$$0 < (n'+n)\theta - (m'+m) < \frac{1}{n}$$

so that $(n'+n)\theta \bmod 1$ lies in $(0, \frac{1}{8})$. Then $\gamma(n') = 1 + 2i$ and $\gamma(n+n') = 1 - 2i$, see Figure 1, so the index $j = n' + n \leq (AC+1)n$ is n-irregular. Thus we can take $B = AC + 1$. □

We will apply Proposition 3.12 to the quantity $E_n(\alpha)$ defined in (3.1):

$$E_n(\alpha) := 2|1-\alpha^n|^2 \operatorname{Re}(F(\alpha) - F_n(\alpha)) = 2\operatorname{Re}\left((1-\bar{\alpha}^n)\sum_{j>n}(\gamma(j) - \gamma(j-n))\alpha^j\right) = \sum_{i+j>n} e_{ij}\alpha^i\bar{\alpha}^j, \quad (3.2)$$

where for each non-zero coefficient one of the indices i or j is n-irregular (and hence $> n$) and the other is equal to 0 or n. The next result says that for suitable n, the indices (i, j) with non-vanishing coefficients e_{ij} in the above sum are well-separated. Note that the coefficients e_{ij} actually depend on n.

Corollary 3.13. *If θ is badly approximable, then there exist $\delta = \delta(\theta) > 0$ and arbitrarily large n such that if $e_{ij}, e_{i'j'} \neq 0$, then*

(i) *$(i, j) = (i', j')$ or $|i - i'| + |j - j'| \geq \delta n$; and*
(ii) *$i + j = i' + j'$ or $|(i + j) - (i' + j')| \geq \delta n$.*

Given $C \geq 1$, one can further arrange for some integer $r = r(\theta, C) \geq 0$ that precisely r of the coefficients e_{ij} with $i + j \in (n, Cn]$ are non-vanishing.

Proof. The existence of δ satisfying (i)–(ii) follows from Proposition 3.12 (i)–(iii). The fourth assertion of the same proposition implies that the number of non-vanishing e_{ij} for $i + j \in (n, Cn]$ is bounded above uniformly in n, so we obtain the last statement of the corollary by taking r to be e.g. the smallest number of non-vanishing e_{ij} that occurs for infinitely many n. $\square$

Continuing to suppose that θ is badly approximable, we pick C large enough, and let $\delta = \delta(\theta) > 0$, $r = r(\theta, C) \geq 0$ and n be as in Corollary 3.13. Pick $\rho \in (0, \delta)$. We will apply Theorem 3.4 to the vector

$$\mathbf{x} = (x_1, x_2, \ldots, x_{r+2}) \in \mathcal{O}_S^{r+2},$$

where $x_{r+1} = -2|1-\alpha^n|^2$, $x_{r+2} = 2|1-\alpha^n|^2 \operatorname{Re} F_n(\alpha)$ and $x_1, \ldots, x_r$ are the non-vanishing terms $e_{ij}\alpha^i\bar{\alpha}^j$ with $i + j \leq Cn$ in the power series (3.2) defining $E_n(\alpha)$. Thus

$$x_1 + \cdots + x_{r+2} = -\sum_{i+j>Cn} e_{ij}\alpha^i\bar{\alpha}^j.$$

Let $p(n)$ denote the maximum value of $i+j$ such that $e_{ij} \neq 0$ and $i+j \leq Cn$; let $q(n)$ denote the minimum value of $i+j$ such that $e_{ij} \neq 0$ and $i+j > Cn$. By Corollary 3.13 we have $q(n) \geq p(n) + \delta n$. Further,

$$\left|\sum_{k=1}^{r+2} x_k\right| \ll |\alpha|^{q(n)}. \tag{3.3}$$

Lemma 3.14. *If $I \subset \{1, 2, \ldots, r+2\}$ is nonempty, then $\sum_{k \in I} x_k \neq 0$.*

Proof. We argue by contradiction, so suppose $\sum_{k\in I} x_k = 0$. By Corollaries 3.10 and 3.13 we cannot have $I \subset \{1, \ldots, r\}$, and the argument at the end of §3.4 shows that if C is large enough, then we cannot have $I \subset \{r+1, r+2\}$ either.

For large n, $|x_{r+1}|, |x_{r+2}| \geq \frac{1}{2}$, whereas $|x_k| \ll |\alpha|^n$ for $1 \leq k \leq r$, so I cannot contain exactly one of $r+1$ and $r+2$, and must therefore contain both $r+1$ and $r+2$. Let $J \subset \{1, \ldots, r\}$ be the complement of I in $\{1, \ldots, r+2\}$.

First suppose J is nonempty. It then follows from Corollary 3.11 and Corollary 3.13 that $|\sum_{k\in J} x_k| \gg |\alpha|^{p(n)+\rho n}$. On the other hand, we have $\sum_{k\in J} x_k = \sum_{k=1}^{r+2} x_k$, so (3.3) gives $|\sum_{k\in J} x_k| \ll |\alpha|^{q(n)}$, a contradiction since $q(n) \geq p(n) + \delta n$ and $\rho < \delta$.

Thus J is empty and $\sum_{k=1}^{r+2} x_k = 0$. This means that

$$0 = \sum_{i+j>Cn} e_{ij}\alpha^i\bar{\alpha}^j = \sum_{i+j=q(n)} e_{ij}\alpha^i\bar{\alpha}^j + \sum_{i+j\geq q(n)+\delta n} e_{ij}\alpha^i\bar{\alpha}^j, \tag{3.4}$$

where we have used Corollary 3.13 (ii) in the last equality.

Recall that if $e_{ij} \neq 0$, then i or j is equal to 0 or n. Therefore, there are at most four terms with $i + j = q(n)$, and $e_{ij} \neq 0$. Further, $q(n) \leq BCn$ by Proposition 3.12 (iv), so Corollary 3.11 implies that

$$\left|\sum_{i+j=q(n)} e_{ij}\alpha^i\bar{\alpha}^j\right| \gg |\alpha|^{q(n)+\rho n}.$$

On the other hand, we have $|\sum_{i+j\geq q(n)+\delta n} e_{ij}\alpha^i\bar{\alpha}^j| \ll |\alpha|^{q(n)+\delta n}$. Together with (3.4) this gives a contradiction since $\rho < \delta$. $\square$

We are now in position to apply Theorem 3.4. As before, we use $T = \{v_0\}$ and S given in the beginning of §3.3. By our choice of S, we have $|\gamma\alpha^i\bar{\alpha}^j|_v = 1$ for all $\gamma \in \Gamma \setminus \{0\}$, $i, j \in \mathbb{N}$, and $v \in S$. In particular $|x_k|_v = 1$ for all $k \leq r$. Hence the product formula and Corollary 3.6 tell us that

$$\prod_{v\in S} |x_1|_v \dots |x_{r+2}|_v = \prod_{v\in S} |x_{r+1}|_v |x_{r+2}|_v \ll R^{4n},$$

since x_{r+1} and x_{r+2} are polynomials in $\alpha, \bar{\alpha}$ of degree $2n$. Further, $x_1, \dots, x_r$ have degree at most Cn, so Corollary 3.7 tells us that

$$H_S(\mathbf{x}) \leq R^{(Cr+4)n}.$$

Since no subset of $\{x_1, \dots, x_{r+2}\}$ adds to zero, see Lemma 3.14, Theorem 3.4 now yields the lower bound

$$|x_1 + \dots + x_{r+2}|^2 \gg \frac{\max\{|x_1|^2, \dots, |x_{r+2}|^2\}}{H_S(\mathbf{x})^\epsilon \left(\prod_{v\in S}\prod_{k=1}^{r+2} |x_k|_v\right)} \gg \frac{1}{R^{(Cr+4)n\epsilon} \cdot R^{4n}} = R^{-(4+(Cr+4)\epsilon)n}.$$

But (3.3) gives

$$|x_1 + \dots + x_{r+2}| \ll |\alpha|^{q(n)} \ll |\alpha|^{Cn},$$

which contradicts the lower bound for large n if $\epsilon > 0$ is chosen small enough and C large enough. We conclude that if θ is badly approximable, α is transcendental.

This completes the proof of the Main Theorem in the introduction.

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