EXCEPTIONAL ISOMORPHISMS BETWEEN COMPLEMENTS OF AFFINE PLANE CURVES

JÉRÉMY BLANC, JEAN-PHILIPPE FURTER, AND MATTIAS HEMMIG

Abstract. This article describes the geometry of isomorphisms between complements of geometrically irreducible curves in the affine plane $\mathbb{A}^2$, over an arbitrary field, which do not extend to an automorphism of $\mathbb{A}^2$.

We show that these isomorphisms are quite exceptional. In particular, they occur only when both curves are isomorphic to open subsets of the affine line $\mathbb{A}^1$. Moreover, the isomorphism is uniquely determined by one of the curves, up to post-composition by an automorphism of $\mathbb{A}^2$, except in the case where the curve is isomorphic to the affine line $\mathbb{A}^1$ or to the punctured line $\mathbb{A}^1\setminus\{0\}$. Finally, we prove that when one curve is isomorphic to a line, then both curves are equivalent to lines and that when one curve is isomorphic to $\mathbb{A}^1\setminus\{0\}$, then both curves are isomorphic to $\mathbb{A}^1\setminus\{0\}$. In addition we provide arbitrarily many non-equivalent embeddings of the punctured line (even infinitely many in characteristic 0) with isomorphic complements.

We also give a construction that provides a large family of examples of non-isomorphic curves of $\mathbb{A}^2$ having isomorphic complements, answering negatively to the Complement Problem raised by Hanspeter Kraft [Kra96]. This yields besides a negative answer to the holomorphic version of this problem in any dimension $n \geq 2$.

Contents

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1. Introduction

In the Bourbaki Seminar *Challenging problems on affine n-space* [Kra96] , Hanspeter Kraft gives a list of eight basic problems related to the affine n-spaces. The sixth one is the following:

Complement Problem. Given two irreducible hypersurfaces $E, F \subset \mathbb{A}^n$ and an isomorphism of their complements, does it follow that E and F are isomorphic?

Recently, Pierre-Marie Poloni gave a negative answer to the problem for any $n \geq 3$ [Pol16]. The construction is given by explicit formulas. There are examples where both E and F are smooth, and examples where E is singular, but F is smooth. This article deals with the case of dimension $n = 2$. As we explain now (Theorem 1 below), the situation is much more rigid than in dimension $n \geq 3$.

We recall that two curves $C, D \subset \mathbb{A}^2$ are *equivalent* if there is an automorphism of $\mathbb{A}^2$ that sends one onto the other, and that a *line* $C \subset \mathbb{A}^2$ is a closed curve of degree 1. A variety defined over any field k is called *geometrically irreducible* if it is irreducible over the algebraic closure of k.

Theorem 1. *Let* k *be any field. Let $C \subset \mathbb{A}^2$ be a geometrically irreducible closed curve and let $\varphi\colon \mathbb{A}^2 \setminus C \xrightarrow{\simeq} \mathbb{A}^2 \setminus D$ be an isomorphism, where $D \subset \mathbb{A}^2$ is also a closed curve.*

If φ extends to an automorphism of $\mathbb{A}^2$, then C is sent onto D. In particular, C and D are equivalent, and thus isomorphic.

If φ does not extend to an automorphism of $\mathbb{A}^2$, then both C and D are isomorphic to open subsets of $\mathbb{A}^1$. More precisely, we have isomorphisms

$$C \simeq \mathrm{Spec}(\mathrm{k}[t, \frac{1}{P}]) \quad \text{and} \quad D \simeq \mathrm{Spec}(\mathrm{k}[t, \frac{1}{Q}])$$

for some polynomials $P, Q \in \mathrm{k}[t]$ without square factors having the same number of roots in k*, and also having the same number of roots in the algebraic closure $\overline{\mathrm{k}}$.*

Moreover, the following holds:

1. *If C is isomorphic to $\mathbb{A}^1$, then both C and D are equivalent to lines.*
2. *If C is not isomorphic to $\mathbb{A}^1$ or to $\mathbb{A}^1 \setminus \{0\}$, the isomorphism φ (not extending to an automorphism of $\mathbb{A}^2$) is uniquely determined by C, up to post-composition by an automorphism of $\mathbb{A}^2$. In particular, there are at most two equivalence classes of curves in $\mathbb{A}^2$ having complements isomorphic to $\mathbb{A}^2 \setminus C$.*

Corollary. *If the ground field* k *is algebraically closed, and*

$$\varphi\colon \mathbb{A}^2 \setminus C \xrightarrow{\simeq} \mathbb{A}^2 \setminus D$$

is an isomorphism between the complements of two irreducible plane closed curves $C, D \subset \mathbb{A}^2$ which does not extend to an automorphism of $\mathbb{A}^2$, then there exist two finite subsets F, G of the affine line $\mathbb{A}^1$ of the same cardinality such that C is isomorphic to $\mathbb{A}^1 \setminus F$ and D is isomorphic to $\mathbb{A}^1 \setminus G$.

If F contains at least 2 points, then φ is uniquely determined by C, up to post-composition by an automorphism of $\mathbb{A}^2$. The equivalence class of D is uniquely determined by C, when F is empty or contains at least 2 points.

Theorem 1 shows in particular that the Complement Problem for $n = 2$ has a positive answer if one of the curves is singular, contrary to the case where $n \geq 3$, as explained before. It is also very different to the case of $\mathbb{P}^2$, where there exist non-isomorphic irreducible curves having isomorphic complements [Bla09, Theorem 1], but where all possible examples have to be singular (see Lemma 5.5 below).

Theorem 1 also shows that the Complement Problem for $n = 2$ has a positive answer if the curve is not rational (which is an easy observation, see Corollary 2.8 below) but more generally when it is not isomorphic to an open subset of $\mathbb{A}^1$. Note that the real circle $x^2 + y^2 = 1$ over $\mathbb{R}$ is an example of a smooth rational affine curve which is not isomorphic to an open subset of $\mathbb{A}^1$. More precisely, Theorem 1 shows that if a curve $C \subset \mathbb{A}^2$ yields a negative answer to the Complement problem, then there is only one non-isomorphic curve $D \subset \mathbb{A}^2$ with $\mathbb{A}^2 \setminus C \simeq \mathbb{A}^2 \setminus D$, and that over the algebraic closure of k the two curves are isomorphic to complements of finite subsets of $\mathbb{A}^1$, both finite subsets having the same cardinality $r \geq 3$, as explained in the corollary.

Theorem 1 also gives strong restrictions on isomorphisms between complements of curves: if C is not isomorphic to $\mathbb{A}^1 \setminus \{0\}$ and there exists an isomorphism $\varphi\colon \mathbb{A}^2 \setminus C \xrightarrow{\simeq} \mathbb{A}^2 \setminus D$ not extending to an automorphism of $\mathbb{A}^2$, then φ is unique (of course up to left composition by an automorphism of $\mathbb{A}^2$), hence the class of D too. This is again quite different to the case of dimension $n \geq 3$ where there are infinitely many hypersurfaces $E \subset \mathbb{A}^n$, up to equivalence, having isomorphic complements [Pol16, Lemma 3.1]. It is also different to the case of $\mathbb{P}^2$, where we can find algebraic families of non-equivalent curves of $\mathbb{P}^2$ having the same complement and thus infinitely many if k is infinite: This follows from a construction made in [Cos12], see Corollary 5.7 below.

All tools necessary to obtain the rigidity result (Theorem 1) are developped in Section 2. The proof is then achieved at the end of the section. Our second statement is the following existence result, which shows the optimality of Theorem 1.

Theorem 2. *Let* k *be any field.*

(1) *There exists a closed curve $C \subset \mathbb{A}^2$, isomorphic to $\mathbb{A}^1 \setminus \{0\}$, whose complement $\mathbb{A}^2 \setminus C$ admits infinitely many open embeddings $\mathbb{A}^2 \setminus C \hookrightarrow \mathbb{A}^2$ into the affine plane, up to automorphisms of $\mathbb{A}^2$. Moreover, the set of equivalence classes of such curves C is infinite.*
(2) *For each integer $n \geq 1$ there exist curves $C_1, \ldots, C_n \subset \mathbb{A}^2$, all isomorphic to $\mathbb{A}^1 \setminus \{0\}$, no two being equivalent, such that all open surfaces $\mathbb{A}^2 \setminus C_1, \ldots, \mathbb{A}^2 \setminus C_n$ are isomorphic. If $\mathrm{char}(\mathrm{k}) = 0$, we can moreover find an infinite sequence of curves $C_i \subset \mathbb{A}^2$, $i \in \mathbb{N}$, pairwise non-equivalent, such that all open surfaces $\mathbb{A}^2 \setminus C_i$, $i \in \mathbb{N}$, are isomorphic.*
(3) *For each polynomial $f \in \mathrm{k}[t]$ of degree ≥ 1, there exist two closed curves $C, D \subset \mathbb{A}^2$, both isomorphic to $\mathrm{Spec}(\mathrm{k}[t, \frac{1}{f}])$, but not equivalent, such that the open surfaces $\mathbb{A}^2 \setminus C$ and $\mathbb{A}^2 \setminus D$ are isomorphic. Moreover, the set of such pairs of closed embeddings of $\mathrm{Spec}(\mathrm{k}[t, \frac{1}{f}])$, up to equivalence, is infinite.*

The proof of Theorem 2, mainly via explicit constructions, is made in Section 3.

We then give counterexamples to the Complement Problem in dimension 2, over any field:

Theorem 3. *For each ground field* k*, there exist two geometrically irreducible closed curves* $C, D \subset \mathbb{A}^2$ *which are not isomorphic but whose complements* $\mathbb{A}^2 \setminus C$ *and* $\mathbb{A}^2 \setminus D$ *are isomorphic. Furthermore, these two curves can be chosen of degree* 7 *if the field admits strictly more than* 2 *elements and of degree* 13 *if the field has* 2 *elements.*

The proof of this result is detailed in Section 4. We first give a geometric construction in Proposition 4.1. Then, we show in Corollary 4.2 that this construction yields, for each polynomial $P \in \mathrm{k}[t]$ of degree $d \geq 1$ and each $\lambda \in \mathrm{k}$ with $P(\lambda) \neq 0$, two closed curves $C, D \subset \mathbb{A}^2$ of degree $d^2 - d + 1$ such that $\mathbb{A}^2 \setminus C$ and $\mathbb{A}^2 \setminus D$ are isomorphic and such that the following isomorphisms hold:

$$C \simeq \mathrm{Spec}\left(\mathrm{k}[t, \frac{1}{P}]\right) \text{ and } D \simeq \mathrm{Spec}\left(\mathrm{k}[t, \frac{1}{Q}]\right), \text{ where } Q(t) = P\left(\lambda + \frac{1}{t}\right) \cdot t^{\deg(P)}.$$

The proof of Theorem 3 follows by providing an appropriate pair (P, λ) for each field. The case of infinite fields is quite easy. Indeed, if k is infinite and $P \in \mathrm{k}[t]$ is a polynomial having at least 3 roots in $\overline{\mathrm{k}}$, then $\mathrm{Spec}(\mathrm{k}[t, \frac{1}{P}])$ and $\mathrm{Spec}(\mathrm{k}[t, \frac{1}{Q}])$ are not isomorphic, when $Q(t) = P(\lambda + \frac{1}{t}) \cdot t^{\deg(P)}$ and $\lambda \in \mathrm{k}$ is a general element, so we can find two curves $C, D \subset \mathbb{A}^2$ having isomorphic complements, such that C is isomorphic to $\mathrm{Spec}(\mathrm{k}[t, \frac{1}{P}])$ but not D (Corollary 4.4). This shows that the isomorphism type of counterexamples to the Complement Problem is as large as possible.

We finish this introduction by giving some easy implications of Theorem 3, detailed in Section 5:

(i) The negative answer to the Complement Problem for $n = 2$ directly yields a negative answer for any $n \geq 3$ (Lemma 5.1): Our construction gives, for each $n \geq 3$, two geometrically irreducible smooth closed hypersurfaces $E, F \subset \mathbb{A}^n$ which are not isomorphic but whose complements $\mathbb{A}^n \setminus E$ and $\mathbb{A}^n \setminus F$ are isomorphic (Corollary 5.2). All the hypersurfaces provided this way are isomorphic to $\mathbb{A}^{n-2} \times C$ for some open subset $C \subset \mathbb{A}^1$. This does not allow to give singular examples like the ones of [Pol16], but gives a different type of examples.

(ii) Choosing $\mathrm{k} = \mathbb{C}$, our construction also gives families of closed complex curves $C, D \subset \mathbb{C}^2$ such that $\mathbb{C}^2 \setminus C$ and $\mathbb{C}^2 \setminus D$ are biholomorphic (because they are isomorphic as algebraic varieties) but C and D are not biholomorphic (Lemma 5.3). This directly provides, for each $n \geq 2$, the existence of algebraic hypersurfaces $H_1, H_2 \subset \mathbb{C}^n$ which are complex manifolds, not biholomorphic but having biholomorphic complements (Corollary 5.4). This is, to our knowledge, the first family of such examples, and yields the answer to a problem asked in [Pol16]. Note that in the counterexamples of [Pol16], if both hypersurfaces are smooth, then they are always biholomorphic (even if they are not isomorphic as algebraic varieties).

2. Geometric description of open embeddings $\mathbb{A}^2 \setminus C \hookrightarrow \mathbb{A}^2$

In the sequel, we work over a field k, which is a priori any field (except when we precise the field). When we say rational, resp. isomorphic, we mean k-rational, resp. k-isomorphic. The word geometrically rational or geometrically irreducible refers to the extension to the algebraic closure $\overline{\mathrm{k}}$, as usual.

2.1. Basic properties.

In order to study isomorphisms between affine surfaces, it is often interesting to see the affine surfaces as open subsets of projective surfaces and to see then the isomorphisms as birational maps between the projective surfaces. The aim of this section is to establish Lemma 2.7, that we use often in the sequel. Its proof relies on some easy results that we recall before: Lemmas 2.2, 2.4, 2.6 and Corollary 2.5.

Example 2.1. The morphism

$$\begin{array}{rcl} \mathbb{A}^2 & \hookrightarrow & \mathbb{P}^2 \\ (x,y) & \mapsto & [x:y:1] \end{array}$$

gives an isomorphism $\mathbb{A}^2 \xrightarrow{\simeq} \mathbb{P}^2 \setminus L_{\mathbb{P}^2}$, where $L_{\mathbb{P}^2} \subset \mathbb{P}^2$ denotes the "line at infinity" given by $z=0$. The above embedding of $\mathbb{A}^2$ into $\mathbb{P}^2$ will be often used in the sequel, and called the *standard embedding*.

With this standard embedding, every line of $\mathbb{A}^2$, given by an equation $ax+by=c$ where a,b,c are elements of k and a,b are not both zero, is the restriction of a line of $\mathbb{P}^2$, given by the equation $ax+by=cz$, and distinct from $L_{\mathbb{P}^2}$.

Lemma 2.2. *Let $\varphi\colon X \dashrightarrow Y$ be a birational map between two projective surfaces and assume that φ restricts to an isomorphism $U \xrightarrow{\simeq} V$ where $U \subset X$ and $V \subset Y$ are two open subsets. Then, any geometrically irreducible closed curve $\Gamma \subset X \setminus U$ is sent either to a point of $Y \setminus V$ or to a curve contained in $Y \setminus V$.*

Proof. The minimal resolution of φ yields a commutative diagram

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow^{\eta} & & \searrow^{\pi} & \\ X & & \overset{\varphi}{\dashrightarrow} & & Y \\ \cup & & & & \cup \\ U & & \xrightarrow{\simeq} & & V \end{array}$$

where η and π blow up respectively the base-points of φ and φ^{-1}, which are by assumption not contained in U and V respectively. This yields $\eta^{-1}(U) = \pi^{-1}(V)$, which implies the result. □

Definition 2.3. For each birational $\varphi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, one defines $J_\varphi \subset \mathbb{P}^2$ to be the reduced curve given by the union of all irreducible $\overline{\mathrm{k}}$-curves contracted by φ.

Lemma 2.4. *Let $\varphi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a birational map.*

(1) *The curve J_φ is defined over k, i.e. is the zero locus of a homogeneous polynomial $f \in \mathrm{k}[x,y,z]$.*
(2) *We have an isomorphism $\mathbb{P}^2 \setminus J_\varphi \to \mathbb{P}^2 \setminus J_{\varphi^{-1}}$. Moreover, the number of $\overline{\mathrm{k}}$-irreducible components of J_φ and $J_{\varphi^{-1}}$ are equal.*

Proof. (1): If $\mathrm{char}(\mathrm{k}) = 0$ one could choose f to be the Jacobian determinant associated to φ. This does not work in positive characteristic as the Jacobian determinant can be zero. We then do as follows: we write φ as $\varphi\colon [x : y : z] \mapsto [s_0(x, y, z) : s_1(x, y, z) : s_2(x, y, z)]$, where $s_0, s_1, s_2 \in \mathrm{k}[x, y, z]$ are homogeneous polynomials of the same degree without common factor and do the same with $\varphi^{-1}\colon [x : y : z] \mapsto [q_0(x, y, z) : q_1(x, y, z) : q_2(x, y, z)]$. We then do the composition and obtain $q_0(s_0, s_1, s_2) = xA$, $q_1(s_0, s_1, s_2) = yA$, $q_2(s_0, s_1, s_2) = zA$, for some polynomial $A \in \mathrm{k}[x, y, z]$ and observe that J_φ is the zero locus of A. Indeed, the polynomial A is zero along an irreducible $\overline{\mathrm{k}}$-curve if and only if this curve is sent by φ onto a base-point of φ^{-1}.

(2) We take a minimal resolution of φ, which yields a commutative diagram

$$\begin{array}{ccc} & X & \\ {}^{\eta}\swarrow & & \searrow^{\pi} \\ \mathbb{P}^2 & \overset{\varphi}{\dashrightarrow} & \mathbb{P}^2 \end{array}$$

where η and π are birational morphisms, the morphism η, resp. π, being the sequence of blow-ups of the $\overline{\mathrm{k}}$-base-points of φ, resp. φ^{-1}.

We can now work over $\overline{\mathrm{k}}$, forgetting the subfield k. Computing the Picard rank of X, we see that η and π contract the same number of irreducible curves of X. Let n be this number. We then denote by $E \subset X$, resp. $F \subset X$, the union of the n irreducible curves contracted by η, resp. π. The map φ restricts then to an isomorphism

$$\mathbb{P}^2 \setminus \eta(E \cup F) \xrightarrow{\simeq} \mathbb{P}^2 \setminus \pi(E \cup F)$$

Let us observe that $\eta(E \cup F) = \eta(F)$. Since $\eta(E)$ consists of finitely many points, it suffices to show that these are contained in the curves of $\eta(F)$. Each point p of $\eta(E)$ corresponds to a connected component of E, which contains at least one (-1)-curve $\mathcal{E} \subset E$. The curve $\mathcal{E}$ is not contracted by π, by minimality, hence sent by π onto a curve $\pi(\mathcal{E}) \subset \mathbb{P}^2$, of self-intersection ≥ 1. This implies that $\mathcal{E}$ intersects F and thus $p \in \eta(F)$. One similarly gets $\pi(E \cup F) = \pi(E)$, and obtains that φ restricts to an isomorphism

$$\mathbb{P}^2 \setminus \eta(F) \xrightarrow{\simeq} \mathbb{P}^2 \setminus \pi(E).$$

It remains to observe that $\eta(F)$ is a closed curve of $\mathbb{P}^2$ (in general not irreducible) and that each of its $\overline{\mathrm{k}}$-component is contracted by φ, so $\eta(F) = J_\varphi$. Similarly, one gets $\pi(E) = J_{\varphi^{-1}}$. Moreover, the number of $\overline{\mathrm{k}}$-irreducible components of $\eta(F)$ is equal to the number of $\overline{\mathrm{k}}$-irreducible components of $\overline{F \setminus E}$, which is equal to the number of $\overline{\mathrm{k}}$-irreducible components of $\overline{E \setminus F}$. This achieves the proof. □

Corollary 2.5. *Let $\varphi\colon \mathbb{P}^2 \setminus \Gamma \hookrightarrow \mathbb{P}^2$ be an open embedding, where Γ is a closed* k*-curve, which is a finite union of r distinct irreducible closed $\overline{\mathrm{k}}$-curves of $\mathbb{P}^2$. Then, there is a unique closed* k*-curve $\Delta \subset \mathbb{P}^2$ such that $\varphi(\mathbb{P}^2 \setminus \Gamma) = \mathbb{P}^2 \setminus \Delta$, and Δ is also a finite union of r distinct irreducible closed $\overline{\mathrm{k}}$-curves of $\mathbb{P}^2$.*

Proof. Let $\hat{\varphi}\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the birational map induced by φ. Lemma 2.4 implies that $J_{\hat{\varphi}} \subset \Gamma$, that $J_{\hat{\varphi}}$ and $J_{\hat{\varphi}^{-1}}$ are finite unions of $s \leq r$ irreducible closed distinct $\overline{\mathrm{k}}$-curves of $\mathbb{P}^2$, and that $\hat{\varphi}$ induces an isomorphism $\mathbb{P}^2 \setminus J_{\hat{\varphi}} \xrightarrow{\simeq} \mathbb{P}^2 \setminus J_{\hat{\varphi}^{-1}}$.

If $s = r$, the proof is over. Otherwise, $\Gamma' = \Gamma \setminus J_{\hat{\varphi}}$ is a closed k-curve of $\mathbb{P}^2 \setminus J_{\hat{\varphi}}$, which is the union of $r - s$ irreducible closed $\overline{\mathrm{k}}$-curves. The closed k-curve $\Delta' = \hat{\varphi}(\Gamma')$ of

$\mathbb{P}^2 \setminus J_{\hat{\varphi}^{-1}}$ is again the union of $r-s$ irreducible closed $\overline{\mathrm{k}}$-curves. The result follows with $\Delta = \Delta' \cup J_{\hat{\varphi}^{-1}}$. □

Lemma 2.6. *Let $\varphi\colon X \dashrightarrow Y$ be a birational map between two smooth projective surfaces (all defined over* k*), such that every irreducible $\overline{\mathrm{k}}$-curve contracted by φ is defined over* k*. Then, each base-point of φ^{-1} is* k*-rational and each curve contracted by φ is* k*-rational.*

Proof. We argue by induction on the number of base-points of φ^{-1}. If there is no such base-point, there is nothing to show. Otherwise, let C be an irreducible $\overline{\mathrm{k}}$-curve contracted by φ onto a point p of Y. Since C is defined over k, so is its image, i.e. p is k-rational (the generic point of C is defined over k and is sent onto the k-point p). Let $\pi\colon Y' \to Y$ be the blow-up at p and let $\varphi' = \pi^{-1} \circ \varphi\colon X \dashrightarrow Y'$. The base-points of $(\varphi')^{-1}$ coincide with the base-points of φ^{-1} from which the point p is removed. Moreover, the curves contracted by φ' are also contracted by φ, and if a curve is contracted by φ and not contracted by φ', then it is sent by φ' onto the exceptional divisor $\pi^{-1}(p)$ and is thus k-rational. Therefore, the result follows by induction. □

In the sequel, we will constantly use the following observation:

Lemma 2.7. *Let $C \subset \mathbb{A}^2$ be a geometrically irreducible closed curve and let $\varphi\colon \mathbb{A}^2 \setminus C \hookrightarrow \mathbb{A}^2$ be an open embedding. Then, there exists a geometrically irreducible closed curve $D \subset \mathbb{A}^2$ such that $\varphi(\mathbb{A}^2 \setminus C) = \mathbb{A}^2 \setminus D$. Denote by $\overline{C}$ and $\overline{D}$ the closures of C and D in $\mathbb{P}^2$, denote as in Example 2.1 by $L_{\mathbb{P}^2} = \mathbb{P}^2 \setminus \mathbb{A}^2$ the line at infinity and denote by $\hat{\varphi}\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ the birational map induced by φ. Then, one of the following three alternatives holds:*

1. *We have $\hat{\varphi}(\overline{C}) = \overline{D}$. Then, the map φ extends to an automorphism of $\mathbb{A}^2 = \mathbb{P}^2 \setminus L_{\mathbb{P}^2}$ sending C onto D.*
2. *We have $\hat{\varphi}(\overline{C}) = L_{\mathbb{P}^2}$. Then, the curve D is a line of $\mathbb{A}^2$, i.e. $\overline{D}$ is a line of $\mathbb{P}^2$ and φ extends to an isomorphism from $\mathbb{A}^2 = \mathbb{P}^2 \setminus L_{\mathbb{P}^2}$ to $\mathbb{P}^2 \setminus \overline{D}$, that sends C onto $L_{\mathbb{P}^2} \setminus \overline{D}$. In particular, C is equivalent to a line via an automorphism of $\mathbb{A}^2$.*
3. *The map $\hat{\varphi}$ contracts the curve $\overline{C}$ onto a* k*-point of $\mathbb{P}^2$. Then, the curve $\overline{C}$ (and therefore, also the curve C) is a rational curve (i.e. is* k*-birational to $\mathbb{P}^1$).*

Proof. The restriction of $\hat{\varphi}$ to $\mathbb{P}^2 \setminus (L_{\mathbb{P}^2} \cup \overline{C}) = \mathbb{A}^2 \setminus C$ gives the open embedding $\varphi\colon \mathbb{A}^2 \setminus C \hookrightarrow \mathbb{A}^2 \hookrightarrow \mathbb{P}^2$. By Corollary 2.5, we obtain an isomorphism $\mathbb{P}^2 \setminus (L_{\mathbb{P}^2} \cup \overline{C}) \xrightarrow{\simeq} \mathbb{P}^2 \setminus \Delta$, for some k-curve $\Delta \subset \mathbb{P}^2$, which is the union of two $\overline{\mathrm{k}}$-irreducible closed curves of $\mathbb{P}^2$. Since $L_{\mathbb{P}^2}$ is included in Δ, there exists an irreducible closed $\overline{\mathrm{k}}$-curve D of $\mathbb{A}^2$ such that $\Delta = L_{\mathbb{P}^2} \cup \overline{D}$. As a conclusion, $\hat{\varphi}$ induces an isomorphism

$$\mathbb{P}^2 \setminus (L_{\mathbb{P}^2} \cup \overline{C}) \xrightarrow{\simeq} \mathbb{P}^2 \setminus (L_{\mathbb{P}^2} \cup \overline{D}).$$

It follows that $\varphi(\mathbb{A}^2 \setminus C) = \mathbb{A}^2 \setminus D$. The equality $D = \mathbb{A}^2 \setminus \varphi(\mathbb{A}^2 \setminus C)$ proves us that the curve D is defined over k and is therefore geometrically irreducible. By Lemma 2.2, one of the following three alternatives holds:

(1) We have $\hat{\varphi}(\overline{C}) = \overline{D}$. Hence, $\hat{\varphi}$ induces an automorphism of $\mathbb{A}^2 = \mathbb{P}^2 \setminus L_{\mathbb{P}^2}$ (Lemma 2.4).

(2) We have $\hat{\varphi}(\overline{C}) = L_{\mathbb{P}^2}$. Then, $\hat{\varphi}$ induces an isomorphism $\mathbb{P}^2 \setminus L_{\mathbb{P}^2} \xrightarrow{\simeq} \mathbb{P}^2 \setminus \overline{D}$ (again by Lemma 2.4). Since the Picard group of $\mathbb{P}^2 \setminus \Gamma$ is isomorphic to $\mathbb{Z}/\deg(\Gamma)\mathbb{Z}$, for each irreducible curve Γ, the curve $\overline{D}$ must be a line of $\mathbb{P}^2$.

(3) The map $\hat{\varphi}$ contracts the curve $\overline{C}$ onto a point of $\mathbb{P}^2$. Then, by Lemma 2.6, this point is necessary a k-point and the curve $\overline{C}$ is k-rational. □

Corollary 2.8. *Let $C \subset \mathbb{A}^2$ be a geometrically irreducible closed curve. If C is not rational (i.e. not* k*-birational to $\mathbb{P}^1$), then every open embedding $\mathbb{A}^2 \setminus C \hookrightarrow \mathbb{A}^2$ extends to an automorphism of $\mathbb{A}^2$.*

Proof. Follows from Lemma 2.7 and the fact that cases (2)-(3) only occur when C is rational. □

Remark 2.9. It follows from Corollary 2.8 that the group of automorphisms of $\mathbb{A}^2 \setminus C$, where C is a non-rational geometrically irreducible curve, is the subgroup of $\mathrm{Aut}(\mathbb{A}^2)$ preserving C. By [BS15, Theorem 2], this group is finite (and in particular conjugate to a subgroup of $\mathrm{GL}_2(\mathrm{k})$ if $\mathrm{char}(\mathrm{k}) = 0$, see for instance [Kam79]).

We find it interesting to observe that case (3) of Lemma 2.7 only occurs when $\overline{C}$ intersects $L_{\mathbb{P}^2}$ in at most two $\overline{\mathrm{k}}$-points, even if this will not be used in the sequel.

Corollary 2.10. *If $C \subset \mathbb{A}^2$ is a closed geometrically irreducible curve such that $\overline{C}$ intersects $L_{\mathbb{P}^2} = \mathbb{P}^2 \setminus \mathbb{A}^2$ in at least three $\overline{\mathrm{k}}$-points, then every open embedding $\mathbb{A}^2 \setminus C \hookrightarrow \mathbb{A}^2$ extends to an automorphism of $\mathbb{A}^2$.*

Proof. We can assume that $\mathrm{k} = \overline{\mathrm{k}}$. Suppose, for contradiction, that the extension $\hat{\varphi} \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ does not restrict to an automorphism of $\mathbb{A}^2$. By Lemma 2.7, the curve $\overline{C}$ is contracted by $\hat{\varphi}$ (because C is not equivalent to a line, so (2) is impossible). We recall that $\hat{\varphi}$ restricts to an isomorphism $\mathbb{A}^2 \setminus C = \mathbb{P}^2 \setminus (L_{\mathbb{P}^2} \cup \overline{C}) \xrightarrow{\simeq} \mathbb{A}^2 \setminus D = \mathbb{P}^2 \setminus (L_{\mathbb{P}^2} \cup \overline{D})$ (Lemma 2.7) and that $\overline{C} \subset J_{\hat{\varphi}} \subset L_{\mathbb{P}^2} \cup \overline{C}$, $J_{\hat{\varphi}^{-1}} \subset L_{\mathbb{P}^2} \cup \overline{D}$, where $J_{\hat{\varphi}}$, $J_{\hat{\varphi}^{-1}}$ have the same number of irreducible components (Lemma 2.4). We take a minimal resolution of $\hat{\varphi}$, which yields a commutative diagram

$$\begin{array}{ccccc} & & X & & \\ & \overset{\eta}{\swarrow} & & \overset{\pi}{\searrow} & \\ \mathbb{P}^2 & & \overset{\hat{\varphi}}{\dashrightarrow} & & \mathbb{P}^2. \end{array}$$

We first observe that the strict transforms $\tilde{L}_{\mathbb{P}^2}, \tilde{C} \subset X$ of $L_{\mathbb{P}^2}, \overline{C}$ by η intersect in at most one point. Indeed, otherwise the curve $\tilde{L}_{\mathbb{P}^2}$ is not contracted by π, because π contracts $\tilde{C}$, and sent onto a singular curve, which has then to be $\overline{D}$. We get $J_{\hat{\varphi}} = \overline{C}$, $J_{\hat{\varphi}^{-1}} = L_{\mathbb{P}^2}$ and get an isomorphism $\mathbb{P}^2 \setminus \overline{C} \to \mathbb{P}^2 \setminus L_{\mathbb{P}^2}$, impossible because $\overline{C}$ has degree at least 3.

Secondly, the fact that $\tilde{L}_{\mathbb{P}^2}, \tilde{C} \subset X$ intersect in at most one point implies that η blows up all points of $\overline{C} \cap L_{\mathbb{P}^2}$ except at most one. Since $J_{\hat{\varphi}^{-1}} \subset D \cup L_{\mathbb{P}^2}$, there are at most two (-1)-curves contracted by η. But $L_{\mathbb{P}^2}$ and $\overline{C}$ intersect in at least three points, so we obtain exactly two proper base-points of $\hat{\varphi}$, corresponding to exactly two (-1)-curves $E_1, E_2 \subset X$ contracted onto two points $p_1, p_2 \in \overline{C} \cap L_{\mathbb{P}^2}$ by η. Moreover, $J_{\hat{\varphi}^{-1}} = D \cup L_{\mathbb{P}^2}$ so $J_{\hat{\varphi}} = C \cup L_{\mathbb{P}^2}$ (Lemma 2.4). Writing $E_i' = \overline{\eta^{-1}(p_i) \setminus E_i}$, we find that π contracts $F = E_1' \cup E_2' \cup \tilde{C} \cup \tilde{L}_{\mathbb{P}^2}$.

Let us show that $E_i \cdot F \geq 2$, for $i = 1, 2$, which will imply that $\pi(E_i)$ is a singular curve for $i = 1, 2$, and yield a contradiction since E_1, E_2 are sent by π onto $L_{\mathbb{P}^2}$ and $\overline{D}$. As $E_i \cup E_i' = \eta^{-1}(p_i)$, it is a tree of rational curves, which intersects both $\tilde{C}$ and $\tilde{L}_{\mathbb{P}^2}$ since $p_i \in \overline{C} \cap L_{\mathbb{P}^2}$. If E_i' is empty, then $E_i \cdot \tilde{C} \geq 1$ and $E_i \cdot \tilde{L}_{\mathbb{P}^2} \geq 1$, whence $E_i \cdot F \geq 2$ as we claimed. If E_i' is not empty, then $E_i \cdot E_i' \geq 1$. The only possibility to get $E_i \cdot F \leq 1$ would thus be that $E_i \cdot E_i' = 1$, $E_i \cdot \tilde{C} = E_i \cdot \tilde{L}_{\mathbb{P}^2} = 0$. The equality $E_i \cdot E_i' = 1$ implies that E_i' is connected, and $E_i \cdot \tilde{C} = E_i \cdot \tilde{L}_{\mathbb{P}^2} = 0$ yields $\tilde{C} \cdot E_i' \geq 1$ and $\tilde{L}_{\mathbb{P}^2} \cdot E_i' \geq 1$. Since $\tilde{L}_{\mathbb{P}^2}$ and $\tilde{C}$ intersect in a point disjoint from E_i', this implies that F contains a loop and thus cannot be contracted. □

Remark 2.11. In case (3) of Lemma 2.7, it is possible that $\overline{C}$ intersects the line $L_{\mathbb{P}^2}$ in two points, as it is the case in most of our examples (see for example Lemma 3.1 or Lemma 3.7). The case of one point is of course also possible (see for instance Lemma 2.13(1)).

We will also need the following basic algebraic observation.

Lemma 2.12. *Let $f \in \mathrm{k}[x,y]$ be a polynomial, irreducible over $\overline{\mathrm{k}}$, and let $C \subset \mathbb{A}^2$ be the curve given by $f = 0$. Then, the ring of functions on $\mathbb{A}^2 \setminus C$ and its subset of invertible elements are equal to*

$$\mathcal{O}(\mathbb{A}^2 \setminus C) = \mathrm{k}[x,y,f^{-1}] \subset \mathrm{k}(x,y),\ \mathcal{O}(\mathbb{A}^2 \setminus C)^* = \{\lambda f^n \mid \lambda \in \mathrm{k}^*, n \in \mathbb{Z}\}.$$

In particular, every automorphism of $\mathbb{A}^2 \setminus C$ exchanges the fibres of the morphism

$$\mathbb{A}^2 \setminus C \to \mathbb{A}^1 \setminus \{0\}$$

given by f.

Proof. The field of rational functions of $\mathbb{A}^2 \setminus C$ is equal to $\mathrm{k}(x,y)$. We can write any element of this field as $\frac{u}{v}$, where $u, v \in \mathrm{k}[x,y]$ are coprime polynomials, $v \neq 0$. The rational function is regular on $\mathbb{A}^2 \setminus C$ if and only if v does not vanish on any $\overline{\mathrm{k}}$-point of $\mathbb{A}^2 \setminus C$. This means that $v = \lambda f^n$, for some $\lambda \in \mathrm{k}^*$, $n \geq 0$. This provides the description of $\mathcal{O}(\mathbb{A}^2 \setminus C)$ and $\mathcal{O}(\mathbb{A}^2 \setminus C)^*$. The last remark follows from the fact that $\mathcal{O}(\mathbb{A}^2 \setminus C)^*$ is generated by k^* and one element, if and only if this element is $\lambda f^{\pm 1}$ for some $\lambda \in \mathrm{k}^*$. □

2.2. **The case of lines.** Lemma 2.7 shows that one needs to study isomorphisms $\mathbb{A}^2 \setminus C \xrightarrow{\simeq} \mathbb{A}^2 \setminus D$, which extend to birational maps of $\mathbb{P}^2$ that contract the curve C onto a point. One can ask whether this point can be a point of $\mathbb{A}^2$ (and thus would be contained in D) or belongs to the boundary line $L_{\mathbb{P}^2} = \mathbb{P}^2 \setminus \mathbb{A}^2$. As we will show (Corollary 2.19), the first possibility only occurs in a very special case, namely when C is equivalent to a line by an automorphism of $\mathbb{A}^2$. The case of lines is special for this reason, and is treated separately here.

Lemma 2.13. *Let $C \subset \mathbb{A}^2$ be the line given by $x = 0$.*

(1) *The group of automorphisms of $\mathbb{A}^2 \setminus C$ is given by:*

$$\mathrm{Aut}(\mathbb{A}^2 \setminus C) = \{(x,y) \mapsto (\lambda x^{\pm 1}, \mu x^n y + s(x,x^{-1})) \mid \lambda, \mu \in \mathrm{k}^*, n \in \mathbb{Z}, s \in \mathrm{k}[x,x^{-1}]\}.$$

(2) *Every open embedding $\mathbb{A}^2 \setminus C \hookrightarrow \mathbb{A}^2$ is equal to $\psi\alpha$, where $\alpha \in \mathrm{Aut}(\mathbb{A}^2 \setminus C)$ and $\psi \colon \mathbb{A}^2 \setminus C \hookrightarrow \mathbb{A}^2$ extends to an automorphism of $\mathbb{A}^2$. In particular, the complement of its image, i.e. the complement of $\psi\alpha(\mathbb{A}^2 \setminus C) = \psi(\mathbb{A}^2 \setminus C)$, is a curve equivalent to a line by an automorphism of $\mathbb{A}^2$.*

Proof. To prove (1), we first observe that each transformation $(x,y) \mapsto (\lambda x^{\pm 1}, \mu x^n y + s(x,x^{-1}))$ actually yields an automorphism of $\mathbb{A}^2 \setminus C$. Then, we only need to show that all automorphisms of $\mathbb{A}^2 \setminus C$ are of this form. An automorphism of $\mathbb{A}^2 \setminus C$ corresponds to an automorphism of $\mathrm{k}[x,y,x^{-1}]$, which sends x onto $\lambda x^{\pm 1}$, where $\lambda \in \mathrm{k}^*$ (Lemma 2.12). Applying the inverse of $(x,y) \mapsto (\lambda x^{\pm 1}, y)$, we can assume that x is fixed. We are left with an R-automorphism of $R[y]$, where R is the ring $\mathrm{k}[x,x^{-1}]$. Such an automorphism is of the form $y \mapsto ay+b$, where $a \in R^*$, $b \in R$. Indeed, if the maps $y \mapsto p(y)$ and $y \mapsto q(y)$ are inverses of each other, the equality $y = p(q(y))$ proves us that $\deg p = \deg q = 1$. This yields the desired form.

To prove (2), we use Lemma 2.7 and write φ as an isomorphism $\mathbb{A}^2 \setminus C \xrightarrow{\simeq} \mathbb{A}^2 \setminus D$ where D is a geometrically irreducible closed curve, and only need to see that D is equivalent to a line by an automorphism of $\mathbb{A}^2$. We write $\psi = \varphi^{-1}$, choose an equation $f = 0$ for D (where $f \in \mathrm{k}[x,y]$ is an irreducible polynomial over $\overline{\mathrm{k}}$) and get an isomorphism $\psi^* \colon \mathcal{O}(\mathbb{A}^2 \setminus C) = \mathrm{k}[x,y,x^{-1}] \to \mathcal{O}(\mathbb{A}^2 \setminus D) = \mathrm{k}[x,y,f^{-1}]$ that sends x onto $\lambda f^{\pm 1}$ for some $\lambda \in \mathrm{k}^*$ (Lemma 2.12). We can thus write ψ as $(x,y) \mapsto (\lambda f(x,y)^{\pm 1}, g(x,y) f(x,y)^n)$, where $n \in \mathbb{Z}$ and $g \in \mathrm{k}[x,y]$. Replacing ψ with its composition with the automorphism $(x,y) \mapsto ((\lambda^{-1}x)^{\pm 1}, y((\lambda^{-1}x)^{\pm 1})^{-n})$ of $\mathbb{A}^2 \setminus C$, we can assume that ψ is of the form $(x,y) \mapsto (f(x,y), g(x,y))$. If g is equal to a constant $\nu \in \mathrm{k}$ modulo f, we apply the automorphism $(x,y) \mapsto (x, (y-\nu)x^{-1})$ and decrease the degree of g. After finitely many steps we obtain an isomorphism $\mathbb{A}^2 \setminus D \to \mathbb{A}^2 \setminus C$ of the form $\psi_0 \colon (x,y) \mapsto (f(x,y), g(x,y))$ where g is not a constant modulo f. The image of D by ψ_0 is then dense in C, which implies that ψ_0 extends to an automorphism of $\mathbb{A}^2$ sending D onto C (Lemma 2.7). □

2.3. Embeddings into Hirzebruch surfaces.

We will not only need embeddings of $\mathbb{A}^2$ into $\mathbb{P}^2$ but also other embeddings of $\mathbb{A}^2$ into smooth projective surfaces, and in particular into Hirzebruch surfaces.

Example 2.14. For $n \geq 1$, the n-th Hirzebruch surface $\mathbb{F}_n$ is

$$\mathbb{F}_n = \{([a:b:c],[u:v]) \in \mathbb{P}^2 \times \mathbb{P}^1 \mid bv^n = cu^n\}$$

and the projection $\pi_n \colon \mathbb{F}_n \to \mathbb{P}^1$ yields a $\mathbb{P}^1$-bundle structure on $\mathbb{F}_n$.

Let $S_n, F_n \subset \mathbb{F}_n$ be the curves given by $[1:0:0] \times \mathbb{P}^1$ and $v = 0$, respectively. The morphism

$$\begin{array}{ccc} \mathbb{A}^2 & \hookrightarrow & \mathbb{F}_n \\ (x,y) & \mapsto & ([x:y^n:1],[y:1]) \end{array}$$

gives an isomorphism $\mathbb{A}^2 \xrightarrow{\sim} \mathbb{F}_n \setminus (S_n \cup F_n)$.

We recall the following classical easy result:

Lemma 2.15. *For each $n \geq 1$, the projection $\pi_n \colon \mathbb{F}_n \to \mathbb{P}^1$ is the unique $\mathbb{P}^1$-bundle structure on $\mathbb{F}_n$, up to automorphism of $\mathbb{P}^1$. The curve S_n is the unique irreducible $\overline{\mathrm{k}}$-curve of $\mathbb{F}_n$ of self-intersection $-n$, and we have $(F_n)^2 = 0$.*

Proof. Since $\mathbb{F}_n \setminus (S_n \cup F_n)$ is isomorphic to $\mathbb{A}^2$, whose Picard group is trivial, one has $\mathrm{Pic}(\mathbb{F}_n) = \mathbb{Z}F_n + \mathbb{Z}S_n$. Moreover, F_n is a fibre of π_n and S_n is a section, so $(F_n)^2 = 0$ and $F_n \cdot S_n = 1$. Denoting by $S_n' \subset \mathbb{F}_n$ the section given by $a = 0$, one finds that S_n' is equivalent to $S_n + nF_n$, by computing the divisor of $\frac{a}{c}$.

Since S_n and S_n' are disjoint, this yields $0 = S_n \cdot (S_n + nF_n) = (S_n)^2 + n$, so $(S_n)^2 = -n$.

To get the result, it suffices to show that an irreducible $\overline{\mathrm{k}}$-curve $C \subset \mathbb{F}_n$ not equal to S_n or to a fibre of π_n has self-intersection at least equal to n. This will show in particular that a general fibre of any morphism $\mathbb{F}_n \to \mathbb{P}^1$ is equal to a fibre of π_n, since this one has self-intersection 0. We write $C = kS_n + lF_n$ for some $k, l \in \mathbb{Z}$ and find $0 < F_n \cdot C = k$ (since C is not contained in a fibre) and $0 \leq C \cdot S_n = l - nk$. This yields in particular $l \geq nk \geq n$ and $C^2 = -nk^2 + 2kl = kl + k(l - nk) \geq kl \geq n$. □

Lemma 2.16. *Let $C \subset \mathbb{A}^2$ be a geometrically irreducible closed curve. Then, there exists an integer $n \geq 1$ and an isomorphism $\iota\colon \mathbb{A}^2 \xrightarrow{\simeq} \mathbb{F}_n \backslash (S_n \cup F_n)$ such that the closure of $\iota(C)$ in $\mathbb{F}_n$ is a curve Γ which satisfies one of the following two possibilities:*

(1) *$\Gamma \cdot F_n = 1$ and $\Gamma \cap F_n \cap S_n = \emptyset$.*
(2) *$\Gamma \cdot F_n \geq 2$ and the following assertions hold:*
    (a) *If $n = 1$, then $2m_p(\Gamma) \leq \Gamma \cdot F_1$ for $\{p\} = S_1 \cap F_1$, and $m_r(\Gamma) \leq \Gamma \cdot S_1$ for each $r \in F_1(\mathrm{k})$.*
    (b) *If $n \geq 2$, then $2m_r(\Gamma) \leq \Gamma \cdot F_n$ for each $r \in F_n(\mathrm{k})$.*

Furthermore, in Case (1), the curve C is equivalent to a curve given by

$$a(y)x + b(y) = 0,$$

where $a, b \in \mathrm{k}[y]$ are coprime polynomials such that $a \neq 0$ and $\deg b < \deg a$. Moreover, the following assertions are equivalent:

(i) *The polynomial a is constant;*
(ii) *The curve C is equivalent to a line by an automorphism of $\mathbb{A}^2$;*
(iii) *The curve C is isomorphic to $\mathbb{A}^1$;*
(iv) *$\Gamma \cdot S_n = 0$.*

Proof. Let us take any fixed isomorphism $\iota\colon \mathbb{A}^2 \xrightarrow{\simeq} \mathbb{F}_n \backslash (S_n \cup F_n)$ for some $n \geq 1$, and denote by Γ the closure of $\iota(C)$.

We first assume that we have $\Gamma \cdot F_n = 1$. This is equivalent to saying that Γ is a section of π_n. We can furthermore assume that $\Gamma \cap F_n \cap S_n = \emptyset$, as otherwise we blow up the point $F_n \cap S_n$, contract the curve F_n, change the embedding to $\mathbb{F}_{n+1}$ and decrease the intersection number of Γ with S_n at the point $S_n \cap F_n$. After finitely many steps we get $\Gamma \cap F_n \cap S_n = \emptyset$, i.e. we are in Case (1).

If $\Gamma \cdot F_n = 0$, then Γ is a fibre of $\pi_n\colon \mathbb{F}_n \to \mathbb{P}^1$. Let ψ be the unique automorphism of $\mathbb{A}^2$ such that $\iota \circ \psi$ is the standard embedding of $\mathbb{A}^2$ into $\mathbb{F}_n$ of Example 2.14. Then, the curve C is equivalent to the curve $\psi^{-1}(C)$, which has equation $y = \lambda$, for some $\lambda \in \mathrm{k}$. This proves that C is equivalent to the line $y = \lambda$, and thus to the line $x = \lambda$, sent by the standard embedding onto a curve satisfying the conditions (1).

It remains to consider the case where $\Gamma \cdot F_n \geq 2$. If Γ satisfies (2), we are done. Otherwise, we have a k-point $p \in F_n$ satisfying one of the following two possibilities:

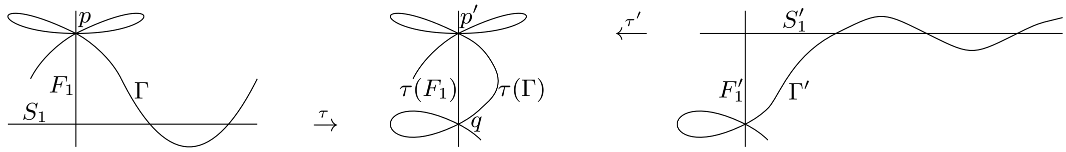
(a) $n = 1$, $m_p(\Gamma) > \Gamma \cdot S_1$, and $p \in F_1$.
(b) $2m_p(\Gamma) > \Gamma \cdot F_n$ and either $n \geq 2$ or $n = 1$ and $p \in S_1 \cap F_1$.

We will replace the isomorphism $\mathbb{A}^2 \xrightarrow{\simeq} \mathbb{F}_n \backslash (S_n \cup F_n)$ with another one, where the singularities of the curve Γ either decrease (all multiplicities have not changed, except one multiplicity which decreases) or stay exactly the same. Moreover, the case where the multiplicities stay the same is only in (a), which cannot appear two consecutive times. We then get the result after finitely many steps.

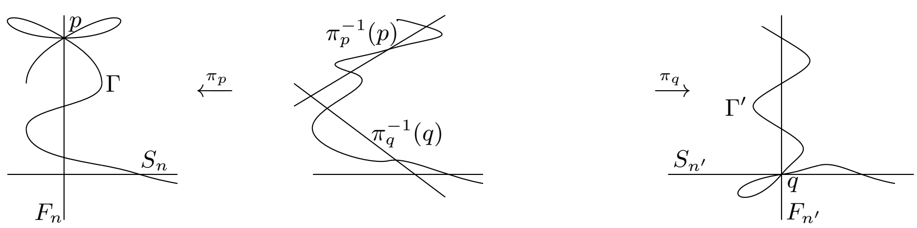
In case (a), we observe that $m_p(\Gamma) > \Gamma \cdot S_1$ implies that $p \notin S_1$. We can then choose p to be a k-point of $F_1 \setminus S_1$ of maximal multiplicity and denote by $\tau\colon \mathbb{F}_1 \to \mathbb{P}^2$ the birational morphism contracting S_1 onto a k-point $q \in \mathbb{P}^2$, observe that $\tau(F_1)$ is a line through q, that $\tau(\Gamma)$ is a curve of multiplicity $\Gamma \cdot S_1$ at q and of multiplicity $m_p(\Gamma) > \Gamma \cdot S_1$ at $p' = \tau(p) \in \tau(F_1)$. Moreover, p' is a k-point of $\tau(F_1)$ of maximal multiplicity on that line. Denote by $\tau'\colon \mathbb{F}_1' \to \mathbb{P}^2$ the birational morphism which is the blow-up at p'. Let S_1' be the exceptional fibre of τ', F_1' the strict transform of $\tau(F_1)$ and Γ' the strict transform of $\tau(\Gamma)$. We then replace the isomorphism $\mathbb{A}^2 \xrightarrow{\simeq} \mathbb{F}_1 \setminus (S_1 \cup F_1)$ with the analogous isomorphism $\mathbb{A}^2 \xrightarrow{\simeq} \mathbb{F}_1' \setminus (S_1' \cup F_1')$ and get

$$\forall\, r \in F_1',\ m_r(\Gamma') \leq \Gamma' \cdot S_1' = m_p(\Gamma).$$

Hence, (a) is not anymore possible. Moreover, the singularities of the new curve Γ' have either decreased or stayed the same. This latter case appears when $m_p(\Gamma) = 1$ and $\Gamma \cdot S_1 = 0$. The situation is illustrated below, for a simple case where $m_p(\Gamma) = 3 > \Gamma \cdot S_1 = 2$.



In case (b), we denote by $\kappa\colon \mathbb{F}_n \dashrightarrow \mathbb{F}_{n'}$ the birational map that blows up the point p and contracts the strict transform of F_n. Call q the point onto which the strict transform of F_n is contracted. We have $\kappa = \pi_q \circ (\pi_p)^{-1}$, where π_p, resp. π_q, are blow-ups of the point p of $\mathbb{F}_n$, resp. the point q of $\mathbb{F}_{n'}$. The drawing below illustrates the situation in a case where $n' = n - 1$. The composition of ι with κ provides a new isomorphism $\mathbb{A}^2 \to \mathbb{F}_{n'} \setminus (S_{n'} \cup F_{n'})$, where $S_{n'}$ is the image of S_n and $F_{n'}$ is the curve corresponding to the exceptional divisor of p. Note that $F_{n'}$ is a fibre of the $\mathbb{P}^1$-bundle $\pi'\colon \mathbb{F}_{n'} \to \mathbb{P}^1$ corresponding to $\pi' = \pi_n \circ \kappa^{-1}$, and that $S_{n'}$ is a section, of self-intersection $-n-1$ if $p \in S_n$ and $-n+1$ if $p \notin S_n$. Hence, since $n \geq 2$ or $n = 1$ and $\{p\} = S_n \cap F_n$, we get that $(S_{n'})^2 = -n' < 0$, and obtain a new isomorphism $\iota'\colon \mathbb{A}^2 \xrightarrow{\simeq} \mathbb{F}_{n'} \setminus (S_{n'} \cup F_{n'})$. The singularity of the new curve Γ' at the point q is equal to $\Gamma \cdot F_n - m_p(\Gamma)$, which is strictly smaller than $m_p(\Gamma)$ by assumption. Moreover $2m_p(\Gamma) > \Gamma \cdot F_n \geq 2$, which implies that p was indeed a singular point of Γ.



Finally, we must now prove the last statement of our lemma, which concerns Case (1). Let ψ be the unique automorphism of $\mathbb{A}^2$ such that $\iota \circ \psi$ is the standard embedding of $\mathbb{A}^2$ into $\mathbb{F}_n$ of Example 2.14. Then, replacing ι by $\iota \circ \psi$ and C by the equivalent curve $\psi^{-1}(C)$, we may assume that $\iota\colon \mathbb{A}^2 \xrightarrow{\simeq} \mathbb{F}_n \backslash (S_n \cup F_n)$ is the standard embedding. This being done, the restriction of $\pi_n\colon \mathbb{F}_n \to \mathbb{P}^1$ to $\mathbb{A}^2$ is $(x,y) \to [y:1]$. The fibres of π_n, equivalent to F_n being given by $y = \mathrm{cst}$, the degree in x of the equation of C is equal to $\Gamma \cdot F_n$ (this can be done for instance by extending the scalars to $\overline{\mathrm{k}}$ and taking a general fibre). Since $\Gamma \cdot F_n = 1$, the equation is of the form $xa(y) + b(y)$ for some polynomials $a, b \in \mathrm{k}[y]$, $a \neq 0$. Since C is geometrically irreducible, the polynomials a and b are coprime. There exist (unique) polynomials $q, \tilde{b} \in \mathrm{k}[x]$ such that $b = aq + \tilde{b}$ with $\deg \tilde{b} < \deg a$. Then, changing the coordinates by applying $(x,y) \mapsto (x + q(y), y)$, one may furthermore assume that $\deg b < \deg a$.

Let us prove that points (i)-(iv) are equivalent. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are obvious. We then prove (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i).

(iii) $\Rightarrow$ (iv): We recall that Γ is a section of $\pi_n\colon \mathbb{F}_n \to \mathbb{P}^1$, so that we have the isomorphisms $\Gamma \simeq \mathbb{P}^1$ and $\Gamma \setminus F_n \simeq \mathbb{A}^1$. The fact that $C = \Gamma \setminus (F_n \cup S_n) \simeq \mathbb{A}^1$ implies that $C \cap (S_n \setminus F_n)$ is empty. Since $\Gamma \cap F_n \cap S_n = \emptyset$ by assumption, one gets $\Gamma \cdot S_n = 0$.

(iv) $\Rightarrow$ (i): We use the open embedding

$$\begin{array}{ccc} \mathbb{A}^2 & \hookrightarrow & \mathbb{F}_n \\ (u,v) & \mapsto & ([1 : uv^n : u], [v:1]). \end{array}$$

The preimages of Γ and S_n by this embedding are the curves of equations $a(v)+b(v)u = 0$ and $u = 0$. Hence $\Gamma \cdot S_n = 0$ implies that a has no $\overline{\mathrm{k}}$-root and thus is a constant. $\square$

2.4. Extension to regular morphisms on $\mathbb{A}^2$.

The following proposition, is the principal tool in the proof of Lemma 2.22, Corollary 2.23 and Proposition 2.25, which themselves give the main part of Theorem 1.

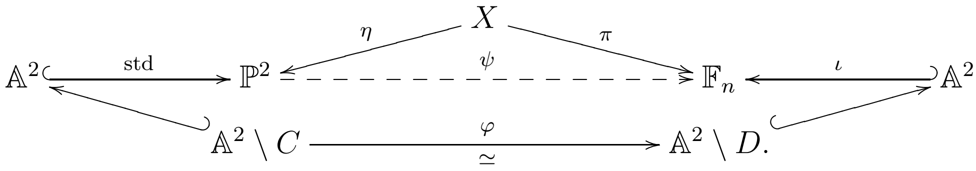
Proposition 2.17. *Let $C \subset \mathbb{A}^2$ be a geometrically irreducible closed curve, not equivalent to a line by an automorphism of $\mathbb{A}^2$, and let $\varphi\colon \mathbb{A}^2 \setminus C \hookrightarrow \mathbb{A}^2$ be an open embedding. Then, there exists an open embedding $\iota\colon \mathbb{A}^2 \hookrightarrow \mathbb{F}_n$, for some $n \geq 1$, such that the rational map $\iota \circ \varphi$ extends to a regular morphism $\mathbb{A}^2 \to \mathbb{F}_n$, and such that $\iota(\mathbb{A}^2) = \mathbb{F}_n \setminus (S_n \cup F_n)$ (where S_n and F_n are as in Example 2.14).*

Proof. By Lemma 2.7, $\varphi(\mathbb{A}^2 \setminus C) = \mathbb{A}^2 \setminus D$ for some geometrically irreducible curve D. If φ extends to an automorphism of $\mathbb{A}^2$ sending C onto D, the result is obvious, by taking any isomorphism $\iota\colon \mathbb{A}^2 \xrightarrow{\simeq} \mathbb{F}_n \setminus (F_n \cup S_n)$, so we can assume that φ does not extend to an automorphism of $\mathbb{A}^2$. Lemma 2.13 implies, since C is not equivalent to a line by an automorphism of $\mathbb{A}^2$, that the same holds for D. Moreover, Lemma 2.7 implies that the extension of φ^{-1} to a birational map $\mathbb{A}^2 \dashrightarrow \mathbb{P}^2$, via the standard embedding $\mathbb{A}^2 \hookrightarrow \mathbb{P}^2$, contracts the curve D (or $\overline{D}$) onto a k-point of $\mathbb{P}^2$. In particular, it does not send D birationally onto C or onto $L_{\mathbb{P}^2}$.

We choose an open embedding $\iota\colon \mathbb{A}^2 \hookrightarrow \mathbb{F}_n$ given by Lemma 2.16, which comes from an isomorphism $\iota\colon \mathbb{A}^2 \xrightarrow{\simeq} \mathbb{F}_n \backslash (S_n \cup F_n)$, such that the closure $\iota(D)$ in $\mathbb{F}_n$ is a curve Γ which satisfies one of the two possibilities (1)-(2) of Lemma 2.16.

We want to show that the open embedding $\iota \circ \varphi\colon \mathbb{A}^2 \setminus C \hookrightarrow \mathbb{F}_n$ extends to a regular morphism on $\mathbb{A}^2$. Using the standard embedding of $\mathbb{A}^2$ into $\mathbb{P}^2$, one gets a birational map $\psi\colon \mathbb{P}^2 \dashrightarrow \mathbb{F}_n$ and needs to show that all base-points of this map are contained in

$L_{\mathbb{P}^2}$. We take as usual a minimal resolution of ψ and obtain a commutative diagram



As we observed before, the map $\psi^{-1}\colon \mathbb{F}_n \dashrightarrow \mathbb{P}^2$ contracts $\Gamma = \overline{\iota(D)}$ onto a k-point, and thus does not send Γ birationally onto C or $L_{\mathbb{P}^2}$. The possible curves contracted by ψ are $L_{\mathbb{P}^2}, \overline{C}$ and the possible curves contracted by ψ^{-1} are Γ, F_n, S_n. Since all these are defined over k, all base-points of ψ, ψ^{-1} are defined over k (Lemma 2.6).

We suppose, for contradiction, that ψ has a base-point q in $\mathbb{A}^2 = \mathbb{P}^2 \setminus L_{\mathbb{P}^2}$, which means that one (-1)-curve $E_q \subset X$ is contracted by η onto q. This curve is the exceptional divisor of a base-point infinitely near to q but not necessarily of q. The minimality of the resolution implies that π does not contract E_q, so $\pi(E_q)$ is a curve of $\mathbb{F}_n$ contracted by ψ^{-1} onto q, which belongs to $\{\Gamma, F_n, S_n\}$.

We observe that ψ has also a base-point p in $L_{\mathbb{P}^2}$. Indeed, otherwise the strict transform of $L_{\mathbb{P}^2}$ would have self-intersection 1 on X and would then not be contracted by π, and would be sent onto a curve of self-intersection ≥ 1, which belongs to $\{\Gamma, F_n, S_n\}$ by Lemma 2.2. As $(F_n)^2 = 0$ and $(S_n)^2 = -n \leq -1$, $L_{\mathbb{P}^2}$ is sent to Γ by ψ. But Γ is not sent birationally onto $L_{\mathbb{P}^2}$ by ψ^{-1}, as we observed before. This contradiction gives a (-1)-curve $E_p \subset X$ contracted by η onto p and not contracted by π. As above, this curve is the exceptional divisor of a base-point infinitely near to p, but not necessarily of p. Again, $\pi(E_p)$ belongs to $\{\Gamma, F_n, S_n\}$.

We thus have at least two of the curves Γ, F_n, S_n that correspond to (-1)-curves of X contracted by η.

We suppose first that S_n corresponds to a (-1)-curve of X contracted by η. The fact that $(S_n)^2 = -n \leq -1$ implies that $n = 1$ and that π does not blow up any point of S_n. As there is another (-1)-curve of X contracted by η, the two curves are disjoint on X, and thus also disjoint on $\mathbb{F}_1$, since π does not blow up any point of S_1. The other curve is then Γ (since $F_1 \cdot S_1 = 1$), and $\Gamma \cdot S_1 = 0$. If moreover $\Gamma \cdot F_1 = 1$ (condition (1) of Lemma 2.16), then the contraction $\mathbb{F}_1 \to \mathbb{P}^2$ of S_1 sends Γ onto a line of $\mathbb{P}^2$, which contradicts the fact that $D \subset \mathbb{A}^2$ is not equivalent to a line. If $\Gamma \cdot F_1 \geq 2$, then condition (2) of Lemma 2.16 implies that $m_r(\Gamma) \leq \Gamma \cdot S_1 = 0$ for each $r \in F_1(\mathrm{k})$. Hence, the intersection of Γ with F_1 (which is not empty since $\Gamma \cdot F_1 \geq 2$) only consists of points not defined over k, which are therefore not blown up by π. The strict transforms $\tilde{\Gamma}$ and $\tilde{F_1}$ on X satisfy then $\tilde{\Gamma} \cdot \tilde{F_1} = \Gamma \cdot F_1 \geq 2$. As $\tilde{\Gamma}$ is contracted by η, the image $\eta(\tilde{F_1})$ is a singular curve and is then equal to $\overline{C}$. This contradicts the fact that ψ contracts $\overline{C}$ onto a point.

The remaining case is when S_n does not correspond to a (-1)-curve of X contracted by η, which implies that $\{\pi(E_p), \pi(E_q)\} = \{F_n, \Gamma\}$, or equivalently that $\{E_p, E_q\} = \{\tilde{F_n}, \tilde{\Gamma}\}$, the strict transforms of F_n and Γ on X. Since $(F_n)^2 = 0$ and $(\tilde{F_n})^2 = -1$, there exists exactly one point $r \in F_n$ (and no infinitely near points) blown up by π, which is then a k-point (as all base-point of π are defined over k). We obtain

$$m_r(\Gamma) = \Gamma \cdot F_n \geq 1 \text{ and } \Gamma \cap F_n = \{r\},$$

since $\tilde{F}_n$ and $\tilde{\Gamma}$ are disjoint on X (and because $\Gamma \cdot F_n \geq 1$, as Γ satisfies the conditions (1)-(2) of Lemma 2.16).

We now prove that $\pi^{-1}(r)$ and $\pi^{-1}(S_n)$ are two disjoint connected sets of rational curves which intersect the two curves $\tilde{F}_n$ and $\tilde{\Gamma}$, i.e. the two curves E_p and E_q. To show this, it suffices to prove that $r \notin S_n$ and that $S_n \cdot \Gamma \geq 1$. Suppose first that $\Gamma \cdot F_n = 1$ (condition (1) of Lemma 2.16). Since $C \cap F_n \cap S_n = \emptyset$, we get $r \in F_n \setminus S_n$. The inequality $\Gamma \cdot S_n > 0$ is provided by the fact that D is not equivalent to a line by an automorphism of $\mathbb{A}^2$ (see again condition (1) of Lemma 2.16). Suppose then that $\Gamma \cdot F_n \geq 2$ (equivalence (iv)-(ii) in case (2) of Lemma 2.16). As $m_r(\Gamma) = \Gamma \cdot F_n \geq 2$, we have $2m_r(\Gamma) > \Gamma \cdot F_n$, which implies that $n = 1$, $r \in F_n \setminus S_n$ and $2 \leq m_r(\Gamma) \leq \Gamma \cdot S_n$ (see again possibility (2) of Lemma 2.16).

We finish by observing that, since $\eta(E_q) = q \in \mathbb{P}^2 \setminus L_{\mathbb{P}^2}$ and $\eta(E_p) = p \in L_{\mathbb{P}^2}$, any connected set of curves of $\eta^{-1}(L_{\mathbb{P}^2} \cup C)$ which touches the two curves E_q and E_p has to contain the strict transform $\tilde{C}$ of $\overline{C}$. This contradicts the fact that $\pi^{-1}(r)$ and $\pi^{-1}(S_n)$ are two disjoint connected sets of rational curves which intersect the two curves $\tilde{F}_n$ and $\tilde{\Gamma}$. □

A direct corollary of Proposition 2.17 is the following, which shows that only smooth curves $C \subset \mathbb{A}^2$ are interesting to study. This follows also from Lemma 2.22 below. Since the proof of Lemma 2.22 is more involved, we prefer to first explain the simple argument that shows how the smoothness follows from Proposition 2.17.

Corollary 2.18. *Let $C \subset \mathbb{A}^2$ be a geometrically irreducible curve and let $\varphi \colon \mathbb{A}^2 \setminus C \hookrightarrow \mathbb{A}^2$ be an open embedding. If C is not smooth, then every open embedding $\mathbb{A}^2 \setminus C \hookrightarrow \mathbb{A}^2$ extends to an automorphism of $\mathbb{A}^2$.*

Proof. By Lemma 2.7, $\varphi(\mathbb{A}^2 \setminus C) = \mathbb{A}^2 \setminus D$ for some geometrically irreducible curve D. We apply Proposition 2.17 and obtain an open embedding $\iota \colon \mathbb{A}^2 \hookrightarrow \mathbb{F}_n$, for some $n \geq 1$, such that the rational map $\iota \circ \varphi$ extends to a regular morphism $\mathbb{A}^2 \to \mathbb{F}_n$. Embedding $\mathbb{A}^2$ into $\mathbb{P}^2$, we get a birational map $\mathbb{P}^2 \dashrightarrow \mathbb{F}_n$ which is regular on $\mathbb{A}^2$. In particular, the singular points of C are not blown up in the minimal resolution of the birational transformation $\mathbb{P}^2 \dashrightarrow \mathbb{F}_n$. So, the curve C is not contracted. By Lemma 2.2, it is then sent onto a singular curve of $\mathbb{F}_n$, not contained in $\mathbb{A}^2 \setminus D$. The three irreducible curves of $\mathbb{F}_n$ not contained in $\mathbb{A}^2 \setminus D$ being $\overline{D}$, S_n, F_n, and since S_n and F_n are smooth, the image of C is $\overline{D}$. Lemma 2.7 then shows that φ extends to an automorphism of $\mathbb{A}^2$. □

Another direct corollary of Proposition 2.17 is the following (Corollary 2.19), which gives in particular a simple proof of the characterisation of birational endomorphisms of $\mathbb{A}^2$ that contract only one irreducible curve, given by Daigle in [Dai91, Theorem 4.11].

Corollary 2.19. *Let $C \subset \mathbb{A}^2$ be a geometrically irreducible closed curve and let $\varphi \colon \mathbb{A}^2 \setminus C \hookrightarrow \mathbb{A}^2$ be an open embedding. The following conditions are equivalent:*

(i) *The extension of φ to a birational self-map of $\mathbb{A}^2$ contracts the curve C onto a point of $\mathbb{A}^2$.*

(ii) *The map φ extends to a birational endomorphism of $\mathbb{A}^2$, which is not an automorphism.*

(*iii*) *There exists automorphisms* α, β *of* $\mathbb{A}^2$ *such that* φ *extends to the birational endomorphism of* $\mathbb{A}^2$ *given by* $\alpha\psi\beta$*, where* $\psi\colon \mathbb{A}^2 \to \mathbb{A}^2$ *is given by* $(x, y) \mapsto (x, x^n y)$*, for some integer* $n \geq 1$.

In particular, if the assertions are satisfied (or equivalently one of the assertions), then $C \subset \mathbb{A}^2$ *is equivalent to a line by an automorphism of* $\mathbb{A}^2$ *and* $\varphi(\mathbb{A}^2 \setminus C) = \mathbb{A}^2 \setminus D$*, where* D *is again a curve equivalent to a line by an automorphism of* $\mathbb{A}^2$.

Proof. (iii)⇒(ii): follows from the fact that, for each $n \geq 1$, the map $\psi\colon (x, y) \mapsto (x, x^n y)$ is a birational endomorphism of $\mathbb{A}^2$ which is not an automorphism, as its inverse $\psi^{-1}\colon (x, y) \mapsto (x, x^{-n}y)$ is not regular.

(ii)⇒(i): Since φ extends to an endomorphism of $\mathbb{A}^2$ which is not an automorphism, the cases (1)-(2) of Lemma 2.7 are not possible. Hence, we are in case (3): C is contracted onto a point of $\mathbb{P}^2$, which is necessarily in $\mathbb{A}^2$ since $\varphi(\mathbb{A}^2) \subset \mathbb{A}^2$.

(i)⇒(iii). By Lemma 2.7, $\varphi(\mathbb{A}^2 \setminus C) = \mathbb{A}^2 \setminus D$ for some geometrically irreducible curve D. Since C is contracted onto a point of $\mathbb{A}^2$, it is not possible to find an open embedding $\iota\colon \mathbb{A}^2 \hookrightarrow \mathbb{F}_n$, for some $n \geq 1$, such that the rational map $\iota \circ \varphi^{-1}$ extends to a regular morphism $\mathbb{A}^2 \to \mathbb{F}_n$. By Proposition 2.17, this implies that D is equivalent to a line by automorphism of $\mathbb{A}^2$. Hence, the same holds for C, by Lemma 2.13. Applying automorphisms of $\mathbb{A}^2$ at the source and the target, we can then assume that $C = D$ is the line $x = 0$ and that φ extends to an endomorphism of $\mathbb{A}^2$ that contracts C onto the origin. By Lemma 2.13(1), the map φ is of the form $(x, y) \mapsto (\lambda x, \mu x^n y + s(x))$, where $\lambda, \mu \in \mathrm{k}^*$, $n \geq 1$ and $s \in \mathrm{k}[x]$ is a polynomial that vanishes at the origin. We then observe that φ is the composition of the automorphism of $\mathbb{A}^2$ given by $(\lambda x, \mu y + s(x\lambda^{-1}))$ with $(x, y) \mapsto (x, x^n y)$. □

2.5. **Completion with two curves and a boundary.** The following technical lemma (Lemma 2.22) is used to prove Corollary 2.23 and Proposition 2.25, which yield almost all statements of Theorem 1.

Definition 2.20. Let X be a smooth projective surface. A reduced closed curve $C \subset X$ is a k-*forest of* X if C is a finite union of closed curves $C_1, \ldots, C_n$, all k-isomorphic to $\mathbb{P}^1$ and if each singular point of C is a k-point lying on exactly two components C_i, C_j intersecting transversally. We moreover ask that C does not contain any loop. If C is connected, we say that C is a k-*tree*.

Remark 2.21. If $\eta\colon X \to Y$ is a birational morphism between smooth projective surfaces such that all base-points of η^{-1} are defined over k, then the exceptional curve of η (union of curves contracted) is a k-forest $E \subset X$. Moreover, the strict transform and the preimage of any k-forest of Y is a k-forest of X. The preimage of a k-tree is a k-tree.

Lemma 2.22. *Let* $C, D \subset \mathbb{A}^2$ *be geometrically irreducible closed curves, not equivalent to lines by automorphisms of* $\mathbb{A}^2$ *and let* $\varphi\colon \mathbb{A}^2 \setminus C \xrightarrow{\simeq} \mathbb{A}^2 \setminus D$ *be an isomorphism which does not extend to an automorphism of* $\mathbb{A}^2$*. Then, there is a smooth projective surface* X *and two open embeddings* $\rho_1, \rho_2\colon \mathbb{A}^2 \hookrightarrow X$ *such that* $\rho_2\varphi = \rho_1$ *and such that the following holds:*

(*i*) *The curves* $\Gamma = \overline{\rho_1(C)} \subset X$*,* $\Delta = \overline{\rho_2(D)} \subset X$ *are isomorphic to* $\mathbb{P}^1$.
(*ii*) *For* $i = 1, 2$*, we have* $\rho_i(\mathbb{A}^2) = X \setminus B_i$ *for some* k*-tree* B_i.
(*iii*) *Writing* $B = B_1 \cap B_2$*, we have* $B_1 = B \cup \Delta$ *and* $B_2 = B \cup \Gamma$.

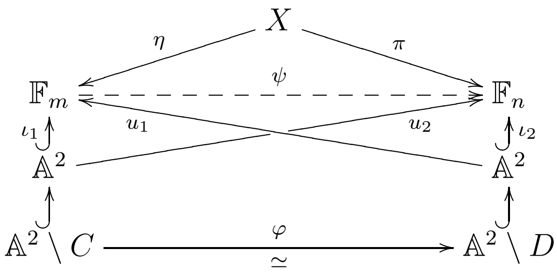
(*iv*) *There is no birational morphism $X \to Y$, where Y is a smooth projective surface, which contracts one component of B, and no other $\overline{\mathrm{k}}$-curve.*

(*v*) *The number of connected components of B is equal to the number of points of $B \cap \Gamma$ and to the number of points of $B \cap \Delta$, and is at most 2.*

Proof. By Proposition 2.17, there exist integers $m, n \geq 1$, and isomorphisms

$$\iota_1\colon \mathbb{A}^2 \xrightarrow{\simeq} \mathbb{F}_m \setminus (S_m \cup F_m), \iota_2\colon \mathbb{A}^2 \xrightarrow{\simeq} \mathbb{F}_n \setminus (S_n \cup F_n)$$

such that both open embeddings $\iota_1\varphi^{-1}\colon \mathbb{A}^2 \setminus D \to \mathbb{F}_m$ and $\iota_2\varphi\colon \mathbb{A}^2 \setminus C \to \mathbb{F}_n$ extend to regular morphisms $u_1\colon \mathbb{A}^2 \to \mathbb{F}_m$, $u_2\colon \mathbb{A}^2 \to \mathbb{F}_n$. Denoting by $\psi\colon \mathbb{F}_m \dashrightarrow \mathbb{F}_n$ the corresponding birational map, equal to $\iota_2(u_1)^{-1} = u_2(\iota_1)^{-1}$, and taking a minimal resolution of the indeterminacies of ψ, we get the following commutative diagram



where η and π are birational morphisms, which are sequences of blow-ups of k-points, being the base-points of ψ and ψ^{-1} respectively (the fact that all points are defined over k is because the curves contracted by ψ and ψ^{-1} are all geometrically irreducible k-curves, since they are contained in S_n, F_n, S_m, F_n or the images of C and D and follows then from Lemma 2.6). Since u_1, u_2 are regular on $\mathbb{A}^2$, the base-points of ψ, resp. ψ^{-1}, belong to $F_m \cup S_m \subset \mathbb{F}_m$, resp. $F_n \cup S_n \subset \mathbb{F}_n$. In particular, we get two open embeddings

$$\rho_1 = \eta^{-1}\iota_1\colon \mathbb{A}^2 \hookrightarrow X, \rho_2 = \pi^{-1}\iota_2\colon \mathbb{A}^2 \hookrightarrow X$$

such that $\rho_2\varphi = \rho_1$ (or more precisely $\rho_2\varphi = \rho_1|_{\mathbb{A}^2\setminus C}$). We have $\rho_1(\mathbb{A}^2) = X \setminus B_1$ and $\rho_2(\mathbb{A}^2) = X \setminus B_2$, where $B_1 = \eta^{-1}(S_m \cup F_m)$ and $B_2 = \pi^{-1}(S_n \cup F_n)$ are k-trees (see Remark 2.21). The fact that φ does not extend to an automorphism of $\mathbb{A}^2$ implies that $B_1 \neq B_2$. We then write $B = B_1 \cap B_2$ and observe that $\eta(B_2 \setminus B)$ is a closed curve of $\mathbb{F}_m \setminus \eta(B) = \iota_1(\mathbb{A}^2)$, sent by ψ outside of $\iota_2(\mathbb{A}^2)$. This shows that $C = \eta(B_2 \setminus B)$, and analogously one obtains $D = \pi(B_1 \setminus B)$. Writing

$$\Gamma = \overline{\rho_1(C)} \subset X, \Delta = \overline{\rho_2(D)} \subset X$$

we then obtain $B_2 = B \cup \Gamma$ and $B_1 = B \cup \Delta$. In particular, since B_1, B_2 are two k-trees, Γ and Δ are isomorphic to $\mathbb{P}^1$ (over k) and intersect transversally B in a finite number of k-points. Moreover, the number of connected components of B is equal to the number of points of $B \cap \Gamma$, and of $B \cap \Delta$ (which are all k-points as said before).

We have then found the surface X together with the embeddings ι_1, ι_2, satisfying conditions (i)–(ii)–(iii). We will then modify X if needed, in order to also get (iv)–(v).

Suppose that the number of connected components of B is $r \geq 3$, and let us show that at least $r-2$ connected components of B are contractible (in the sense that there is a birational morphism $X \to Y$, where Y is a smooth projective rational surface, which contracts one component of B and no other $\overline{\mathrm{k}}$-curve). To show this, we first observe that Γ intersects r distinct curves of B. Since Γ is one of the irreducible components of $B_2 =$

$\pi^{-1}(S_n \cup F_n)$, we can decompose π as $\pi_2 \circ \pi_1$ where $\pi_1(\Gamma)$ is an irreducible component of $(\pi_2)^{-1}(S_n \cup F_n)$ intersecting exactly two other irreducible components R_1, R_2, and such that all points blown up by π_1 are infinitely near points of $\pi_1(\Gamma) \setminus (R_1 \cup R_2)$. This proves that we can contract at least $r-2$ connected components of B.

If one connected component of B is contractible, there exists a morphism $X \to Y$, where Y is a smooth projective rational surface, which contracts this component of B, and no other curve. Since the component intersects Δ transversally in one point, and also Γ in one point, we can replace X with Y, ι_1, ι_2 with the composition with the morphism $X \to Y$ and still have conditions (i)–(ii)–(iii). After finitely many steps, condition (*iv*) is satisfied. By the observation before, the number of connected components of B, after this being done, is at most 2, giving then (v). □

Corollary 2.23. *Let $C, D \subset \mathbb{A}^2$ be geometrically irreducible closed curves and let $\varphi\colon \mathbb{A}^2 \setminus C \xrightarrow{\simeq} \mathbb{A}^2 \setminus D$ be an isomorphism which does not extend to an automorphism of $\mathbb{A}^2$.*

Then, the curves C, D are isomorphic to open subsets of $\mathbb{A}^1$: there exist polynomials $P, Q \in \mathrm{k}[t]$ without square factors, such that $C \simeq \mathrm{Spec}(\mathrm{k}[t, \frac{1}{P}])$ and $D \simeq \mathrm{Spec}(\mathrm{k}[t, \frac{1}{Q}])$. Moreover, the number of $\overline{\mathrm{k}}$-roots of P and Q is the same (i.e. extending the scalars to $\overline{\mathrm{k}}$, the curves C and D become isomorphic to $\mathbb{A}^1$ minus some finite number of points, the same number for both curves). The number of k-roots of P and Q are also the same.

Remark 2.24. When $\mathrm{k} = \mathbb{C}$, this follows from the fact that C and D are isomorphic to open subsets of $\mathbb{A}^1$, which is given by the fact that the curves are rational (Corollary 2.8) and smooth (Corollary 2.18). Indeed, since $\mathbb{A}^2 \setminus C$ and $\mathbb{A}^2 \setminus D$ are isomorphic, they have the same Euler characteristic, so C and D also have the same Euler characteristic.

Proof. If C or D is equivalent to a line, so are both curves (Lemma 2.13), and the result holds. Otherwise, we apply Lemma 2.22 and get a smooth projective surface X and two open embeddings $\rho_1, \rho_2\colon \mathbb{A}^2 \hookrightarrow X$ such that $\rho_2\varphi = \rho_1$ and satisfying the conditions (i)-(ii)-(iii)-(iv)-(v). In particular, C is isomorphic to $\Gamma \setminus B_1 = \Gamma \setminus ((\Gamma \cap B) \cup (\Gamma \cup \Delta))$. Since Γ is isomorphic to $\mathbb{P}^1$ and $\Gamma \cap B$ consists of one or two k-points, this shows that Γ is isomorphic to an open subset of $\mathbb{A}^1$. Doing the same for D, we get isomorphisms $C \simeq \mathrm{Spec}(\mathrm{k}[t, \frac{1}{P}])$ and $D \simeq \mathrm{Spec}(\mathrm{k}[t, \frac{1}{Q}])$ where $P, Q \in \mathrm{k}[t]$ are polynomials, that we can assume without square factors.

The number of $\overline{\mathrm{k}}$-roots of P is equal to the number of $\overline{\mathrm{k}}$-points of $\Gamma \cap B_1$ plus 1. Similarly, the number of $\overline{\mathrm{k}}$-roots of Q is equal to the number of $\overline{\mathrm{k}}$-points of $\Delta \cap B_2$ plus 1. To see that these numbers are equal, we observe that $\Gamma \cap B_1 = (\Gamma \cap B) \cup (\Gamma \cap \Delta)$, that $\Delta \cap B_2 = (\Delta \cap B) \cup (\Delta \cap \Gamma)$, and that the number of points of $\Gamma \cap B$ is the same as the number of points of $\Delta \cap B$ (follows from (v)). As each point of $\Gamma \cap B$ that is also contained in $\Gamma \cap \Delta$ is also contained in $\Delta \cap B$, this shows that P and Q have the same number of $\overline{\mathrm{k}}$-roots. As each $\overline{\mathrm{k}}$-point of $\Gamma \cap B_1$ or $\Delta \cap B_2$ which is not a k-point is contained in $\Gamma \cap \Delta$, the number of k-roots of P and Q is also the same. □

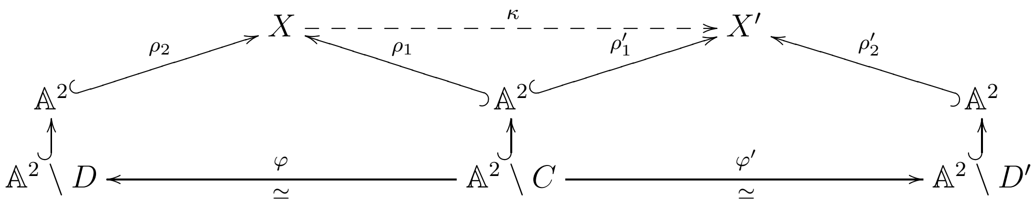
Proposition 2.25. *Let $C, D, D' \subset \mathbb{A}^2$ be geometrically irreducible closed curves, not equivalent to lines by automorphisms of $\mathbb{A}^2$, and let $\varphi\colon \mathbb{A}^2 \setminus C \xrightarrow{\simeq} \mathbb{A}^2 \setminus D$, $\varphi'\colon \mathbb{A}^2 \setminus C \xrightarrow{\simeq} \mathbb{A}^2 \setminus D'$ be isomorphisms which do not extend to an automorphism of $\mathbb{A}^2$. Then, one of the following holds:*

(*a*) *The map $\varphi'(\varphi)^{-1}$ extends to an automorphism of $\mathbb{A}^2$ (sending D to D');*

(*b*) *The three curves* C, D, D' *are* k*-isomorphic to* $\mathbb{A}^1$*;*
(*c*) *The three curves* C, D, D' *are* k*-isomorphic to* $\mathbb{A}^1 \setminus \{0\}$*.*

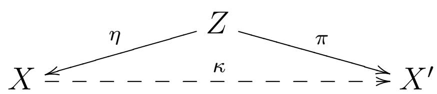
Remark 2.26. Case (b) never happens, as we will show after. Indeed, since C is not equivalent to a line, the existence of φ, φ' is prohibited (Proposition 2.28 below).

Proof. If $C \simeq \mathbb{A}^1$ or $C \simeq \mathbb{A}^1 \setminus \{0\}$, then $D \simeq C \simeq D'$ by Corollary 2.23. We can thus assume that C is not isomorphic to $\mathbb{A}^1$ or $\mathbb{A}^1 \setminus \{0\}$. We apply Lemma 2.22 with φ and φ' and get smooth projective surfaces X, X' and open embeddings $\rho_1, \rho_2, \rho_1', \rho_2' \colon \mathbb{A}^2 \hookrightarrow X$ such that $\rho_2\varphi = \rho_1$, $\rho_2'\varphi' = \rho_1'$ and satisfying the conditions (i)-(ii)-(iii)-(iv)-(v). In particular, we obtain an isomorphism $\kappa \colon X \setminus (B \cup \Gamma \cup \Delta) \xrightarrow{\simeq} X' \setminus (B' \cup \Gamma' \cup \Delta')$ (where $\Gamma = \overline{\rho_1(C)} \subset X$, $\Delta = \overline{\rho_2(D)} \subset X$, $\Gamma' = \overline{\rho_1'(C)} \subset X'$, $\Delta' = \overline{\rho_2'(D')} \subset X'$) and a commutative diagram



By construction, κ sends birationally $\Gamma = \overline{\rho_1(C)}$ onto $\Gamma' = \overline{\rho_1'(C)}$. If κ also sends Δ birationally onto Δ', then $\varphi'\varphi^{-1}$ extends to a birational map that sends birationally D onto D' and then extends to an automorphism of $\mathbb{A}^2$ (Lemma 2.7). It remains then to show that this holds.

Note that all base-points of κ and κ^{-1} are defined over k, since all $\overline{\text{k}}$-curves contracted by κ and κ^{-1} are defined over k (Lemma 2.6). We take a minimal resolution of the indeterminacies of κ:



and observe, as before, that all points blown up by η and π are defined over k. We want to show that the strict transforms $\tilde{\Delta}$ and $\tilde{\Delta}'$ of $\Delta \subset X$, $\Delta' \subset X'$ are equal. We will do this by studying the strict transform $\tilde{\Gamma} = \tilde{\Gamma}'$ of Γ and Γ' and its intersection with $\tilde{\Delta}$ and $\tilde{\Delta}'$ and with the other components of $B_Z = \eta^{-1}(B \cup \Gamma \cup \Delta) = \pi^{-1}(B' \cup \Gamma' \cup \Delta')$.

Recall that $B_1 = B \cup \Delta$, $B_2 = B \cup \Gamma$, $B_1' = B' \cup \Delta'$, $B_2' = B' \cup \Gamma'$ are k-trees and that C is isomorphic to $\Gamma \setminus B_1$ and $\Gamma' \setminus B_1'$ (Lemma 2.22).

(*i*) Suppose first that $\Gamma \cap B_1$ contains some $\overline{\text{k}}$-points which are not defined over k. None of these points is thus a base-point of κ, so $\tilde{\Gamma} \cap \tilde{\Delta}$ contains $\overline{\text{k}}$-points not defined over k. Since B_2' is a k-tree, $\pi^{-1}(B_2')$ is a k-tree, so $\tilde{\Gamma} = \tilde{\Gamma}'$ intersects all irreducible components of B_Z into k-points, except maybe $\tilde{\Delta}'$. This yields $\tilde{\Delta} = \tilde{\Delta}'$ as we wanted.

(*ii*) We can now assume that all $\overline{\text{k}}$-points of $\Gamma \cap B_1$ are defined over k, which implies that all intersections of irreducible components of B_Z are defined over k. We will say that an irreducible component of B_Z is *separating* if the union of all other irreducible components is a k-forest (see Definition 2.20).

Since $B_1 = B \cup \Delta$ is a k-tree, its preimage on B_Z is a k-tree. The union of all components of B_Z distinct from $\tilde{\Gamma}$ being equal to the disjoint union of $\eta^{-1}(B_1)$ with some k-forest contracted onto points of $\Gamma \setminus B_1$, we find that $\tilde{\Gamma}$ is separating. The same argument shows that $\tilde{\Delta}$ and $\tilde{\Delta}'$ are also separating.

It remains then to show that any irreducible component $E \subset B_Z$ which is not equal to $\tilde{\Delta}$ or $\tilde{\Gamma}$ is not separating. We use for this the fact that $C \simeq \Gamma \setminus B_1$ is not isomorphic to $\mathbb{A}^1$ or $\mathbb{A}^1 \setminus \{0\}$, so the set $\Gamma \cap B_1$ contains at least 3 points. If $\eta(E)$ is a point q, then the complement of $\eta^{-1}(q)$ in B_Z contains a loop, since Γ intersects the k-tree B_1 into at least two points distinct from q. If $\eta(E)$ is not a point, it is one of the components of B. We denote by F the union of all irreducible components of $B \cup \Gamma \cup \Delta$ not equal to $\eta(E)$, and prove that F is not a k-forest, since it contains a loop. This is true if $\Delta \cap \Gamma$ contains at least 2 points. If $\Delta \cap \Gamma$ contains one or less points, then $\Delta \cap B$ contains at least two points, so contains exactly two points, on the two connected components of B which both intersect Γ and Δ (see Lemma 2.22(v)). We again get a loop on the union of Γ, Δ and of the connected component of B not containing $\eta(E)$. The fact that F contains a loop implies that $\eta^{-1}(F)$ contains a loop, and achieves to prove that E is not separating. □

2.6. **The case of curves isomorphic to $\mathbb{A}^1$ and the proof of Theorem 1.** To finish the proof of Theorem 1, one still needs to do the case of curves isomorphic to $\mathbb{A}^1$. The case of lines has been already treated in Lemma 2.13. In characteristic zero, this finishes the study by the Abyhankar-Moh-Suzuki theorem, but in positive characteristic, there are many closed curves of $\mathbb{A}^2$ which are isomorphic to $\mathbb{A}^1$ but are not equivalent to lines (these curves are sometimes called "bad lines" in the literature). We will show that an open embedding $\mathbb{A}^2 \setminus C \hookrightarrow \mathbb{A}^2$ always extends to $\mathbb{A}^2$ if C is isomorphic to $\mathbb{A}^1$ but not equivalent to a line.

Lemma 2.27. *Let $n \geq 1$ and let $\Gamma \subset \mathbb{F}_n$ be a closed geometrically irreducible curve, such that $\Gamma \cdot F_n \geq 2$. If there exists a birational map $\mathbb{F}_n \dashrightarrow \mathbb{P}^2$ that contracts Γ onto a point (and maybe contracts some other curves), then Γ is geometrically rational and singular. Moreover, one of the following occurs:*

(a) *There exists a point $p \in \mathbb{F}_n(\overline{\mathrm{k}})$ such that $2m_p(\Gamma) > \Gamma \cdot F_n$.*
(b) *We have $n = 1$ and there exists a point $p \in \mathbb{F}_1(\overline{\mathrm{k}}) \setminus S_1$ such that $m_p(\Gamma) > \Gamma \cdot S_1$.*

Proof. We can assume that $\mathrm{k} = \overline{\mathrm{k}}$. Denote by $\psi\colon \mathbb{F}_n \dashrightarrow \mathbb{P}^2$ the birational map that contracts C onto a point (and maybe some other curves). The minimal resolution of this map yields a commutative diagram

$$\begin{array}{ccccc} & & X & & \\ & \overset{\eta}{\swarrow} & & \overset{\pi}{\searrow} & \\ \mathbb{F}_n & & \overset{\varphi}{\dashrightarrow} & & \mathbb{P}^2 \end{array}$$

In $\mathrm{Pic}(\mathbb{F}_n) = \mathbb{Z}F_n \bigoplus \mathbb{Z}S_n$ we write

$$\begin{aligned} \Gamma &= aS_n + bF_n \\ -K_{\mathbb{F}_n} &= 2S_n + (2+n)F_n \end{aligned}$$

for some integers a, b. Note that $a = \Gamma \cdot F_n \geq 2$ and that $b - an = \Gamma \cdot S_n \geq 0$. By hypothesis, the strict transform $\tilde{\Gamma}$ of Γ on X is a smooth curve contracted by π. In particular, Γ is rational and the divisor $2\tilde{\Gamma} + aK_X$ is not effective, since $\pi(2\tilde{\Gamma} + aK_X) = aK_{\mathbb{P}^2}$ is not effective.

Denoting by $E_1, \ldots, E_r \in \mathrm{Pic}(X)$ the pull-backs of the exceptional divisors blown up by η (which satisfy $(E_i)^2 = -1$ for each i and $E_i \cdot E_j = 0$ for $i \neq j$) we have

$$\begin{array}{rcllcl} \tilde{\Gamma} & = & a\eta^*(S_n) & + & b\eta^*(F_n) & - \sum_{i=1}^r m_i E_i \\ -K_X & = & 2\eta^*(S_n) & + & (2+n)\eta^*(F_n) & - \sum_{i=1}^r E_i \\ 2\tilde{\Gamma} + aK_X & = & & & (2b - a(2+n))\eta^*(F_n) & + \sum_{i=1}^r (a - 2m_i)E_i \end{array}$$

which implies, since $2\tilde{\Gamma} + aK_X$ is not effective, that either $2b < a(2+n)$ or $2m_i > a$ for some i. If $2m_i > a$ for some i, we get (a), since the m_i are the multiplicities of $\tilde{\Gamma}$ at the points blown up by η.

It remains to study the case where $2m_i \leq a$ for each i, and where $2b < a(2+n)$. Remembering that $b - an = \Gamma \cdot S_n \geq 0$, one finds $n \leq \frac{b}{a} < \frac{2+n}{2}$, whence $n = 1$ and thus $2b < 3a$. We then compute

$$3\tilde{\Gamma} + bK_X \;=\; (3a - 2b)\eta^*(S_n) \;+\; \textstyle\sum_{i=1}^r (b - 3m_i)E_i$$

which is again not effective, since $\pi(3\tilde{\Gamma} + bK_X) = bK_{\mathbb{P}^2}$ and $b \geq an = a \geq 2$. This implies that there exists i such that $3m_i > b$. Since $2m_i \leq a$, one finds $m_i > b - a = \Gamma \cdot S_1$, which yields (b). □

Proposition 2.28. *Let $C \subset \mathbb{A}^2$ be a closed curve, isomorphic to $\mathbb{A}^1$ (i.e. isomorphic to $\mathbb{A}^1$ over* k*). The following are equivalent:*

(*a*) *The curve C is equivalent to a line by an automorphism of $\mathbb{A}^2$.*
(*b*) *There exists an open embedding $\mathbb{A}^2 \setminus C \hookrightarrow \mathbb{A}^2$ which does not extend to an automorphism of $\mathbb{A}^2$.*
(*c*) *Embedding $\mathbb{A}^2$ into $\mathbb{P}^2$, via the canonical embedding, there exists a birational map of $\mathbb{P}^2$ that contracts the curve C onto a point (and maybe contracts other curves).*

Proof. The implications (*a*) ⇒ (*b*) and (*a*) ⇒ (*c*) can be observed, for example by taking the map $(x, y) \mapsto (x, xy)$, which is an open embedding of $\mathbb{A}^2 \setminus \{x = 0\}$ into $\mathbb{A}^2$, which does not extend to an automorphism of $\mathbb{A}^2$, and whose extension to $\mathbb{P}^2$ contracts the line $x = 0$ onto a point.

To prove (*b*) ⇒ (*c*), we take an open embedding $\varphi\colon \mathbb{A}^2 \setminus C \hookrightarrow \mathbb{A}^2$ which does not extend to an automorphism of $\mathbb{A}^2$ and look at the extension to $\mathbb{P}^2$. By Lemma 2.7, either this one contracts C, or C is equivalent to a line, in which case (*c*) is true as was shown before.

It remains to prove (*c*) ⇒ (*a*). We apply Lemma 2.16, and obtain an isomorphism $\iota\colon \mathbb{A}^2 \xrightarrow{\simeq} \mathbb{F}_n \backslash (S_n \cup F_n)$ such that the closure of $\iota(C)$ in $\mathbb{F}_n$ is a curve Γ which satisfies one of the two cases (1)-(2) of Lemma 2.16. In case (1), the curve is equivalent to a line as it is isomorphic to $\mathbb{A}^1$ (equivalence (*ii*) − (*iii*) of Lemma 2.16). It remains to study the case where Γ satisfies the conditions (2) of Lemma 2.16 (in particular $\Gamma \cdot F_n \geq 2$), and to show that these, together with (*c*), yield a contradiction. We prove that there is no point $p \in \mathbb{F}_n(\overline{\mathrm{k}})$ such that $2m_p(\Gamma) > \Gamma \cdot F_n$. Indeed, since $\Gamma \cdot F_n \geq 2$, the point would be a singular point of Γ, and since $\Gamma \setminus (F_n \cup E_n) = C$ is isomorphic to $\mathbb{A}^1$, p is a k-point and is the unique $\overline{\mathrm{k}}$-point of $\Gamma \cap (F_n \cup E_n)$. Moreover, as $\Gamma \cdot F_n \geq 2$, we find that $p \in F_n$. Since $2m_p(\Gamma) > \Gamma \cdot F_n$ and because Γ satisfies the conditions (2) of Lemma 2.16, the only possibility is that $n = 1$, $p \in F_1 \setminus S_1$ and $0 < m_p(\Gamma) \leq \Gamma \cdot S_1$. This is impossible as it contradicts the fact that $\Gamma \cap (S_1 \cup F_1)$ contains only one $\overline{\mathrm{k}}$-point.

Denote by $\psi_0\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ the birational map that contracts C (and maybe other curves) onto a point. Observe that $\psi_0 \circ \iota^{-1}$ yields a birational map $\psi\colon \mathbb{F}_n \dashrightarrow \mathbb{P}^2$ which contracts Γ onto a point. As there is no point $p \in \mathbb{F}_n(\overline{\mathrm{k}})$ such that $2m_p(\Gamma) > \Gamma \cdot F_n$, Lemma 2.27 implies that $n = 1$ and that there exists a point $p \in \mathbb{F}_1(\overline{\mathrm{k}}) \setminus S_1$ such that $m_p(\Gamma) > \Gamma \cdot S_1$. Again, this point is a k-point, since C is k-isomorphic to $\mathbb{A}^1$. This contradicts the conditions (2) of Lemma 2.16. □

Remark 2.29. If the field k is perfect, then every curve that is geometrically isomorphic to $\mathbb{A}^1$ (i.e. over $\overline{\mathrm{k}}$) is also isomorphic to $\mathbb{A}^1$. This can be seen by embedding the curve in $\mathbb{P}^1$ and looking at the complement point, necessarily defined over k. For non-perfect fields, there exist closed curves $C \subset \mathbb{A}^2$ geometrically isomorphic to $\mathbb{A}^1$ but not isomorphic to $\mathbb{A}^1$ (see [Rus70]). Corollary 2.23 shows that every open embedding $\mathbb{A}^2 \setminus C \hookrightarrow \mathbb{A}^2$ extends to an automorphism of $\mathbb{A}^2$ for all such curves.

We can now finish the section by proving Theorem 1:

Proof of Theorem 1. We recall the hypothesis of the theorem: we have a geometrically irreducible closed curve $C \subset \mathbb{A}^2$ and an isomorphism $\varphi\colon \mathbb{A}^2 \setminus C \xrightarrow{\simeq} \mathbb{A}^2 \setminus D$ for some closed curve $C \subset \mathbb{A}^2$. Assume that φ does not extend to an automorphism of $\mathbb{A}^2$.

(1): If C is isomorphic to $\mathbb{A}^1$, then Proposition 2.28(a) shows that C is equivalent to a line and Lemma 2.13(2) implies that the same holds for D. In particular, the curves C and D are isomorphic.

If C is isomorphic to $\mathbb{A}^1 \setminus \{0\}$ then so is D by Corollary 2.23.

(2): if C is not isomorphic to $\mathbb{A}^1$ or to $\mathbb{A}^1 \setminus \{0\}$, then Proposition 2.25 shows that the isomorphism $\varphi\colon \mathbb{A}^2 \setminus C \xrightarrow{\simeq} \mathbb{A}^2 \setminus D$ (not extending to an automorphism of $\mathbb{A}^2$) is uniquely determined by C, up to post-composition by an automorphism of $\mathbb{A}^2$. In particular, there are at most two equivalence classes of curves of $\mathbb{A}^2$ having complements isomorphic to $\mathbb{A}^2 \setminus C$. Corollary 2.23 achieves the proof by giving the existence of isomorphisms $C \simeq \mathrm{Spec}(\mathrm{k}[t, \frac{1}{P}])$ and $D \simeq \mathrm{Spec}(\mathrm{k}[t, \frac{1}{Q}])$ for some polynomials $P, Q \in \mathrm{k}[t]$ without square factors having the same number of roots in k, and having also the same number of roots in the algebraic closure of k. □

3. Families of different embeddings given by sections of line bundles

3.1. A construction using elements of $\mathrm{SL}_2(\mathrm{k}[y])$.

As we observed in Lemma 2.16 and its applications, the curves of $\mathbb{A}^2$ given by

$$a(y)x + b(y) = 0$$

for some coprime polynomials $a, b \in \mathrm{k}[y]$, $a \neq 0$ (where we can always assume that $\deg b < \deg a$), yield a natural family which plays an important role. We study this family here. Recall that such curves are equivalent to a line if and only $a(y)$ is a constant (Lemma 2.16(i)-(ii)), which is a case already treated in Lemma 2.13. We can thus assume that $\deg(a) \geq 1$, and thus that $\deg(b) \geq 0$ (i.e. that $b \neq 0$) since the polynomial $a(y)x + b(y)$ has to be irreducible.

Lemma 3.1. *For each matrix* $\begin{pmatrix} a(y) & b(y) \\ c(y) & d(y) \end{pmatrix} \in \mathrm{SL}_2(\mathrm{k}[y])$*, we have an isomorphism*

$$\begin{array}{cccc} \varphi\colon & \mathbb{A}^2 \setminus C & \xrightarrow{\simeq} & \mathbb{A}^2 \setminus D \\ & (x,y) & \mapsto & \left(\frac{c(y)x+d(y)}{a(y)x+b(y)}, y\right) \end{array}$$

where $C, D \subset \mathbb{A}^2$ *are given by* $a(y)x + b(y) = 0$ *and* $a(y)x - c(y) = 0$ *respectively.*

Proof. Note first that φ is a birational transformation of $\mathbb{A}^2$, with inverse $\psi\colon (x,y) \mapsto (\frac{-b(y)x+d(y)}{a(y)x-c(y)}, y)$. It remains to prove that the isomorphism $\varphi^*\colon \mathrm{k}(x,y) \to \mathrm{k}(x,y)$, $x \mapsto \frac{cx+d}{ax+b}$, $y \mapsto y$ induces an isomorphism $\mathrm{k}[x, y, \frac{1}{ax-c}] \to \mathrm{k}[x, y, \frac{1}{ax+b}]$. This follows from $\varphi^*(\{x, y, \frac{1}{ax-c}\}) = \{\frac{cx+d}{ax+b}, y, ax+b\}$ and $\psi^*(\{x, y, \frac{1}{ax+b}\}) = \{\frac{-bx+d}{ax-c}, y, ax-b\}$. □

The curves C and D of Lemma 3.1 are always isomorphic, but are in general not equivalent, as we now prove.

Lemma 3.2. *Let* $C, D \subset \mathbb{A}^2$ *be two irreducible curves given by*

$$a(y)x + b(y) = 0 \text{ and } \tilde{a}(y)x + \tilde{b}(y) = 0$$

respectively, for some $a, b, \tilde{a}, \tilde{b} \in \mathrm{k}[y]$*, such that* $\deg(a) > \deg(b) \geq 0$ *and* $\deg(\tilde{a}) > \deg(\tilde{b}) \geq 0$*.*

(1) *The curves* C *and* D *are isomorphic to* $\mathrm{Spec}(\mathrm{k}[y, \frac{1}{a}])$ *and* $\mathrm{Spec}(\mathrm{k}[y, \frac{1}{\tilde{a}}])$ *respectively.*
(2) *The curves* C *and* D *are equivalent if and only if there exist* $\alpha, \lambda, \mu \in \mathrm{k}^*$*,* $\beta \in \mathrm{k}$ *such that*

$$\tilde{a}(y) = \lambda \cdot a(\alpha y + \beta), \quad \tilde{b}(y) = \mu \cdot b(\alpha y + \beta).$$

Proof. The proof of (1) is given by

$$\mathrm{k}[C] = \mathrm{k}[x,y]/(a(y)x + b(y)) \simeq \mathrm{k}[y, -b(y)/a(y)] = \mathrm{k}[y, 1/a(y)],$$

where the last equality comes from the fact that $a, b \in \mathrm{k}[y]$ are coprime (since C is irreducible), so there exist $c, d \in \mathrm{k}[y]$ with $ad - bc = 1$, which implies that $\frac{1}{a} = \frac{ad-bc}{a} = d - c \cdot \frac{b}{a} \in \mathrm{k}[y, \frac{b}{a}]$. The isomorphism $D \simeq \mathrm{Spec}(\mathrm{k}[y, \frac{1}{\tilde{a}}])$ is analogous.

To prove (2), we first observe that if $\tilde{a}(y) = \lambda \cdot a(\alpha y + \beta)$ and $\tilde{b}(y) = \mu \cdot b(\alpha y + \beta)$ for some $\alpha, \lambda, \mu \in \mathrm{k}^*$, $\beta \in \mathrm{k}$, then the automorphism $(x, y) \mapsto (\frac{\lambda}{\mu} x, \alpha y + \beta)$ of $\mathbb{A}^2$ sends D to C.

Conversely, we assume the existence of $\varphi \in \mathrm{Aut}(\mathbb{A}^2)$ that sends D onto C and want to find $\alpha, \lambda, \mu \in \mathrm{k}^*$, $\beta \in \mathrm{k}$ as above. Writing φ as $(x, y) \mapsto (f(x, y), g(x, y))$ for some $f, g \in \mathrm{k}[x, y]$, one gets

$$\text{(A)} \qquad \mu(a(g)f + b(g)) = \tilde{a}x + \tilde{b}$$

for some $\mu \in \mathrm{k}^*$.

(*i*) If $g \in \mathrm{k}[y]$, the fact that $\mathrm{k}[f, g] = \mathrm{k}[x, y]$ implies that $g = \alpha y + \beta, f = \gamma x + s$ for some $\alpha, \gamma \in \mathrm{k}^*, \beta \in \mathrm{k}, s \in \mathrm{k}[y]$, and yields $a(g)f + b(g) = a(g)(\gamma x + s) + b(g)$, so Equation (A) yields:

$$\tilde{a} = \mu\gamma \cdot a(g), \ \tilde{b} = \mu \cdot (a(g)s + b(g)).$$

This shows in particular that $\deg a = \deg \tilde{a}$, whence $\deg \tilde{b} < \deg a(g)$. Since $\deg(b(g)) < \deg a(g)$, we find that $s = 0$, and thus that $\tilde{b} = \mu \cdot b(g)$, as desired. This ends the proof, by choosing $\lambda = \mu\gamma$.

(ii) It remains to study the case where $g \notin \mathrm{k}[y]$, which corresponds to $\deg_x(g) \geq 1$ and yields $\deg_x(a(g)) = \deg(a) \cdot \deg_x(g) > \deg(b) \cdot \deg_x(g) = \deg_x(b(g))$, which implies that $\deg_x(a(g)f + b(g)) = \deg(a) \cdot \deg_x(g) + \deg_x(f)$. Equation (A) shows that this degree is 1, and since $\deg(a) \geq 1$, we find $\deg(a) = 1$. Similarly, the automorphism sending C to D satisfies the same condition, so $\deg(\tilde{a}) = 1$. This implies that $b, \tilde{b} \in \mathrm{k}^*$. There exist thus some $\alpha, \lambda, \mu \in \mathrm{k}^*$, $\beta \in \mathrm{k}$ such that $\tilde{a}(y) = \lambda \cdot a(\alpha y + \beta)$, $\tilde{b}(y) = \mu \cdot b(\alpha y + \beta)$. □

Corollary 3.3. *For each polynomial $f \in \mathrm{k}[t]$ of degree ≥ 1, there exist two closed curves $C, D \subset \mathbb{A}^2$ both isomorphic to $\mathrm{Spec}(\mathrm{k}[t, \frac{1}{f}])$, but not equivalent, such that $\mathbb{A}^2 \setminus C \simeq \mathbb{A}^2 \setminus D$. Moreover, the set of such pairs of closed embeddings, up to equivalence, is infinite.*

Proof. We choose an irreducible polynomial $b \in \mathrm{k}[t]$ which does not divide f. We then choose, for each $n \geq 1$ such that $\deg(f^n) > 2\deg(b)$, two polynomials $c, d \in \mathrm{k}[t]$ such that $f^n d - bc = 1$.

The curves $C_n, D_n \subset \mathbb{A}^2$ given by $f^n x + b = 0$ and $f^n x - c = 0$ are both isomorphic to $\mathrm{Spec}(\mathrm{k}[t, \frac{1}{f^n}]) = \mathrm{Spec}(\mathrm{k}[t, \frac{1}{f}])$ (Lemma 3.2(1)) and have isomorphic complements (Lemma 3.1). Moreover, as $\deg bc = \deg(f^n d - 1) \geq \deg(f^n) > 2\deg(b)$, we find that $\deg c > \deg b$, which implies that C_n and D_n are not equivalent (Lemma 3.2(2)). Moreover, the curves C_n are all non-equivalent, again by Lemma 3.2(2). □

3.2. **Curves isomorphic to $\mathbb{A}^1 \setminus \{0\}$.** We consider now families of curves of $\mathbb{A}^2$ of the form $xy^d + yP(y) + 1 = 0$ for some $d \geq 1$ and some polynomial $P \in \mathrm{k}[y]$ (which can be chosen to be of degree $\leq d-2$). Note that all these curves are isomorphic to $\mathrm{Spec}(\mathrm{k}[t, \frac{1}{t^d}]) = \mathrm{Spec}(\mathrm{k}[t, \frac{1}{t}]) \simeq \mathbb{A}^1 \setminus \{0\}$ (Lemma 3.2(1)).

Lemma 3.4. *Let $d \geq 1$ be an integer. For each polynomial $P \in \mathrm{k}[y]$ of degree $\leq d-2$, one defines $C_P \subset \mathbb{A}^2$ to be the curve given by*

$$xy^d + yP(y) + 1 = 0.$$

(1) *Fixing some polynomial $P \in \mathrm{k}[y]$ of degree $\leq d-2$, there exists, for each integer $m \geq 1$, a unique polynomial $Q \in \mathrm{k}[y]$ of degree $\leq d-2$ satisfying*

$$P(y) \equiv (yP(y)+1)^m Q(y(yP(y)+1)^m) \pmod{y^{d-1}}. \tag{B}$$

There exist then $f_m \in \mathrm{k}[x, y]$ and $\lambda \in \mathrm{k}$ that give an isomorphism

$$\begin{array}{rccc} \psi_m\colon & \mathbb{A}^2 \setminus C_P & \xrightarrow{\simeq} & \mathbb{A}^2 \setminus C_Q \\ & (x,y) & \mapsto & \left(\frac{x+\lambda+yf_m(x,y)}{(xy^d+yP(y)+1)^{md}}, y(xy^d+yP(y)+1)^m\right). \end{array}$$

(2) *For all integers $m > m' \geq 1$, the open embeddings $\psi_m, \psi_{m'}\colon \mathbb{A}^2 \setminus C_P \hookrightarrow \mathbb{A}^2$ are not equal, up to automorphisms of $\mathbb{A}^2$.*

Proof. (1): We first observe that if $d = 1$, then (1) is trivially true, with $P = Q = 0$ (these are the only polynomials of degree $\leq d-2$), by choosing $f_m = x$. Indeed, the map $(x,y) \mapsto \left(\frac{x}{(xy+1)^m}, y(xy+1)^m\right)$ yields an automorphism of $\mathbb{A}^2 \setminus C_P = \mathbb{A}^2 \setminus C_Q$, whose inverse is $(x,y) \mapsto \left(x(xy+1)^m, \frac{y}{(xy+1)^m}\right)$. We then assume that $d \geq 2$.

For each polynomial $R \in \mathrm{k}[x]$ of degree $\leq d-2$, one defines

$$\varphi_R\colon (x,y) \mapsto (xy^d + yR(y) + 1, y)$$

which is a birational morphism $\psi_R\colon \mathbb{A}^2 \to \mathbb{A}^2$, that contracts the line $L_y \subset \mathbb{A}^2$ given by $y = 0$ onto the point $(1, 0)$, and induces an automorphism of $\mathbb{A}^2 \setminus L_y$. Indeed, the inverse $(\varphi_R)^{-1}\colon \mathbb{A}^2 \dashrightarrow \mathbb{A}^2$ is given by $(\varphi_R)^{-1}\colon (x, y) \mapsto \left(\frac{x-yR(y)-1}{y^d}, y\right)$, so $(\varphi_R)^* \in \mathrm{Aut}(\mathrm{k}(x, y))$ restricts to an automorphism of $\mathrm{k}[x, y, \frac{1}{y}]$.

We fix some $P \in \mathrm{k}[x]$ of degree $\leq d - 2$, choose some integer $m \geq 1$ and observe that $(yP(y) + 1)^m$ is invertible in $\mathrm{k}[y]/(y^{d-1})$ since $yP(y)$ is nilpotent. This shows that Equation (B) has a unique solution in $\mathrm{k}[y]/(y^{d-1})$, so we get a unique polynomial Q of degree $\leq d - 2$ satisfying it. We then consider the birational map

$$\psi_m = (\varphi_Q)^{-1}\tau\varphi_P$$

where $\tau\colon \mathbb{A}^2 \to \mathbb{A}^2$ is the birational endomorphism given by $(x, y) \mapsto (x, x^m y)$ (that fixes the point $(1, 0)$ and induces a local isomorphism around this point). The map ψ_m is constructed so that it induces an isomorphism $\mathbb{A}^2 \setminus (L_y \cup C_P) \xrightarrow{\simeq} \mathbb{A}^2 \setminus (L_y \cup C_Q)$ (this does not depend on the choice of Q). It remains to see that the choice of Q implies that ψ_m extends to an isomorphism $\mathbb{A}^2 \setminus C_P \xrightarrow{\simeq} \mathbb{A}^2 \setminus C_Q$ of the desired form. Since Q satisfies Equation (B), we find

$$\text{(C)} \qquad P(y) \equiv (yP(y) + 1)^m Q(y(yP(y) + 1)^m) + \lambda y^{d-1} \pmod{y^d}$$

for some $\lambda \in \mathrm{k}$. We then compute $\psi(x, y)$, and find

$$\left(\frac{xy^d + y\left(P(y) - (xy^d + yP(y) + 1)^m Q(y(xy^d + yP(y) + 1)^m)\right)}{y^d(xy^d + yP(y) + 1)^{md}}, y(xy^d + yP(y) + 1)^m\right)$$

which can be written, using Equation (C), as

$$\psi(x, y) = \left(\frac{x + \lambda + yf_m(x, y)}{(xy^d + yP(y) + 1)^{md}}, y(xy^d + yP(y) + 1)^m\right)$$

for some polynomial $f_m \in \mathrm{k}[x, y]$. This shows that ψ_m restricts to the automorphism $x \mapsto x + \lambda$ on L_y and then restricts to an isomorphism $\mathbb{A}^2 \setminus C_P \xrightarrow{\simeq} \mathbb{A}^2 \setminus C_Q$ of the desired form, achieving the proof of (1).

To prove (2), we observe that, since $x + \lambda + yf_m(x, y) \equiv x + \lambda \pmod{y}$, the polynomial $x + \lambda + yf_m(x, y)$ is not divisible by $xy^d + yP(y) + 1$. In particular, applying any automorphism of $\mathbb{A}^2$ to ψ_m, we cannot decrease the degree of the denominators, and thus do not get any $\psi_{m'}$, for $m' < m$. □

Remark 3.5. Geometrically, the construction of Lemma 3.4 can be interpreted as follows: the birational morphism $\varphi_P\colon (x, y) \mapsto (xy^d + yP(y) + 1, y)$ contracts the line $y = 0$ onto the point $(1, 0)$. If $n = 1$ then φ_P just sends the line onto the exceptional divisor of $(1, 0)$. If $n \geq 2$, it sends the line onto the exceptional divisor of a point in the $(n - 1)$-th neighbourhood of $(1, 0)$. The coordinates of these points are determined by P. The fact that $\tau\colon (x, y) \mapsto (x, x^m y)$ contracts $x = 0$ implies that ψ_m contracts the curve $xy^d + yP(y) + 1 = 0$. Moreover, τ fixes the point $(1, 0)$ and induces a local isomorphism around it, so acts on the set of infinitely near points. This action changes the polynomial P and replaces it with another one, which is the polynomial Q provided by Lemma 3.4.

Corollary 3.6. *For each field* k*, there exists an infinite sequence of curves* $C_i \subset \mathbb{A}^2$*,* $i \in \mathbb{N}$*, all pairwise not equivalent under automorphisms, all isomorphic to* $\mathbb{A}^1 \setminus \{0\}$ *and* *such that for each i there are infinitely many open embeddings $\mathbb{A}^2 \setminus C_i \hookrightarrow \mathbb{A}^2$, up to automorphisms of $\mathbb{A}^2$.*

Proof. It suffices to choose the curve C_i given by $xy^{i+2}+y+1$, for each $i \geq 2$. These curves are all isomorphic to $\mathbb{A}^1 \setminus \{0\}$ (Lemma 3.2(1)) and are pairwise not equivalent, under automorphisms of $\mathbb{A}^2$ (Lemma 3.2(2)). The existence of infinitely many open embeddings $\mathbb{A}^2 \setminus C \hookrightarrow \mathbb{A}^2$, up to automorphisms of $\mathbb{A}^2$, is then provided by Lemma 3.4. □

One can compute the polynomial Q provided by Lemma 3.4, in terms of P, m and d, and find explicit formulas. We find in particular the following result:

Lemma 3.7. *For each $\mu \in \mathrm{k}$, we denote by $C_\mu \subset \mathbb{A}^2$ the curve given by*

$$xy^3 + \mu y^2 + y + 1 = 0.$$

Then, there exists an isomorphism $\mathbb{A}^2 \setminus C_\mu \to \mathbb{A}^2 \setminus C_{\mu-1}$, for each $\mu \in \mathrm{k}$. In particular, if $\mathrm{char}(\mathrm{k}) = 0$, we obtain infinitely many closed curves of $\mathbb{A}^2$, not equivalent under automorphism of $\mathbb{A}^2$, which have isomorphic complements.

Proof. To get the isomorphism $\mathbb{A}^2 \setminus C_\mu \to \mathbb{A}^2 \setminus C_{\mu-1}$, it suffices to apply Lemma 3.4 with $d = 3$, $P = \mu y + 1$, $m = 1$, and check that $Q = (\mu - 1)y + 1$. One needs to see that

$$P(y) \equiv (yP(y) + 1)Q\big(y(yP(y) + 1)\big) \pmod{y^2}.$$

(see Equation (B) in Lemma 3.4). Indeed, one finds, since $yP(y) \equiv y \pmod{y^2}$, that

$$(yP(y) + 1)Q\big(y(yP(y) + 1)\big) \equiv (y + 1)Q(y) \equiv Q(y) + y \equiv P(y) \pmod{y^2}.$$

To get the last statement, one assumes that $\mathrm{char}(\mathrm{k}) = 0$ and observes that the affine surfaces $\mathbb{A}^2 \setminus C_n$ are all isomorphic for each $n \in \mathbb{Z}$. To show that the curves C_n, $n \in \mathbb{Z}$ are pairwise non-equivalent, we apply Lemma 3.2(2): for $m, n \in \mathbb{Z}$, the curves C_m and C_n are equivalent only if there exist $\alpha, \lambda, \mu \in \mathrm{k}^*$, $\beta \in \mathrm{k}$ such that

$$y^3 = \lambda \cdot (\alpha y + \beta)^3, \; my^2 + y + 1 = \mu \cdot (n(\alpha y + \beta)^2 + (\alpha y + \beta) + 1).$$

The first equality yields $\beta = 0$, which yields, together with the second equation $my^2 + y + 1 = \mu \cdot (n\alpha^2 y^2 + \alpha y + 1)$, so $\mu = 1$, $\alpha = 1$ and thus $m = n$, as we wanted. □

If $\mathrm{char}(\mathrm{k}) = p > 0$, Lemma 3.7 only gives p non-equivalent curves having the same complement. We can get more curves applying Lemma 3.2 to polynomials of higher degree:

Lemma 3.8. *For each integer $n \geq 1$ there exist curves $C_1, \ldots, C_n \subset \mathbb{A}^2$, all isomorphic to $\mathbb{A}^1 \setminus \{0\}$, no two being equivalent, such that all surfaces $\mathbb{A}^2 \setminus C_1, \ldots, \mathbb{A}^2 \setminus C_n$ are isomorphic.*

Proof. The case where $\mathrm{char}(\mathrm{k}) = 0$ is provided by Lemma 3.7 so we can assume that $\mathrm{char}(\mathrm{k}) = p \geq 2$. Set $d = p^n + 2$. For each integer i satisfying $1 \leq i \leq n$, there exists a unique polynomial $P_i \in \mathrm{k}[y]$ of degree at most $d - 2 = p^n$ satisfying

$$P_i(y) \equiv \big(1 + yP_i(y)\big)^{p^i} \pmod{y^{d-1}}. \tag{D}$$

Actually, the coefficients of P_i are inductively determined: Setting $y = 0$ in (D), we get $P_i(0) = 1$; assuming that the first m coefficients $u_0, \ldots, u_{m-1}$ of $P_i = \sum_{j=0}^{d-2} u_j y^j$ are known, Equation (D) yields us

$$\sum_{j=0}^{m} u_j y^j \equiv \big(1 + y \sum_{j=0}^{m-1} u_j y^j\big)^{p^i} \pmod{y^{m+1}}$$

which determines u_m in terms of $u_0, \dots, u_{m-1}$. In particular, for $m = p^i$, we have

$$\sum_{j=0}^{p^i} u_j y^j \equiv 1 + y^{p^i}\big(\sum_{j=0}^{p^i-1} u_j y^j\big)^{p^i} \equiv 1 + y^{p^i} \pmod{y^{p^i+1}}$$

showing that $P_i = 1 + y^{p^i} + (\text{terms of higher order})$. In fact, we could show that

$$P_i = 1 + \sum_k y^{p^{ki}},$$

where the sum is taken over all odd integers k satisfying $0 \le ki \le n$, but we will not need this result. As in Lemma 3.4, for each polynomial $P \in \mathrm{k}[x]$, denote by $C_P \subset \mathbb{A}^2$ the closed curve given by the equation

$$xy^d + yP(y) + 1 = 0.$$

By Lemma 3.4(1) applied with $(P, Q) = (P_i, 1)$ and $m = p^i$, the curves C_{P_i} and C_1 of $\mathbb{A}^2$ have isomorphic complements. To show that the curves C_{P_i}, $1 \le i \le n$ are pairwise non-equivalent, we apply Lemma 3.2(2): for $i, j \in \{1, \dots, n\}$, the curves C_{P_i} and C_{P_j} are equivalent only if there exist $\alpha, \lambda, \mu \in \mathrm{k}^*$, $\beta \in \mathrm{k}$ such that

$$y^d = \lambda \cdot (\alpha y + \beta)^d, \quad yP_j(y) + 1 = \mu \cdot \big((\alpha y + \beta)P_i(\alpha y + \beta) + 1\big).$$

The first equality yields $\beta = 0$. Replacing in the second equation, we get $yP_j(y) + 1 = \mu\big(\alpha y P_i(\alpha y) + 1\big)$, which is equivalent to $\mu = 1$ and $P_j(y) = \alpha P_i(\alpha y)$. Setting $y = 0$, this gives us $\alpha = 1$. Therefore, we get $P_j = P_i$, so that $i = j$, as we wanted. □

The proof of Theorem 2 is now finished:

Proof of Theorem 2. Part (1) corresponds to Corollary 3.6. Part (2) is given by Corollary 3.7 ($\mathrm{char(k)} = 0$) and Lemma 3.8 ($\mathrm{char(k)} > 0$). Part (3) corresponds to Corollary 3.3. □

4. Non-isomorphic curves with isomorphic complements

4.1. A geometric construction.

Proposition 4.1. *Let $d \ge 3$ be an integer, let $\Gamma \subset \mathbb{P}^2$ be a cuspidal curve of equation*

$$y^{d-1}z = P_d(x, y),$$

(for some homogenous polynomial P_d of degree d, not multiple of y), singular at $q_1 = [0:0:1] \in \mathbb{P}^2(\mathrm{k})$, let Δ, L be two lines of $\mathbb{P}^2$ such that $L \cap \Gamma = \{p_1, q_1\}$ for some point $p_1 \in \mathbb{P}^2(\mathrm{k}) \setminus \{q_1\}$, and assume that Δ does not pass through p_1 nor q_1.

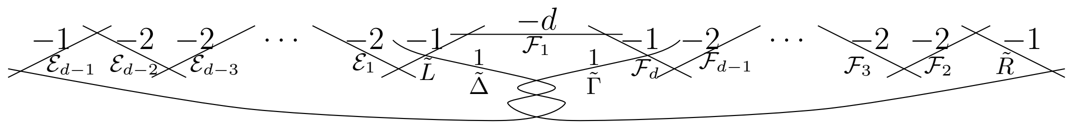
Denoting by $\pi\colon X \to \mathbb{P}^2$ the birational morphism given by the blow-up of p_1, q_1, followed by the blow-up of the points $p_2, \dots, p_{d-1}$ and $q_2, \dots, q_d$ infinitely near p_1 and q_1 respectively and all belonging to the strict transform of Γ, and denoting by $\tilde{\Gamma}, \tilde{\Delta}, \tilde{L}$,

$\mathcal{E}_1, \ldots, \mathcal{E}_{d-1}$, $\mathcal{F}_1, \ldots, \mathcal{F}_d \subset X$ the strict transforms of Γ, Δ, L and of the exceptional divisors of $p_1, \ldots, p_{d-1}, q_1, \ldots, q_d$ we obtain that the two surfaces

$$X \setminus \left(\tilde{\Gamma} \cup \tilde{L} \cup \bigcup_{i=1}^{d-2} \mathcal{E}_i \cup \bigcup_{i=1}^{d} \mathcal{F}_i\right) \text{ and } X \setminus \left(\tilde{\Delta} \cup \tilde{L} \cup \bigcup_{i=1}^{d-2} \mathcal{E}_i \cup \bigcup_{i=1}^{d} \mathcal{F}_i\right)$$

are isomorphic to $\mathbb{A}^2$. Moreover, the situation on X is given as follows,

(E)



where all curves are isomorphic to $\mathbb{P}^1$, all intersections indicated are transversal in exactly one k*-point, except for $\tilde{\Gamma} \cap \tilde{\Delta}$, which can be more complicated (the picture just shows the case where we get 3 points with transversal intersection), and where the numbers indicated are the self-intersections. The curve $\tilde{R}$ is the strict transform of the line R of equation $y = 0$ (tangent to Γ at q_1).*

In particular, this construction provides an isomorphism $\mathbb{A}^2 \setminus C \simeq \mathbb{A}^2 \setminus D$, where $C, D \subset \mathbb{A}^2$ are closed curves isomorphic to $\Gamma \setminus (\Delta \cup \{q\})$ and $\Delta \setminus (\Gamma \cup L)$ respectively, both of degree $d^2 - d + 1$. Moreover, the singularities of the closure of C are infinitely near two points of the line at infinity and consist of d points of multiplicity $d-1$, $d-1$ points of multiplicity d and one point of multiplicity $(d-1)^2$.

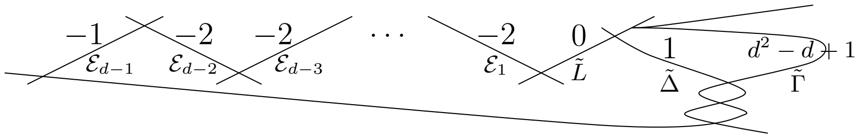
Proof. Blowing up the singular point q_1 of Γ, the strict transform is isomorphic to $\mathbb{P}^1$ and intersects the exceptional divisor in one point (corresponding to the tangent direction given by the line R), being tangent of the order $d-1$. Hence, all points $q_2, \ldots, q_d$ belong to the strict transform of the exceptional divisor of q_1. This gives the self-intersections of $\mathcal{F}_1, \ldots, \mathcal{F}_d$ and their configurations, as in the above diagram. As p_1 is a smooth point of Γ, the curves $\mathcal{E}_1, \ldots, \mathcal{E}_{d-1}$ form a chain of curves as above. The rest of the diagram is checked by looking at the definitions of the curves Δ, Γ, L, R.

Let us write $B = \tilde{L} \cup \bigcup_{i=1}^{d-2} \mathcal{E}_i \cup \bigcup_{i=1}^{d} \mathcal{F}_i$. In order to prove the result, one only needs to show the existence of isomorphisms

$$\psi_1 \colon X \setminus (B \cup \tilde{\Delta}) \xrightarrow{\simeq} \mathbb{A}^2 \text{ and } \psi_2 \colon X \setminus (B \cup \tilde{\Gamma}) \xrightarrow{\simeq} \mathbb{A}^2$$

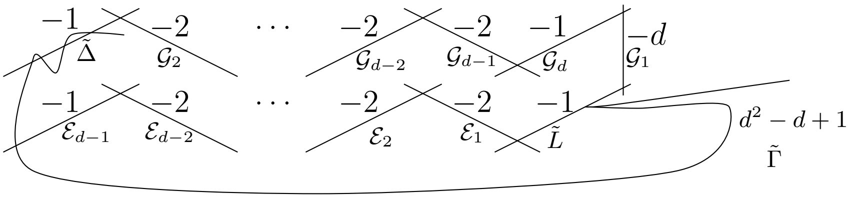
such that $C = \psi_1(\tilde{\Gamma} \setminus (B \cup \tilde{\Delta}))$ and $D = \psi_2(\tilde{\Delta} \setminus (B \cup \tilde{\Gamma}))$ are of degree $d^2 - d + 1$. Indeed, the existence of ψ_1, ψ_2 implies that $\mathbb{A}^2 \setminus C$ and $\mathbb{A}^2 \setminus D$ are isomorphic, and that C and D are respectively isomorphic to $\tilde{\Gamma} \setminus (B \cup \tilde{\Delta})$ and $\tilde{\Delta} \setminus (B \cup \tilde{\Gamma})$. The morphism π gives then isomorphisms of these curves with $\Gamma \setminus (\Delta \cup \{q\})$ and $\Delta \setminus (\Gamma \cup L)$ respectively.

We first show that ψ_1 exists. We observe that since π is the blow-up of $2d-1$ k-points, the Picard group of X is of rank $2d$, over k and over its algebraic closure $\overline{\mathrm{k}}$. We contract the curves $\mathcal{F}_1, \ldots, \mathcal{F}_d$ and obtain a smooth projective surface Y of Picard rank $d-1$ (again over k and $\overline{\mathrm{k}}$). The configuration of the image of the curves $\mathcal{E}_1, \ldots, \mathcal{E}_{d-1}, \tilde{L}, \tilde{\Gamma}$ is then as follows (we omit to write the curve $\tilde{R}$ as we will not need it):



(In fact, Y is just the blow-up of the points $p_1, \ldots, p_d$ in $\mathbb{P}^2$).

In order to show that $X \setminus (B \cup \tilde{\Delta}) \simeq Y \setminus (\tilde{\Delta} \cup \tilde{L} \cup \mathcal{E}_1 \cup \cdots \cup \mathcal{E}_{d-2})$ is isomorphic to $\mathbb{A}^2$, we construct a birational map $\hat{\psi}_1 \colon Y \dashrightarrow \mathbb{P}^2$ which restricts to an isomorphism $Y \setminus (\tilde{\Delta} \cup \tilde{L} \cup \mathcal{E}_1 \cup \cdots \cup \mathcal{E}_{d-2}) \to \mathbb{P}^2 \setminus \mathcal{L}$ for some line $\mathcal{L}$. We now describe this map more precisely. We blow up the point $r_1 = \tilde{\Delta} \cap \tilde{L} \in X$ and then $d-1$ infinitely near points, all belonging to the exceptional curve of r_1, the first one on $\tilde{\Delta}$, and obtain a birational morphism $\theta \colon Z \to Y$ and the following configuration on Y (we again use the same name for a curve on X and its strict transform on Z, and denote by $\mathcal{G}_i \subset Z$ the strict transform of the exceptional divisor of r_i):



We can then contract the curves $\tilde{\Delta}, \mathcal{G}_2, \ldots, \mathcal{G}_{d-1}, \tilde{L}, \mathcal{E}_1, \ldots, \mathcal{E}_{d-1}, \mathcal{G}_1$ and obtain a birational morphism $\rho \colon Z \to \mathbb{P}^2$ (the image of the target is $\mathbb{P}^2$ because it has Picard rank 1). The birational map $\hat{\psi}_1 \colon Y \dashrightarrow \mathbb{P}^2$ given by $\hat{\psi}_1 = \rho\theta^{-1}$ is the desired birational map. The closure of $C \subset \mathbb{A}^2$ in $\mathbb{P}^2$ is then equal to the image of Γ by ρ. The multiplicities at the points where the curves above are contracted are then d for $\tilde{\Delta}, \mathcal{G}_2, \ldots, \mathcal{G}_{d-1}$, are $d-1$ for $\tilde{L}, \mathcal{E}_1, \ldots, \mathcal{E}_{d-2}$ and d and $(d-1)^2$ for $\mathcal{G}_1$.

The self-intersection is then $(d^2-d+1) + (d-1) \cdot d^2 + (d-1) \cdot (d-1)^2 + ((d-1)^2)^2 = (d^2-d+1)^2$, which implies that the curve has degree d^2-d+1.

The case of ψ_2 is similar, as the diagram is symmetric. □

Corollary 4.2. *For each polynomial $P \in \mathrm{k}[t]$ of degree $d \geq 1$ and each $\lambda \in \mathrm{k}$ with $P(\lambda) \neq 0$, there are two closed curves $C, D \subset \mathbb{A}^2$ of degree d^2-d+1 such that $\mathbb{A}^2 \setminus C$ and $\mathbb{A}^2 \setminus D$ are isomorphic and such that $C \simeq \mathrm{Spec}(\mathrm{k}[t, \frac{1}{P}])$ and $D \simeq \mathrm{Spec}(\mathrm{k}[t, \frac{1}{Q}])$, where $Q(t) = P(\lambda + \frac{1}{t}) \cdot t^{\deg(P)}$.*

Proof. Write $P_d(x,y) = P(\frac{x}{y})y^d \in \mathrm{k}[x,y]$, which is a homogeneous polynomial of degree d, such that $P_d(x,1) = P(x)$. We then choose $\Gamma, \Delta, L \subset \mathbb{P}^2$ to be the curve given by the following equations

$$\Gamma : y^{d-1}z = P_d(x,y), \quad \Delta : z = 0, \quad L : x = \lambda y.$$

By construction, P_d is not divisible by y, so Γ is irreducible and singular at $q = [0:0:1]$. Moreover, the two lines L and Δ satisfy $L \cap \Gamma = \{p_1, q\}$ where $p_1 = [\lambda : 1 : P(\lambda)]$ and Δ does not pass through p_1 or q.

We apply Proposition 4.1, and obtain an isomorphism $\mathbb{A}^2 \setminus C \simeq \mathbb{A}^2 \setminus D$, where $C, D \subset \mathbb{A}^2$ are closed curves isomorphic to $\Gamma \setminus (\Delta \cup \{q\})$ and $\Delta \setminus (\Gamma \cup L)$ respectively and both have degree d^2-d+1.

Since $\Gamma \setminus \{q\}$ is isomorphic to $\mathbb{A}^1$ via $t \mapsto [t : 1 : P_d(t, 1)] = [t : 1 : P(t)]$, we obtain that $C \simeq \Gamma \setminus (\Delta \cup \{q\})$ is isomorphic to $\mathrm{Spec}(\mathrm{k}[t, \frac{1}{P}])$.

We then take the isomorphism $\mathbb{A}^1 \to \Delta \setminus L = \Delta \setminus \{[\lambda : 1 : 0]\}$ given by $t \mapsto [\lambda t+1 : t : 0]$. The pull-back of $\Delta \cap \Gamma$ corresponds to the zeroes of $P_d(\lambda t+1, t) = t^d P_d(\lambda + \frac{1}{t}, 1) = Q(t)$. Hence, D is isomorphic to $\mathrm{Spec}(\mathrm{k}[t, \frac{1}{Q}])$ as desired. □

Corollary 4.3. *For each $d \geq 1$ and each distinct $\mu_1, \dots, \mu_d, \lambda \in \mathrm{k}$, there are two closed curves $C, D \subset \mathbb{A}^2$ such that $\mathbb{A}^2 \setminus C$ and $\mathbb{A}^2 \setminus D$ are isomorphic and such that $C \simeq \mathbb{A}^1 \setminus \{\mu_1, \dots, \mu_d\}$ and $D \simeq \mathbb{P}^1 \setminus \{[\mu_1 : 1], \dots, [\mu_d : 1], [\lambda : 1]\}$.*

Proof. We apply Corollary 4.2 with $P = \prod_{i=1}^d (t - \mu_i)$ and get two closed curves $C, D \subset \mathbb{A}^2$ such that $\mathbb{A}^2 \setminus C$ and $\mathbb{A}^2 \setminus D$ are isomorphic and such that $C \simeq \mathrm{Spec}(\mathrm{k}[t, \frac{1}{P}]) \simeq \mathbb{A}^1 \setminus \{\mu_1, \dots, \mu_d\}$ and $D \simeq \mathrm{Spec}(\mathrm{k}[t, \frac{1}{Q}])$, where $Q(t) = P(\lambda + \frac{1}{t}) \cdot t^d$. It remains to see that D is isomorphic to $\mathbb{P}^1 \setminus \{[\mu_1 : 1], \dots, [\mu_d : 1], [\lambda : 1]\}$ via $t \mapsto [\lambda t + 1 : t]$. □

Corollary 4.4. *If k is infinite and $P \in \mathrm{k}[t]$ is a polynomial having at least 3 roots in $\overline{\mathrm{k}}$, then $\mathrm{Spec}(\mathrm{k}[t, \frac{1}{P}])$ and $\mathrm{Spec}(\mathrm{k}[t, \frac{1}{Q}])$ are not isomorphic, when $Q(t) = P(\lambda + \frac{1}{t}) \cdot t^{\deg(P)}$ and $\lambda \in \mathrm{k}$ is a general element.*

In particular, we can find two curves $C, D \subset \mathbb{A}^2$ having isomorphic complements, such that C is isomorphic to $\mathrm{Spec}(\mathrm{k}[t, \frac{1}{P}])$ but not D.

Proof. Let $\lambda_1, \dots, \lambda_d \in \overline{\mathrm{k}}$ be the single roots of P. It is enough to check that for a general λ, there exists no automorphism of $\mathbb{P}^1$ sending $\{\lambda_1, \dots, \lambda_d, \infty\}$ to $\{\frac{1}{\lambda_1 - \lambda}, \dots, \frac{1}{\lambda_d - \lambda}, \infty\}$, or equivalently no automorphism sending $\{\lambda_1, \dots, \lambda_d, \infty\}$ to $\{\lambda_1, \dots, \lambda_d, \lambda\}$. An automorphism of $\mathbb{P}^1$ being determined by the image of 3 points, the set $\mathcal{A}$ of automorphisms φ such that $\varphi^{-1}(\{\lambda_1, \lambda_2, \lambda_3\}) \subset \{\lambda_1, \dots, \lambda_d, \infty\}$ has at most $6\binom{d+1}{3} = (d+1)d(d-1)$ elements. If λ does not belong to the following set

$$\Lambda = \{\varphi(\mu),\ \varphi \in \mathcal{A},\ \mu \in \{\lambda_1, \dots, \lambda_d, \infty\}\},$$

then no automorphism of $\mathbb{P}^1$ sends $\{\lambda_1, \dots, \lambda_d, \infty\}$ to $\{\lambda_1, \dots, \lambda_d, \lambda\}$. This gives the result, together with Corollary 4.2. □

Remark 4.5. If k is a finite field (with at least 3 elements), then the above result is false, by taking $P = \prod_{\alpha \in \mathrm{k}} (x - \alpha)$. Indeed, if $C, D \subset \mathbb{A}^2$ are two curves such that C is isomorphic to $\mathrm{Spec}(\mathrm{k}[t, \frac{1}{P}])$ and $\mathbb{A}^2 \setminus C$ is isomorphic to $\mathbb{A}^2 \setminus D$, then D is isomorphic to $\mathrm{Spec}(\mathrm{k}[t, \frac{1}{Q}])$ for some polynomial Q without square factors having the same number of roots in k and in $\overline{\mathrm{k}}$ as P (Theorem 1(2)). This implies that Q is equal to μP for some $\mu \in \mathrm{k}^*$ and thus that C and D are isomorphic.

A similar argument holds for $P = \prod_{\alpha \in \mathrm{k}^*} (x - \alpha)$ and $P = \prod_{\alpha \in \mathrm{k} \setminus \{0,1\}} (x - \alpha)$ (when the field has at least 4, respectively 5 elements).

Corollary 4.6. *For each ground field k having more than 27 elements, one gets two geometrically irreducible curves $C, D \subset \mathbb{A}^2$ of degree 7 which are not isomorphic but such that $\mathbb{A}^2 \setminus C$ and $\mathbb{A}^2 \setminus D$ are isomorphic.*

Proof. We fix some $\zeta \in \mathrm{k} \setminus \{0, 1\}$. For each $\lambda \in \mathrm{k} \setminus \{0, 1, \zeta\}$, one can apply Corollary 4.3 with $d = 3$, $\mu_1 = 0, \mu_2 = 1, \mu_3 = \zeta$ and get two closed curves $C, D \subset \mathbb{A}^2$ such that

$\mathbb{A}^2 \setminus C$ and $\mathbb{A}^2 \setminus D$ are isomorphic and such that $C \simeq \mathbb{A}^1 \setminus \{0, 1, \zeta\} = \mathbb{P}^1 \setminus \{[0 : 1], [1 : 1], [\zeta : 1], [1 : 0]\}$ and $D \simeq \mathbb{P}^1 \setminus \{[0 : 1], [1 : 1], [\zeta : 1], [\lambda : 1]\}$. It remains to see that one can find at least one λ such that C and D are not isomorphic. Note that C and D are isomorphic if and only if there is an element of $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2(\mathrm{k})$ that sends $\{[0 : 1], [1 : 1], [\zeta : 1], [\lambda : 1]\}$ onto $\{[0 : 1], [1 : 1], [\zeta : 1], [1 : 0]\}$. The image of this element is determined by the image of $[0 : 1], [1 : 1], [\zeta : 1]$, so one gets at most 24 automorphisms to avoid, hence at most 24 elements of $\mathrm{k} \setminus \{0, 1, \zeta\}$ to avoid. The field k having at least 28 elements, we find at least one λ having the right property. □

We can now give the proof of Theorem 3.

Proof of Theorem 3. If the field is infinite (or simply more than 27 elements), it follows from Corollary 4.6. Let us therefore assume that k is a finite field. We again apply Corollary 4.2 (with $\lambda = 0$). Therefore, if $|\mathrm{k}| > 2$ (resp. $|\mathrm{k}| = 2$), it is enough to give a polynomial $P \in \mathrm{k}[t]$ of degree 3 (resp. 4) such that $P(0) \neq 0$ and such that if we set $Q := P(\frac{1}{t})t^{\deg P}$, then the k-algebras $\mathrm{k}[t, \frac{1}{P}]$ and $\mathrm{k}[t, \frac{1}{Q}]$ are not isomorphic.

We begin with the case where the characteristic of k is odd. Then, the kernel of the morphism of groups $\mathrm{k}^* \to \mathrm{k}^*$, $x \mapsto x^2$ is equal to $\{-1, 1\}$, so that this map is not surjective. Let us pick an element $\alpha \in \mathrm{k}^* \setminus (\mathrm{k}^*)^2$. Let us check that we can take $P = (t-1)((t-1)^2 - \alpha)$. Indeed, up to a multiplicative constant, we have $Q = (t-1)((t-1)^2 - \alpha t^2)$. Let us assume by contradiction that the algebras $\mathrm{k}[t, \frac{1}{P}]$ and $\mathrm{k}[t, \frac{1}{Q}]$ are isomorphic. Then, these algebras would still be isomorphic if we replace P and Q by

$$\tilde{P} = P(t+1) = t\big(t^2 - \alpha\big) \text{ and } \tilde{Q} = Q(t+1) = t\big(t^2 - \alpha(t+1)^2\big).$$

This would yield an automorphism of $\mathbb{P}^1$, via the embedding $t \mapsto [t : 1]$, which sends the polynomial $uv(u^2 - \alpha v^2)$ onto $uv(u^2 - \alpha(u+v)^2)$. This automorphism preserving the set of k-roots: $\{[0 : 1], [1 : 0]\}$, it is either of the form $[u : v] \mapsto [\mu u : v]$ or $[u : v] \mapsto [\mu v : u]$ where $\mu \in \mathrm{k}^*$. The polynomial $u^2 - \alpha v^2$ has to be sent to $u^2 - \alpha(u+v)^2$, which is not possible because of the term uv.

We now handle the case where k has characteristic 2. We divide into three cases, depending wether the cube homomorphism of groups $\mathrm{k}^* \to \mathrm{k}^*$, $x \mapsto x^3$ is injective or not (which corresponds to ask if 4 divides $|\mathrm{k}|$ or not), and putting the field with two elements apart.

If the cube homomorphism is not surjective, we can pick an element $\alpha \in \mathrm{k}^* \setminus (\mathrm{k}^*)^3$. Let us check that we can take the irreducible polynomial $P = t^3 - \alpha \in \mathrm{k}[t]$. Indeed, up to a multiplicative constant, we have $Q = t^3 - \alpha^{-1}$. Let us assume by contradiction that the algebras $\mathrm{k}[t, \frac{1}{P}]$ and $\mathrm{k}[t, \frac{1}{Q}]$ are isomorphic. Then, there should exist constants $\lambda, \mu, c \in \mathrm{k}$ with $\lambda c \neq 0$ such that

$$c(t^3 - \alpha^{-1}) = (\lambda t + \mu)^3 - \alpha.$$

This gives us $\mu = 0$ and $\lambda^3 = c = \alpha^2$. The square homomorphism of groups $\mathrm{k}^* \to \mathrm{k}^*$, $x \mapsto x^2$ being bijective, there is a unique square root for each element of k^*. Taking the square root of the equality $\alpha^2 = \lambda^3$, we obtain $\alpha = (\nu)^3$, where ν is the square root of λ. This is impossible since α was chosen not to be a cube.

If the cube homomorphism is surjective, then 1 is the only root of $t^3 - 1 = (t-1)(t^2 + t + 1)$, so $t^2 + t + 1 \in \mathrm{k}[t]$ is irreducible. If moreover k has more than 2 elements, we can choose $\alpha \in \mathrm{k} \setminus \{0,1\}$ and take $P = (t-\alpha)(t^2+t+1)$. Up to a multiplicative constant, we have $Q = (t-\alpha^{-1})(t^2+t+1)$. Let us assume by contradiction that the algebras $\mathrm{k}[t, \frac{1}{P}]$ and $\mathrm{k}[t, \frac{1}{Q}]$ are isomorphic. Then, these algebras would still be isomorphic if we replace P and Q by

$$\tilde{P} = P(t+\alpha) = t(t^2+t+\alpha^2+\alpha+1) \text{ and } \tilde{Q} = Q(t+\alpha^{-1}) = t(t^2+t+\alpha^{-2}+\alpha^{-1}+1).$$

This would yield an automorphism of $\mathbb{P}^1$, via the embedding $t \mapsto [t:1]$, which sends the polynomial $uv(u^2+uv+(\alpha^2+\alpha+1)v^2)$ onto $uv(u^2+uv+(\alpha^{-2}+\alpha^{-1}+1)v^2)$. The same argument as before yields $\alpha^2+\alpha+1 = \alpha^{-2}+\alpha^{-1}+1$, i.e. $\alpha^2+\alpha+1 = \alpha^{-2}(\alpha^2+\alpha+1)$. This is impossible since $\alpha^2+\alpha+1 \neq 0$ and $\alpha^2 \neq 1$.

The last case is when $\mathrm{k} = \{0,1\}$ is the field with two elements. Here the construction does not work with polynomials of degree 3: the only ones which are not symmetric and do not vanish in 0 are t^3+t^2+1 and t^3+t+1, which are equivalent via $t \mapsto t+1$. We then choose for P the irreducible polynomial $P = t^4+t+1$ (it has not root and is not equal to $(t^2+t+1)^2 = t^4+t^2+1$). This gives $Q = t^4+t^3+1$. Let us assume by contradiction that the algebras $\mathrm{k}[t, \frac{1}{P}]$ and $\mathrm{k}[t, \frac{1}{Q}]$ are isomorphic. Then, there should exist constants $\lambda, \mu, c \in \mathrm{k}$ with $\lambda c \neq 0$ such that

$$c(t^4+t^3+1) = (\lambda t+\mu)^4 + (\lambda t+\mu) + 1.$$

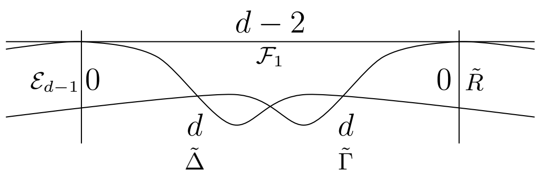
This is impossible since $(\lambda t+\mu)^4 + (\lambda t+\mu)+1 = \lambda^4 t^4 + \lambda t + (\mu^4+\mu+1)$. □

4.2. **Getting explicit formulas.** To obtain the equation of the curves C, D and the isomorphism $\mathbb{A}^2 \setminus C \xrightarrow{\simeq} \mathbb{A}^2 \setminus D$ given by Proposition 4.1, one can follow the construction and explicit the birational maps described. The lemma yields the existence of isomorphisms

$$\psi_1\colon X \setminus (B \cup \tilde{\Delta}) \xrightarrow{\simeq} \mathbb{A}^2 \text{ and } \psi_2\colon X \setminus (B \cup \tilde{\Gamma}) \xrightarrow{\simeq} \mathbb{A}^2$$

such that $C = \psi_1(\tilde{\Gamma} \setminus (B \cup \tilde{\Delta}))$ and $D = \psi_2(\tilde{\Delta} \setminus (B \cup \tilde{\Gamma}))$ are of degree d^2-d+1, where $B = \tilde{L} \cup \bigcup_{i=1}^{d-2} \mathcal{E}_i \cup \bigcup_{i=1}^{d} \mathcal{F}_i$, and ψ_1, ψ_2 are given by blow-ups and blow-downs, so it is possible to explicit $\psi_i \pi^{-1}\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ with formulas (looking at the linear systems), and then get the isomorphism $\psi_2\pi^{-1} \circ (\psi_1\pi^{-1})^{-1}\colon \mathbb{A}^2 \setminus C \to \mathbb{A}^2 \setminus D$. The formulas for $\psi_1\pi^{-1}, \psi_2\pi^{-1}$ are however quite complicated.

Another possibility can be done as follows: we choose a birational morphism $X \to W$ that contracts $\tilde{L}, \mathcal{E}_1, \ldots, \mathcal{E}_{d-2}$ and $\mathcal{F}_d, \ldots, \mathcal{F}_2$ onto two smooth points of W, passing through the image of $\mathcal{F}_1$ (possible, see Diagram (E)). The situation of the image of the curves $\tilde{R}, \mathcal{E}_{d-1}, \mathcal{F}_1, \tilde{\Gamma}, \tilde{\Delta}$ (that we again denote by the same name) in W is as follows:



Computing the dimension of the Picard group, one finds that W is a Hirzebruch surface. Hence, the curves $\mathcal{E}_{d-1}, \tilde{R}$ are fibres of a $\mathbb{P}^1$-bundle $W \to \mathbb{P}^1$ and $\mathcal{F}_1, \tilde{\Delta}, \tilde{\Gamma}$ are sections of self-intersection $d-2, d, d$. One can then find plenty of examples in $\mathbb{F}_1$ and $\mathbb{F}_0$ (depending on the parity of d) but also in $\mathbb{F}_m$ for $m \geq 2$ if the polynomial chosen at the beginning is special enough.

The case where $d = 3$ corresponds to curves of degree 7 in $\mathbb{A}^2$ (Proposition 4.1) which is the first interesting case, as it gives non-isomorphic curves for almost each field (Theorem 3). When $d = 3$, one finds that $\mathcal{F}_1$ is a section of self-intersection 1 in $W = \mathbb{F}_1$, so $\mathbb{F}_1 \setminus \mathcal{F}_1$ is isomorphic to the blow-up of $\mathbb{A}^2$ at one point, and $\tilde{\Gamma}, \tilde{\Delta}$ are sections of self-intersecion 3 and are thus strict transforms of parabolas passing through the point blown up. This explains how the following result is derived from Proposition 4.1. However, the result can also be read independently of Proposition 4.1, since the proof that we give does not use the rest of the article:

Proposition 4.7. *Let us fix some $a_0, a_3 \in \mathrm{k}^*$, $a_1, a_2 \in \mathrm{k}$, which define two irreducible polynomials $P, Q \in \mathrm{k}[x,y]$ of degree 2 given by*

$$P = x^2 - a_2 x - a_3 y, Q = y^2 + a_0 x + a_1 y \in \mathrm{k}[x,y].$$

(1) *Denoting by $\eta\colon \hat{\mathbb{A}}^2 \to \mathbb{A}^2$ the blow-up of the origin and writing $\tilde{\Gamma}, \tilde{\Delta} \subset \hat{\mathbb{A}}^2$ the strict transforms of the curves $\Gamma, \Delta \subset \mathbb{A}^2$ given $P = 0$ and $Q = 0$, the rational maps*

$$\begin{array}{rccc} \varphi_P\colon & \mathbb{A}^2 & \dashrightarrow & \mathbb{A}^2 \\ & (x,y) & \mapsto & \left(-\frac{x}{P(x,y)}, P(x,y)\right) \end{array} \qquad \begin{array}{rccc} \varphi_Q\colon & \mathbb{A}^2 & \dashrightarrow & \mathbb{A}^2 \\ & (x,y) & \mapsto & \left(\frac{y}{Q(x,y)}, Q(x,y)\right) \end{array}$$

are birational maps that induce isomorphisms

$$\psi_P = (\varphi_P \eta)|_{\hat{\mathbb{A}}^2 \setminus \tilde{\Gamma}}\colon \quad \hat{\mathbb{A}}^2 \setminus \tilde{\Gamma} \xrightarrow{\simeq} \mathbb{A}^2 \qquad \psi_Q = (\varphi_Q \eta)|_{\hat{\mathbb{A}}^2 \setminus \tilde{\Delta}}\colon \quad \hat{\mathbb{A}}^2 \setminus \tilde{\Delta} \xrightarrow{\simeq} \mathbb{A}^2.$$

(2) *We then get an isomorphism $\psi\colon \mathbb{A}^2 \setminus C \to \mathbb{A}^2 \setminus D$, where $C = \psi_Q(\tilde{\Gamma} \setminus \tilde{\Delta})$ and $D = \psi_P(\tilde{\Delta} \setminus \tilde{\Gamma})$ are given by $f = 0$ and $g = 0$, where $f, g \in \mathrm{k}[x,y]$ are:*

$$\begin{array}{rcl} f & = & (1 - x(xy + a_1))(y(1 - x(xy + a_1)) - a_0 a_2) - x(a_0)^2 a_3, \\ g & = & (1 - x(xy + a_2))(y(1 - x(xy + a_2)) - a_1 a_3) - x a_0 (a_3)^2. \end{array}$$

The curves C, D are isomorphic to $\mathrm{Spec}(\mathrm{k}[t, \frac{1}{\sum_{i=0}^3 a_i t^i}])$ and $\mathrm{Spec}(\mathrm{k}[t, \frac{1}{\sum_{i=0}^3 a_{3-i} t^i}])$ respectively. Moreover, ψ and ψ^{-1} are given by

$$\begin{array}{rccc} \psi\colon & (x,y) & \mapsto & \left(\frac{a_0(x(xy+a_1)-1)}{f(x,y)}, \frac{y f(x,y)}{(a_0)^2}\right) \\ & \left(\frac{a_3(x(xy+a_2)-1)}{g(x,y)}, \frac{y g(x,y)}{(a_3)^2}\right) & \leftarrow\!\shortmid & (x,y). \end{array}$$

Proof. (1): Let us first prove that φ_P is birational and that $\varphi_P \eta$ induces an isomorphism $\hat{\mathbb{A}}^2 \setminus \tilde{\Gamma} \xrightarrow{\simeq} \mathbb{A}^2$. We observe that $\kappa\colon (x,y) \mapsto (x, x^2 - a_2 x - a_3 y)$ is an automorphism of $\mathbb{A}^2$ that sends Γ onto the line $L_Y \subset \mathbb{A}^2$ of equation $y = 0$. Moreover $\tilde{\varphi}_P = \kappa \varphi_P \kappa^{-1}\colon (x,y) \mapsto (-\frac{x}{y}, y)$ is birational, so φ_P is birational. Since κ fixes the origin, $\eta^{-1}\kappa\eta$ is an automorphism of $\hat{\mathbb{A}}^2$, that sends $\tilde{\Gamma}$ onto the strict transform $\tilde{L}_Y \subset \hat{\mathbb{A}}^2$ of L_Y. The fact that $\tilde{\varphi}_P \eta$ induces an isomorphism $\hat{\mathbb{A}}^2 \setminus \tilde{L}_X \xrightarrow{\simeq} \mathbb{A}^2$ can be checked in local coordinates, and corresponds to the classical local description of a blow-up as $(x,y) \mapsto (xy, y)$. This yields (1), the case of φ_Q and $\varphi_Q \eta$ being done similarly, with $(x,y) \mapsto (y^2 + a_0 x + a_1 y, y)$.

(2): Now that (1) is proven, one gets two isomorphisms

$$\psi_P|_U\colon U \xrightarrow{\simeq} \mathbb{A}^2 \setminus D, \ \psi_Q|_U\colon U \xrightarrow{\simeq} \mathbb{A}^2 \setminus C,$$

where $U = \hat{\mathbb{A}}^2 \setminus (\tilde{\Gamma} \cup \tilde{\Gamma})$, $D = \psi_P(\tilde{\Delta} \setminus \tilde{\Gamma})$ and $C = \psi_Q(\tilde{\Gamma} \setminus \tilde{\Delta})$. Remembering that $\Gamma \subset \mathbb{A}^2$ is given by $x(x-a_2) = a_3 y$, we have an isomorphism

$$\begin{array}{rccc} \rho\colon & \mathbb{A}^1 & \xrightarrow{\simeq} & \Gamma \\ & t & \mapsto & (ta_3 + a_2, t(ta_3 + a_2)) \\ & \frac{1}{a_3}(x - a_2) & \leftarrow\!\shortmid & (x, y). \end{array}$$

Replacing $\rho(t)$ in the equation $Q(x,y) = xa_0 + ya_1 + y^2$ of Δ, one finds

$$Q(ta_3 + a_2, t(ta_3 + a_2)) = (ta_3 + a_2)(t^3 a_3 + t^2 a_2 + ta_1 + a_0).$$

The root of $ta_3 + a_2$ corresponds to the point sent on $(0,0)$, blown up by η. Hence, $V = \mathrm{Spec}(\mathrm{k}[t, \frac{1}{\sum_{i=0}^{3} t^i a_i}]) \subset \mathbb{A}^1$ is isomorphic to $\tilde{\Gamma} \backslash \tilde{\Delta}$ via $\eta^{-1}\rho$. Applying $\psi_Q = (\varphi_Q \eta)|_{\hat{\mathbb{A}}^2 \backslash \tilde{\Delta}}$, one gets an isomorphism $\theta = (\varphi_Q \rho)|_V \colon V \xrightarrow{\simeq} C$. Since $(\varphi_Q)^{-1}$ is given by

$$(\varphi_Q)^{-1} \colon (x,y) \mapsto \left(\frac{y(1 - x(xy + a_1))}{a_0}, xy \right),$$

one can give explicitly θ and its inverse:

$$\begin{array}{rccc} \theta\colon & \mathrm{Spec}(\mathrm{k}[t, \frac{1}{\sum_{i=0}^{3} t^i a_i}]) & \xrightarrow{\simeq} & C \\ & t & \mapsto & \left(\frac{t}{\sum_{i=0}^{3} t^i a_i}, (ta_3 + a_2)(t^3 a_3 + t^2 a_2 + ta_1 + a_0) \right) \\ & \frac{1}{a_3}(\frac{y(1 - x(xy - a_1))}{a_0} - a_2) & \leftarrow\!\shortmid & (x, y). \end{array}$$

Computing the extension of θ to a morphism $\mathbb{P}^1 \to \mathbb{P}^2$, one sees that the curve $C \subset \mathbb{A}^2$ has degree 7. To get its equation, we can compute $((\varphi_Q)^{-1})^*(P)$: since $(a_0)^2 P(x,y) = (a_0 x)(a_0 x - a_0 a_2) - (a_0)^2 a_3 y$, one gets

$$\begin{array}{rcl} (a_0)^2((\varphi_Q)^{-1})^*(P) & = & (a_0)^2 P(\frac{y(1 - x(xy + a_1))}{a_0}, xy) \\ & = & y(1 - x(xy + a_1))(y(1 - x(xy + a_1)) - a_0 a_2) - xy(a_0)^2 a_3 \\ & = & yf(x,y), \end{array}$$

where

$$f = (1 - x(xy + a_1))(y(1 - x(xy + a_1)) - a_0 a_2) - x(a_0)^2 a_3 \in \mathrm{k}[x,y]$$

yields the equation of C (note that the polynomial $y = 0$ appears here because it corresponds to the line contracted by $(\psi_Q)^{-1}$, corresponding to the exceptional divisor of $\hat{\mathbb{A}^2} \to \mathbb{A}^2$ via the isomorphism $\mathbb{A}^2 \to \hat{\mathbb{A}^2} \setminus \hat{\Delta}$). The linear involution of $\mathbb{A}^2$ given by $(x,y) \mapsto (-y, -x)$ exchanges the polynomials P and Q and the maps ψ_P and φ_Q, by replacing a_0, a_1, a_2, a_3 with a_3, a_2, a_1, a_0 respectively. This shows that $D \subset \mathbb{A}^2$ has equation $g = 0$ where

$$g = (1 - x(xy + a_2))(y(1 - x(xy + a_2)) - a_1 a_3) - xa_0(a_3)^2 \in \mathrm{k}[x,y]$$

and is isomorphic to $\mathrm{Spec}(\mathrm{k}[t, \frac{1}{\sum_{i=0}^{3} \alpha_{3-i} t^i}])$. It remains to compute the isomorphism $\psi \colon \mathbb{A}^2 \setminus C \to \mathbb{A}^2 \setminus D$, which is by construction equal to the birational maps $\psi_P(\psi_Q)^{-1} = \varphi_P(\varphi_Q)^{-1}$. Using the equation $(a_0)^2 P(\frac{y(1 - x(xy + a_1))}{a_0}, xy) = yf(x,y)$, one gets:

$$\begin{array}{rcl} \psi((x,y)) & = & \varphi_P(\left(\frac{y(1 - x(xy + a_1))}{a_0}, xy \right)) = \left(-\frac{y(1 - x(xy + a_1))}{a_0 P(\frac{y(1 - x(xy + a_1))}{a_0}, xy)}, P(\frac{y(1 - x(xy + a_1))}{a_0}, xy) \right) \\ & = & \left(\frac{a_0(x(xy + a_1) - 1)}{f(x,y)}, \frac{yf(x,y)}{(a_0)^2} \right). \end{array}$$

By symmetry, the inverse of ψ is given by $(x,y) \mapsto \left(\frac{a_3(x(xy+a_2)-1)}{g(x,y)}, \frac{yg(x,y)}{(a_3)^2}\right)$. □

Remark 4.8. Proposition 4.7, yields an isomorphism $\psi^*\colon \mathrm{k}[x,y,\frac{1}{g}] \xrightarrow{\simeq} \mathrm{k}[x,y,\frac{1}{f}]$ which sends the invertible elements onto invertible elements and thus sends g onto $\lambda f^{\pm 1}$ for some $\lambda \in \mathrm{k}^*$ (see Lemma 2.12). This corresponds to say that ψ induces an isomorphism between the two fibrations

$$\mathbb{A}^2 \setminus C \xrightarrow{f} \mathbb{A}^1 \setminus \{0\},\ \mathbb{A}^2 \setminus D \xrightarrow{g} \mathbb{A}^1 \setminus \{0\},$$

possibly exchanging the fibres. To study these fibrations, one uses the equalities

$$(\varphi_Q)^*(f) = \frac{(a_0)^2 P}{Q},\ (\varphi_P)^*(g) = \frac{(a_3)^2 Q}{P}, \tag{F}$$

which can either be checked directly, or observed as follows: the first equality follows from $((\varphi_Q)^{-1})^*(P) = \frac{yf(x,y)}{(a_0)^2}$, applying $(\varphi_Q)^*$, and the second is obtained by symmetry.

Note that Equation (F) yields $\psi^*(g) = \frac{(a_0a_3)^2}{f}$, since $\psi = \varphi_P(\varphi_Q)^{-1}$.

For each $\mu \in \mathrm{k}$, the fibre $C_\mu \subset \mathbb{A}^2$ given $f(x,y) = \mu$ is an algebraic curve isomorphic to its preimage by the isomorphism $\psi_Q = (\varphi_Q\eta)|_{\hat{\mathbb{A}}^2\setminus\tilde{\Delta}}\colon \hat{\mathbb{A}}^2 \setminus \tilde{\Delta} \xrightarrow{\simeq} \mathbb{A}^2$ of Proposition 4.7(1). By construction, $(\psi_Q)^{-1}(C_\mu)$ is equal to $\tilde{\Gamma}_\mu \setminus \tilde{\Delta}$, where $\tilde{\Gamma}_\mu \subset \hat{\mathbb{A}}^2$ is the strict transform of the curve $\Gamma_\mu \subset \mathbb{A}^2$ given by $(a_0)^2P - \mu Q = 0$ (follows from Equation (F)). The closure of Γ_μ in $\mathbb{P}^2$ is a conic given by

$$(a_0)^2x^2 - \mu y^2 - z(a_0(\mu + a_0a_2)x - (\mu a_1 + (a_0)^2a_3)y)$$

which passes through $[0:0:1]$ and is irreducible for a general μ. Projecting from the point one obtains an isomorphism with $\mathbb{P}^1$. The curve $\tilde{\Gamma}_\mu \setminus \tilde{\Delta}$ is then isomorphic to $\mathbb{P}^1$ minus three $\overline{\mathrm{k}}$-points of $\tilde{\Delta}$, which are fixed and do not depend on μ, and minus the two points at infinity, which correspond to $(a_0)^2x^2 - \mu y^2 = 0$.

When the field is algebraically closed, one thus gets that the general fibres of f are isomorphic to $\mathbb{P}^1$ minus 4 points, whereas the zero fibre is isomorphic to $\mathbb{P}^1$ minus 3 points (if $\sum_{i=0}^3 a_it^i$ is chosen to have three distint roots). Moreover, the two points of intersection with the line at infinity say that this curve is a *horizontal curve of degree* 2, or equivalently a *horizontal curve which is not a section* (in the usual notation of polynomials and components on boundary, see [NN02, AC96]), so the polynomials f and g are rational but not of simple type (see [NN02]). When $\mathrm{k} = \mathbb{C}$, this implies that the polynomial has non-trivial monodromy [ACD98].

5. Related questions

5.1. **Providing higher dimensional counterexamples.** As one now explains, the negative answer to the Complement Problem for $n = 2$ also yields a negative answer for any $n \geq 3$. This follows from the following easy observation:

Lemma 5.1. *Let $C, D \subset \mathbb{A}^2$ be two closed geometrically irreducible curves having isomorphic complements. Then for each $m \geq 1$, the varieties $H_C = C \times \mathbb{A}^m$ and $H_D = D \times \mathbb{A}^m$ are closed hypersurfaces of $\mathbb{A}^2 \times \mathbb{A}^m = \mathbb{A}^{m+2}$ having isomorphic complements. Moreover, C and D are isomorphic if and only if $C \times \mathbb{A}^m$ and $D \times \mathbb{A}^m$ are isomorphic.*

Proof. Denoting by $f, g \in \mathrm{k}[x, y]$ the geometrically irreducible polynomials that define the curves C, D, the varieties $H_C, H_D \subset \mathbb{A}^2 \times \mathbb{A}^m = \mathbb{A}^{m+2}$ are given by the same polynomials and are thus again closed geometrically irreducible hypersurfaces. The isomorphism $\mathbb{A}^2 \setminus C \xrightarrow{\simeq} \mathbb{A}^2 \setminus D$ extends then naturally to an isomorphism $\mathbb{A}^{m+2} \setminus H_C \xrightarrow{\simeq} \mathbb{A}^{m+2} \setminus H_D$.

If C and D are isomorphic, then H_C and H_D are isomorphic. The converse also holds, and is proven in [AHE72, Corollary (3.4)]. □

Corollary 5.2. *For each ground field* k *and each integer* $n \geq 3$*, there exist two geometrically irreducible smooth closed hypersurfaces* $E, F \subset \mathbb{A}^n$ *which are not isomorphic but whose complements* $\mathbb{A}^n \setminus E$ *and* $\mathbb{A}^n \setminus F$ *are isomorphic. Furthermore, the hypersurfaces can be given by polynomials* $f, g \in \mathrm{k}[x_1, x_2] \subset \mathrm{k}[x_1, \ldots, x_n]$ *of degree* 7 *if the field admits strictly more than* 2 *elements and of degree* 13 *if the field has* 2 *elements. The hypersurfaces* E, F *are isomorphic to* $C \times \mathbb{A}^{n-2}$ *and* $D \times \mathbb{A}^{n-2}$ *for some smooth closed curves* $C, D \subset \mathbb{A}^2$ *of the same degree.*

Proof. It suffices to choose for f, g the equations of the curves $C, D \subset \mathbb{A}^2$ given by Theorem 3. The result then follows from Lemma 5.1. □

5.2. The holomorphic case.

Lemma 5.3. *For each* $d+1$ *distinct points* $a_1, \ldots, a_d, a_{d+1} \in \mathbb{C}$*, with* $d \geq 3$*, there exist two closed algebraic curves* $C, D \subset \mathbb{C}^2$ *of degree* $d^2 - d + 1$ *such that* C *and* D *are algebraically isomorphic to* $\mathbb{C} \setminus \{a_1, \ldots, a_{d-1}, a_d\}$ *and* $\mathbb{C} \setminus \{a_1, \ldots, a_{d-1}, a_{d+1}\}$ *respectively, and such that* $\mathbb{C}^2 \setminus C$ *and* $\mathbb{C}^2 \setminus D$ *are algebraically isomorphic.*

In particular, choosing the points generally enough, the curves C *and* D *are not biholomorphic, but their complements are biholomorphic.*

Proof. The existence of C, D directly follows from Corollary 4.2. It remains to observe that C and D are not biholomorphic if the points are general enough. If $f \colon C \to D$ is a biholomorphism, then f extends to a holomorphic map $\mathbb{CP}^1 \to \mathbb{CP}^1$, as it cannot have essential singularities. The same holds for f^{-1}, so f is just an element of $\mathrm{PGL}_2(\mathbb{C})$, hence an algebraic automorphism of the projective complex line. Removing at least 4 points of $\mathbb{CP}^1$ (this is the case since $d \geq 3$) and moving one of them gives then infinitely many curves up to biholomorphism. □

Corollary 5.4. *For each* $n \geq 2$*, there are algebraic hypersurfaces* $H_1, H_2 \subset \mathbb{C}^n$ *which are complex manifolds, not biholomorphic but having biholomorphic complements.*

Proof. It suffices to take polynomials $f, g \in \mathbb{C}[x_1, x_2]$ provided by Lemma 5.3, whose zero sets are smooth algebraic curves $C, D \subset \mathbb{C}^2$ not biholomorphic but having holomorphic complements. We then use the same polynomials to define $H_1, H_2 \subset \mathbb{C}^n$, which are smooth complex manifolds having biholomorphic complements and being biholomorphic to $C \times \mathbb{C}^{n-2}$ and $D \times \mathbb{C}^{n-2}$ respectively. It remains to observe that $C \times \mathbb{C}^{n-2}$ and $D \times \mathbb{C}^{n-2}$ are not biholomorphic. Denote by $p_C \colon C \times \mathbb{C}^{n-2} \to C$ and $p_D \colon D \times \mathbb{C}^{n-2} \to D$ the projections on the first factor. If $\psi \colon \mathbb{C}^{n-2} \times C \to \mathbb{C}^{n-2} \times D$ is a biholomorphism, then $p_D \circ \psi \colon \mathbb{C}^{n-2} \times C \to D$ induces, for each $c \in C$, a holomorphic map $\mathbb{C}^{n-2} \to D$ which has to be constant by Picard's theorem (since it avoids at least two values of $\mathbb{C}$). Therefore, the map $p_D \circ \psi$ factors through a holomorphic map $\chi \colon C \to D$: we have $p_D \circ \varphi = \chi \circ p_C$.

We get analogously a holomorphic map $\theta\colon D \to C$, which is by construction the inverse of χ, so C and D are biholomorphic, a contradiction. □

Appendix: The case of $\mathbb{P}^2$

In this appendix, we describe some results on the question of complements of curves in $\mathbb{P}^2$ explained in the introduction. These are not directly related to the rest of the text and only serve as comparison with the affine case.

We recall the following simple argument, known to specialists, for lack of reference:

Lemma 5.5. *Let $C, D \subset \mathbb{P}^2$ be two geometrically irreducible closed curves such that $\mathbb{P}^2 \setminus C$ and $\mathbb{P}^2 \setminus D$ are isomorphic. If C and D are not equivalent, up to automorphism of $\mathbb{P}^2$, then C and D are singular rational curves.*

Proof. Denote by $\varphi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ a birational map which restricts to an isomorphism from $\mathbb{P}^2 \setminus C$ to $\mathbb{P}^2 \setminus D$. If φ is an automorphism of $\mathbb{P}^2$, then C and D are equivalent. Otherwise, the same argument as in Lemma 2.7 shows that both C and D are rational (this also follows from [Bla09, Lemma 2.2]). If C and D are singular, we are finished, so we can assume that one of them is smooth, and then has degree 1 or 2. The Picard group of $\mathbb{P}^2 \setminus C$ being $\mathbb{Z}/\deg(C)\mathbb{Z}$, one finds that C and D have the same degree. This implies that C and D are equivalent under automorphisms of $\mathbb{P}^2$. The case of lines is obvious. For conics, it is enough to check that a rational conic over any field is necessarily equivalent to the conic of equation $xy+z^2 = 0$. Actually, one may always assume that the rational conic contains the point $[1 : 0 : 0]$, since it contains a rational point. One may furthermore assume that the tangent at this point has equation $y = 0$. This means that the equation of the conic is of the form $xy+u(y, z)$, where u is a homogenous polynomial of degree 2. Using a change of variables of the form $(x, y, z) \mapsto (x + ay + bz, y, z)$, where $a, b \in \mathrm{k}$, one may assume that the equation is of the form $xy + cz^2 = 0$, where $c \in \mathrm{k}^*$. Then, using the change of variables $(x, y, z) \mapsto (cx, y, z)$, one finally gets, as announced, the equation $xy + z^2 = 0$. □

In order to get families of (singular) curves of $\mathbb{P}^2$ having the same complement, we explicit here the construction of Paolo Costa [Cos12], that provides examples of unicuspidal curves of $\mathbb{P}^2$ having isomorphic complements but being non-equivalent under the action of $\mathrm{Aut}(\mathbb{P}^2)$. We provide equations and give the details of the proof for self-containedness, and because the results below are not explicitly stated in [Cos12].

Lemma 5.6. *To each homogeneous polynomial $P \in \mathrm{k}[x, y]$ of degree $d \geq 1$, one associates a homogeneous irreducible polynomial $f_P \in \mathrm{k}[x, y, z]$ of degree $4d + 1$:*

$$f_P = z(xz - y^2)^{2d} + 2yP(x^2, xz - y^2)(xz - y^2)^d + x(P(x^2, xz - y^2))^2$$

and defines $C_P \subset \mathbb{P}^2$ and $V_P \subset \mathbb{A}^3$ to be the varieties given by $f_P = 0$ (note that V_P is the cone over C_P). We also denote by $\mathcal{L}, \mathcal{Q} \subset \mathbb{P}^2$ the curves of equation $z = 0$ and $xz - y^2 = 0$ and by $V_\mathcal{L}, V_\mathcal{Q} \subset \mathbb{A}^3$ their corresponding cones (given by the same equations).

(1) *For each homogeneous $P \in \mathrm{k}[x, y]$, the rational map ψ_P given by*

$$\psi_P\colon (x, y, z) \mapsto \left(x, y + xP(\frac{x^2}{xz - y^2}, 1), z + 2yP(\frac{x^2}{xz - y^2}, 1) + x(P(\frac{x^2}{xz - y^2}, 1))^2\right)$$

is a birational map of $\mathbb{A}^3$ that restricts to isomorphisms

$$\mathbb{A}^3 \setminus V_{\mathcal{Q}} \xrightarrow{\simeq} \mathbb{A}^3 \setminus V_{\mathcal{Q}},\ V_P \setminus V_{\mathcal{Q}} \xrightarrow{\simeq} V_{\mathcal{L}} \setminus V_{\mathcal{Q}} \text{ and } \mathbb{A}^3 \setminus (V_{\mathcal{Q}} \cup V_P) \xrightarrow{\simeq} \mathbb{A}^3 \setminus (V_{\mathcal{Q}} \cup V_{\mathcal{L}}).$$

In particular, since ψ_P is homogeneous, the same formula induces a birational map of $\mathbb{P}^2$ that restricts to isomorphisms

$$\mathbb{P}^2 \setminus \mathcal{Q} \xrightarrow{\simeq} \mathbb{P}^2 \setminus \mathcal{Q},\ C_P \setminus \mathcal{Q} \xrightarrow{\simeq} \mathcal{L} \setminus \mathcal{Q} \text{ and } \mathbb{P}^2 \setminus (\mathcal{Q} \cup C_P) \xrightarrow{\simeq} \mathbb{P}^2(\mathcal{Q} \cup \mathcal{L}).$$

(2) *For each $\lambda \in \mathrm{k}^*$, the rational map*

$$\varphi_\lambda \colon (x,y,z) \mapsto \left(x + (\lambda-1)\frac{xz-y^2}{z}, y, z\right)$$

is a birational map of $\mathbb{A}^3$ that restricts to automorphisms of $\mathbb{A}^3 \setminus V_{\mathcal{L}}$, $V_{\mathcal{Q}} \setminus V_{\mathcal{L}}$ and $\mathbb{A}^3 \setminus (V_{\mathcal{L}} \cup V_{\mathcal{Q}})$. The same formula yields then automorphisms of $\mathbb{P}^2 \setminus \mathcal{L}$, $\mathcal{Q} \setminus \mathcal{L}$ and $\mathbb{P}^2 \setminus (\mathcal{L} \cup \mathcal{Q})$.

(3) *For each $\lambda \in \mathrm{k}^*$ and each homogeneous polynomial $P \in \mathrm{k}[x,y]$, the rational map $\kappa = (\psi_{\tilde{P}})^{-1}\varphi_\lambda\psi_P$ restricts to an isomorphism $\mathbb{A}^3 \setminus V_P \xrightarrow{\simeq} \mathbb{A}^3 \setminus V_{\tilde{P}}$, with $\tilde{P}(x,y) = P(\lambda x, y)$. In particular, κ also induces an isomorphism $\mathbb{P}^2 \setminus C_P \xrightarrow{\simeq} \mathbb{P}^2 \setminus C_{\tilde{P}}$.*

(4) *For each P, $\tilde{P} \in \mathrm{k}[x,y]$, the curves C_P and $C_{\tilde{P}}$ are equivalent, up to automorphism of $\mathbb{P}^2$, if and only if there exists $\rho \in \mathrm{k}^*, \mu \in \mathrm{k}$ such that $\tilde{P}(x,y) = \rho P(x\rho^2, y) + \mu y^{\deg P}$.*

Proof. (1): The rational map ψ_P corresponds to a k-endomorphism $(\psi_P)^*$ of $\mathrm{k}(x,y,z)$ that sends x, y, z onto $x, y + P(\frac{x^2}{xz-y^2}, 1)x, z + 2P(\frac{x^2}{xz-y^2}, 1)y + (P(\frac{x^2}{xz-y^2}, 1))^2 z$. One observes that $(\psi_P)^*$ fixes x and $xz - y^2$. In particular, $(\psi_P)^* \circ (\psi_{-P})^* = \mathrm{id}$, so ψ_P is a birational map of $\mathbb{A}^3$ and is moreover an isomorphism of $\mathbb{A}^3 \setminus V_{\mathcal{Q}}$, because $(\psi_P)^*(xz - y^2) = xz - y^2$ and since the coordinates of ψ_P have only denominators which are powers of $xz - y^2$. One then observes that $(\psi_P)^*(z) = \frac{f_P}{(xz-y^2)^{2d}}$, so ψ_P restricts to an isomorphism $V_P \setminus V_{\mathcal{Q}} \xrightarrow{\simeq} V_{\mathcal{L}} \setminus V_{\mathcal{Q}}$. This implies that V_P and C_P are rational, and that f_P is geometrically irreducible, and also that ψ_P restricts to an isomorphism $\mathbb{A}^3 \setminus (V_{\mathcal{Q}} \cup V_P) \xrightarrow{\simeq} \mathbb{A}^3 \setminus (V_{\mathcal{Q}} \cup V_L)$. As ψ_P is homogeneous, one gets the analogous results for $\mathbb{P}^2$, $\mathcal{Q}$, $\mathcal{L}$, C_P.

(2): One checks that $\varphi_\lambda \circ \varphi_{\lambda^{-1}} = \mathrm{id}$, so φ_λ is a birational map of $\mathbb{A}^3$, which restricts to an automorphism of $\mathbb{A}^3 \setminus V_{\mathcal{L}}$, since the denominators only involve z. Moreover, $(\varphi_\lambda)^*(xz - y^2) = \lambda(xz - y^2)$ (defining $(\varphi_{\lambda,\theta})^*$ similarly as before), so the surface $V_{\mathcal{Q}} \setminus V_{\mathcal{L}}$ is preserved, hence φ_λ restricts to automorphisms of $\mathbb{A}^3 \setminus V_{\mathcal{L}}$, $V_{\mathcal{Q}} \setminus V_{\mathcal{L}}$ and $\mathbb{A}^3 \setminus (V_{\mathcal{L}} \cup V_{\mathcal{Q}})$. Since φ_λ is homogeneous, the same formula yields then automorphisms of $\mathbb{P}^2 \setminus \mathcal{L}$, $\mathcal{Q} \setminus \mathcal{L}$ and $\mathbb{P}^2 \setminus (\mathcal{L} \cup \mathcal{Q})$.

(3): By (1)-(2), $\kappa = (\psi_Q)^{-1}\varphi\psi_P$ restricts to an isomorphism $\mathbb{A}^3 \setminus (V_{\mathcal{Q}} \cup V_P) \xrightarrow{\simeq} \mathbb{A}^3 \setminus (V_{\mathcal{Q}} \cup V_{\tilde{P}})$, for all homogeneous polynomials $P, \tilde{P} \in \mathrm{k}[x,y]$. We then choose $\tilde{P}$ as $\tilde{P}(x,y) = P(\lambda x, y)$ and prove that κ restricts in this case to an isomorphism $\mathbb{A}^3 \setminus V_P \xrightarrow{\simeq} \mathbb{A}^3 \setminus V_{\tilde{P}}$, by proving that the restriction of κ yields the identity automorphism on $V_{\mathcal{Q}} \setminus V_P = V_{\mathcal{Q}} \setminus V_{\tilde{P}} = V_{\mathcal{Q}} \setminus V_{\mathcal{L}}$. We compute

$$\varphi_\lambda\psi_P(x,y,z) = \left(x + (\lambda-1)\frac{(xz-y^2)^{2d+1}}{f_P(x,y,z)}, y + x\frac{P(x^2, xz-y^2)}{(xz-y^2)^d}, \frac{f_P}{(xz-y^2)^{2d}}\right)$$

which satisfies $(\varphi_\lambda\psi_P)^*(xz-y^2)=(\varphi_\lambda)^*(xz-y^2)=\lambda(xz-y^2)$. To simplify the notation, we write $\delta=(\lambda-1)\frac{(xz-y^2)^{2d+1}}{f_P(x,y,z)}$ and get that $\kappa(x,y,z)=(\psi_{\tilde{P}})^{-1}\varphi_\lambda\psi_P(x,y,z)$ is equal to

$$\left(x+\delta, y+x\frac{P(x^2,xz-y^2)}{(xz-y^2)^d}-(x+\delta)\tilde{P}(\frac{(x+\delta)^2}{\lambda(xz-y^2)},1), z+\rho\right)$$

for some $\rho\in \mathrm{k}(x,y,z)$. Since $\tilde{P}(x,y)=P(\lambda x,y)$, the second component is

$$\kappa^*(y)=y+\frac{xP(x^2,xz-y^2)-P((x+\delta)^2,xz-y^2)(x+\delta)}{(xz-y^2)^d}.$$

As $(xz-y^2)^{d+1}$ divides the numerator of δ, one can write $\kappa^*(y)$ as $y+\frac{(xz-y^2)R}{(f_P)^n}$, for some $R\in \mathrm{k}[x,y,z]$ and $n\geq 0$. Similarly, $\kappa^*(x)=x+\frac{(xz-y^2)S}{f_P}$, where $S\in\mathrm{k}[x,y,z]$. Since $\kappa^*(xz-y^2)=\lambda(xz-y^2)$, we get

$$\lambda(xz-y^2)=(x+\frac{(xz-y^2)S}{f_P})(z+\rho)-(y+\frac{(xz-y^2)R}{(f_P)^n})^2$$

which shows that $\rho(x+(xz-y^2)\frac{S}{f_P})=\frac{(xz-y^2)\tilde{T}}{(f_P)^{\tilde{m}}}$ for some $\tilde{T}\in\mathrm{k}[x,y,z]$, $\tilde{m}\geq 0$, hence one can write $\kappa^*(z)=z+\rho=z+\frac{(xz-y^2)T}{(f_P)^m}$ for some $T\in\mathrm{k}[x,y,z]$ and $m\geq 0$. This shows that κ is well defined on $V_{\mathcal{Q}}\setminus V_P=V_{\mathcal{Q}}\setminus V_{\tilde{P}}=V_{\mathcal{Q}}\setminus V_{\mathcal{L}}$ and restricts to the identity on this surface.

Since κ is homogeneous, the isomorphism $\mathbb{A}^3\setminus V_P\xrightarrow{\simeq}\mathbb{A}^3\setminus V_{\tilde{P}}$ also induces an isomorphism $\mathbb{P}^2\setminus C_P\xrightarrow{\simeq}\mathbb{P}^2\setminus C_{\tilde{P}}$, which fixes pointwise the curve $\mathcal{Q}\setminus C_P=\mathcal{Q}\setminus C_{\tilde{P}}$.

(4) Suppose first that $\tilde{P}(x,y)=\rho P(x\rho^2,y)+\mu y^{\deg P}$ for some $\rho\in\mathrm{k}^*,\mu\in\mathrm{k}$. The element $\alpha\in\mathrm{GL}_3(\mathrm{k})$ given by $(x,y,z)\mapsto(x,\rho y-\mu x,\rho^2z-2\rho\mu y+\mu^2x)$ satisfies

$$\begin{aligned}\alpha^*(xz-y^2)&=\rho^2(xz-y^2)\\ \alpha^*((\psi_{\tilde{P}})^*(y))&=\alpha^*(y+x\tilde{P}(\tfrac{x^2}{xz-y^2},1))=\rho y-\mu x+x\tilde{P}(\tfrac{x^2}{\rho^2(xz-y^2)},1)\\ &=\rho y+\rho xP(\tfrac{x^2}{xz-y^2},1)=\rho(\psi_P)^*(y).\end{aligned}$$

Since $\alpha^*(\psi_{\tilde{P}}^*(f))=(\psi_P)^*(f)$ for each $f\in\{x,y,xz-y^2\}$ and because $\mathrm{k}(x,y,z)=\mathrm{k}(x,y,xz-y^2)$, we obtain $\alpha^*(f_{\tilde{P}})=\alpha^*(\psi_{\tilde{P}}^*(z))=(\psi_P)^*(z)=f_P$, which shows that α induces an automorphism of $\mathbb{P}^2$ that sends C_P onto $C_{\tilde{P}}$.

Conversely, suppose that there exists $\tau\in\mathrm{Aut}(\mathbb{P}^2)$ sending C_P onto $C_{\tilde{P}}$. Note that the conic $\mathcal{Q}\subset\mathbb{P}^2$ intersects C_P in exactly one point, the unique singular point $[0:0:1]$ of C_P. Moreover, there is no other such conic, as otherwise the curve C_P would have to be contained in the pencil $\mathbb{P}^2\dashrightarrow\mathbb{P}^1$ generated by the two conics, impossible since C_P is irreducible of degree ≥ 5. The same holds for $C_{\tilde{P}}$, so τ preserves the conic $\mathcal{Q}$. Since the line of $\mathbb{P}^2$ given by $x=0$ is the tangent direction of C_P and $C_{\tilde{P}}$ it is also invariant by τ.

Denote by $\alpha\in\mathrm{GL}_3(\mathrm{k})$ the lift of $\tau\in\mathrm{Aut}(\mathbb{P}^2)=\mathrm{PGL}_3(\mathrm{k})$ which satisfies $\alpha^*(x)=x$ and let us write $s=\psi_{\tilde{P}}\alpha(\psi_P)^{-1}\in\mathrm{Bir}(\mathbb{A}^3)$. By construction, one has

$$s^*(x)=x, s^*(xz-y^2)=\nu(xz-y^2), s^*(z)=\xi z, \tag{G}$$

for some $\nu,\xi\in\mathrm{k}^*$. Moreover, s induces a birational map $\hat{s}$ of $\mathbb{P}^2$ which is an automorphism of $\mathbb{P}^2\setminus\mathcal{Q}$, since the same holds for α, ψ_P and $\psi_{\tilde{P}}$. Let us observe that $\hat{s}$ is in fact an automorphism of $\mathbb{P}^2$. Indeed, otherwise $\hat{s}$ would contract $\mathcal{Q}$ onto one point, impossible since $\hat{s}$ preserves the two pencil of conics given by $[x:y:z]\mapsto[xz-y^2:x^2]$

and $[x : y : z] \mapsto [xz - y^2 : z^2]$, which have distinct base-points. Hence, (G) implies that s is a linear map such that $s^*(y) = \rho y$ for some $\rho \in \mathrm{k}^*$ which satisfies $\rho^2 = \nu = \xi$, which yields

$$s\colon (x, y, z) \mapsto (x, \rho y, \rho^2 z).$$

Note that the equation $\psi_{\tilde{P}}\alpha = s\psi_P$ yields $\alpha\psi_{-P} = \psi_{-\tilde{P}}s$. Since the denominators of the three coordinates of $\psi_{-\tilde{P}}s$ have increasing degrees, and the same holds for ψ_{-P}, for the same degrees, one obtains that α is triangular. Since $\alpha^*(x) = x$ and $\alpha^*(xz - y^2) = \nu(xz - y^2) = \rho^2(xz - y^2)$, we find that α is of the form

$$\alpha\colon (x, y, z) \mapsto (x, ay - \mu x, a^2 z - 2a\mu y + \mu^2 x)$$

for some $a \in \mathrm{k}^*$, $\mu \in \mathrm{k}$, with $a^2 = \nu$. The second coordinate of $\psi_{\tilde{P}}\alpha = s\psi_P$ is equal to

$$(ay - \mu x) + x\tilde{P}(\frac{x^2}{a^2(xz - y^2)}, 1) = (\psi_{\tilde{P}}\alpha)^*(y) = (s\psi_P)^*(y) = \rho(y + xP(\frac{x^2}{xz - y^2}, 1))$$

which yields $a = \rho$ and $\tilde{P}(\frac{x^2}{\rho^2(xz-y^2)}, 1) = \rho P(\frac{x^2}{xz-y^2}, 1) + \mu$. Setting $z = \frac{y^2+\rho^{-2}xy}{x}$ we obtain with we get $\tilde{P}(\frac{x}{y}, 1) = \rho P(\rho^2\frac{x}{y}, 1) + \mu$, whence $\tilde{P}(\frac{x}{y}, 1) = \rho P(\rho^2\frac{x}{y}, 1) + \mu$, which is equivalent to $\tilde{P}(x, y) = \rho P(\rho^2 x, y) + \mu y^{\deg P}$, as we wanted. □

The construction of Lemma 5.6 yields, for each $d \geq 1$, families of curves of degree $4d + 1$ having the same complement. These are equivalent for $d = 1$, at least when k is algebraically closed (Lemma 5.6(4)), but not for $d \geq 2$. One can now easily provide explicit examples:

Corollary 5.7. *Let $d \geq 2$ be an integer, and let $P = x^d + x^{d-1}y \in \mathrm{k}[x, y]$. All curves of $\mathbb{P}^2$ given by*

$$z(xz - y^2)^{2d} + 2yP(\lambda x^2, xz - y^2)(xz - y^2)^d + x(P(\lambda x^2, xz - y^2))^2$$

for $\lambda \in \mathrm{k}^*$, have the same complement and are pairwise not equivalent, up to automorphism of $\mathbb{P}^2$.

Proof. The curves correspond to the curves $C_{P(\lambda x,y)}$ of Lemma 5.6 and have thus isomorphic complements by Lemma 5.6(3). It remains to show that if $C_{P(\lambda x,y)}$ is equivalent to $C_{P(\tilde{\lambda}x,y)}$, then $\lambda = \tilde{\lambda}$. Lemma 5.6(3) yields the existence of $\rho \in \mathrm{k}^*, \mu \in \mathrm{k}$ such that $P(\tilde{\lambda}x, y) = \rho P(\rho^2\lambda x, y) + \mu y^d$. Since $d \geq 2$, both $P(\tilde{\lambda}x, y)$ and $\rho P(x\rho^2\lambda, y)$ do not have component with y^d, so $\mu = 0$. We then compare the coefficients of x^d and $x^{d-1}y$ and get

$$\tilde{\lambda}^d = \rho(\rho^2\lambda)^d, \quad \tilde{\lambda}^{d-1} = \rho(\rho^2\lambda)^{d-1},$$

which yields $\tilde{\lambda} = \rho^2\lambda$, so $\tilde{\lambda}^d = (\rho^2\lambda)^d$, whence $\rho = 1$ and $\tilde{\lambda} = \lambda$ as desired. □

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Jérémy Blanc, Universität Basel, Departement Mathematik und Informatik, Spiegelgasse 1, CH-4051 Basel, Switzerland

E-mail address: jeremy.blanc@unibas.ch

Jean-Philippe Furter, Universität Basel, Spiegelgasse 1, CH-4051 Basel, Switzerland, On leave from Dpt. of Math., Univ. of La Rochelle, av. Crépeau, 17000 La Rochelle, France

E-mail address: jpfurter@univ-lr.fr

Mattias Hemmig, Universität Basel, Departement Mathematik und Informatik, Spiegelgasse 1, CH-4051 Basel, Switzerland

E-mail address: mattias.hemmig@gmail.com