QUOTIENTS OF HIGHER DIMENSIONAL CREMONA GROUPS

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Abstract. We study large groups of birational transformations $\mathrm{Bir}(X)$, where X is a variety of dimension at least 3, defined over $\mathbf{C}$ or a subfield of $\mathbf{C}$. Two prominent cases are when X is the projective space $\mathbb{P}^n$, in which case $\mathrm{Bir}(X)$ is the Cremona group of rank n, or when $X \subset \mathbb{P}^{n+1}$ is a smooth cubic hypersurface. In both cases, and more generally when X is birational to a conic bundle, we produce infinitely many distinct group homomorphisms from $\mathrm{Bir}(X)$ to $\mathbf{Z}/2$. As a consequence we also obtain that the Cremona group of rank $n \geqslant 3$ is not generated by linear and Jonquières elements.

Contents

Date: January 15, 2019.

2010 *Mathematics Subject Classification.* 14E07, 14E30, 20F05; 20L05, 14J45, 14E05.

Key words and phrases. Cremona groups; normal subgroups; conic bundles; Sarkisov links; BAB conjecture.

The first author acknowledges support by the Swiss National Science Foundation Grant "Birational transformations of threefolds" 200020_178807. The second author was partially supported by the UMI-CRM 3457 of the CNRS in Montréal, and by the Labex CIMI. The third author was supported by Projet PEPS 2018 "JC/CJ" and is supported by the ANR Project FIBALGA ANR-18-CE40-0003-01.

1. Introduction

1.A. **Higher rank Cremona groups.** The *Cremona group of rank* n, denoted by $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^n)$, or simply $\mathrm{Bir}(\mathbb{P}^n)$ when the ground field $\mathbf{k}$ is implicit, is the group of birational transformations of the projective space.

The classical case is $n = 2$, where the group is already quite complicated but is now well described, at least when $\mathbf{k}$ is algebraically closed. In this case the Noether-Castelnuovo Theorem [Cas01, Alb02] asserts that $\mathrm{Bir}(\mathbb{P}^2)$ is generated by $\mathrm{Aut}(\mathbb{P}^2)$ and a single quadratic transformation. This fundamental result, together with the strong factorisation of birational maps between surfaces helps to have a good understanding of the group.

The dimension $n \geqslant 3$ is more difficult, as we do not have any analogue of the Noether-Castelnuovo Theorem (see §1.C for more details) and also no strong factorisation. Here is an extract from the article "Cremona group" in the Encyclopedia of Mathematics, written by V. Iskovskikh in 1987 (who uses the notation $\mathrm{Cr}(\mathbb{P}^n_{\mathbf{k}})$ for the Cremona group):

> *One of the most difficult problems in birational geometry is that of describing the structure of the group* $\mathrm{Cr}(\mathbb{P}^3_{\mathbf{k}})$*, which is no longer generated by the quadratic transformations. Almost all literature on Cremona transformations of three-dimensional space is devoted to concrete examples of such transformations. Finally, practically nothing is known about the structure of the Cremona group for spaces of dimension higher than* 3. [Isk87]

Thirty years later, there are still very few results about the group structure of $\mathrm{Bir}(\mathbb{P}^n)$ for $n \geqslant 3$, even if there were exciting recent developments using a wide range of techniques. After the pioneering work [Dem70] on the algebraic subgroups

of rank n in $\mathrm{Bir}(\mathbb{P}^n)$, we should mention the description of their lattices via p-adic methods [CX18], the study of the Jordan property [PS16], and the fact that Cremona groups of distinct ranks are non-isomorphic [Can14].

For $n = 3$, there is also a classification of the connected algebraic subgroups [Ume85, BFT17], and partial classification of finite subgroups [Pro11, Pro12, Pro14]. There are also numerous articles devoted to the study of particular classes of examples of elements in $\mathrm{Bir}(\mathbb{P}^n)$, especially for n small (we do not attempt to start a list here, as it would always be very far from exhaustive).

The question of the non-simplicity of Cremona groups of higher rank was up to now left open. Using modern tools such as the Minimal model programme and factorisation into Sarkisov links, we will be able in this text to give new insight on the structure of the Cremona groups $\mathrm{Bir}(\mathbb{P}^n)$ and of its quotients.

1.B. **Normal subgroups.** The question of the non-simplicity of $\mathrm{Bir}(\mathbb{P}^n)$ for each $n \geqslant 2$ was also mentioned in the article of V. Iskovskikh in the Encyclopedia:

> *It is not known to date* (1987) *whether the Cremona group is simple.*
> [Isk87]

The question was in fact asked much earlier, and is explicitly mentioned in a book by F. Enriques in 1895:

> *Tuttavia altre questioni d'indole gruppale relative al gruppo Cremona nel piano* (*ed a più forte ragione in* S_n $n > 2$) *rimangono ancora insolute; ad esempio l'importante questione se il gruppo Cremona contenga alcun sottogruppo invariante* (*questione alla quale sembra probabile si debba rispondere negativamente*).
> [Enr95, p. 116][1]

The feeling expressed by F. Enriques that the Cremona group should be simple was perhaps supported by the analogy with biregular automorphism groups of projective varieties, such as $\mathrm{Aut}(\mathbb{P}^n) = \mathrm{PGL}_{n+1}(\mathbf{k})$. In fact in the trivial case of dimension $n = 1$, we have $\mathrm{Bir}(\mathbb{P}^1) = \mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2(\mathbf{k})$, which is indeed a simple group when the ground field $\mathbf{k}$ is algebraically closed. Pushing further the analogy with algebraic groups, it was proved by the first author that when considered as a topological group, the Cremona group $\mathrm{Bir}(\mathbb{P}^2)$ is simple, in the sense that any proper Zariski closed normal subgroup must be trivial [Bla10]. This result was recently extended to arbitrary dimension by the first and third authors [BZ18].

The non-simplicity of $\mathrm{Bir}(\mathbb{P}^2)$ as an abstract group was proven, over an algebraically closed field, by S. Cantat and the second author [CL13]. The idea of proof was to apply small cancellation theory to an action of $\mathrm{Bir}(\mathbb{P}^2)$ on a hyperbolic space. A first instance of roughly the same idea was [Dan74], in the context of plane polynomial automorphisms (see also [FL10]). The modern small cancellation machinery as developed in [DGO17] allowed A. Lonjou to prove the non simplicity of $\mathrm{Bir}(\mathbb{P}^2)$ over an arbitrary field, and the fact that every countable group is a subgroup of a quotient of $\mathrm{Bir}(\mathbb{P}^2)$ [Lon16].

Another source of normal subgroups for $\mathrm{Bir}(\mathbb{P}^2)$, of a very different nature, was discovered by the third author, when the ground field is $\mathbf{R}$ [Zim18]. In contrast

[1] "However, other group-theoretic questions related to the Cremona group of the plane (and, even more so, of $\mathbb{P}^n$, $n > 2$) remain unsolved; for example, the important question of whether the Cremona group contains any normal subgroup (a question which seems likely to be answered negatively)."

with the case of an algebraically closed field where the Cremona group of rank 2 is a perfect group, she proved that the abelianisation of $\mathrm{Bir}_{\mathbf{R}}(\mathbb{P}^2)$ is an uncountable direct sum of $\mathbf{Z}/2$. Here the main idea is to use an explicit presentation by generators and relations. In fact a presentation of $\mathrm{Bir}(\mathbb{P}^2)$ over an arbitrary perfect field is available since [IKT93], but because they insist in staying inside the group $\mathrm{Bir}(\mathbb{P}^2)$, they obtain very long lists. In contrast, if one accepts to consider birational maps between non-isomorphic varieties, the Sarkisov programme provides more tractable lists of generators. Using this idea together with results of A.-S. Kaloghiros [Kal13], the existence of abelian quotients for $\mathrm{Bir}(\mathbb{P}^2)$ was extended to the case of many non-closed perfect fields by the second and third authors [LZ17].

The present paper is a further extension in this direction, this time in arbitrary dimension, and over any ground field $\mathbf{k}$ which is a subfield of $\mathbf{C}$. Our first result is the following:

Theorem A. *For each subfield $\mathbf{k} \subseteq \mathbf{C}$ and each $n \geqslant 3$, there is a group homomorphism*

$$\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^n) \twoheadrightarrow \bigoplus_I \mathbf{Z}/2$$

where the indexing set I has the same cardinality as $\mathbf{k}$, and such that the restriction to the subgroup of birational dilatations given locally by

$$\{(x_1, \dots, x_n) \dashmapsto (x_1\alpha(x_2, \cdots, x_n), x_2, \dots, x_n) \mid \alpha \in \mathbf{k}(x_2, \dots, x_n)^*\}$$

is surjective. In particular, the Cremona group $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^n)$ is not perfect and thus not simple.

We give below a few immediate comments, and a quick preview of the rest of the introduction where we will present several statements that generalise or complement Theorem A in different directions.

First we emphasise that this result contrasts with the situation in dimension 2 (over $\mathbf{C}$). Indeed, as $\mathrm{Bir}_{\mathbf{C}}(\mathbb{P}^2)$ is generated by the simple group $\mathrm{Aut}(\mathbb{P}^2) = \mathrm{PGL}_3(\mathbf{C})$ and one quadratic map birationally conjugated to a linear map, every non-trivial quotient of $\mathrm{Bir}_{\mathbf{C}}(\mathbb{P}^2)$ is non-abelian and uncountable.

Another intriguing point at first sight is the indexing set I. We shall be more precise later, but the reader should think of I as a kind of moduli space for some varieties of dimension $n-2$. Indeed to construct the group homomorphism we will see $\mathbb{P}^n$ as being birational to a $\mathbb{P}^1$-bundle over $\mathbb{P}^{n-1}$, and each factor $\mathbf{Z}/2$ is related to the choice of a general hypersurface in $\mathbb{P}^{n-1}$ of sufficiently high degree, up to some equivalence.

The next natural question is to understand the kernel of the group homomorphism. As will soon become clear, it turns out that $\mathrm{Aut}(\mathbb{P}^n) = \mathrm{PGL}_{n+1}(\mathbf{k})$ is contained in the kernel. This implies that the normal subgroup generated by $\mathrm{Aut}(\mathbb{P}^n)$ and any finite subset of elements in $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^n)$ is proper. Theorem C below will be a stronger version of this fact.

One can also ask about the possibility to get a homomorphism to a free product of $\mathbf{Z}/2$, instead of a direct sum. We will see that is is indeed possible, and is related to the existence of many conic bundle models for $\mathbb{P}^n$ which are not pairwise square birational. See Theorems D and E below.

Finally, one can ask about replacing $\mathbb{P}^n$ by a nonrational variety. In this direction, we will prove the following result about the group $\mathrm{Bir}(X)$ of birational transformations of a conic bundle X/B.

Theorem B. *Let $B \subseteq \mathbb{P}^m$ be a smooth projective complex variety, $P \longrightarrow \mathbb{P}^m$ a decomposable $\mathbb{P}^2$-bundle (projectivisation of a decomposable rank 3 vector bundle) and $X \subset P$ a smooth closed subvariety such that the projection to $\mathbb{P}^m$ gives a conic bundle $\eta\colon X \longrightarrow B$. Then there exists a group homomorphism*

$$\mathrm{Bir}(X) \twoheadrightarrow \bigoplus_{\mathbf{Z}} \mathbf{Z}/2$$

which is surjective in restriction to $\mathrm{Bir}(X/B) = \{\varphi \in \mathrm{Bir}(X) \mid \eta \circ \varphi = \eta\}$.

Moreover, if there exists a subfield $\mathbf{k} \subseteq \mathbf{C}$ over which X, B and η are defined, the image of elements of $\mathrm{Bir}(X/B)$ defined over $\mathbf{k}$ is also infinite.

Theorem B applies to any product $X = \mathbb{P}^1 \times B$, to smooth cubic hypersurfaces $X \subseteq \mathbb{P}^n$ (see Section 8.E and in particular Corollary 8.8 and Proposition 8.9), and to many other varieties of dimension $n \geqslant 3$ which are very far from being rational (see for instance [Kol17, Theorem 3] and [AO18, Theorem 1.1 and Corollary 1.2]). Of course it also includes the case of $X = \mathbb{P}^1 \times \mathbb{P}^{n-1}$ which is birational to $\mathbb{P}^n$, but observe that Theorem A is slightly stronger in this case, since there the set indexing the direct sum has the same cardinality as the ground field, and also because we can give an explicit subgroup, easy to describe, whose image is surjective.

1.C. **Generators.** As already mentioned, the Noether-Castelnuovo theorem provides simple generators of $\mathrm{Bir}(\mathbb{P}^2)$ when $\mathbf{k}$ is algebraically closed. Using Sarkisov links, there are also explicit (long) lists of generators of $\mathrm{Bir}(\mathbb{P}^2)$ for each field $\mathbf{k}$ of characteristic zero or more generally for each perfect field $\mathbf{k}$ [Isk91, Isk96]. In dimension $n \geqslant 3$, we do not have a complete list of all Sarkisov links and thus are far from having an explicit list of generators for $\mathrm{Bir}(\mathbb{P}^n)$. The lack of an analogue to the Noether-Castelnuovo Theorem for $\mathrm{Bir}(\mathbb{P}^n)$ and the question of finding good generators was already cited in the article of the Encyclopedia above, in [HM13, Question 1.6], and also in the book of Enriques:

> *Questo teorema non è estendibile senz'altro allo S_n dove $n > 2$; resta quindi insoluta la questione capitale di assegnare le più semplici trasformazioni generatrici dell'intiero gruppo Cremona in S_n per $n > 2$.* [Enr95, p. 115][2]

A classical result, due to H. Hudson and I. Pan [Hud27, Pan99], says that $\mathrm{Bir}(\mathbb{P}^n)$, for $n \geqslant 3$, is not generated by $\mathrm{Aut}(\mathbb{P}^n)$ and finitely many elements, or more generally by any set of elements of $\mathrm{Bir}(\mathbb{P}^n)$ of bounded degree. The reason is that one needs at least, for each irreducible variety Γ of dimension $n-2$, one birational map that contracts a hypersurface birational to $\mathbb{P}^1 \times \Gamma$. These contractions can be realised in $\mathrm{Bir}(\mathbb{P}^n)$ by *Jonquières elements*, i.e. elements that preserve a family of lines through a given point, which form a subgroup

$$\mathrm{PGL}_2(\mathbf{k}(x_2, \ldots, x_n)) \rtimes \mathrm{Bir}(\mathbb{P}^{n-1}) \subseteq \mathrm{Bir}(\mathbb{P}^n).$$

Hence, it is natural to ask whether the group $\mathrm{Bir}(\mathbb{P}^n)$ is generated by $\mathrm{Aut}(\mathbb{P}^n)$ and by Jonquières elements (a question for instance asked in [PS15]).

We answer this question by the negative, in the following stronger form:

[2] "This theorem can not be easily extended to $\mathbb{P}^n$ where $n > 2$; therefore, the main question of finding the most simple generating transformations of the entire Cremona group of $\mathbb{P}^n$ for $n > 2$ remains open."

Theorem C. *Let* $\mathbf{k}$ *be a subfield of* $\mathbf{C}$*, and* $n \geqslant 3$*. Let* S *be a set of elements in the Cremona group* $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^n)$ *that has cardinality smaller than the one of* $\mathbf{k}$ *(for example* S *finite, or* S *countable if* $\mathbf{k}$ *is uncountable), and let* $G \subseteq \mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^n)$ *be the subgroup generated by* $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^n)$*, by all Jonquières elements and by* S*.*

Then, G *is contained in the kernel of a surjective group homomorphism*

$$\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^n) \twoheadrightarrow \mathbf{Z}/2.$$

In particular G *is a proper subgroup of* $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^n)$*, and the same is true for the normal subgroup generated by* G*.*

It is interesting to make a parallel between this statement and the classical Tame Problem in the context of the affine Cremona group $\mathrm{Aut}(\mathbb{A}^n)$, or group of polynomial automorphisms. This is one of the "challenging problems" on the affine spaces, described by H. Kraft in the Bourbaki seminar [Kra96]. Recall that the tame subgroup $\mathrm{Tame}(\mathbb{A}^n) \subseteq \mathrm{Aut}(\mathbb{A}^n)$ is defined as the subgroup generated by affine automorphisms and by the subgroup of elementary automorphisms of the form $(x_1, \dots, x_n) \mapsto (ax_1 + P(x_2, \dots, x_n), x_2, \dots, x_n)$. This elementary subgroup is an analogue of the $\mathrm{PGL}_2(\mathbf{k}(x_1, \dots, x_n))$ factor in the Jonquières group, and of course the affine group is $\mathrm{PGL}_{n+1}(\mathbf{k}) \cap \mathrm{Aut}(\mathbb{A}^n)$. The Tame Problem asks whether the inclusion $\mathrm{Tame}(\mathbb{A}^n) \subseteq \mathrm{Aut}(\mathbb{A}^n)$ is strict in dimension $n \geqslant 3$. It was solved in dimension 3 over a field of characteristic zero in [SU04], and remains an open problem otherwise.

On the one hand, one could say that our Theorem C is much stronger, since we consider the *normal* subgroup generated by these elements, and we allow some extra generators. It is not known (even if not very likely) whether one can generate $\mathrm{Aut}(\mathbb{A}^3)$ with linear automorphisms, elementary automorphisms and one single automorphism, and not even whether the normal subgroup generated by these is the whole group $\mathrm{Aut}(\mathbb{A}^3)$ (this last statement, even without the extra automorphism, seems more plausible).

On the other hand, even in dimension 3 we should stress that Theorem C does not recover a solution to the Tame Problem. Indeed, it seems plausible that the whole group $\mathrm{Aut}(\mathbb{A}^n)$ lies in the kernel of the group homomorphism to $\mathbf{Z}/2$ of Theorem C. In fact, every element of $\mathrm{Bir}(\mathbb{P}^n)$ that admits a decomposition into Sarkisov links that contract only rational varieties (or more generally varieties birational to $\mathbb{P}^2 \times B$ for some variety B of dimension $n-3$) is in the kernel of all our group homomorphisms (all are given by the construction of Theorem D below), and it seems natural to expect that elements of $\mathrm{Aut}(\mathbb{A}^n)$ are of this type, but we leave this as an open question. In fact we are not aware of any element of $\mathrm{Aut}(\mathbb{A}^3)$ which has been proved to lie outside the group generated, in $\mathrm{Bir}(\mathbb{P}^3)$, by linear and Jonquières maps: see [BH15, Proposition 6.8] for the case of the Nagata automorphism, which can be generalised to any other automorphism given by a $\mathbb{G}_a$ action, as all algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^3)$ isomorphic to $\mathbb{G}_a$ are conjugate [BFT18].

1.D. **Overwiew of the strategy.** By the Minimal model programme, every variety Z which is covered by rational curves is birational to a Mori fibre space, and every birational map between two Mori fibre spaces is a composition of simple birational maps, called *Sarkisov links* (see Definition 3.8 and Theorem 4.29). We associate to such a variety Z the groupoid $\mathrm{BirMori}(Z)$ of all birational maps between Mori fibre spaces birational to Z. The main idea is that even if we are primarily interested in describing morphisms from the group $\mathrm{Bir}(Z)$ to $\mathbf{Z}/2$, it turns

out to be easier to first define such a morphism on the larger groupoid $\mathrm{BirMori}(Z)$, and then restrict to $\mathrm{Bir}(Z)$.

We concentrate on some special Sarkisov links, called *Sarkisov links of conic bundles of type* II (see Definitions 3.8 and 3.9). To each such link, we associate a marked conic bundle, which is a pair $(X/B, \Gamma)$, where X/B is a conic bundle (a terminal Mori fibre space with $\dim B = \dim X - 1$) and $\Gamma \subset B$ is an irreducible hypersurface (see Definition 3.22 and Lemma 3.23). We also define a natural equivalence relation between marked conic bundles (Definition 3.22).

For each variety Z, we denote by $\mathsf{CB}(Z)$ the set of equivalence classes of conic bundles X/B with X birational to Z, and for each class of conic bundles $C \in \mathsf{CB}(Z)$ we denote by $\mathsf{M}(C)$ the set of equivalence classes of marked conic bundles $(X/B, \Gamma)$, where C is the class of X/B.

The Sarkisov programme is established in every dimension [HM13] and relations among them are described in [Kal13]. Inspired by the latter, we define *rank r fibrations* X/B (see Definition 3.1); rank 1 fibrations are Mori fibre spaces and rank 2 fibrations dominate Sarkisov links (see Lemma 3.7). We prove that the relations among Sarkisov links are generated by *elementary relations* (Definition 4.4), which we define as relations dominated by rank 3 fibrations (see Theorem 4.29).

We associate to each of these Sarkisov links χ an integer $\mathrm{cov.gon}(\chi)$ that measures the degree of irrationality of the base locus of χ (see §2.G). The BAB conjecture, proven in [Bir16a] and [Bir16b], tells us that the set of weak Fano terminal varieties of dimension n form a bounded family and the degree of their images by a (universal) multiple of the anticanonical system is bounded by a (universal) integer d (see Proposition 5.1). As a consequence, we show that any Sarkisov link χ of conic bundles of type II appearing in an elementary relation over a base of small dimension satisfies $\mathrm{cov.gon}(\chi) \leqslant d$ (see Proposition 5.3). This and the description of the elementary relations over a base of maximal dimension and including a Sarkisov link of conic bundles of type II (Proposition 5.5) allows us to prove the following statement in §5.C.

Theorem D. *Let $n \geqslant 3$. There is an integer $d > 1$ depending only on n, such that for every conic bundle X/B, where X is a terminal variety of dimension n, we have a groupoid homomorphism*

$$\mathrm{BirMori}(X) \longrightarrow \underset{C \in \mathsf{CB}(X)}{\circledast} \left(\underset{\mathsf{M}(C)}{\oplus} \mathbf{Z}/2 \right)$$

that sends each Sarkisov link of conic bundles χ of type II *with* $\mathrm{cov.gon}(\chi) > \max\{d, 8\,\mathrm{conn.gon}(X)\}$ *onto the generator indexed by its associated marked conic bundle, and all other Sarkisov links and all automorphisms of Mori fibre spaces birational to X onto zero.*

Moreover it restricts to group homomorphisms

$$\mathrm{Bir}(X) \longrightarrow \underset{C \in \mathsf{CB}(X)}{\circledast} \left(\underset{\mathsf{M}(C)}{\oplus} \mathbf{Z}/2 \right), \qquad \mathrm{Bir}(X/B) \longrightarrow \underset{\mathsf{M}(X/B)}{\oplus} \mathbf{Z}/2.$$

In order to deduce Theorem A, we study the image of the group homomorphism from $\mathrm{Bir}(X)$ and $\mathrm{Bir}(X/B)$ provided by Theorem D, for some conic bundle X/B. In particular, we must check that these restrictions are not the trivial morphism. We give a criterion to compute the image in §6.A. We apply this criterion to show that the image is very large if the generic fibre of X/B is $\mathbb{P}^1$ (or equivalently if

X/B has a rational section, or is equivalent to $(\mathbb{P}^1 \times B)/B$). This is done in §6.B and allows us to prove Theorem A. Then in §6.C we study the more delicate case where the generic fibre X/B is not $\mathbb{P}^1$ (or equivalently if X/B has no rational section), and show that for each conic bundle X/B, the image of $\mathrm{Bir}(X/B)$ by the group homomorphism of Theorem D contains an infinite direct sum of $\mathbf{Z}/2$ (Proposition 6.9). This allows to prove Theorem B.

1.E. **Non-equivalent conic bundle structures.** Coming back to the case of $\mathbb{P}^n$, we study the free product structure appearing in Theorem D. We want to prove that the indexing set $\mathsf{CB}(\mathbb{P}^n)$ is large. This is equivalent to the question of existence of many non-equivalent conic bundle structures on $\mathbb{P}^n$. Using conic bundles over $\mathbb{P}^2$ with discriminant an elliptic curve, we manage to produce such examples, and we get the following.

Theorem E. *Let $n \geqslant 3$ and let $\mathbf{k} \subseteq \mathbf{C}$ be a subfield. There is a surjective group homomorphism*

$$\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^n) \twoheadrightarrow \mathop{\ast}_{J} \mathbf{Z}/2,$$

where the indexing set J has the same cardinality as $\mathbf{k}$. In particular, every group generated by a set of involutions with cardinality smaller or equal than $|\mathbf{k}|$ is a quotient of $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^n)$. Moreover, the group homomorphism that we construct admits a section, so $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^n)$ is a semi-direct product with one factor being a free product.

A first consequence is Theorem C. Other complements are given in Section 8.

First we get the SQ-universality of $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^n)$, meaning that any countable group is a subgroup of a quotient of $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^n)$. But in fact, many natural subgroups are quotients of $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^n)$, with no need to passing to a subgroup: this includes dihedral and symmetric groups, linear groups, and the Cremona group of rank 2 (see §8.A).

Another consequence of our results is that the group $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^n)$ is not hopfian if it is generated by involutions, for each subfield $\mathbf{k} \subseteq \mathbf{C}$ and each $n \geqslant 3$ (Corollary 8.5). This is another difference with the dimension 2, as $\mathrm{Bir}_{\mathbf{C}}(\mathbb{P}^2)$ is hopfian and generated by involutions (see §8.B).

All our results hold over any field abstractly isomorphic to a subfield of $\mathbf{C}$ (§8.C). This is the case of most field of characteristic zero that are encountered in algebraic geometry. For instance, any field of rational functions of any algebraic variety defined over a subfield of $\mathbf{C}$.

Another feature of the Cremona groups in higher dimension is that the group $\mathrm{Bir}_{\mathbf{C}}(\mathbb{P}^n)$ is a free product of uncountably many distinct subgroups, amalgamated over the intersection of the subgroups, which is the same for any two subgroups. This strong version of an amalgamated product (Theorem 8.6) is again very different from $\mathrm{Bir}_{\mathbf{C}}(\mathbb{P}^2)$ (which is not a non-trivial amalgam, as already explained) and generalises to other varieties as soon as they have two non-equivalent conic bundle structures. Again this result can be generalised to subfields of $\mathbf{C}$.

Theorem 8.6 implies that $\mathrm{Bir}(\mathbb{P}^n)$ acts non-trivially on a tree. More generally, for each conic bundle X/B, we provide a natural action of $\mathrm{Bir}(X)$ on a graph constructed from rank r fibrations birational to X (see §8.F).

Aknowledgements. We thank Hamid Ahmadinezhad, Marcello Bernardara, Caucher Birkar, Christian Böhnig, Hans-Christian Graf von Bothmer, Serge Cantat, Ivan Cheltsov, Tom Ducat, Andrea Fanelli, Enrica Floris, Jean-Philippe Furter, Philipp Habegger, Anne-Sophie Kaloghiros, Vladimir Lazić, Zsolt Patakfalvi, Yuri

Prokhorov, Miles Reid and Christian Urech for interesting discussions related to this project.

2. Preliminaries

Unless explicitly stated otherwise, all ambient varieties are assumed to be projective, irreducible, reduced and defined over the field $\mathbf{C}$ of complex numbers.

This restriction on the ground field comes from the fact that this is the setting of many references that we use, such as [BCHM10, HM13, Kal13, KKL16]. It seems to be folklore that all the results in these papers are also valid over any algebraically closed field of characteristic zero, but we let the reader take full responsibility if he is willing to deduce that our results automatically hold over such a field. However, in Sections 6 and 7, see also §8.C, we will show how to work over fields that can be embedded in $\mathbf{C}$.

General references for this section are [KM98, Laz04, BCHM10].

2.A. **Divisors and curves.** Let X be a normal variety, $\mathrm{Div}(X)$ the group of Cartier divisors, and $\mathrm{Pic}(X) = \mathrm{Div}(X)/\sim$ the Picard group of divisors modulo linear equivalence. The *Néron-Severi* space $N^1(X) = \mathrm{Div}(X) \otimes \mathbf{R}/\equiv$ is the space of $\mathbf{R}$-divisors modulo numerical equivalence. This is a finite-dimensional vector space whose dimension $\rho(X)$ is called the *Picard rank* of X. We denote $N_1(X)$ the dual space of 1-cycles with real coefficients modulo numerical equivalence. We have a perfect pairing $N^1(X) \times N_1(X) \longrightarrow \mathbf{R}$ induced by intersection. If we need to work with coefficients in $\mathbf{Q}$ we will use notation such as $N^1(X)_{\mathbf{Q}} := \mathrm{Div}(X) \otimes \mathbf{Q}/\equiv$ or $\mathrm{Pic}(X)_{\mathbf{Q}} := \mathrm{Pic}(X) \otimes \mathbf{Q}$. We say that a Weil divisor D on X is $\mathbf{Q}$-*Cartier* if mD is Cartier for some integer $m > 0$. The variety X is $\mathbf{Q}$-*factorial* if all Weil divisors on X are $\mathbf{Q}$-Cartier. An element in $\mathrm{Div}(X) \otimes \mathbf{Q}$ is called a $\mathbf{Q}$-divisor.

First we recall a few classical geometric notions attached to a $\mathbf{Q}$-divisor D. Let m be a sufficiently large and divisible integer. D is effective, denoted $D \geqslant 0$, if all coefficients of D are non negative, and D is *movable* if the base locus of the linear system $|mD|$ has codimension at least 2. D is *big* if the map associated with the linear system $|mD|$ is birational. Similarly, D is *semiample* if $|mD|$ is base point free, and D is *ample* if furthermore the associated map is an embedding. Finally, D is *nef* if for any curve C we have $D \cdot C \geqslant 0$.

Now we recall how the numerical counterparts of these notions define cones in $N^1(X)$. The effective cone $\mathrm{Eff}(X) \subseteq N^1(X)$ is the cone generated by effective divisors on X. Its closure $\overline{\mathrm{Eff}}(X)$ is the cone of *pseudo-effective* classes. Similarly we denote $NE(X) \subseteq N_1(X)$ the cone of effective 1-cycles, and $\overline{NE}(X)$ its closure. By Kleiman's criterion, a divisor D is ample if and only if $D \cdot C > 0$ for any 1-cycle $C \in \overline{NE}(X)$. It follows that the cone $\mathrm{Ample}(X)$ of ample classes if the interior of the closed cone $\mathrm{Nef}(X) \subseteq N^1(X)$ of nef classes. Similarly, the interior of the pseudo-effective cone $\overline{\mathrm{Eff}}(X)$ is the big cone $\mathrm{Big}(X)$: Indeed a class D is big if and only if $D \equiv A + E$ with A ample and E effective. A class is semiample if it is the pull-back of an ample class by a morphism. Finally the *movable* cone $\overline{\mathrm{Mov}}(X)$ is the closure of the cone spanned by movable divisors, and we will denote by $\mathrm{IntMov}(X)$ its interior.

One should keep in mind the following inclusions between all these cones:

$$\begin{array}{ccccccccc} \mathrm{Ample}(X) & \subseteq & \mathrm{Semiample}(X) & \subseteq & \mathrm{Nef}(X) & \subseteq & \overline{\mathrm{Mov}}(X) & \subseteq & \overline{\mathrm{Eff}}(X) \\ & & & & \| & & & & \| \\ & & & & \overline{\mathrm{Ample}}(X) & & & & \overline{\mathrm{Big}}(X) \end{array}$$

We say that a 1-cycle $C \in \overline{NE}(X)$ is *extremal* if any equality $C = C_1 + C_2$ inside $\overline{NE}(X)$ implies that C, C_1, C_2 are proportional.

2.B. **Maps.** Let $\pi\colon X \longrightarrow Y$ be a surjective morphism between normal varieties. We shall also denote X/Y such a situation. The *relative Picard group* is the quotient $\mathrm{Pic}(X/Y) := \mathrm{Pic}(X)/\pi^*\,\mathrm{Pic}(Y)$.

We say that a curve $C \subseteq X$ is *contracted* by π if $\pi(C)$ is a point. The subsets $NE(X/Y) \subseteq N_1(X/Y) \subseteq N_1(X)$ are respectively the cone and the subspace generated by curves contracted by π. The relative Néron-Severi space $N^1(X/Y)$ is the quotient of $N^1(X)$ by the orthogonal of $N_1(X/Y)$. The dimension $\rho(X/Y)$ of $N^1(X/Y)$, or equivalently $N_1(X/Y)$, is the *relative Picard rank* of π. If π has connected fibres, then $\rho(X/Y) = 0$ if and only if π is an isomorphism, because a bijective morphism between normal varieties is an isomorphism.

We denote by $\mathrm{Eff}(X/Y)$, $\mathrm{Nef}(X/Y)$, $\mathrm{Ample}(X/Y)$, $\mathrm{Big}(X/Y)$, $\overline{\mathrm{Mov}}(X/Y)$ the images of the corresponding cones of $N^1(X)$ in the quotient $N^1(X/Y)$. If $D \in N^1(X)$ is a class that projects to an element in $\mathrm{Nef}(X/Y)$, we says that D is π-nef. Equivalently, D is π-nef if $D\cdot C \geqslant 0$ for any $C \in NE(X/Y)$. Similarly, we define the notion of π-ample, π-big, π-effective. In particular a class D is π-ample if $D\cdot C > 0$ for any $C \in \overline{NE}(X/Y)$.

Geometrically, a $\mathbf{Q}$-divisor D is π-ample if the restriction of D to each fibre is ample, and D is *π-big* if the restriction of D to the generic fibre is big. We have the following characterisation for this last notion:

Lemma 2.1. *Let $\pi\colon X \longrightarrow Y$ be a surjective morphism between normal varieties. A divisor D on X is π-big if and only if we can write D as a sum*

$$D = \pi\text{-}ample + effective.$$

Proof. Let D be a divisor on X, and $F = \pi^{-1}(p)$ a general fibre. By [Nak04, Corollary 5.17] we have

$$\begin{aligned} D \text{ is } \pi\text{-big} \iff D|_F \text{ is big} \iff D|_F = \text{ample} + \text{effective} \\ \iff D = \pi\text{-ample} + \pi\text{-effective}. \end{aligned}$$

We conclude by using the equivalence

$$\pi\text{-effective} \iff \pi\text{-numerically trivial} + \text{effective},$$

and absorbing the π-numerically trivial divisor in the π-ample factor to get the result. □

When the morphism $\pi\colon X \longrightarrow Y$ is birational, the *exceptional locus* $\mathrm{Ex}(\pi)$ is the set covered by all contracted curves. Assume moreover that $\rho(X/Y) = 1$, and that X is $\mathbf{Q}$-factorial. Then we are in one of the following situations [KM98, Prop 2.5]: either $\mathrm{Ex}(\pi)$ is a prime divisor, and we say that π is a *divisorial contraction*, or $\mathrm{Ex}(\pi)$ has codimension at least 2 in X, and we say that π is a *small contraction*. In this case, Y is not $\mathbf{Q}$-factorial.

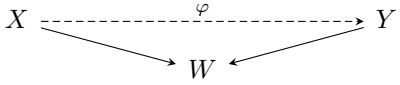
In the context of a birational morphism, we have the following classical lemma. Observe that here a class is π-effective if it can be represented by an effective divisor with no component of the support in the exceptional locus of π.

Lemma 2.2 (Negativity Lemma, see [BCHM10, 3.6.2] or [Mat02, 13-1-4])**.** *Let $\pi\colon X \longrightarrow Y$ be a birational morphism from a smooth variety X, E a divisor with support in the exceptional locus of φ, and assume that*

$$E \equiv \pi\text{-}nef + \pi\text{-}effective.$$

Then $-E \geqslant 0$.

Given three normal varieties X, Y, W together with surjective morphisms X/W, Y/W, we say that $\varphi\colon X \dashrightarrow Y$ is a *rational map over W* if we have a commutative diagram



Now let $\varphi\colon X \dashrightarrow Y$ be a birational map. Any Weil divisor D on X is sent to a well-defined cycle $\varphi(D)$ on Y, and by removing all components of codimension $\geqslant 2$ we obtain a well-defined divisor $\varphi_* D$: one says that φ induces a map in codimension 1. If $\operatorname{codim} \varphi(D) \geqslant 2$ (and so $\varphi_* D = 0$), we say that φ contracts the divisor D. A *birational contraction* is a birational map such that the inverse does not contract any divisor, or equivalently a birational map which is surjective in codimension 1. A *pseudo-isomorphism* is a birational map which is an isomorphism in codimension 1. Birational morphisms and pseudo-isomorphisms (and compositions of those) are examples of birational contractions.

We use a dashed arrow $\dashrightarrow$ to denote a rational (or birational) map, a plain arrow $\longrightarrow$ for a morphism, and a dotted arrow $\dashrightarrow$, or simply a dotted line $\cdots\cdots$, to indicate a pseudo-isomorphism.

We denote by $\operatorname{Bir}(X)$ the group of birational transformations of X. Given a surjective morphism $\eta\colon X \longrightarrow B$, we denote by $\operatorname{Bir}(X/B)$ the subgroup of $\operatorname{Bir}(X)$ consisting of all birational transformations over B, i.e.

$$\operatorname{Bir}(X/B) := \{\varphi \in \operatorname{Bir}(X) \mid \eta \circ \varphi = \eta\} \subseteq \operatorname{Bir}(X).$$

2.C. **Mori dream spaces and Cox sheaves.** We shall use a relative version of the usual definition of Mori dream space (compare with [KKL16, Definition 2.2]). Before giving the definition we recall the following notions.

Let $\pi\colon X \longrightarrow Y$ be a surjective morphism, and $\mathcal{F}$ a sheaf on X. The *higher direct images* of $\mathcal{F}$ are the sheaves $R^i\pi_*\mathcal{F}$, $i \geqslant 0$, which are defined on each affine subset $U \subset Y$ as $R^i\pi_*\mathcal{F}(U) = H^i(\pi^{-1}(U), \mathcal{F})$.

We say that a normal variety Y has *rational singularities* if for some (hence any) desingularisation $\pi\colon X \longrightarrow Y$, we have $R^i\pi_*\mathcal{O}_X = 0$ for all $i > 0$.

Recall also that a variety is *rationally connected* if any two general points are contained in a rational curve (see [Kol96, IV.3]).

Definition 2.3. Let $\eta\colon X \longrightarrow B$ be a surjective morphism between normal varieties. We say that X/B is a *Mori dream space* if the following conditions hold:

(MD1) X is $\mathbf{Q}$-factorial, and both X, B have rational singularities.

(MD2) A general fibre of η is rationally connected and has rational singularities.

(MD3) $\operatorname{Nef}(X/B)$ is the convex cone generated by finitely many semiample divisors;

(MD4) There exist finitely many pseudo-isomorphisms $f_i\colon X \dashrightarrow X_i$ over B, such that each X_i is a $\mathbf{Q}$-factorial variety satisfying (MD3), and

$$\overline{\mathrm{Mov}}(X/B) = \bigcup f_i^*(\mathrm{Nef}(X_i/B)).$$

Lemma 2.4. *Let $\eta\colon X \longrightarrow B$ be a surjective morphism between normal varieties, and F a general fibre. Assume that X and B have rational singularities, and assume:*

(i) F is rationally connected and has rational singularities.

Then the following properties hold true:

(ii) $H^i(F, \mathcal{O}_F) = 0$ for all $i > 0$;
(iii) $\eta_*\mathcal{O}_X = \mathcal{O}_B$ and $R^i\eta_*\mathcal{O}_X = 0$ for all $i > 0$;
(iv) $H^1(\eta^{-1}(U), \mathcal{O}_{\eta^{-1}(U)}) = 0$ for each affine open set $U \subset B$;
(v) $\mathrm{Pic}(X/B)_{\mathbf{Q}} = N^1(X/B)_{\mathbf{Q}}$.

Remark 2.5. Condition (i) from Lemma 2.4 is our condition (MD2). The lemma implies that we would obtain a more general definition replacing (MD2) by Condition (iv), which is the choice of [BCHM10], or by Condition (v), which is a relative version of the choice made in [KKL16]. However our more restrictive definition suits to our purpose and seems easier to check in practice.

Proof. (i) $\Longrightarrow$ (ii). Consider a resolution of singularities $\pi\colon \hat{F} \longrightarrow F$. Since F has rational singularities, we have $R^i\pi_*\mathcal{O}_{\hat{F}} = 0$ for $i > 0$. Then [Har77, III, Ex.8.1] implies that $H^i(\hat{F}, \mathcal{O}_{\hat{F}}) \simeq H^i(F, \pi_*\mathcal{O}_{\hat{F}}) = H^i(F, \mathcal{O}_F)$ for all $i \geqslant 0$. Finally $H^i(F, \mathcal{O}_F) = H^i(\hat{F}, \mathcal{O}_{\hat{F}}) = 0$ by [Kol96, IV.3.8].

(i) $\Longrightarrow$ (iii). Since X has rational singularities, without loss in generality we can replace X by a desingularisation and assume X smooth. We just saw that $H^i(F, \mathcal{O}_F) = 0$ for all $i > 0$, and since we assume that B has rational singularities, the result follows from [Kol86, Theorem 7.1].

(iii) $\Longrightarrow$ (iv). This is just the definition of $R^1\eta_*\mathcal{O}_X = 0$.

(iii) $\Longrightarrow$ (v). Let $D \in \mathrm{Div}(X)_{\mathbf{Q}}$ a divisor which is numerically trivial against the contracted curves. We want to show that D is trivial in $\mathrm{Pic}(X/B)_{\mathbf{Q}}$, that is, a multiple of D is a pull-back. This is exactly the content of [KM92, 12.1.4]. Observe that here again we only need the vanishing assumption for $i = 1$. □

Let $\eta\colon X \longrightarrow B$ be a surjective morphism between normal varieties, and $L_1, \ldots, L_r$ some $\mathbf{Q}$-divisors on X. We define the divisorial sheaf $R(X/B; L_1, \ldots, L_r)$ to be the sheaf of graded $\mathcal{O}_B$-algebras defined on every open affine set $U \subset B$ as

$$R(X/B; L_1, \ldots, L_r)(U) = \bigoplus_{(m_1,\ldots,m_r)\in\mathbf{N}^r} H^0(\eta^{-1}(U)/U, m_1L_1 + \cdots + m_rL_r),$$

where for any $D \in \mathrm{Pic}(X)_{\mathbf{Q}}$

$$H^0(\eta^{-1}(U)/U, D) = \\ \big\{f \in \mathbf{k}(\eta^{-1}(U))^* \mid \exists L \in \mathrm{Pic}_{\mathbf{Q}}(U), \mathrm{div}(f) + D + \eta^*L \geqslant 0\big\} \cup \{0\}.$$

If moreover $\mathrm{Eff}(X/B) \subseteq \sum \mathbf{R}_+L_i$, which ensures that we would get the same algebras using a $\mathbf{Z}^r$-grading instead of $\mathbf{N}^r$, then we say that the sheaf is a *Cox sheaf*, and we denote

$$\mathrm{Cox}(X/B; L_1, \ldots, L_r) := R(X/B; L_1, \ldots, L_r).$$

We say that a divisorial sheaf $R(X/B; L_1, \dots, L_r)$ is finitely generated if for every affine set U the $\mathbf{N}^r$-graded $\mathcal{O}_B(U)$-algebra $R(X/B; L_1, \dots, L_r)(U)$ is finitely generated.

As the following lemma shows, for Cox sheaves this property of finite generation is independent of the choice of the L_i, and therefore we shall usually omit the reference to such a choice and denote a Cox sheaf simply by $\mathrm{Cox}(X/B)$.

Lemma 2.6. *Let $\eta\colon X \longrightarrow B$ be a surjective morphism between normal varieties, $L_1, \dots, L_r \in \mathrm{Pic}(X)_{\mathbf{Q}}$ such that $\mathrm{Eff}(X/B) \subseteq \sum \mathbf{R}^+ L_i$, and $\mathrm{Cox}(X/B; L_1, \dots, L_r)$ the associated Cox sheaf. Let $L'_1, \dots, L'_s \in \mathrm{Pic}(X)_{\mathbf{Q}}$. If $\mathrm{Cox}(X/B; L_1, \dots, L_r)$ is finitely generated, then the divisorial sheaf $R(X/B; L'_1, \dots, L'_s)$ also is finitely generated. In particular, the property of finite generation of a Cox sheaf of X/B does not depend on the choice of the L_i.*

Proof. As already observed we can use a $\mathbf{Z}^r$-grading and write

$$\mathrm{Cox}(X; L_1, \dots, L_r)(U) = \bigoplus_{(n_1,\dots,n_r)\in\mathbf{Z}^r} H^0(\eta^{-1}(U)/U, n_1L_1 + \cdots + n_rL_r)$$

Replacing the L_i par $\frac{1}{n}L_i$ for some sufficiently divisible n, which by [ADHL15, I.1.2.2] does not affect the finite generation, we can assume that each L'_j is of the form $L'_j = n_1L_1 + \cdots + n_rL_r + \eta^*V$ for some $n_i \in \mathbf{Z}$ and some $\mathbf{Q}$-divisor V on B. So

$$R(X; L'_1, \dots, L'_r)(U) = \bigoplus_{(m_1,\dots,m_r)\in\mathbf{N}^r} H^0(\eta^{-1}(U)/U, m_1L'_1 + \cdots + m_rL'_r)$$

is a Veronese subalgebra of $\mathrm{Cox}(X; L_1, \dots, L_r)(U)$, and is finitely generated again by [ADHL15, I.1.2.2]. □

Lemma 2.7. *Let X/B be a surjective morphism between normal varieties, whose general fibres are rationally connected. Assume that X is $\mathbf{Q}$-factorial, and that X, B and the general fibres have rational singularities. Then X/B is a Mori dream space if and only if its Cox sheaf is finitely generated.*

Proof. The proof is similar to the proofs in the non-relative setting of [KKL16, Corollaries 4.4 and 5.7]. □

Example 2.8. Standard examples of Mori dream spaces in the non relative case (i.e. when B is a point) are toric varieties and Fano varieties. Both of these classes of varieties are special examples of log Fano varieties, which are Mori dream spaces by [BCHM10, Corollary 1.3.2]. If F is a log Fano variety, and B is any smooth variety, then $(F \times B)/B$ is a basic example of relative Mori dream space.

2.D. **Minimal model programme.** Let X be a normal variety, and $C \in \overline{NE}(X)$ an extremal class. We say that the *contraction* of C exists (and in that case it is unique), if there exists a surjective morphism $\pi\colon X \longrightarrow Y$ with connected fibres to a normal variety Y, with $\rho(X/Y) = 1$, and such that any curve contracted by π is numerically proportional to C. If π is a small contraction, we say that the *log-flip* of C exists (and again, in that case it is unique) is there exists $X \dashrightarrow X'$ a pseudo-isomorphism over Y which is not an isomorphism, such that X' is normal and $X' \longrightarrow Y$ is a small contraction that contracts curves proportional to a class C'. For each $D \in N^1(X)$, if D' is the image of D under the pseudo-isomorphism, we have a sign change between $D \cdot C$ and $D' \cdot C'$. We say that $X \dashrightarrow X'$ is a *D-flip*, resp. a *D-flop*, resp. a *D-antiflip* when $D \cdot C < 0$, resp. $D \cdot C = 0$, resp. $D \cdot C > 0$.

If D is nef on X, we say that X is a *D-minimal model*. If there exists a contraction $X \longrightarrow Y$ with $\rho(X/Y) = 1$, $\dim Y < \dim X$ and $-D$ relatively ample, we say that X/Y is a *D-Mori fibre space*.

A *step* in the D-Minimal Model Programme (or in the D-MMP for short) is the removal of an extremal class C with $D \cdot C < 0$, either via a divisorial contraction, or via a D-flip. In this paper we will ensure the existence of each step in a D-MMP by working in one the following contexts. Either $D = K_X + \Delta$ will be an adjoint divisor with Δ ample and we can apply the main result of [BCHM10], or we will assume that X is a Mori dream space, and rely on Lemma 2.9 below (which is the reason for the name). By *running a D-MMP from X*, we mean performing a sequence of such steps, replacing each time D by its image, until reaching one of the following two possible outputs: a D-minimal model or a D-Mori fibre space. In particular, observe that for us the output of a D-MMP is always of the same dimension as the starting variety, and the whole process makes sense even for D not pseudo-effective (in contrast with another possible convention which would be to define the output of a D-MMP as $\mathrm{Proj}(\oplus_n H^0(X, nD))$.

We will often work in a relative setting where all steps are maps over a base variety B, and we will indicate such a setting by saying that we run a D-MMP *over* B.

When $D = K_X$ is the canonical divisor, we usually omit the mention of the divisor in the previous notations. So for instance given a small contraction contracting the class of a curve C, we speak of the flip of C only if $K_X \cdot C < 0$, of the D-flip of C if $D \cdot C < 0$, and of the log-flip of C when we do not want to emphasise the sign of the intersection against any divisor.

Lemma 2.9 (see [HK00, Proposition 1.11] or [KKL16, Theorem 5.4])**.** *If X/B is a Mori dream space, then for any class $D \in N^1(X)$ one can run a D-MMP from X over B, and there are only finitely many possible outputs for such MMP.*

2.E. **Singularities.** We have recalled the basic terminology of the MMP without assumption on singularities, but as usual in practice we will make some drastic restriction on the allowed singularities. A first basic fact is that $\mathbf{Q}$-factoriality is preserved under all operations of the MMP. Precisely, assume that X is a normal $\mathbf{Q}$-factorial variety. If $\pi\colon X \longrightarrow Y$ is a divisorial contraction or a Mori fibre space, then Y is $\mathbf{Q}$-factorial. If $\pi\colon X \longrightarrow Y$ is a small contraction and $X \dashrightarrow X'$ is the associated log-flip, then X' is $\mathbf{Q}$-factorial, but Y is not (see [KM98, 3.36 & 3.37]).

Now let X be a normal variety such that the canonical divisor K_X is $\mathbf{Q}$-Cartier, and let $\pi\colon Z \longrightarrow X$ be a resolution of singularities, with exceptional divisors E_1, $\dots$, E_r. We say that X has *terminal singularities*, or that X is terminal, if in the ramification formula

$$K_Z = \pi^* K_X + \sum a_i E_i,$$

we have $a_i > 0$ for each i. Similarly we say that X has *Kawamata log terminal* (*klt* for short) singularities, or that X is klt, if $a_i > -1$ for each i. Each coefficient a_i, which is often called the *discrepancy* of E_i, does not depend on a choice of resolution in the sense that it is an invariant of the geometric valuation associated to E_i. Let Δ an effective $\mathbf{Q}$-divisor on X. We call (X, Δ) a klt pair if $K_X + \Delta$ is $\mathbf{Q}$-Cartier and if for a (and hence any) resolution of singularities $\pi\colon Z \longrightarrow X$ such that the divisor $(\pi^{-1})_*\Delta \cup \mathrm{Ex}(\pi)$ has normal crossing support we have

$$K_Z = \pi^*(K_X + \Delta) + \sum a_i E_i$$

where $\pi_*(\sum a_i E_i) + \Delta = 0$ and $a_i > -1$ for all i. Observe that if (X, Δ) is a klt pair, then for any $\Delta \geqslant \Delta' \geqslant 0$ the pair (X, Δ') also is klt. In particular taking $\Delta' = 0$ we get that X is klt.

Lemma 2.10. *Let $\pi\colon X \longrightarrow Y$ be a divisorial contraction with exceptional divisor $E = \mathrm{Ex}(\pi)$ that contracts the class of a curve C. If $D \in \mathrm{Div}(X)$ and $D' = \pi_* D$, then in the ramification formula*

$$D = \pi^* D' + aE,$$

the numbers a and $D \cdot C$ have opposite signs. In particular, if X is $\mathbf{Q}$-factorial and terminal, then Y is $\mathbf{Q}$-factorial and terminal if and only if $K_X \cdot C < 0$.

Proof. We have $D \cdot C = aE \cdot C$, so the claim follows from $E \cdot C < 0$. For this, see for instance [Mat02, proof of 8-2-1(i)]. The last assertion follows by taking $D = K_X$ and $D' = K_Y$. □

If we start with a $\mathbf{Q}$-factorial terminal variety and we run the classical MMP (that is, relatively to the canonical divisor), then each step (divisorial contraction or flip) of the MMP keeps us in the category of $\mathbf{Q}$-factorial terminal varieties (for divisorial contractions, this follows from Lemma 2.10). Moreover, when one reaches a Mori fibre space X/B, the base B is $\mathbf{Q}$-factorial as mentioned above, but might not be terminal. However by the following result B has at worst klt singularities.

Proposition 2.11 ([Fuj99, Corollary 4.6])**.** *Let X/B be a Mori fibre space, where X is a $\mathbf{Q}$-factorial klt variety. Then B also is a $\mathbf{Q}$-factorial klt variety.*

We will also use the following related result:

Proposition 2.12 ([Fuj15, Theorem 1.5])**.** *Let (X, Δ) be a klt pair, and set*

$$Y = \mathrm{Proj}\Big(\bigoplus_m H^0(X, m(K_X + \Delta))\Big)$$

where the sum is over all positive integers m such that $m(K_X + \Delta)$ is Cartier. Then Y is klt.

The following class of Mori fibre spaces will be of special importance to us.

Definition 2.13. A *conic bundle* is a $\mathbf{Q}$-factorial terminal Mori fibre space X/B with $\dim B = \dim X - 1$. The *discriminant locus* of X/B is defined as the union of irreducible hypersurfaces $\Gamma \subset B$ such that the preimage of a general point of Γ is not irreducible. We emphasise that the terminology of conic bundle is often used in a broader sense (for instance, for any morphism whose general fibre is isomorphic to $\mathbb{P}^1$, with no restriction on the singularities of X or on the relative Picard rank), but for our purpose we will stick to the above more restricted definition.

We say that two conic bundles X/B and X'/B' are *equivalent* if there exists a commutative diagram

$$\begin{array}{ccc} X & \overset{\psi}{\dashrightarrow} & X' \\ \downarrow & & \downarrow \\ B & \overset{\theta}{\dashrightarrow} & B' \end{array}$$

where ψ, θ are birational.

The singular locus of a terminal variety has codimension at least 3 ([KM98, 5.18]). This fact is crucial in the following result.

Lemma 2.14. *Let $\pi\colon X \longrightarrow Y$ be a divisorial contraction between $\mathbf{Q}$-factorial terminal varieties, with exceptional divisor E, and assume that $\Gamma = \pi(E)$ has codimension 2 in Y. Then π is the blow-up of the symbolic powers of the sheaf of ideals $\mathcal{I}$ defining the reduced scheme Γ. In particular, the fibre f over a general point of Γ (precisely, a point that is smooth for both Γ and Y) is a smooth rational curve such that $K_X \cdot f = E \cdot f = -1$.*

Proof. The fact that π is the blow-up of the sheaf of ideals $\mathcal{I}$ follows from the universal property of blowing-up ([Har77, II.7.14]) and the assumption $\rho(X/Y) = 1$. For the last assertion, it suffices to notice that since X and Y are terminal, there exists a codimension 3 closed subset $S \subset Y$ such that $\Gamma \smallsetminus S$, $Y \smallsetminus S$ and $X \smallsetminus \pi^{-1}(S)$ are smooth, so that the restriction of π to $X \smallsetminus \pi^{-1}(S)$ is the ordinary blow-up of a smooth subvariety. □

Lemma 2.15. *Let $\eta\colon X \longrightarrow B$ be a morphism between normal varieties with X terminal (resp. klt). Then for a general point $p \in B$, the fibre $\eta^{-1}(p)$ also is terminal (resp. klt), so in particular it has rational singularities.*

Proof. The fact that $\eta^{-1}(p)$ is terminal (resp. klt) follows from [Kol97, 7.7] by taking successive hyperplane sections on B locally defining p. As already mentioned klt singularities are rational, see [KM98, 5.22]. □

Lemma 2.16.

(1) *Let (X, Δ) be a klt pair, and $\pi\colon X \longrightarrow Y$ be a morphism with connected fibres such that $-(K_X + \Delta)$ is π-big and π-nef. Then for every $p \in Y$ the fibre $\pi^{-1}(p)$ is covered by rational curves, and for a general $p \in Y$ the fibre $\pi^{-1}(p)$ is rationally connected with klt singularities.*

(2) *Let Y be a klt variety, and $\pi\colon X \longrightarrow Y$ a birational morphism. Then every fibre of π is covered by rational curves.*

(3) *Let $\varphi\colon X \dashrightarrow X'$ be a sequence of log-flips between klt varieties, and $\Gamma \subset X$ a codimension 2 subvariety contained in the base locus of φ. Then Γ is covered by rational curves.*

Proof. We first prove (1). By [HM07, Corollary 1.3(1)], every connected component of every fibre of π is rationally chain connected. As π has connected fibres, every fibre f is rationally chain connected. This implies that f is uniruled [Kol96, IV.3.3.4] and thus that f is covered by rational curves [Kol96, IV.1.4.4]. The fact that a general fibre f is klt is Lemma 2.15. As f is rationally chain connected, it is also rationally connected by [HM07, Corollary 1.5(2)].

(2): By [HM07, Corollary 1.5(1)], each fibre f of $Y \longrightarrow X$ is rationally chain connected. By the same argument as before, this implies that f is uniruled and then covered by rational curves.

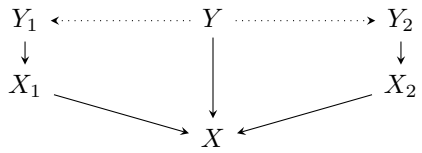
Finally we prove (3). It is sufficient to consider the case of a single log-flip $X \dashrightarrow X'$, associated to a small contraction $X \longrightarrow Y$, and to prove that the exceptional locus of X/Y is covered by rational curves. If $-K_X$ is relatively ample, this follows directly from (1), applied with $\Delta = 0$. Otherwise, denoting C, resp. C', a curve contracted by X/Y, resp. by X'/Y, we have $K_X \cdot C \geqslant 0$ and $K_{X'} \cdot C' \leqslant 0$. We choose an effective ample divisor Δ' on X' such that $(K_{X'} + \Delta') \cdot C' > 0$ and (X', Δ') is a klt pair. Denote by Δ the strict transform of Δ' on X. Then (X, Δ) is a klt pair and $(K_X + \Delta) \cdot C < 0$ because a log-flip changes the sign of intersection, and again we can apply (1). □

Lemma 2.17. *Let X/Y be a Mori dream space that factorises as a composition of two morphisms X/W and W/Y via a $\mathbf{Q}$-factorial klt variety W. Then W/Y is a Mori dream space.*

Proof. The general fibres of W/Y are rationally connected because they are images of the rationally connected fibres of X/Y, and they have rational singularities by Lemma 2.15. For any affine open subset $U \subset Y$, the algebra $\mathrm{Cox}(W/Y)(U)$ embeds by pull-back as a subalgebra of $\mathrm{Cox}(X/Y)(U)$, hence is finitely generated by Lemma 2.6. We conclude by Lemma 2.7. □

2.F. **Two-rays game.** A reference for the notion of two-rays game is [Cor00, §2.2]. We use a slightly different setting in the discussion below. Namely, first we ensure that all moves do exist by putting a Mori dream space assumption, and secondly we do not put strong restrictions on singularities (this will come later in Definition 3.1).

Let $Y \longrightarrow X$ be a surjective morphism between normal varieties, with $\rho(Y/X) = 2$. Assume also that there exists a morphism X/B such that Y/B is a Mori dream space. In particular, by Lemma 2.9 for any divisor D on Y one can run a D-MMP over B, hence a fortiori over X. Then $NE(Y/X)$ is a closed 2-dimensional cone, generated by two extremal classes represented by curves C_1, C_2. Let $D = -A$ where A is an ample divisor on Y, so that a D-minimal model does not exist. Then by Lemma 2.9 for each $i = 1, 2$ we can run a D-MMP from Y over X, which starts by the divisorial contraction or log-flip of the class C_i, and produce a commutative diagram that we call the *two-rays game* associated to Y/X (and which does not depend on the choice of D):

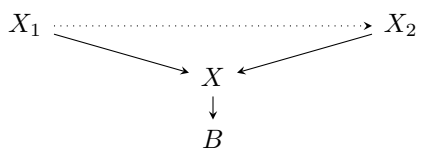


Here $Y \dashrightarrow Y_i$ is a (possibly empty) sequence of D-flips, and $Y_i \longrightarrow X_i$ is either a divisorial contraction or a D-Mori fibre space.

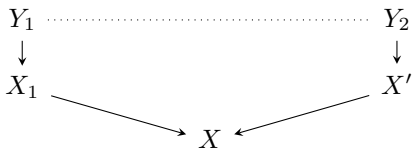
Now we give a few direct consequences of the two-rays game construction.

Lemma 2.18. *Let Y_1/B be a Mori dream space, $Y_1 \longrightarrow X_1$ a morphism over B with $\rho(Y_1/X_1) = 1$, and $X_1 \dashrightarrow X_2$ a sequence of relative log-flips over B. Then there exists a sequence of log-flips $Y_1 \dashrightarrow Y_2$ over B such that the induced map $Y_2 \longrightarrow X_2$ is a morphism, of relative Picard rank 1 by construction. Moreover if Y_1/X_1 is a divisorial contraction (resp. a Mori fibre space), then Y_2/X_2 also is.*

Proof. By induction, it is sufficient to consider the case where $X_1 \dashrightarrow X_2$ is a single log-flip over a non $\mathbf{Q}$-factorial variety X dominating B, given by a diagram



In this situation, the two-rays game Y_1/X gives a diagram



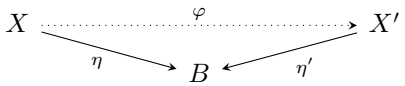
where $Y_1 \dashrightarrow Y_2$ is a sequence of log-flips and $Y_2 \longrightarrow X'$ is a morphism of relative Picard rank 1, with X' a $\mathbf{Q}$-factorial variety. If Y_1/X_1 is a divisorial contraction, then Y_2/X' must be birational hence also is a divisorial contraction. On the other hand if Y_1/X_1 is a Mori fibre space, then Y_2/X' cannot be birational, otherwise X'/X would be a D-Mori fibre space for some divisor D; impossible since X' is $\mathbf{Q}$-factorial but not X. By uniqueness of the log-flip associated to the small contraction $X_1 \longrightarrow X$, we conclude in both cases that $X' = X_2$. □

We now recall the following result of [Cor95], which follows from the Negativity Lemma 2.2, and then deduce from it Corollaries 2.20 and 2.21, similar to [Cor95, Proposition 3.5].

Lemma 2.19 ([Cor95, Proposition 2.7])**.** *Let $\varphi\colon Y \dashrightarrow Y'$ be a pseudo-isomorphism between $\mathbf{Q}$-factorial varieties. If there is an ample divisor H on Y such that φ_*H is ample on Y', then φ is an isomorphism.*

In the following Corollary 2.20, the assumption $\rho(X/B) = 1$, which is part of the definition of a Mori fibre space, is crucial. For instance if $\rho(X/B) = 2$, then any Sarkisov diagram whose pseudo-isomorphism on the top row is not an isomorphism provides a counter example (see Definition 3.8 and Figure 1).

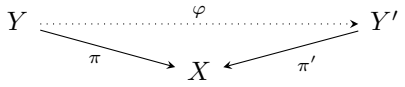
Corollary 2.20. *Let X/B and X'/B be Mori fibre spaces over the same base B, and $\varphi\colon X \dashrightarrow X'$ a pseudo-isomorphism over B, that is, the following diagram commutes:*



Then φ is an isomorphism.

Proof. Let A_B, A_X be ample divisors respectively on B and X, and $0 < \varepsilon \ll 1$. If C is a general curve contracted by X/B, then φ is an isomorphism in a neighborhood of C, hence φ_*A_X is relatively ample on X'/B. Then $\eta^*A_B + \varepsilon A_X$ and $\eta'^*A_B + \varphi_*\varepsilon A_X$ are both ample, and we conclude by Lemma 2.19. □

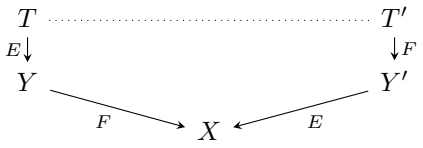
Corollary 2.21. *Consider a commutative diagram*



where X, Y, Y' are $\mathbf{Q}$-factorial varieties, π and π' are divisorial contractions, and φ is a pseudo-isomorphism. Then φ is an isomorphism.

Proof. Let E and E' be the exceptional divisors of π and π', respectively. Observe that $\varphi_*E = E'$. Pick A a general ample divisor on X and $0 < \varepsilon \ll 1$, and consider $H = \pi^*A - \varepsilon E$, $H' = \pi'^*A - \varepsilon E'$. Both H and H' are ample, and we have $H' = \varphi_*H$, so by Lemma 2.19 we conclude that $Y \dashrightarrow Y'$ is an isomorphism. □

Lemma 2.22. *Let $T \longrightarrow Y$ and $Y \longrightarrow X$ be two divisorial contractions between $\mathbf{Q}$-factorial varieties, with respective exceptional divisors E and F. Assume that there exists a morphism $X \longrightarrow B$ such that T/B is a Mori dream space. Then there exist two others $\mathbf{Q}$-factorial varieties T' and Y', with a pseudo-isomorphism $T \dashrightarrow T'$ and birational contractions $T' \longrightarrow Y' \longrightarrow X$, with respective exceptional divisors the strict transforms of F and E, such that the following diagram commutes:*



Proof. The diagram comes from the two-rays game associated to T/X. The only thing to prove is that the divisors are not contracted in the same order on the two sides of the two-rays game. Assume that both $\pi\colon Y \longrightarrow X$ and $\pi'\colon Y' \longrightarrow X$ contract the strict transforms of the same divisor F. Then $T \longrightarrow Y$ and $T' \longrightarrow Y'$ both contract a same divisor E and $T \dashrightarrow T'$ descends to a pseudo-isomorphism $Y \dashrightarrow Y'$. By Corollary 2.21 the pseudo-isomorphism $Y \dashrightarrow Y'$ is an isomorphism. Then applying again Corollary 2.21 to the two divisorial contractions from T, T' to $Y \simeq Y'$, with same exceptional divisor E, we obtain that $T \dashrightarrow T'$ also is an isomorphism. The morphisms T/Y and T/Y' are then divisorial contractions of the same extremal ray, contradicting the assumption that the diagram was produced by a two-rays game. □

2.G. **Gonality and covering gonality.** Recal that the *gonality* $\mathrm{gon}(C)$ of a (possibly singular) curve C is defined to be the least degree of the field extension associated to a dominant rational map $C \dashrightarrow \mathbb{P}^1$.

Note that $\mathrm{gon}(C) = 1$ if and only if C is rational. Moreover, for each smooth curve $C \subset \mathbb{P}^2$ of degree > 1 we have $\mathrm{gon}(C) = \deg(C) - 1$. Indeed, the inequality $\mathrm{gon}(C) \leqslant \deg(C) - 1$ is given by the projection from a general point of C and the other inequality is given by a result of Noether (see for instance [BDE+17]).

The following definitions are taken from [BDE+17] (with a slight change, see Remark 2.24).

Definition 2.23. For each variety X we define the *covering gonality* of X to be

$$\mathrm{cov.\,gon}(X) = \min\left\{ c > 0 \;\middle|\; \begin{array}{c}\text{There is a dense open subset } U \subseteq X \text{ such}\\ \text{that each point } x \in U \text{ is contained in an}\\ \text{irreducible curve } C \subseteq X \text{ with } \mathrm{gon}(C) \leqslant c.\end{array} \right\}.$$

Similarly we define the *connecting gonality* of X to be

$$\mathrm{conn.\,gon}(X) = \min\left\{ c > 0 \;\middle|\; \begin{array}{c}\text{There is a dense open subset } U \subseteq X \text{ such}\\ \text{that any two points } x, y \in U \text{ are contained}\\ \text{in an irreducible curve } C \subseteq X \text{ with}\\ \mathrm{gon}(C) \leqslant c.\end{array} \right\}.$$

Remark 2.24.

(1) Our definitions of the covering and connecting gonality slightly differ from those of [BDE+17], as we ask $\mathrm{gon}(C) \leqslant c$ where they ask $\mathrm{gon}(C) = c$. Lemma 2.26 shows that the covering gonality is the same for both definitions. A similar argument should also work for the connecting gonality, but we do not need it here, as we will not use any result of [BDE+17] involving the connecting gonality.

(2) The covering gonality and connecting gonality are integers which are invariant under birational maps.

(3) For each variety X, we have

$$\mathrm{cov.\,gon}(X) \leqslant \mathrm{conn.\,gon}(X).$$

Moreover, if $\dim(X) = 1$, then $\mathrm{cov.\,gon}(X) = \mathrm{conn.\,gon}(X) = \mathrm{gon}(X)$.

(4) If $\mathrm{cov.\,gon}(X) = 1$ one says that X is *uniruled*. This corresponds to asking that the union of all rational curves on X contains an open subset of X. Similarly, X is said to be *rationally connected* if $\mathrm{conn.\,gon}(X) = 1$. As already mentioned in §2.C, this corresponds to asking that a rational curve passes through two general points.

(5) Each rationally connected variety is uniruled. However, the converse does not hold in general. Indeed, for each variety B, we have $\mathrm{cov.\,gon}(B \times \mathbb{P}^n) = 1$ for each $n \geqslant 1$, but $\mathrm{conn.\,gon}(B \times \mathbb{P}^n) = \mathrm{conn.\,gon}(B)$ as the following lemma shows: Lemma 2.25(2) applied to $B \times \mathbb{P}^n / B$ gives $\mathrm{conn.\,gon}(B \times \mathbb{P}^n) \geqslant \mathrm{conn.\,gon}(B)$, and the other inequality is given by taking sections in $B \times \mathbb{P}^n$ of curves in B.

We recall the following classical facts:

Lemma 2.25. *Let X, Y be varieties and $\varphi\colon X \dashrightarrow Y$ a surjective morphism.*

(1) *If X and Y have dimension 1, then $\mathrm{gon}(X) \geqslant \mathrm{gon}(Y)$.*

(2) *We have $\mathrm{conn.\,gon}(X) \geqslant \mathrm{conn.\,gon}(Y)$ (but not $\mathrm{cov.\,gon}(X) \geqslant \mathrm{cov.\,gon}(Y)$ in general, see Remark 2.24(5)).*

(3) *If $\dim X = \dim Y$, denote by $\deg(\varphi)$ the degree of the associated field extension $\mathbf{C}(Y) \subseteq \mathbf{C}(X)$. Then*

$$\mathrm{cov.\,gon}(X) \leqslant \mathrm{cov.\,gon}(Y) \cdot \deg(\varphi).$$

(4) *If $X \subseteq \mathbb{P}^n$ is a closed subvariety, then $\mathrm{cov.\,gon}(X) \leqslant \deg(X)$.*

Proof. (1). See for instance [Poo07, Proposition A.1(vii)].

(2). We take two general points $y_1, y_2 \in Y$, choose then two general points $x_1, x_2 \in X$ with $\varphi(x_i) = y_i$ for $i = 1, 2$, and take an irreducible curve $C \subset X$ of gonality $\leqslant \mathrm{conn.\,gon}(X)$ which contains x_1 and x_2. The image is an irreducible curve $\varphi(C)$ of gonality $\leqslant \mathrm{conn.\,gon}(X)$ (by (1)), containing y_1 and y_2.

(3). By definition of $\mathrm{cov.\,gon}(Y)$, the union of irreducible curves C of Y with $\mathrm{gon}(C) \leqslant \mathrm{cov.\,gon}(Y)$ covers a dense open subset of Y. Taking the preimages of general such curves, we obtain a covering of a dense open subset of X by irreducible curves D of X with $\mathrm{gon}(D) \leqslant \mathrm{cov.\,gon}(Y) \cdot \deg(\varphi)$.

(4). If $X \subseteq \mathbb{P}^n$ is a closed subvariety, we apply (3) to the projection onto a general linear subspace $Y \subseteq \mathbb{P}^n$ of dimension $\dim(Y) = \dim(X)$. □

Lemma 2.26. *Let X be a variety with $\mathrm{cov.\,gon}(X) = c$. There is a smooth projective morphism $\mathcal{C} \to T$ over a quasi-projective irreducible base variety T, with irreducible fibres of dimension one and of gonality c, together with a dominant morphism $\mathcal{C} \dashrightarrow X$ such that a general fibre of $\mathcal{C}/T$ is birational to its image in X. In particular, there is a dense open subset U of X such that through every point $p \in U$ there is an irreducible curve $C \subseteq X$ with $\mathrm{gon}(C) = c$.*

Proof. The proof is analogue to the one of [GK17, Lemma 2.1]. We consider the Hilbert Scheme $\mathcal{H}$ of all one-dimensional subschemes of X, which is not of finite type, but has countably many components. One of the irreducible components

contains enough curves of gonality $\leqslant \mathrm{cov.gon}(X)$ to get a dominant map to X. We then look at the set of gonality i for each i and obtain algebraic varieties parametrising these, as in [GK17, Lemma 2.1]. Having finitely many constructible subsets in the image, at least one integer $i \leqslant \mathrm{cov.gon}(X)$ gives a dominant map to X parametrising curves of gonality i. By definition of $\mathrm{cov.gon}(X)$, this integer i has to be equal to $\mathrm{cov.gon}(X)$. □

The following result gives a bound from below that complements the easy bound from above from Lemma 2.25.

Theorem 2.27 ([BDE+17, Theorem A])**.** *Let $X \subset \mathbb{P}^{n+1}$ be an irreducible hypersurface of degree $d \geqslant n+2$ with canonical singularities. Then,* $\mathrm{cov.gon}(X) \geqslant d-n$.

We now recall the following definition of [BDE+17], which is a birational version of the classical p-very ampleness criterion, which asks that every subscheme of length $p+1$ imposes independent conditions on the sections of a line bundle.

Definition 2.28. Let X be variety and let $p \geqslant 0$ be an integer.

A line bundle L on X *satisfies property* BVA_p if there exists a proper Zariski-closed subset $Z = Z(L) \subsetneq X$ depending on L such that the restriction map $H^0(X, L) \longrightarrow H^0(X, L \otimes \mathcal{O}_\xi)$ is surjective for every finite subscheme $\xi \subset X$ of length $p+1$ whose support is disjoint from Z.

The line bundle is moreover *p-very ample* if one can choose Z to be empty.

This notion is related to the covering gonality via the following result:

Theorem 2.29 ([BDE+17, Theorem 1.10])**.** *Let X be a variety, and $p \geqslant 0$ an integer. If K_X satisfies* BVA_p*, then* $\mathrm{cov.gon}(X) \geqslant p+2$.

We will use the following observations of [BDE+17] to check the hypothesis of Theorem 2.29:

Lemma 2.30. *Let X be a variety, L a line bundle on X and $p \geqslant 0$ an integer.*

(1) *If L satisfies* BVA_p *and E is an effective divisor on X, then $\mathcal{O}_X(L+E)$ satisfies* BVA_p.

(2) *Suppose that $f\colon Y \longrightarrow X$ is a morphism which is birational onto its image, that L satisfies* BVA_p *and that the closed set $Z \subseteq X$ from Definition 2.28 does not contain the image of f. Then, f^*L satisfies* BVA_p.

(3) *For each $d \geqslant 0$, $\mathcal{O}_{\mathbb{P}^n}(p)$ is p-very ample, i.e. satisfies* BVA_p *with an empty closed set $Z \subseteq \mathbb{P}^n$.*

Proof. The three assertions follow from the definition of BVA_p, as mentioned in [BDE+17, Example 1.2]. □

3. Rank r fibrations and Sarkisov links

In this section we introduce the notion of rank r fibration, recovering the notion of Sarkisov link for $r = 2$. Then we focus on rank r fibrations and Sarkisov links with general fibre a curve.

3.A. Rank r fibrations.

The notion of rank r fibration is a key concept in this paper. Essentially these are (relative) Mori dream spaces with strong constraints on singularities. The cases of $r = 1, 2, 3$ will allow us to recover respectively the notion of terminal Mori fibre spaces, of Sarkisov links, and of elementary relations between those. The precise definition is as follows.

Definition 3.1. Let $r \geqslant 1$ be an integer. A morphism $\eta\colon X \longrightarrow B$ is a *rank r fibration* if the following conditions hold:

(RF1) X/B is a Mori dream space (see Definition 2.3);

(RF2) $\dim X > \dim B \geqslant 0$ and $\rho(X/B) = r$;

(RF3) X is $\mathbf{Q}$-factorial and terminal, and for any divisor D on X, the output of any D-MMP from X over B is still $\mathbf{Q}$-factorial and terminal (recall that such an output has the same dimension as X by definition, see §2.D);

(RF4) B is klt.

(RF5) The anticanonical divisor $-K_X$ is η-big (see Lemma 2.1).

We say that a rank r fibration X/B *factorises through* a rank r' fibration X'/B', or that X'/B' *is dominated by* X/B, if the fibrations X/B and X'/B' fit in a commutative diagram

$$\begin{array}{ccccc} X & & \dashrightarrow & & B \\ & \searrow & & \nearrow & \\ & & X' & \longrightarrow & B' \end{array}$$

where $X \dashrightarrow X'$ is a birational contraction, and $B' \longrightarrow B$ is a morphism with connected fibres. This implies $r \geqslant r'$.

Example 3.2.

(1) If X is a $\mathbf{Q}$-factorial terminal Fano variety of rank r, then X/pt is a rank r fibration. Indeed as already mentioned in Example 2.8, X is a Mori dream space, and moreover for any divisor D the output of a D-MMP is $\mathbf{Q}$-factorial and terminal. Both assertions follow from the fact that we can pick a small rational number $\varepsilon > 0$ such that $-K_X + \varepsilon D$ is ample, and then writing $\varepsilon D = K_X + (-K_X + \varepsilon D)$ we see that a D-MMP is also a $(K_X+$ ample$)$-MMP.

(2) Let p_1, p_2 be two distinct points on a fibre f of $\mathbb{P}^1 \times \mathbb{P}^1/\mathbb{P}^1$, and consider $S \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ the blow-up of p_1 and p_2. Then S is a weak del Pezzo toric surface of Picard rank 4, hence in particular S/pt is a Mori dream space. However S/pt is *not* a rank 4 fibration, because when contracting the strict transform of f one gets a singular point (hence non terminal as we work here with surfaces), which is forbidden by condition (RF3) of Definition 3.1.

Other basic examples are Mori fibre spaces:

Lemma 3.3. *Let $\eta\colon X \longrightarrow B$ a surjective morphism between normal varieties. Then X/B is a rank 1 fibration if and only if X/B is a terminal Mori fibre space.*

Proof. Observe that if $\rho(X/B) = 1$, the notions of η-ample and η-big are equivalent. So the implication

$$X/B \text{ is a rank 1 fibration} \implies X/B \text{ is a Mori fibre space}$$

is immediate from the definitions, and we need to check the converse.

Assume that X/B is a Mori fibre space. Then $\dim X > \dim B$ and $\rho(X/B) = 1$, which is (RF2), by Proposition 2.11 the base B is klt, which is (RF4), and $-K_X$ is η-ample, which gives (RF5).

We now prove that X/B is a Mori dream space, which is (RF1). Condition (MD1) holds by assumption. By Lemma 2.16(1) the general fibre of X/B is rationally connected with rational singularities, which gives (MD2). Moreover since $\rho(X/B) = 1$, we have $\mathrm{Ample}(X/B) = \mathrm{Nef}(X/B) = \overline{\mathrm{Mov}}(X/B)$ equal to a single ray, and so conditions (MD3) and (MD4) are immediate.

Finally we prove (RF3). By assumption X is terminal and $\mathbf{Q}$-factorial. For any divisor D, either D is η-nef and X/B is a D-minimal model, or $-D$ is η-ample and X/B is a D-Mori fibre space. So X is the only possible output for a D-MMP, which proves the claim. □

Lemma 3.4. *Let X/B be a rank r fibration.*

(1) *If X' is obtained from X by performing a log-flip (resp. a divisorial contraction) over B, then X'/B is a rank r fibration (resp. a rank $(r-1)$-fibration).*

(2) *Assume that X/B factorizes through a rank s fibration X'/B' such that the birational map $X \longrightarrow X'$ is a morphism. Let $t = \rho(X/B')$. Then X/B' is a rank t fibration.*

Proof. (1). Let $\pi\colon X \longrightarrow X'$ be a divisorial contraction over B, with exceptional divisor E (the case of a log-flip, which is similar and easier, is left to the reader).

(RF1). The general fibre of X'/B remains rationally connected, and is terminal by Lemma 2.15, so it remains to show that a Cox sheaf of X'/B is finitely generated (Lemma 2.7).

Let $L_1, \dots, L_p \in \mathrm{Pic}(X)_{\mathbf{Q}}$ and $L'_1, \dots, L'_q \in \mathrm{Pic}_{\mathbf{Q}}(X')$ such that $\mathrm{Eff}(X/B) \subseteq \sum \mathbf{R}_+ L_i$ and $\mathrm{Eff}(X'/B) \subseteq \sum \mathbf{R}_+ L'_i$. For each open set $U \subseteq B$, by pulling-back we get an injective morphism of algebras

$$\mathrm{Cox}(X'/B; L'_1, \dots, L'_q)(U) \hookrightarrow \mathrm{Cox}(X/B; E, \pi^* L'_1, \dots, \pi^* L'_q, L_1, \dots, L_p)(U).$$

Since X/B is a rank r fibration, its Cox sheaf is finitely generated by Lemma 2.7, and so $\mathrm{Cox}(X'/B; L'_1, \dots, L'_q)$ also is finitely generated by Lemma 2.6.

(RF2). By definition of a divisorial contraction we have $\dim X' = \dim X > \dim B$, and $\rho(X') = \rho(X) - 1$, so $\rho(X'/B) = r - 1$.

(RF3). The output of any MMP from X' also is the output of a MMP from X, and so is $\mathbf{Q}$-factorial and terminal by assumption.

(RF4) holds by assumption.

(RF5). Follows from the fact that the image of a big divisor by a birational morphism is still big.

(2). The conditions of (RF2) and (RF4) hold by assumption. (RF3) follows because any MMP over B' also is a MMP over B. For (RF5) we observe that a curve contracted by X/B' also is contracted by X/B, so a divisor relatively ample for X/B also is relatively ample for X/B'. Then we can restrict a decomposition $-K_X = \eta$-ample + effective for X/B to get a similar decomposition for X/B'.

Finally we show (RF1). Let $L_1, \dots, L_r$ be $\mathbf{Q}$-divisors on X such that $\mathrm{Eff}(X/B) \subseteq \sum \mathbf{R}_+ L_i$, which implies $\mathrm{Eff}(X/B') \subseteq \sum \mathbf{R}_+ L_i$. Let $\varphi\colon B' \longrightarrow B$ the morphism given by assumption. Then for each affine open set $U' \subset B'$, we have

$$\mathrm{Cox}(X/B'; L_1, \dots, L_r)(U') = \mathrm{Cox}(X/B; L_1, \dots, L_r)(\varphi(U')),$$

and the latter is finitely generated by assumption. A general fibre of X/B' is rationally connected because it is birational to a fibre of X'/B', and it has rational singularities by Lemma 2.15. We conclude by Lemma 2.7. □

Lemma 3.5. *Any rank r fibration X/B is pseudo-isomorphic, via a sequence of log-flips over B, to another rank r fibration Y/B such that $-K_Y$ is relatively nef and big over B.*

Proof. We run a $(-K)$-MMP from X over B (recall that by Lemma 2.9, one can run a D-MMP for an arbitrary divisor D). It is not possible to have a divisorial

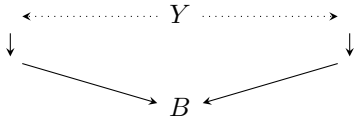
contraction, because by Lemma 2.10 the resulting singularity would not be terminal, in contradiction with assumption (RF3) in the definition of rank r fibration. If there exists an extremal class that gives a small contraction, we anti-flip it. After finitely many such steps, either $-K$ is relatively nef, or there exists a fibration such that K is relatively ample. But this last situation contradicts the assumption (RF5) that the anti-canonical divisor is big over B. So finally $-K$ is also relatively nef over B, as expected. □

Corollary 3.6. *Let $\eta\colon Y \longrightarrow B$ be a rank r fibration. Then for a general point $p \in B$, the fibre $Y_p := \eta^{-1}(p)$ is pseudo-isomorphic to a weak Fano terminal variety, and the curves in Y_p that are non-positive against the canonical divisor cover a subset of codimension at least 2 in Y_p.*

Proof. By Lemma 3.5, by performing a sequence of log-flips over B, which only affects a general fibre along a codimension 2 subset, we can assume that $-K_Y$ is relatively nef and big over B. The fact that Y_p is terminal is Lemma 2.15. Let $\Gamma \subset Y$ be the subset covered by curves contracted by Y/B that are trivial against the canonical divisor. Then consider the rational map $\varphi := |-mK_Y| \times \eta\colon Y \dashrightarrow \mathbb{P}^N \times B$. By [Kol93, Theorem 1.1], φ is a morphism, and it is a birational contraction onto its image. If Γ contains a divisor E, then E is contracted by φ, and by Lemma 2.10 this would produce a non-terminal singularity, in contradiction with the definition of rank r fibration. So Γ has codimension at least 2 in Y, hence $\Gamma_p = \Gamma \cap Y_p$ has codimension at least 2 in Y_p for a general p. Since by adjunction $K_{Y_p} = K_Y|_{Y_p}$, Γ_p is exactly the locus of contracted curves in Y_p with trivial intersection against K_{Y_p}. The fact that $-K_{Y_p}$ is big over B follows from Lemma 2.1, by restricting to Y_p a decomposition $-K_Y = \eta\text{-ample} + \text{effective}$. □

3.B. **Sarkisov links.** The notion of rank 2 fibration recovers the notion of Sarkisov link:

Lemma 3.7. *Let Y/B be a rank 2 fibration. Then Y/B factorises through exactly two rank 1 fibrations $X_1/B_1, X_2/B_2$, which both fit into a diagram*



where the top dotted arrows are sequences of log-flips, and the other four arrows are morphisms of relative Picard rank 1.

Proof. The diagram comes from the two-rays game associated to Y/B, as explained in §2.F. Morever, since $\dim Y > \dim B$, on each side of the diagram exactly one of the two descending arrows corresponds to a morphisms $X_i \longrightarrow B_i$ with $\dim Y = \dim X_i > \dim B_i$. If $B_i = B$ then X_i/B_i is a rank 1 fibration by Lemma 3.4(1). If $\rho(B_i/B) = 1$, we can use Lemma 3.4(2), or alternatively use the following simpler argument. Since $-K_{X_i}$ is relatively big over B we have $-K_{X_i} \cdot C > 0$ for a general contracted curve of X_i/B_i (write $-K_{X_i} = A + E$ with A relatively ample and E effective, and take C not contained in E.) So $-K_{X_i}$ is relatively ample over B_i, hence X_i/B_i is a Mori fibre space, or equivalently a rank 1 fibration (Lemma 3.3). □

Definition 3.8. In the situation of Lemma 3.7, we say that the birational map $\chi\colon X_1 \dashrightarrow X_2$ is a *Sarkisov link*. The diagram is called a *Sarkisov diagram*. Observe that a rank 2 fibration uniquely defines a Sarkisov diagram, but the Sarkisov link is only defined up to taking inverse.

If a rank r fibration factorises through Y/B, we equivalently say that it *factorises through* the Sarkisov link associated to Y/B.

We say that the Sarkisov link associated with a rank 2 fibration Y/B is a *Sarkisov link of conic bundles* if $\dim B = \dim X - 1$. Observe that in this situation both X_1/B_1 and X_2/B_2 are indeed conic bundles in the sense of Definition 2.13.

Definition 3.9. In the diagram of Lemma 3.7, there are two possibilities for the sequence of two morphisms on each side of the diagram: either the first arrow is already a Mori fibre space, or it is divisorial and in this case the second arrow is a Mori fibre space. This gives 4 possibilities, which correspond to the usual definition of *Sarkisov links of type* I, II, III *and* IV, as illustrated on Figure 1.

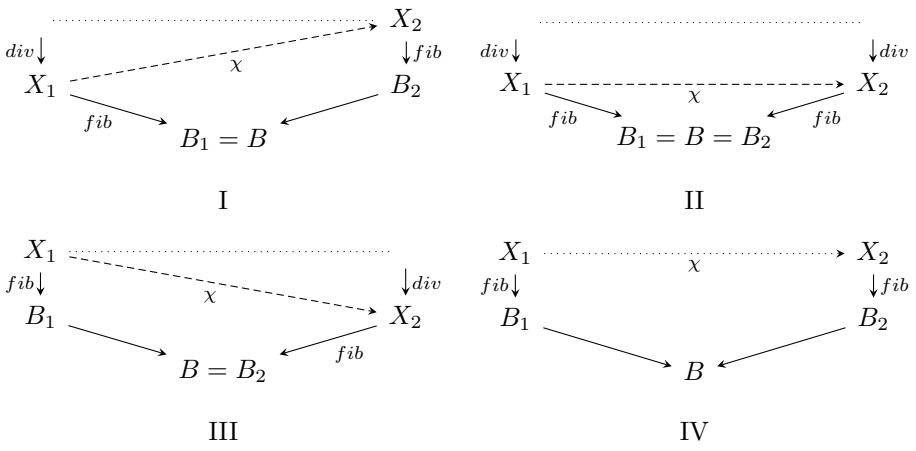
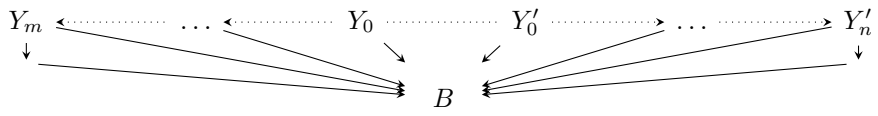


FIGURE 1. The four types of Sarkisov links.

Remark 3.10. The definition of a Sarkisov link in the literature is usually not very precise about the pseudo-isomorphism involved in the top row of the diagram. An exception is [CPR00, Definition 3.1.4(b)], but even there they do not make clear that there is at most one flop, and that all varieties admit morphisms to a common B.

In fact, it follows from the definition that there are strong constraints about the sequence of anti-flips, flops and flips (that is, about the sign of the intersection of the exceptional curves against the canonical divisor). Precisely, the top row of a Sarkisov diagram has the following form:



where $Y_0 \dashrightarrow Y_0'$ is a flop over B (or an isomorphism), $m, n \geqslant 0$, and each $Y_i \dashrightarrow Y_{i+1}$, $Y_i' \dashrightarrow Y_{i+1}'$ is a flip over B. This follows from the fact that for $Y = Y_i$ or Y_i', a general contracted curve C of the fibration Y/B satisfies $K_Y \cdot C < 0$, hence at least one of the two extremal rays of the cone $NE(Y/B)$ is strictly negative against K_Y.

Observe also that both Y_0/B and Y_0'/B are relatively weak Fano (or Fano if the flop is an isomorphism) over B, as predicted by Lemma 3.5. All other Y_i/B and Y_i'/B are not weak Fano over B, but still each is a rank 2 fibration that uniquely defines the Sarkisov diagram.

Example 3.11. We give some simple examples of Sarkisov links of each type in dimension 3. Here all varieties are smooth, and the pseudo-isomorphisms in the top rows of the Sarkisov diagrams are isomorphisms. For more complicated (and typical) examples, see §4.D. Observe that (1) and (2) are examples of Sarkisov links of conic bundles, while (3) and (4) are not.

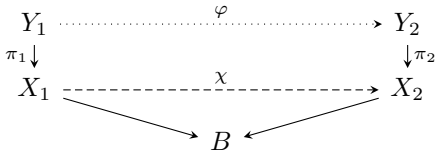
(1) Let $X_1/B_1 = \mathbb{P}^1 \times \mathbb{P}^2/\mathbb{P}^2$, and let $X_2 \longrightarrow X_1$ be the blow-up of one fibre. Then $X_2 = \mathbb{P}^1 \times \mathbb{F}_1$ is a Mori fibre space over the Hirzebruch surface $B_2 = \mathbb{F}_1$. The map $\chi\colon X_1/B_1 \dashrightarrow X_2/B_2$ is a link of type I, or equivalently $\chi^{-1}\colon X_2/B_2 \longrightarrow X_1/B_1$ is a link of type III.

(2) Take again $X_1/B_1 = \mathbb{P}^1 \times \mathbb{P}^2/\mathbb{P}^2$, let $L \subset \mathbb{P}^2$ be a line, and $\Gamma = \{0\} \times L \subset X_1$. Let $Y \longrightarrow X_1$ be the blow-up of Γ, and denote by D the strict transform on Y of $\mathbb{P}^1 \times L \subset X_1$. Then there is a divisorial contraction $Y \longrightarrow X_2$ that contracts D to a curve, and $X_2/\mathbb{P}^2$ is still a $\mathbb{P}^1$-bundle (but not a trivial product). The map $\chi\colon X_1/\mathbb{P}^2 \dashrightarrow X_2/\mathbb{P}^2$ is a link of type II.

(3) A general cubo-cubic map in $\mathrm{Bir}(\mathbb{P}^3)$ provides an example of link of type II with X_1, X_2 equal to $\mathbb{P}^3$ and $B_1 = B_2 = \mathrm{pt}$ a point. Indeed the resolution of such a map consists in blowing-up a smooth curve of genus 3 and degree 6 in X_1, and then contracting a divisor onto a curve of the same kind in X_2. This is the only example of a link of type II from $\mathbb{P}^3$ to $\mathbb{P}^3$ starting with the blow-up of a smooth curve where the pseudo-isomorphism is in fact an isomorphism: see [Kat87].

(4) Finally, take $X_1 = X_2 = \mathbb{P}^1 \times \mathbb{P}^2$, $B_1 = \mathbb{P}^1$, $B_2 = \mathbb{P}^2$, and let X_1/B_1 and X_2/B_2 be respectively the first and second projection. Then the identity map $\mathrm{id}\colon X_1/B_1 \longrightarrow X_2/B_2$ is a link of type IV.

Lemma 3.12. *Consider a Sarkisov link of type* II*:*



and denote E_1, E_2 the respective exceptional divisors of π_1, π_2. Then $\varphi_* E_1 \neq E_2$.

Proof. Assume that $\varphi_* E_1 = E_2$. Then $\chi\colon X_1 \dashrightarrow X_2$ is a pseudo-isomorphism, hence an isomorphism by Corollary 2.20. Then Corollary 2.21 implies that the pseudo-isomorphism $\varphi\colon Y_1 \dashrightarrow Y_2$ also is an isomorphism. The morphisms Y_1/X_1 and Y_1/X_2 are then divisorial contractions of the same extremal ray, contradicting the assumption that the diagram was the result of the two-rays game from Y_1/B. □

Lemma 3.13. *Let X/B be a rank 2 fibration that factorises through a rank 1 fibration $\sigma\colon X \longrightarrow B'$, with $\dim X - 1 = \dim B' > \dim B$. Then $\eta\colon B' \longrightarrow B$ is a klt*

Mori fibre space, and in particular for each $p \in B$, the fibre $\eta^{-1}(p)$ is covered by rational curves.

Proof. Recall that B' is klt by (RF4). We need to show that $-K_{B'}$ is η-ample, and then the fibre $\eta^{-1}(p)$ is covered by rational curves for each $p \in B$ by Lemma 2.16(1), applied with $\Delta = 0$.

By assumption $\rho(B'/B) = 1$, so we only need to show that there exists a contracted curve $C \subseteq B'$ such that $-K_{B'} \cdot C > 0$. Since $\dim B' > \dim B$, the contracted curves cover B', so we can choose C sufficiently general in a fibre $\eta^{-1}(q)$ of a general point $q \in B$ such that the following holds:

(*i*) C is not contained in the discriminant locus $\Delta' \subset B'$ of the conic bundle $\sigma\colon X \longrightarrow B'$;

(*ii*) The surface $\sigma^{-1}(C)$ does not contain any of the curves $C' \subseteq X$ contracted by $\eta \circ \sigma$ with $-K_X \cdot C' \leqslant 0$.

(*iii*) The fibre $F = (\eta \circ \sigma)^{-1}(q)$ of $\eta \circ \sigma\colon X \longrightarrow B$ containing the surface $\sigma^{-1}(C)$ is general, so that $(-K_X)|_F$ is big

More precisely, for (*i*) is suffices to choose $\eta^{-1}(q)$ not contained in the hypersurface $\Delta' \subset B'$. We can ensure (*ii*) because by Corollary 3.6 such curves cover at most a codimension 2 subset of F. Finally for (*iii*) recall first that since X/B is a rank 2 fibration, $-K_X$ is relatively big by (RF5). Moreover the intersection $(-K_X)|_F \cdot \sigma^{-1}(C)$ is a non-trivial effective 1-cycle. Indeed, since $(-K_X)|_F$ is big, we can take a large integer $m > 0$ and find that $(-mK_X)|_F$ induces a rational morphism contracting no curve on the complement of a divisor of F. It suffices then to choose C such that $\sigma^{-1}(C)$ is not contained in this divisor.

As in [MM85, Corollary 4.6], we have $-4K_{B'} \equiv \sigma_*(-K_X)^2 + \Delta$. Intersecting with C, we obtain

$$\begin{aligned} -4K_{B'} \cdot C &= \sigma_*(-K_X)^2 \cdot C + \Delta \cdot C \\ &\geqslant (-K_X)|_F \cdot (-K_X)|_F \cdot \sigma^* C \\ &> 0 \text{ by our choice of } C. \end{aligned}$$

□

3.C. **Rank r fibrations with general fibre a curve.** Let $\eta\colon T \longrightarrow B$ be a rank r fibration, with $\dim B = \dim T - 1$. If $\Gamma \subset B$ is an irreducible hypersurface, we define $\eta^\sharp(\Gamma) \subseteq T$ to be the Zariski closure of all fibres of dimension 1 over Γ. The reason for introducing this notion is twofold: first B might not be $\mathbf{Q}$-factorial, so we cannot consider the pull-back of Γ as a $\mathbf{Q}$-Cartier divisor, and second the preimage $\eta^{-1}(\Gamma)$ might contain superfluous components (see Example 3.15).

Now we distinguish two classes of special divisors in T, and we shall show in Proposition 3.16 below that they account for the relative rank of T/B. Let $D \subset T$ be a prime divisor. If $\eta(D)$ has codimension at least 2 in B, we say that D is a *divisor of type* I. If $\eta(D)$ is a divisor in B, and the inclusion $D \subsetneq \eta^\sharp(\eta(D))$ is strict, we say that D is a *divisor of type* II.

Remark 3.14. The similarity between the terminology for Sarkisov links and for special divisors of type I or II is intentional. See Lemma 3.19(2) below.

Example 3.15. We give an example illustrating the definitions above, which also shows that the inclusion $\eta^\sharp(\Gamma) \subseteq \eta^{-1}(\Gamma)$ might be strict. For B an arbitrary smooth variety, consider $Y = \mathbb{P}^1 \times B$ with Y/B the second projection. Let $\Gamma \subset B$ be any irreducible smooth divisor, $D = \mathbb{P}^1 \times \Gamma$ the pull-back of Γ in Y, $\Gamma' = \{t\} \times \Gamma \subset D$

a section and $p \in D \smallsetminus \Gamma'$ a point. Let $T \longrightarrow Y$ be the blow-up of Γ' and p, with respective exceptional divisors D' and E, and denote again D the strict transform of $\mathbb{P}^1 \times \Gamma$ in T. Then one can check that the induced morphism $\eta\colon T \longrightarrow B$ is a rank 3 fibration (see Example 4.34 for the case $B = \mathbb{P}^2$), E is a divisor of type I, $D \cup D'$ is a pair of divisors of type II, and

$$\eta^\sharp(\Gamma) = D \cup D' \subsetneq D \cup D' \cup E = \eta^{-1}(\Gamma).$$

Proposition 3.16. *Let $\eta\colon T \longrightarrow B$ be a rank r fibration, with $\dim B = \dim T - 1$.*

(1) *For any rank r' fibration T'/B' such that T/B factorises through T'/B', any divisor contracted by the birational contraction $T \dashrightarrow T'$ is a divisor of type* I *or* II *for T/B.*

(2) *Divisors of type* II *always come in pairs: for each divisor D_1 of type* II*, there exists another divisor D_2 of type* II *such that*

$$D_1 \cup D_2 = \eta^\sharp(\eta(D_1)) = \eta^\sharp(\eta(D_2)).$$

(3) *If $D_1 \cup D_2$ is a pair of divisors of type* II*, and p a general point of $\eta(D_1) = \eta(D_2)$, then $\eta^{-1}(p) = f_1 \cup f_2$ with $f_i \subseteq D_i$, $i = 1, 2$, some smooth rational curves satisfying*

$$K_T \cdot f_i = -1, \qquad D_i \cdot f_i = -1, \qquad D_1 \cdot f_2 = D_2 \cdot f_1 = 1.$$

(4) *Let $D \subset T$ be a divisor of type* I *or* II*. Then there exists a birational contraction over B*

$$T \dashrightarrow X \longrightarrow B$$

that contracts D and such that $\rho(X) = \rho(T) - 1$.

(5) *Assume B is $\mathbf{Q}$-factorial. Let d_1 (resp. d_2) be the number of divisors of type* I *(resp. the number of pairs of divisors of type* II*). Then*

$$r = 1 + d_1 + d_2.$$

Proof. (1). Let D be a prime divisor contracted by $T \dashrightarrow T'$ and suppose that it is neither of type I nor of type II for T/B. By running a D-MMP over B we produce a sequence of log-flips (which do not change the type of special divisors) and then a divisorial contraction. Replacing T by the result of the sequence of log-flips, and T' by the image of the divisorial contraction, we can assume $T \longrightarrow T'$ is a divisorial contraction. By Lemma 2.14, a general fibre f in the exceptional divisor D is a smooth irreducible rational curve. Now $\eta(D) \subset B$ is a divisor because D is not of type I, and $D = \eta^\sharp(\eta(D))$ because D is a prime divisor and is not of type II. But then $f = \eta^{-1}(p)$ for some $p \in \eta(D)$, so f is proportional to a general fibre of η, in contradiction with the fact that the extremal contraction of f is divisorial.

(2) and (3). Let D_1 be a divisor of type II, and let $D_2, \dots, D_s$ be the other divisors of type II such that

$$\eta^\sharp(\eta(D_1)) = D_1 \cup \dots \cup D_s.$$

By definition we have $s \geqslant 2$, and want to prove $s = 2$. By definition of $\eta^\sharp$, each D_i is a hypersurface and the general fibres of $D_i \longrightarrow \eta(D_i)$ are curves. Hence, the image of D_i is a hypersurface in B. In particular, $\eta(D_i) = \eta(D_j)$ for all $i, j \in \{1, \dots, s\}$. Let $p \in \eta(D_1)$ be a general point, and write $\eta^{-1}(p) = f_1 \cup \dots \cup f_s$ with f_i a curve in D_i. Let f be a general fibre of η. We have $D_i \cdot f = 0$ for each i, $D_i \cdot f_j > 0$ for at least one j (because $\eta^{-1}(p)$ is connected) and $f \equiv f_1 + \dots + f_s$, which gives

$$D_i \cdot f_i < 0.$$

Then by running a D_i-MMP from T over B, one obtains a sequence of log-flips that does not affect the general fibre $\eta^{-1}(p)$, and then a divisorial contraction between $\mathbf{Q}$-factorial and terminal varieties, with exceptional divisor D_i and center of codimension 2. By Lemma 2.14, this implies that f_i is smooth with $K_T \cdot f_i = D_i \cdot f_i = -1$. But $K_T \cdot f = -2$, so we conclude that $s = 2$ as expected. The equality $D_1 \cdot f_2 = 1$ (and similarly $D_2 \cdot f_1 = 1$) follows from $D_1 \cdot f = 0$, $f \equiv f_1 + f_2$ and $D_1 \cdot f_1 = -1$.

To prove (4), it suffices to show that the divisor D is covered by curves ℓ such that $D \cdot \ell < 0$, since then we can get the expected birational contraction by running a D-MMP. When D has type II we already showed in (3) that D is covered by such curves. Now let D be a divisor of type I, p a general point in $\eta(D)$, and let $d \geqslant 0$ be the dimension of $\eta(D)$. By definition of a divisor of type I we have $n - 3 \geqslant d$, where $n = \dim T$ (In particular divisors of type I do not exist on surfaces). Now consider the surface $S \subset T$ obtained by taking the following intersection of $n-2$ divisors:

$$S = \bigcap_{i=1}^{n-2-d} H_i \cap \bigcap_{j=1}^{d} \eta^* H'_j,$$

where the H_i are general hyperplane sections of T, and the H'_j general hyperplane sections of B through p. By construction, one of the irreducible components of $S \cap D$ is a curve ℓ contracted to p by η. Moreover $\eta(S)$ is a surface; indeed the general fibres of η are curves and H_1 is transverse to the fibres so the morphism from H_1 to $\eta(H_1)$ has finite fibres. The same then holds for S, as $n-2-d \geqslant 1$. We obtain $D \cdot \ell = (\ell \cdot \ell)_S < 0$ as expected (a curve contracted by a morphism between two surfaces has negative self-intersection).

To prove (5), observe first that the contraction of a divisor of type I does not affect the other divisors of type I or II, and the contraction of a divisor of type II only affects the other divisor in the pair, which is not a divisor of type II anymore. So after applying several times (4), we may assume $d_1 = d_2 = 0$, and we want to show $r = 1$, or equivalently, that T/B is a Mori fibre space. Then we run a MMP from T over B. A flip does not change d_1 nor d_2, so we can assume that we have a divisorial contraction or a Mori fibre space. A divisorial contraction would produce a divisor of type I or II by (1) (depending on the codimension of the centre), in contradiction with our assumption $d_1 = d_2 = 0$. On the other hand, if $T \longrightarrow B'$ is a Mori fibre space, then both B' and B are $(n-1)$ dimensional varieties, and B' is $\mathbf{Q}$-factorial klt by Proposition 2.11. If the birational morphism $B' \longrightarrow B$ is not an isomorphism, it must contracts at least one divisor D because B is $\mathbf{Q}$-factorial by assumption. By Lemma 2.17 B'/B is a Mori dream space, so we can run a D-MMP from B' over B. After a sequence of D-flips this has to produce a divisorial contraction, hence a divisor of type I in T by pulling-back, and again a contradiction. In conclusion, $B' \xrightarrow{\sim} B$ is an isomorphism and we have a Mori fibre space T/B, as expected. □

Lemma 3.17. *Let $\eta\colon T \longrightarrow B$ be a rank r fibration with $\dim B = \dim T - 1$. Assume that D is a divisor of type* II *for T/B, with $\mathrm{cov.gon}(\eta(D)) > 1$. Then for any rank r' fibration T'/B' that factorises through T/B, with $\dim B' = \dim T' - 1 = \dim B$, the strict transform of D is a divisor of type* II *for T'/B'.*

Proof. Recall that $T' \dashrightarrow T$ is a birational contraction and $\pi\colon B \longrightarrow B'$ is a morphism with connected fibres between klt varieties (Definition 3.1), which in our

situation is birational as $\dim(B) = \dim(B')$. We write $D = D_1$ and by Proposition 3.16(2) we have a pair $D_1 \cup D_2$ of divisors of type II for T/B, where $\Gamma = \eta(D_1) = \eta(D_2)$ is a divisor of B and $D_1 \cup D_2 = \eta^\sharp(\Gamma)$.

We first observe that the image of Γ in B' is again a divisor $\Gamma' \subset B'$. Indeed otherwise, the divisor $\Gamma \subset B$ is one of the divisors contracted by the birational morphism $\pi\colon B \longrightarrow B'$. By Lemma 2.16(2), this implies that Γ is covered by rational curves, in contradiction with our assumption $\mathrm{cov.\,gon}(\Gamma) > 1$.

Writing $\eta'\colon T' \longrightarrow B'$ the rank r' fibration, one observe that the strict transforms $\tilde{D}_1$ and $\tilde{D}_2$ of D_1 and D_2 are such that $\tilde{D}_1 \cup \tilde{D}_2 \subseteq \eta^\sharp(\Gamma')$. Hence, $\tilde{D}_1$ and $\tilde{D}_2$ are divisors of type II for T'/B'. □

Lemma 3.18. *Let T/B be a rank r fibration with $\dim B = \dim T - 1$ and B **Q**-factorial. Assume that for each divisor D of type* II *for T/B, we have*

$$\mathrm{cov.\,gon}(\eta(D)) > 1.$$

Then T/B factorises through a rank 1 fibration T'/B' such that $T \dashrightarrow T'$ is a pseudo-isomorphism if and only if T/B does not admit any divisor of type II.

If this holds, then $\dim B' = \dim T - 1$, $B' \longrightarrow B$ is a birational morphism and $\rho(B'/B) = r - 1$.

Proof. If T/B factorises through a rank r' fibration T'/B' such that $T \dashrightarrow T'$ is a pseudo-isomorphism, first observe that $\rho(B'/B) = r - r'$, and $B' \longrightarrow B$ is birational, since $\dim(B) = \dim(B')$, which follows from

$$\dim(T) = \dim(T') > \dim(B') \geqslant \dim(B) = \dim(T) - 1.$$

If $D_1 \cup D_2$ is a pair of divisors of type II for T/B, then their strict transforms $\tilde{D}_1, \tilde{D}_2$ have the same image in B', which is a divisor because $B' \longrightarrow B$ is birational. So if T/B admits at least one divisor of type II, then by Proposition 3.16(3) some fibres of T'/B' have the form $f_1 + f_2$ with f_1, f_2 non proportional. In particular $r' = \rho(T'/B') \geqslant 2$ and so T'/B' is not a Mori fibre space.

To prove the converse, we assume that T/B does not admit any divisor of type II, and we proceed by induction on the number d_1 of divisors of type I. If $d_1 = 0$ then by Proposition 3.16(5), T/B is already a rank 1 fibration, so we just take $T'/B' = T/B$. Now if $d_1 > 0$, by Proposition 3.16(4) there exists a birational contraction over B, $T \dashrightarrow X_1 \longrightarrow B$, which contracts one divisor D of type I. Since the contraction is obtained by running a D-MMP, in fact it factorises as $T \dashrightarrow T_1 \longrightarrow X_1$, where $T \dashrightarrow T_1$ is a sequence of D-flips and $T_1 \longrightarrow X_1$ is a divisorial contraction. Then by induction hypothesis X_1/B factorises through a rank 1 fibration X_2/B_2 with $X_1 \dashrightarrow X_2$ a pseudo-isomorphism (here we use Lemma 3.17, which shows that X_1/B does not admit any divisor of type II). By Lemma 2.18, there exist a pseudo-isomorphism $T_1 \dashrightarrow T_2$ and a divisorial contraction $T_2 \longrightarrow X_2$ that makes the diagram on Figure 2 commute. Finally we play the two-rays game T_2/B_2. Since T_2/B_2 admits one divisor of type I and no divisor of type II (by our assumption on the covering gonality and by Lemma 3.17), the other side of the two-rays game must be a Mori fibre space, which gives the expected rank 1 fibration T'/B'. □

3.D. **Sarkisov links of conic bundles.** In this subsection, by applying Proposition 3.16 to the case $r = 2$, we classify Sarkisov links of conic bundles.

Lemma 3.19. *Let Y/B be a rank 2 fibration with $\dim B = \dim Y - 1$, and χ the associated Sarkisov link, well-defined up to taking inverse.*

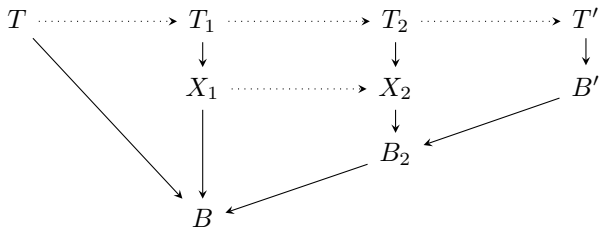


FIGURE 2

(1) *χ has type* IV *if and only if B is not* **Q***-factorial.*

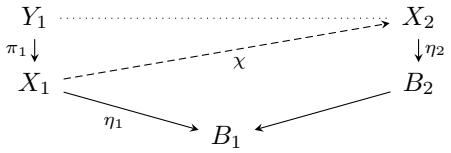
(2) *If B is* **Q***-factorial, let d_1 (resp. d_2) be the number of special divisors of type* I (*resp. of type* II) *for Y/B. Then*

- *χ has type* I *or* III *if and only if $(d_1, d_2) = (1, 0)$.*
- *χ has type* II *if and only if $(d_1, d_2) = (0, 1)$.*

Proof. (1). If B is not **Q**-factorial, then it follows directly that χ has type IV, from the fact that the base of a terminal Mori fibre space if always **Q**-factorial (Proposition 2.11), and by inspection of the diagrams in Figure 1. Conversely, assuming that $\chi\colon X_1/B_1 \dashrightarrow X_2/B_2$ is a link of type IV, we show that B is not **Q**-factorial. As $\dim B = \dim Y - 1$, the morphisms B_1/B, B_2/B are birational. If B is **Q**-factorial, then B_1/B and B_2/B are birational contractions with respective exceptional divisors E_1 and E_2. If the birational map $B_1 \dashrightarrow B_2$ sends E_1 onto E_2, then the map is a pseudo-isomorphism, hence an isomorphism by Corollary 2.21, and then $X_1 \dashrightarrow X_2$ also is an isomorphism by Corollary 2.20, a contradiction. Otherwise, the pull-backs of E_1, E_2 together with the choice of any ample divisor give three independent classes in $N^1(Y/B)$, in contradiction with $\rho(Y/B) = 2$.

To prove (2), first we observe that Proposition 3.16(5) gives $d_1 + d_2 = 1$, hence the two possibilities $(d_1, d_2) = (1, 0)$ or $(0, 1)$. Recall also from Proposition 3.16(1) that any divisor contracted by a birational contraction from Y over B must be of type I or II. If the link χ is of type II, then Lemma 3.12 gives two birational contractions from Y contracting distinct prime divisors, and this is possible only in the case $(d_1, d_2) = (0, 1)$ where there is a pair of divisors of type II available. Conversely, if $(d_1, d_2) = (0, 1)$, we have two distinct prime divisors, that we can contract via two distinct birational contractions (Proposition 3.16(4)). These are the two starting moves of a 2-ray game which provides a link of type II. □

Corollary 3.20. *Let χ be a Sarkisov link of conic bundles of type* I*:*



Let E_1 be the exceptional divisor of the divisorial contraction π_1. Then $\eta_1 \circ \pi_1(E_1)$ has codimension at least 2 *in B_1.*

Proof. Follows from the fact that E_1 is a divisor of type I for Y_1/B_1. □

Remark 3.21. There are examples of link of type IV as in Lemma 3.19(1) only when $\dim B \geqslant 3$, hence $\dim Y \geqslant 4$. See the discussion on the two subtypes of type IV links in [HM13, p. 391 after Theorem 1.5]. For instance, take B_1 and B_2 that differ by a log-flip, and B the non $\mathbf{Q}$-factorial target of the associated small contractions. Then the birational map from $(\mathbb{P}^1 \times B_1)/B_1$ to $(\mathbb{P}^1 \times B_2)/B_2$ induced by the log-flip is a link of type IV.

Now we focus on the case of Sarkisov links of conic bundles of type II. First we introduce the following definition.

Definition 3.22. A *marked conic bundle* is a triple $(X/B, \Gamma)$, where X/B is a conic bundle in the sense of Definition 2.13, and $\Gamma \subset B$ is an irreducible hypersurface, not contained in the discriminant locus of X/B (i.e. the fibre of a general point of Γ is isomorphic to $\mathbb{P}^1$). The *marking* of the marked conic bundle is defined to be Γ.

We say that two marked conic bundles $(X/B, \Gamma)$, and $(X'/B', \Gamma')$ are *equivalent* if there exists a commutative diagram

$$\begin{array}{ccc} X & \overset{\psi}{\dashrightarrow} & X' \\ \downarrow & & \downarrow \\ B & \overset{\theta}{\dashrightarrow} & B' \end{array}$$

where ψ, θ are birational and such that the restriction of θ induces a birational map $\Gamma \dashrightarrow \Gamma'$ between the markings. In particular, if $(X/B, \Gamma)$, and $(X'/B', \Gamma')$ are equivalent, then the conic bundles X/B and X'/B' are equivalent in the sense of Definition 2.13.

For each variety Z, we denote by $\mathsf{CB}(Z)$ the set of equivalence classes of conic bundles X/B with X birational to Z and denote, for each class of conic bundles $C \in \mathsf{CB}(Z)$ by $\mathsf{M}(C)$ the set of equivalence classes of marked conic bundles $(X/B, \Gamma)$ where C is the class of X/B.

The next lemma explains how a Sarkisov link of conic bundles of type II gives rise to an equivalence class of marked conic bundles.

Lemma 3.23. *Let χ be a Sarkisov link of conic bundles of type* II *between varieties of dimension $n \geqslant 2$. Recall that χ fits in a commutative diagram of the form*

$$\begin{array}{ccccc} Y_1 & & \overset{\varphi}{\cdots\cdots\cdots\cdots\cdots\cdots} & & Y_2 \\ {\scriptstyle \pi_1}\downarrow & & & & \downarrow{\scriptstyle \pi_2} \\ X_1 & & \overset{\chi}{\dashrightarrow\dashrightarrow\dashrightarrow\dashrightarrow\dashrightarrow\dashrightarrow} & & X_2 \\ & {\scriptstyle \eta_1}\searrow & & \swarrow{\scriptstyle \eta_2} & \\ & & B & & \end{array}$$

where X_1, X_2, Y_1, Y_2 are $\mathbf{Q}$-factorial terminal varieties of dimension n, B is a $\mathbf{Q}$-factorial klt variety of dimension $n-1$, φ is a sequence of log-flips over B, and each π_i is a divisorial contraction with exceptional divisor $E_i \subset Y_i$ and centre $\Gamma_i = \pi_i(E_i) \subset X_i$.

Then there exists an irreducible hypersurface $\Gamma \subset B$ (of dimension $n-2$) such that

(1) *for $i = 1, 2$, the centre $\Gamma_i = \pi_i(E_i)$ has codimension 2 in X_i, and the restriction $\eta_i|_{\Gamma_i} : \Gamma_i \longrightarrow \Gamma$ is birational. In particular, for each i we have $\eta_i \circ \pi_i(E_i) = \Gamma$, and the marked conic bundles $(X_1/B, \Gamma)$ and $(X_2/B, \Gamma)$ are equivalent.*

(2) *Let Y be equal to Y_1, Y_2, or any one of the intermediate varieties in the sequence of log-flips φ. Then $E_1 \cup E_2$ is a pair of divisors of type* II *for Y/B.*

(3) Γ *is not contained in the discriminant locus of* η_1*, or equivalently of* η_2*, which means that a general fibre of* $\eta_i\colon \eta_i^{-1}(\Gamma)\longrightarrow\Gamma$ *is isomorphic to* $\mathbb{P}^1$*.*

(4) *At a general point* $x\in\Gamma_i$*, the fibre of* X_i/B *through* x *is transverse to* Γ_i*.*

Proof. (1) and (2). By Lemma 3.19, Y_1/B admits no divisor of type I, and exactly one pair of divisors of type II. By Lemma 3.12 we have $\varphi_*E_1\neq E_2$, so the birational contractions $Y_1\dashrightarrow X_1$ and $Y_1\dashrightarrow X_2$ contract distinct divisors. It follows from Proposition 3.16 that the pair of divisors of type II is $E_1\cup E_2$. So by definition E_1 and E_2 projects to the same hypersurface $\Gamma\subset B$. By Proposition 3.16(3) both finite maps $\Gamma_i\longrightarrow\Gamma$ are birational, otherwise the fibre in Y_i over a general point of Γ would have more than two components.

(3) and (4) follow from Proposition 3.16(3). Indeed if Γ was in the discriminant locus of η_1 then the preimage in Y_1 of a general point $p\in B$ would have 3 irreducible components, instead of 2. Moreover writing $f_1\cup f_2$ the fibre through x, with $f_i\subseteq E_i$, the fact that the fibre is transverse to Γ_i is equivalent to $f_1\cdot E_2=f_2\cdot E_1=1$. □

Definition 3.24. By Lemma 3.23(1), to each Sarkisov link of conic bundles of type II $\chi\colon X_1\dashrightarrow X_2$, we can associate the equivalence class of the marked conic bundle $(X_1/B,\Gamma)$ given in this lemma. We define the *marking* of χ to be $\Gamma\subset B$. We say that two Sarkisov links of conic bundles of type II are *equivalent* if their corresponding marked conic bundles are equivalent.

We also extend the notion of covering gonality (see §2.G) to Sarkisov links of conic bundles of type II.

Definition 3.25. Let χ be a Sarkisov of conic bundles of type II between varieties of dimension $n\geqslant 3$. We define $\operatorname{cov.gon}(\chi)$ to be $\operatorname{cov.gon}(\Gamma)$, where Γ is the marking of χ.

Remark 3.26. If two Sarkisov links of conic bundles of type II are equivalent, then their markings are birational to each other. In particular the number $\operatorname{cov.gon}(\chi)$ only depends on the equivalence class of χ.

The above definition makes sense if the varieties X_i have dimension $\geqslant 2$, but it is not a very good invariant if the dimension is 2, as the centre is always a point, and there is only one class of marked conic bundles, given by a point in the base of a Hirzebruch surface. However, the analogue definition over $\mathbf{Q}$ or over a finite field, instead of over $\mathbf{C}$, is interesting even for surfaces.

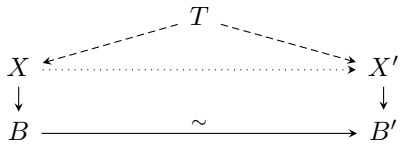
4. Relations between Sarkisov links

The fact that one can give a definition of Sarkisov links in terms of relative Mori dream spaces of Picard rank 2 as in the previous section was independently observed in [AZ16, §2] and [LZ17, §2.3]. Our next aim is to extend this observation to associate some relations between Sarkisov links to each rank 3 fibration. First we define elementary relations, and then we relate this notion to the work of A.-S. Kaloghiros about relations in the Sarkisov programme.

4.A. **Elementary relations.**

Definition 4.1. Let X/B and X'/B' be two rank r fibrations, and $T\dashrightarrow X$, $T\dashrightarrow X'$ two birational maps from the same variety T. We say that X/B and

X'/B' are T-*equivalent* (the birational maps being implicit) if there exist a pseudo-isomorphism $X \dashrightarrow X'$ and an isomorphism $B \xrightarrow{\sim} B'$ such that all these maps fit in a commutative diagram:



Lemma 4.2. *Let X_3/B_3 be a rank 3 fibration that factorises through a rank 1 fibration X_1/B_1. Then up to X_3-equivalence there exist exactly two rank 2 fibrations that factorise through X_1/B_1, and that are dominated by X_3/B_3.*

Proof. We distinguish three cases according to $\rho(B_1/B_3)$.

If $\rho(B_1/B_3) = 2$, then B_1 – being the base of a Mori fibre space – is $\mathbf{Q}$-factorial klt (Proposition 2.11), and B_1/B_3 is a Mori dream space by Lemma 2.17. The associated two-rays game yields exactly two non-isomorphic B_2, B_2' with $\rho(B_2/B_3) = \rho(B_2'/B_3) = 1$. Then Lemma 2.18 provides sequences of log-flips over B_3, $X_1 \dashrightarrow X_2$ and $X_1 \dashrightarrow X_2'$, such that X_2/B_2, X_2'/B_2' are the expected rank 2 fibrations.

If $\rho(B_1/B_3) = 1$, then the base B_2 of any of the expected rank 2 fibrations must be equal to B_1 or B_3, because by assumption we have morphisms $B_1 \longrightarrow B_2 \longrightarrow B_3$. By Lemma 3.4(1) X_1/B_3 is the first expected rank 2 fibration, and up to equivalence it is the only one with base B_3, because any rank 2 fibration X_2/B_3 satisfies $\rho(X_2) = \rho(X_1)$, so the birational contraction $X_2 \dashrightarrow X_1$ is a pseudo-isomorphism. Let D be the pull-back on X_3 of an ample divisor on X_1. The birational contraction $X_3 \dashrightarrow X_1$ is a D-MMP over B_3, and as $\rho(X_3) - \rho(X_1) = 1$, it decomposes as a sequence of D-flips $X_3 \dashrightarrow X_3'$, a divisorial contraction $X_3' \longrightarrow X_1'$, and a sequence of D-flips $X_1' \dashrightarrow X_1$. Then Lemma 2.18 provides a sequence of log-flips over B_3, $X_3' \dashrightarrow X_2$, such that $X_2 \longrightarrow X_1$ is a divisorial contraction, and by Lemma 3.4 X_2/B_1 is the second expected rank 2 fibrations. Any other rank 2 fibration X_2'/B_1 satisfying the lemma is equivalent to X_2/B_1, because as before the condition on Picard numbers forces $X_2 \dashrightarrow X_2'$ to be a pseudo-isomorphism.

If $\rho(B_1/B_3) = 0$, then $\rho(X_3) - \rho(X_1) = 2$, and $B_1 = B_3$ must be the base of any of the expected rank 2 fibrations. By applying several times Lemma 2.18 we construct a sequence of log-flips over B_3, $X_3 \dashrightarrow X_3'$, such that $X_3' \longrightarrow X_1$ is a morphism. The associated two-rays game yields exactly two divisorial contractions $X_2 \longrightarrow X_1$ and $X_2' \longrightarrow X_1$. Moreover X_2 and X_2' are not pseudo-isomorphic by Lemma 2.22, and are uniquely determined up to equivalence by Corollary 2.21. Then X_2/B_1 and X_2'/B_1 are the expected rank 2 fibrations. □

Proposition 4.3. *Let T/B be a rank 3 fibration. Then there are only finitely many Sarkisov links χ_i dominated by T/B, and they fit in a relation*

$$\chi_t \circ \cdots \circ \chi_1 = \mathrm{id}.$$

Proof. Since T/B is a Mori dream space, by Lemma 2.9 there are only finitely many rank 1 or 2 fibrations dominated by T/B. We construct a bicolored graph Γ as follows. Vertices are rank 1 or 2 fibrations dominated by T/B, up to T-equivalence, and we put an edge between X_2/B_2 and X_1/B_1 if X_2/B_2 is a rank 2 fibration that factorises through the rank 1 fibration X_1/B_1. By construction, two vertices of rank 1 of Γ are at distance 2 if and only if there is a Sarkisov link

between them. Then by Lemmas 3.7 and 4.2 we obtain that Γ is a circular graph, giving the expected relation. □

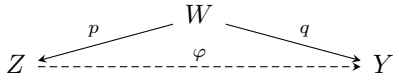
Definition 4.4. In the situation of Proposition 4.3, we say that

$$\chi_t \circ \cdots \circ \chi_1 = \mathrm{id}$$

is an *elementary relation* between Sarkisov links, coming from the rank 3 fibration T/B. Observe that the elementary relation is uniquely defined by T/B, up to taking the inverse, cyclic permutations and insertion of isomorphisms.

4.B. **Geography of ample models.** In this section, we recall some preliminary material from [BCHM10, HM13]. The aim is to explain the construction of a polyhedral complex attached with the choice of some ample divisors on a smooth variety, and to state some properties (Proposition 4.10 and Lemma 4.15) that we will use in the next section to understand relations between Sarkisov links.

Definition 4.5 ([BCHM10, Definition 3.6.5])**.** Let Z be a terminal $\mathbf{Q}$-factorial variety, D be a $\mathbf{R}$-divisor on Z and $\varphi\colon Z \dashrightarrow Y$ a dominant rational map to a normal variety Y. We take a resolution



where W is smooth, p is a birational morphism and q a morphism with connected fibres. We say that φ *is an ample model of* D if there exists an ample divisor H on Y such that p^*D is linearly equivalent to $q^*H + E$ where $E \geqslant 0$, and if for each effective $\mathbf{R}$-divisor R linearly equivalent to p^*D we have $R \geqslant E$.

If φ is a birational contraction, we say that φ is *a semiample model of* D if $H = \varphi_*D$ is semiample (hence in particular $\mathbf{R}$-Cartier) and if $p^*D = q^*H + E$ where $E \geqslant 0$ is q-exceptional.

Lemma 4.6 ([BCHM10, Lemma 3.6.6])**.** *Let Z be a terminal $\mathbf{Q}$-factorial variety and D a $\mathbf{R}$-divisor on Z.*

(1) *If $\varphi_i\colon Z \dashrightarrow Y_i$, $i = 1, 2$, are two ample models of D, there exists an isomorphism $\theta\colon Y_1 \xrightarrow{\sim} Y_2$ such that $\varphi_2 = \theta \circ \varphi_1$.*

(2) *If a birational map $\psi\colon Z \dashrightarrow X$ is a semiample model of D, the ample model $\varphi\colon Z \dashrightarrow Y$ exists and $\varphi = \theta \circ \psi$ for some morphism $\theta\colon X \longrightarrow Y$. Moreover, $\psi_*D = \theta^*H$, where H is the ample divisor $H = \varphi_*D$.*

(3) *A birational map $\varphi\colon Z \dashrightarrow Y$ is the ample model of D if and only if it is a semiample model of D and φ_*D is ample.*

Remark 4.7. Note that composing with an isomorphism of the target does not change the notion of ample or semiample model, so it is natural to say that two ample or semiample models $\varphi_1\colon Z \dashrightarrow Y_1$, $\varphi_2\colon Z \dashrightarrow Y_2$ are equivalent if there is an isomorphism $\theta\colon Y_1 \xrightarrow{\sim} Y_2$ such that $\varphi_2 = \theta \circ \varphi_1$. Then Lemma 4.6(1) says that up to equivalence, if an ample model exists then it is unique. This justifies that we can speak of *the* ample model of a divisor D.

Definition 4.8. We say that two divisors D and D' are *Mori equivalent* if they have the same ample model.

Remark 4.9. For a $\mathbf{Q}$-divisor, the ample model of D, if it factorises through a semiample model, is the rational map φ_D associated with the linear system $|mD|$

for $m \gg 0$, whose image is $Z_D = \mathrm{Proj}(\oplus_m H^0(Z, mD))$, where the sum is over all positive integers m such that mD is Cartier (see [KKL16, Remark 2.4(ii)]). It does exist if the ring $\oplus_m H^0(Z, mD)$ is finitely generated, which is for instance true if $D = K_Z + A$ for some ample $\mathbf{Q}$-divisor A (follows from [BCHM10, Corollary 1.1.2]).

Let $V_{\mathbf{Q}}$ be a $\mathbf{Q}$-vector space, and $V_{\mathbf{R}} = V_{\mathbf{Q}} \otimes \mathbf{R}$ the associated real vector space. Recall that a *rational polytope* in $V_{\mathbf{R}}$ is the convex hull of finitely many points lying in $V_{\mathbf{Q}}$. In particular, it is convex and compact.

Proposition 4.10. *Let Z be a smooth variety with K_Z not pseudo-effective and let $A_1, \dots, A_s$ be ample $\mathbf{Q}$-divisors that generate the $\mathbf{R}$-vector space $N^1(Z)$. Assume that there exist ample effective $\mathbf{Q}$-divisors $A, A'_1, \dots, A'_s$ such that for each i, $A_i = A + A'_i$. Define*

$$\mathcal{C} = \Big\{ D \in \mathrm{Div}(Z)_{\mathbf{R}} \;\Big|\; D = a_0 K_Z + \sum_{i=1}^{s} a_i A_i, \quad a_0, \dots, a_s \geqslant 0 \text{ and } D \text{ is pseudo-effective} \Big\}.$$

Then, every element of $\mathcal{C}$ has an ample model, and the Mori equivalence classes give a finite partition

$$\mathcal{C} = \coprod_{i \in I} \mathcal{A}_i.$$

We denote by $\varphi_i \colon Z \dashrightarrow Z_i$ the common ample model of all $D \in \mathcal{A}_i$. Then, the following holds (each i, j is always assumed to be in I in the next statements):

(1) *The set $\mathcal{C}$ is a cone over a rational polytope.*
(2) *Each $\mathcal{A}_i$ is a finite union of relative interiors of cones over rational polytopes.*
(3) *For each i, the following are equivalent:*
   (*i*) *The image of $\mathcal{A}_i$ in $N_1(Z)$ has non-empty interior;*
   (*ii*) *φ_i is birational and Z_i is $\mathbf{Q}$-factorial;*
   (*iii*) *φ_i is a birational contraction that is the output of a $(K_Z + \Delta)$-MMP for some $K_Z + \Delta \in \mathcal{C}$;*

(4) *If φ_j is birational, then $\overline{\mathcal{A}_j}$ is a cone over a rational polytope. Taking a resolution $p \colon W \longrightarrow Z$, $q \colon W \longrightarrow Z_j$ of φ_j, we have*

$$\mathcal{A}_j = \{D \in \mathcal{C} \mid \varphi_{j*}D \text{ is ample and } p^*D - q^*\varphi_{j*}D \geqslant 0\},$$
$$\overline{\mathcal{A}_j} = \{D \in \mathcal{C} \mid \varphi_{j*}D \text{ is nef and } p^*D - q^*\varphi_{j*}D \geqslant 0\}.$$

Moreover, for each $D \in \overline{\mathcal{A}_j}$, the divisor $\varphi_{j*}D$ is semiample, so we also have

$$\overline{\mathcal{A}_j} = \{D \in \mathcal{C} \mid \varphi_j \text{ is a semiample model of } D\}.$$

(5) *If i, j are such that $\overline{\mathcal{A}_j} \cap \mathcal{A}_i \neq \emptyset$, there exists a morphism $\varphi_{ji} \colon Z_j \longrightarrow Z_i$ with connected fibres such that $\varphi_i = \varphi_{ji} \circ \varphi_j$.*
   (*i*) *If moreover φ_j is birational, we obtain*

$$\overline{\mathcal{A}_j} \cap \overline{\mathcal{A}_i} = \{D \in \overline{\mathcal{A}_j} \mid \varphi_{j*}D \cdot C = 0 \text{ for each } C \in N_1(Z_j/Z_i)\},$$
$$\overline{\mathcal{A}_j} \cap \mathcal{A}_i = \{D \in \overline{\mathcal{A}_j} \cap \overline{\mathcal{A}_i} \mid \varphi_{i*}D \text{ is ample }\}.$$

   (*ii*) *If furthermore φ_i is birational, we also have*

$$\overline{\mathcal{A}_j} \cap \overline{\mathcal{A}_i} = \{D \in \overline{\mathcal{A}_i} \mid \varphi_{j*}D \cdot C = 0 \text{ for each } C \in N_1(Z_j/Z_i)\}.$$

(6) *Each variety Z_i is normal and klt. In particular, it has rational singularities.*

(7) *For each numerically equivalent divisors $D, D' \in \mathcal{C}$ and each i, we have*

$$D \in \mathcal{A}_i \iff D' \in \mathcal{A}_i \qquad \text{and} \qquad D \in \overline{\mathcal{A}_i} \iff D' \in \overline{\mathcal{A}_i}.$$

Proof. The finite partition, the assertions (2), (3), the fact that $\overline{\mathcal{A}_i}$ is a cone over a rational polytope if φ_i is birational (first part of (4)) and the existence of the morphism φ_{ji} in (5) are given by [HM13, Theorem 3.3]. Indeed, we can apply their result with (in their notation) the affine subspace $V \subset \mathrm{Div}(Z)_{\mathbf{R}}$ generated by $A'_1, \dots, A'_s$ and $-A$. Observe that they normalise their divisors by $a_0 = 1$, so they work with an affine section of our cone $\mathcal{C}$.

As the set of pseudo-effective divisors is a closed convex cone, so is $\mathcal{C}$. Moreover, $\mathcal{C}$ is rational and polyhedral, as it is the union of finitely many rational polyhedral cones. This gives (1).

We now prove the remaining part of (4). By Lemma 4.6(3), for each $D \in \mathcal{C}$, φ_j is the ample model of D if and only if it is a semiample model of D and $\varphi_{j*}D$ is ample. This corresponds exactly to asking that $\varphi_{j*}D$ is ample and $p^*D - q^*\varphi_*D$ is effective. The closure of ample divisors being nef divisors, we obtain the explicit description of $\mathcal{A}_j$ and the first description of $\overline{\mathcal{A}_j}$ given in (4).

The fact that $\varphi_{j*}D$ is semiample for each $D \in \overline{\mathcal{A}_j}$ is part of the proof (and in fact the main point) of the existence of $\varphi_{ji}\colon Z_j \longrightarrow Z_i$ given in [HM13, Theorem 3.3]. It implies that $\overline{\mathcal{A}_j} = \{D \in \mathcal{C} \mid \varphi_j \text{ is a semiample model of } D\}$.

We now prove the remaining part of (5). If $i = j$, everything follows with $\varphi_{ii} = \mathrm{id}_{Z_i}$, so we can assume that $i \neq j$.

We first prove (5)(*i*). For each element $D \in \overline{\mathcal{A}_j}$, we denote by H_j the divisor $H_j = \varphi_{j*}D$, which is semiample by (4), and obtain $p^*D = q^*H_j + E$, where E is effective and q-exceptional. We then use the resolution of φ_i given by $p\colon W \longrightarrow Z$ and $q' = \varphi_{ji} \circ q\colon W \longrightarrow Z_i$ to determine if D belongs to $\mathcal{A}_i$ or its closure.

(*a*) Suppose first that $D \in \mathcal{A}_i$, which means that φ_i is the ample model of D. There exists then an ample divisor H_i on Z_i such that p^*D is linearly equivalent to $(\varphi_{ji} \circ q)^*H_i + E'$ where $E' \geqslant 0$ and where $R \geqslant E'$ for each effective $\mathbf{R}$-divisor R linearly equivalent to p^*D. As $p^*D = q^*H_j + E$ and as q^*H_j has no fixed component (because H_j is semiample), we obtain $E \geqslant E'$, so E' is q-exceptional. Hence, $\varphi_{j*}D$ is linearly equivalent to $q_*((\varphi_{ji} \circ q)^*H_i + E') = \varphi_{ji}^*H_i$ and thus satisfies $\varphi_{j*}D \cdot C = 0$ for each $C \in N_1(Z_j/Z_i)$. Moreover, $\varphi_{i*}D$ is linearly equivalent to H_i and is thus ample.

(*b*) Conversely, we suppose that $\varphi_{j*}D \cdot C = 0$ for each $C \in N_1(Z_j/Z_i)$ and that $\varphi_{i*}D$ is ample, and prove that $D \in \mathcal{A}_i$.

We denote by H_i the ample divisor $\varphi_{i*}D$. Since the nef divisor $H_j = \varphi_{j*}D$ on Z_j satisfies $H_j \cdot C = 0$ for each $C \in N_1(Z_j/Z_i)$ and satisfies $\varphi_{ji*}H_j = H_i$, the divisor $H_j - \varphi_{ji}^*H_i$ is numerically trivial. We obtain $p^*D \equiv (\varphi_{ji} \circ q)^*H_i + E$. It remains to see that if R is an effective divisor linearly equivalent to p^*D, then $R \geqslant E$. We write

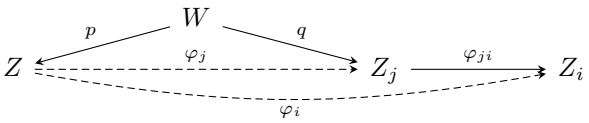
$$M + F = R \equiv (\varphi_{ji} \circ q)^*H_i + E$$

with $|M|$, F respectively the mobile and fixed part of the linear system $|R|$. The divisor $(\varphi_{ji} \circ q)^*H_i = q^*(\varphi_{ji}^*H_i)$ is q-trivial and mobile, so $E - F \geqslant 0$ with support in the exceptional locus of q. So we have

$$M + F = R = q\text{-trivial} + E \implies E - F = q\text{-trivial} + M$$

and we conclude by the Negativity Lemma 2.2 that $F - E \geqslant 0$, hence $F - E = 0$, and $R - E = M \geqslant 0$. This achieves the proof of (5)(*i*).

We now prove (5)(*ii*). The fact that $\varphi_{j*}D \cdot C = 0$ for each $C \in N_1(Z_j/Z_i)$ and each $D \in \overline{\mathcal{A}_j} \cap \overline{\mathcal{A}_i}$ follows from (5)(*i*). Conversely, we take $D \in \overline{\mathcal{A}_i}$ that satisfies $\varphi_{j*}D \cdot C = 0$ for each $C \in N_1(Z_j/Z_i)$, and prove that $D \in \overline{\mathcal{A}_j}$, using the diagram



As $D \in \overline{\mathcal{A}_i}$, by (4) the divisor $H_i = \varphi_{i*}D$ is nef and $p^*D = q^*\varphi_{ji}^*H_i + E_i$ where $E_i \geqslant 0$ is $(\varphi_{ji} \circ q)$-exceptional. Hence, $H_j = \varphi_{j*}D = q_*p^*D = \varphi_{ji}^*H_i + q_*E_i$, where $q_*E_i \geqslant 0$ is φ_{ji}-exceptional. As $H_j \cdot C = 0$ for each $C \in N_1(Z_j/Z_i)$, we also obtain $q_*E_i \cdot C = 0$ for each $C \in N_1(Z_j/Z_i)$. The Negativity Lemma 2.2 gives $q_*E_i \leqslant 0$, so $q_*E_i = 0$. This implies that $H_j = {\varphi_{ji}}^*H_i$ is nef (because H_i is nef) and that $p^*D = q^*H_j + E_i$, where $E_i \geqslant 0$ is q-exceptional, so (4) gives $D \in \overline{\mathcal{A}_j}$ as expected.

Finally we prove (6) and (7), for a given index i. The variety Z_i is normal by definition of an ample model. If $\mathcal{A}_i$ satisfies the equivalent conditions of (3), then Z_i is terminal (hence has rational singularities) as the output of a $(K_Z + \Delta)$-MMP, and the fact that belonging to $\mathcal{A}_i$ or $\overline{\mathcal{A}_i}$ is a numerical condition follows from (4). To see this, we take $D \in \mathcal{C}$, and write $p^*D - q^*\varphi_{j*}D = \sum_{i=1}^{l} a_iE_i$ where $E_1, \dots, E_l$ are the divisors contracted by the birational morphism $q\colon W \longrightarrow Z_j$. Then we observe that the real numbers a_i can be computed by intersecting E with linear combinations of curves on the exceptional divisors. Indeed there does not exist any $(\mu_1, \dots, \mu_l) \in \mathbf{R}^l \smallsetminus \{0\}$ such that $E = \sum_{i=1}^{l} \mu_iE_i$ intersects trivially each element of $N_1(W/Z_j)$. This follows from the Negativity Lemma 2.2 applied to E and $-E$.

If $\mathcal{A}_i$ does not satisfies the equivalent conditions of (3), there exists a chamber $\mathcal{A}_j$ satisfying them, such that $\overline{\mathcal{A}_j} \cap \mathcal{A}_i \neq \emptyset$. By (5) there is a contraction $\varphi_{ji}\colon Z_j \longrightarrow Z_i$ and Z_j is $\mathbf{Q}$-factorial and terminal again by (3). Assertion (6) follows from Proposition 2.12 and Remark 4.8. Assertion (7) follows from (5). □

Lemma 4.11. *Assume the setting of Proposition 4.10. Let $\mathcal{A}_j \subseteq \mathcal{C}$ be a Mori equivalence class such that $\varphi_j\colon Z \dashrightarrow Z_j$ is birational.*

(1) *If $\{D_i\}$ is a finite collection of classes in $N^1(Z)$ such that φ_j is a semiample model of each, then φ_j is a semiample model for any convex combination of the D_i.*

(2) *Let $D_j \in \mathrm{Nef}(Z_j)$ and set $D = \varphi_j^*D_j \in N^1(Z)$. Then φ_j is a semiample model of D.*

(3) *Let E be a prime divisor contracted by φ_j. Then φ_j is a semiample model of E.*

(4) *The cone $\overline{\mathcal{A}_j}$ is the intersection of $\mathcal{C}$ with the closed convex cone generated by $\varphi_j^*\,\mathrm{Nef}(Z_j)$ and by the divisors contracted by φ_j.*

Proof. (1). For each i we write $p^*D_i = q^*H_i + E_i$ where $H_i = \varphi_{j*}D_i$ is semiample, and $E_i \geqslant 0$ is q-exceptional. For any choice of coefficients $a_i \geqslant 0$ we have

$$p^*\Big(\sum a_iD_i\Big) = q^*\Big(\sum a_iH_i\Big) + \sum a_iE_i$$

where $\sum a_iH_i = \varphi_{j*}(\sum a_iD_i)$ is semiample and $\sum a_iE_i$ is q-exceptional and effective.

(2). We take $p\colon W \longrightarrow Z$ and $q\colon Z \longrightarrow Z_j$ a resolution of φ_j. We obtain $D = p_*q^*D_j$. The divisor $E = p^*D - q^*D_j$ is p-exceptional, and also q-exceptional since φ_j is a birational contraction, hence $\varphi_{j*}D = q_*p^*D = D_j$. Moreover, $-E = q^*D_j - p^*D$ is p-nef, so $E \geqslant 0$ by the Negativity Lemma 2.2.

(3). The trivial divisor $H = \varphi_{j*}E = 0$ is semiample, and $p^*E - q^*H = p^*E$ is effective, as E is effective.

(4). As φ_j is birational, by Proposition 4.10(4) we have

$$\overline{\mathcal{A}_j} = \{D \in \mathcal{C} \mid \varphi_j \text{ is a semiample model of } D\}.$$

By (2) and (3), φ_j is a semiample model of every element of $\varphi_j^* \operatorname{Nef}(Z_j)$ and of every prime divisor contracted by φ_j. So, by (1), $\overline{\mathcal{A}_j}$ contains the intersection of $\mathcal{C}$ with the closed convex cone generated by $\varphi_j^* \operatorname{Nef}(Z_j)$ and by the divisors contracted by φ_j. Conversely, let $D \in \overline{\mathcal{A}_j}$, and set $D_j = \varphi_{j*}D$. The divisor

$$E = D - {\varphi_j}^*D_j = p_*p^*D - p_*q^*D_j = p_*(p^*D - q^*D_j)$$

is effective and φ_j-exceptional, so $D = {\varphi_j}^*D_j + E$ is the sum of an element in $\varphi_j^* \operatorname{Nef}(Z_j)$ and an effective φ_j-exceptional divisor. □

Set-Up 4.12. Let Z be a smooth variety with K_Z not pseudo-effective and let $A_1, \dots, A_s$ be ample $\mathbf{Q}$-divisors that generate the $\mathbf{R}$-vector space $N^1(Z)$. We still denote

$$\mathcal{C} = \Big\{ D \in N^1(Z) \;\Big|\; D = a_0K_Z + \sum_{i=1}^{s} a_iA_i, \\ a_0, \dots, a_s \geqslant 0 \text{ and } D \text{ is pseudo-effective} \Big\}.$$

This is the image under the natural map $\operatorname{Div}(Z)_{\mathbf{R}} \longrightarrow N^1(Z)$ of the cone from Proposition 4.10, for some choice of ample effective $\mathbf{Q}$-divisors $A, A'_1, \dots, A'_s$ such that for each i, $A_i \equiv A + A'_i$. By Proposition 4.10(7), the decomposition $\mathcal{C} = \coprod_{i\in I} \mathcal{A}_i$ (hence also its image in $N^1(Z)$) does not depend on such a choice of effective representatives. So from now on we will work directly in the finite dimensional $\mathbf{R}$-vector space $N^1(Z)$, and use the notation $\mathcal{C}, \mathcal{A}_i$ in this context only.

Remark 4.13. One advantage of working up to numerical equivalence is that we can always assume that the pairs (Z, Δ) in Set-Up 4.12 are klt with arbitrary small discrepancies, where $\Delta = \frac{1}{a_0}\sum_{i=1}^{s} a_iA_i$. Indeed, by expressing each A_i as $A_i \equiv \frac{1}{N}\sum_{j=1}^{N} H_{i,j}$ for some large integer N and some general members $H_{i,j} \in |A_i|$, we can ensure that the union of the supports of the $H_{i,j}$ is a simply normal crossing divisor and that all coefficients appearing in the convex combination Δ are positive and very small.

Assuming Set-Up 4.12, we introduce some terminology. We say that a chamber $\mathcal{A}_i$ has *maximal dimension* if it has non-empty interior in $N^1(Z)$, which corresponds to the equivalent assertions of Proposition 4.10(3). We say that a chamber $\mathcal{A}_i$ is *big* if all divisors (or equivalently, one divisor) in $\mathcal{A}_i$ are big. We call *face* of $\mathcal{C}$ any face of a polyhedral cone $\overline{\mathcal{A}_i} \subseteq \mathcal{C} \subseteq N^1(Z)$, for any big chamber $\mathcal{A}_i$. By the *codimension* of a face in $\mathcal{C}$ we always mean the codimension in $N^1(Z)$ of the smallest vector subspace containing it. We will usually denote $\mathcal{F}^r$ a face of codimension r in $\mathcal{C}$, and $\mathring{\mathcal{F}}^r$ its relative interior.

We denote by $\partial^+\mathcal{C}$ the set of non-big divisors in $\mathcal{C}$. As $\partial^+\mathcal{C}$ is the intersection of $\mathcal{C}$ with the boundary of the pseudo-effective cone, the set $\partial^+\mathcal{C}$ is a closed subset of the boundary of $\mathcal{C}$. We have $\mathcal{A}_i \subseteq \partial^+\mathcal{C}$ if $\dim Z_i < Z$ and $\mathcal{A}_i \subseteq \mathcal{C} \smallsetminus \partial^+\mathcal{C}$ if $\dim Z_i = Z$.

By definition, the cone $\mathcal{C} \subset N^1(Z)$ is equal to the intersection of two convex closed cones, namely $\mathcal{C} = \mathcal{C}' \cap \overline{\mathrm{Eff}}(Z)$ with $\mathcal{C}'$ the convex cone generated by K_Z and the A_i. We will say that a face $\mathcal{F} \subseteq \mathcal{C}$ is *inner* if it meets the interior of $\mathcal{C}'$. In particular, $\mathcal{F}$ is inner if for any $D' \in \mathring{\mathcal{F}}$, there exists a neighborhood V of D' in $N^1(Z)$ such that $\overline{\mathrm{Eff}}(Z) \cap V = \mathcal{C} \cap V$. Equivalently, a face is inner if it meets either the interior of $\mathcal{C}$ or the relative interior of $\partial^+\mathcal{C}$.

Remark 4.14. If $\mathcal{F}$ is an inner face, then for any $D \in \overline{\mathrm{Eff}}(Z)$ and any $D' \in \mathring{\mathcal{F}}$, we have $D' + \varepsilon D \in \mathcal{C}$ for sufficiently small $\varepsilon \geqslant 0$. Indeed with the notation above one can choose $V \subset \mathcal{C}'$ a neighborhood of D' such that $\overline{\mathrm{Eff}}(Z) \cap V = \mathcal{C} \cap V$. Then it suffices to choose ε such that $D' + \varepsilon D \in V$. Since D, D' are both pseudo-effective, the segment $[D, D']$ also is contained in the convex cone $\overline{\mathrm{Eff}}(Z)$, and the claims follows.

Lemma 4.15. *Let $\mathcal{A}_j$ be a big inner chamber.*

(1) *Let $\mathcal{A}_i$ be a chamber such that $\overline{\mathcal{A}_j} \cap \mathcal{A}_i \neq \emptyset$. Then $\mathcal{F}_{ji} := \overline{\mathcal{A}_j} \cap \overline{\mathcal{A}_i}$ has the following properties:*

- (*i*) *$\mathcal{F}_{ji}$ is a face of $\overline{\mathcal{A}_j}$.*
- (*ii*) *If $\mathcal{A}_i$ is a big inner chamber, then $\mathcal{F}_{ji}$ is also a face of $\overline{\mathcal{A}_i}$.*
- (*iii*) *If $\overline{\mathcal{A}_j}$ has maximal dimension, then the vector space*

$$V_{ji} := \{D \in N^1(Z) \mid \varphi_{j*}D \cdot C = 0 \text{ for each } C \in N_1(Z_j/Z_i)\}$$

  is spanned by $\mathrm{Ex}(\varphi_j)$ and $\varphi_i^* \mathrm{Nef}(Z_i)$, and has codimension $\rho(Z_j/Z_i)$ in $N^1(Z)$. Moreover if $\mathcal{F}_{ji}$ is inner, then V_{ji} is spanned by $\mathcal{F}_{ji}$.

- (*iv*) *$\mathring{\mathcal{F}}_{ji} \subseteq \mathcal{A}_i$.*

(2) *Let $\mathcal{A}_i$, $\mathcal{A}_k$ be two chambers such that $\overline{\mathcal{A}_j} \cap \mathcal{A}_i \neq \emptyset$, $\overline{\mathcal{A}_j} \cap \mathcal{A}_k \neq \emptyset$ and $\overline{\mathcal{A}_i} \cap \mathcal{A}_k \neq \emptyset$. Then the face $\mathcal{F}_{jk}$ is a subface of the face $\mathcal{F}_{ji}$, and a strict subface if $i \neq k$.*

(3) *Conversely, any inner face $\mathcal{F}^r \subseteq \mathcal{C}$ is of the form $\mathcal{F}^r = \mathcal{F}_{ki} := \overline{\mathcal{A}_k} \cap \overline{\mathcal{A}_i}$ for some chamber $\mathcal{A}_k$ of maximal dimension, and some chamber $\mathcal{A}_i$ containing $\mathring{\mathcal{F}}^r$.*

Proof. We first prove Assertions (*i*)-(*iii*) of (1). By Proposition 4.10(5)(*i*),

$$\overline{\mathcal{A}_j} \cap \overline{\mathcal{A}_i} = \{D \in \overline{\mathcal{A}_j} \mid \varphi_{j*}D \cdot C = 0 \text{ for each } C \in N_1(Z_j/Z_i)\}.$$

So each curve $C \in N_1(Z_j/Z_i)$ defines a supporting hyperplane of $\overline{\mathcal{A}_j}$, and $\overline{\mathcal{A}_j} \cap \overline{\mathcal{A}_i}$ is a face of $\overline{\mathcal{A}_j}$ as the intersection of a cone over a polytope with a collection of supporting hyperplanes. This proves (*i*). The same argument using Proposition 4.10(5)(*ii*) gives (*ii*).

We now prove (*iii*). We denote by $E_j, V_j, V_i \subseteq N^1(Z)$ the vector spaces generated by $\mathrm{Ex}(\varphi_j)$, $\varphi_j^* \mathrm{Nef}(Z_j)$ and $\varphi_i^* \mathrm{Nef}(Z_i)$ respectively. As $\overline{\mathcal{A}_j}$ has maximal dimension, φ_j is a birational contraction that is the output of a $(K_Z + \Delta)$-MMP for some $K_Z + \Delta \in \mathcal{C}$ (Proposition 4.10(3)). This gives $N^1(Z) = E_j \oplus V_j$. Now, $\varphi_j^* \mathrm{Nef}(Z_j)$ contains $\varphi_i^* \mathrm{Nef}(Z_i) = \varphi_j^* \varphi_{ji}^* \mathrm{Nef}(Z_i)$, and the codimension of $\varphi_i^* \mathrm{Nef}(Z_i)$ in $\varphi_j^* \mathrm{Nef}(Z_j)$ is the same as the codimension of $\varphi_{ji}^* \mathrm{Nef}(Z_i)$ in $\mathrm{Nef}(Z_j)$ (indeed, the group homomorphism $\varphi_j^* \colon N_1(Z_j) \longrightarrow N_1(Z)$ is injective as φ_j is a birational contraction). We obtain $\dim(V_j) - \dim(V_i) = \rho(Z_j/Z_i)$. The vector

space V_{ji} is of codimension $\dim N_1(Z_j/Z_i) = \rho(Z_j/Z_i)$ since φ_j is birational. Since $E_j \oplus V_i \subseteq V_{ji}$ and both vector space have the same dimension, we obtain

$$V_{ji} = E_j \oplus V_i.$$

It remains to prove the last sentence of *(iii)*. To do this, we assume moreover that $\mathcal{F}_{ji}$ is inner, and show that $\mathcal{F}_{ji}$ spans V_{ji} in this case. Writing

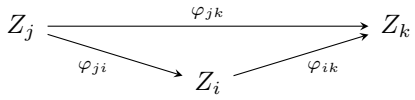
$$S_j = \{D \in N^1(Z) \mid \varphi_j \text{ is a semiample model of } D\},$$

Proposition 4.10(4)&(5)(*i*) yields

$$\overline{\mathcal{A}_j} = \mathcal{C} \cap S_j, \quad \overline{\mathcal{A}_j} \cap \overline{\mathcal{A}_i} = \mathcal{F}_{ji} = V_{ji} \cap \mathcal{C} \cap S_j.$$

In particular, we have $\mathcal{F}_{ji} \subseteq V_{ji}$. To show that $\mathcal{F}_{ji}$ spans $V_{ji} = E_j \oplus V_i$, it suffices to prove that every divisor E which is either an exceptional divisor of φ_j or an element of $\varphi_i^* \operatorname{Nef}(Z_i)$ lies in $\operatorname{span}(\mathcal{F}_{ji})$. We fix some $D' \in \mathring{\mathcal{F}}_{ji}$ and may assume that $D' + E \in \mathcal{C}$; indeed, we may replace E with εE for some small $\varepsilon > 0$, and then the result follows from $E \in \overline{\operatorname{Eff}}(Z)$ and Remark 4.14. This implies that $E \in \overline{\mathcal{A}_j}$ (Lemma 4.11(4)), and thus that $E \in \mathcal{F}_{ji} = V_{ji} \cap \mathcal{C} \cap S_j$, since $\overline{\mathcal{A}_j} = \mathcal{C} \cap S_j$ and $E \in V_{ji}$.

(2). By Proposition 4.10(5) we have a commutative diagram



Any curve contracted by φ_{ji} is also contracted by φ_{jk}, so the collection of supporting hyperplanes defining the face $\mathcal{F}_{ji}$ is a subcollection of the one defining $\mathcal{F}_{jk}$ and moreover a strict subcollection if $N_1(Z_j/Z_i) \subsetneq N_1(Z_j/Z_k)$.

We can now prove (1)(*iv*). Let $D \in \mathcal{F}_{ji}$, and assume that $D \in \mathcal{A}_k$ with $k \neq i$. By (2), the divisor D is in $\mathcal{F}_{jk}$ which is a strict subface of $\mathcal{F}_{ji}$, so $D \notin \mathring{\mathcal{F}}_{ji}$. This proves $\mathring{\mathcal{F}}_{ji} \subseteq \mathcal{A}_i$.

(3). If $r = 0$, then $\mathcal{F}^r$ is equal to $\overline{\mathcal{A}_i}$ for some chamber i of maximal dimension, so we simply choose $i = k$. Now assume that $\mathcal{F}^r$ is an inner face of codimension $r \geqslant 1$. By definition $\mathcal{F}^r$ is a face of $\overline{\mathcal{A}_k}$ for some big chamber $\mathcal{A}_k$. Among all possible choices for such a chamber $\mathcal{A}_k$ we pick one such that $\rho(Z_k)$ is maximal. Let $D \in \mathring{\mathcal{F}}^r$, and $\mathcal{A}_i$ the chamber containing D. If D is not big, then $i \neq k$. If D is big, then since $\mathcal{F}^r$ is inner there exists at least one big chamber $\mathcal{A}_l$ distinct from $\mathcal{A}_i$ such that $D \in \overline{\mathcal{A}_l} \cap \mathcal{A}_i$. This implies that $\mathcal{F}^r \subseteq \mathcal{F}_{li}$ is a face of $\overline{\mathcal{A}_l}$, and by Proposition 4.10(5) we have a morphism $Z_l \longrightarrow Z_i$. Since $l \neq i$ we get $\rho(Z_l) > \rho(Z_i)$, and the maximality of $\rho(Z_k)$ ensures that $i \neq k$. By (1) and *(iv)* we get $\mathcal{F}^r \subseteq \mathcal{F}_{ki}$. If $r = 1$, this gives $\mathcal{F}^r = \mathcal{F}_{ki}$.

Finally we consider the case where $r \geqslant 2$, and prove by contradiction that the inclusion $\mathcal{F}^r \subseteq \mathcal{F}_{ki}$ is an equality. If $\mathcal{F}^r \subsetneq \mathcal{F}_{ki}$ is a strict subface, then there exists an inner codimension 1 face $\mathcal{F}^1$ of $\overline{\mathcal{A}_k}$ such that $\mathcal{F}^r \subseteq \mathcal{F}^1 \cap \mathcal{F}_{ki} \subsetneq \mathcal{F}_{ki}$. By the first part of the proof, there exists a chamber $\mathcal{A}_l$ such that either $\mathcal{F}^1 = \mathcal{F}_{kl}$ or $\mathcal{F}^1 = \mathcal{F}_{lk}$. But in the latter case, we would have $\overline{\mathcal{A}_l} \cap \mathcal{A}_k \neq \emptyset$, hence a morphism $\varphi_{lk} \colon Z_l \longrightarrow Z_k$, contradicting the maximality of $\rho(Z_k)$. So $\mathcal{F}^1 = \mathcal{F}_{kl}$, so we have $\overline{\mathcal{A}_k} \cap \mathcal{A}_l \neq \emptyset$, and moreover since $D \in \overline{\mathcal{A}_k} \cap \overline{\mathcal{A}_l} \cap \mathcal{A}_i$, we also have $\overline{\mathcal{A}_k} \cap \mathcal{A}_i \neq \emptyset$ and $\overline{\mathcal{A}_l} \cap \mathcal{A}_i \neq \emptyset$. By (2), $\mathcal{F}_{ki}$ is a subface of $\mathcal{F}_{kl}$, so we get $\mathcal{F}_{kl} \cap \mathcal{F}_{ki} = \mathcal{F}_{ki}$, which contradicts $\mathcal{F}^1 \cap \mathcal{F}_{ki} \subsetneq \mathcal{F}_{ki}$, since $\mathcal{F}_{kl} = \mathcal{F}^1$. □

Notation 4.16. Lemma 4.15 provides the following indexing system for faces. Any inner face can be written $\mathcal{F}_{ji} := \overline{\mathcal{A}_j} \cap \overline{\mathcal{A}_i}$, for some chamber $\mathcal{A}_j$ of maximal dimension and some chamber $\mathcal{A}_i$ such that $\mathring{\mathcal{F}}_{ji} \subseteq \mathcal{A}_i$. The index i is uniquely defined by this last property, but there might be several possible choices for the index j. For instance, if we have a log-flip from Z_j to Z_k, over a non $\mathbf{Q}$-factorial Z_i, we have $\mathcal{F}_{ji} = \mathcal{F}_{ki}$.

Example 4.17. We illustrate the definition of Mori chambers and faces on the simple example of the blow-up $Z \longrightarrow \mathbb{P}^2$ at two distinct points p_1 and p_2. Using the notation above, there are eight Mori chambers $\mathcal{A}_0, \dots, \mathcal{A}_7$, corresponding to morphisms $\varphi_i \colon Z \longrightarrow Z_i$, $i = 0, \dots, 7$ to the varieties $Z_0 = Z$, $Z_1 = Z_2 = \mathbb{F}_1$, $Z_3 = \mathbb{F}_0$, $Z_4 = \mathbb{P}^2$, $Z_5 = Z_6 = \mathbb{P}^1$ and $Z_7 = \mathrm{pt}$ in the commutative diagram on Figure 3 (φ_0 being the identity). The two morphisms $\varphi_{14}, \varphi_{24} \colon \mathbb{F}_1 \longrightarrow \mathbb{P}^2$ are the blow-ups of $p_1, p_2 \in \mathbb{P}^2$ respectively, and $\varphi_1, \varphi_2 \colon Z \longrightarrow \mathbb{F}_1$ are the blow-ups of the images of p_1 and p_2. The morphisms $\varphi_{15}, \varphi_{26} \colon \mathbb{F}_1 \longrightarrow \mathbb{P}^1$ correspond to the $\mathbb{P}^1$-bundle of $\mathbb{F}_1$ and $\varphi_3 = \varphi_5 \times \varphi_6 \colon Z \longrightarrow \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$.

We give the detail of the relation between these Mori chambers and the faces of the cone $\mathcal{C}$ in Figure 3. We denote by $E_1, E_2 \subset Z$ the curves contracted onto $p_1, p_2 \in \mathbb{P}^2$ respectively, by L the strict transform of the line through p_1 and p_2, and by $H = L + E_1 + E_1$ the pull-back of a general line. The cone $\mathrm{Eff}(Z)$ is the closed convex cone generated by E_1, E_2 and L, which are the only (-1)-curves on Z, while the cone $\mathrm{Nef}(Z)$ is the closed convex cone generated by $H, H - E_1$ and $H - E_2$. The anti-canonical divisor $-K_Z = 3H - E_1 - E_2 = 3L + 2E_1 + 2E_2$ is ample. In the figure we represent an affine section of the cone, and all divisors must be understood up to rescaling by an adequate homothety: for instance this is really $-\frac{1}{7}K_Z$ that is in the same affine section as E_1, E_2 and L, but for simplicity we write $-K_Z$.

The faces $\mathcal{F}_i^0 = \overline{\mathcal{A}_i}$, $i = 0, \dots, 4$ are the faces of maximal dimension, the faces $\mathcal{F}_{ji}$ (written $\mathcal{F}_{ji}^r$ where r is the codimension as usual) are as above $\mathcal{F}_{ji} = \overline{\mathcal{A}_j} \cap \overline{\mathcal{A}_i}$. Every face of $\mathcal{C} = \mathrm{Eff}(Z)$ is inner.

We can notice that the ample chamber $\mathcal{A}_0$ is the only open one and that $\mathcal{A}_7$ is the only closed one. Moreover, as a hint that the behaviour of non maximal Mori chambers can be quite erratic, observe that $\mathcal{A}_7 = \overline{\mathcal{A}_7}$ is not connected, and that neither $\overline{\mathcal{A}_5}$ nor $\overline{\mathcal{A}_6}$ is a single face.

This example will be continued in Example 4.26 below.

As a warm-up before the next section, we let the reader check that Proposition 4.10 implies the following facts about codimension 1 faces of $\mathcal{C}$.

Remark 4.18. Let $\mathcal{F}^1$ be an inner codimension 1 face of the cone $\mathcal{C} \subseteq N^1(Z)$ from Set-Up 4.12, and $\mathcal{A}_i$ the Mori chamber containing $\mathring{\mathcal{F}}^1$ given by Lemma 4.15(*iv*). Then $\mathcal{F}^1$ is contained in the closure of exactly one or two chambers of maximal dimension, depending whether $\mathcal{F}^1$ is in the boundary of $\mathcal{C}$ or not.

(1) Assume first that $\mathcal{F}^1 \subset \overline{\mathcal{A}_j}$ for a unique chamber $\mathcal{A}_j$ of maximal dimension, so $\mathcal{F}^1$ is in the boundary of $\mathcal{C}$. Moreover since $\mathcal{F}^1$ is inner we have $\mathcal{F}^1 \subseteq \partial^+\mathcal{C}$, so $\dim Z_i < \dim Z_j$. The associated map $\varphi_{ji} \colon Z_j \longrightarrow Z_i$ satisfies $\rho(Z_j/Z_i) = 1$. Moreover $-K_{Z_j}$ is relatively ample, so that Z_j/Z_i is a Mori fibre space ([Kal13, Lemma 3.2], see also Proposition 4.25 below for a generalisation).

(2) Now consider the case where $\mathcal{F}^1 = \overline{\mathcal{A}_j} \cap \overline{\mathcal{A}_k}$ for some distinct chambers $\mathcal{A}_j, \mathcal{A}_k$ of maximal dimension. We distinguish two subcases.

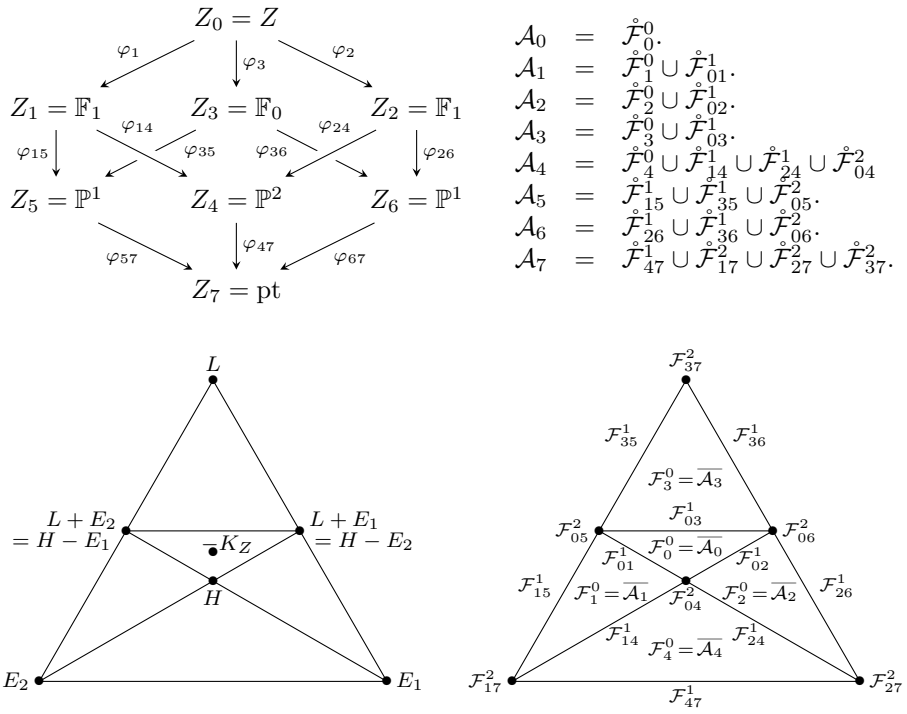


FIGURE 3. Ample models and faces in Example 4.17.

(*i*) If $\mathcal{A}_i$ is of maximal dimension, up to renumbering we can assume $\mathcal{A}_i = \mathcal{A}_k$, so that $\overline{\mathcal{A}_j} \cap \mathcal{A}_i \supseteq \mathring{\mathcal{F}}^1$. In this situation both Z_j and Z_i are $\mathbf{Q}$-factorial and terminal, so the morphism $\varphi_{ji} \colon Z_j \longrightarrow Z_i$ with relative Picard rank 1 given by Proposition 4.10 is a birational contraction.

(*ii*) Finally if $\mathcal{A}_i$ is not of maximal dimension, both birational morphisms φ_{ji} and φ_{ki} given by Proposition 4.10 have relative Picard 1 and target variety Z_i which is not $\mathbf{Q}$-factorial, so φ_{ji} and φ_{ki} are small contractions. By uniqueness of the log-flip, the induced birational map $Z_j \dashrightarrow Z_k$ must be the associated log-flip.

Remark 4.19. Let $\Delta \in \mathcal{C}$ be an ample divisor. Then the successive chambers of maximal dimension that are cut by the segment $[\Delta, K_Z]$ can be interpreted as successive steps in a K_Z-MMP from Z. In [BCHM10, Remark 3.10.10] this is called a K_Z-MMP *with scaling of* Δ. Moreover by perturbing Δ we can assume that the segment is transverse to the polyhedral decomposition. Then as mentioned in Remark 4.18, each intermediate face of codimension 1 that the segment meets corresponds either to a flip or to a divisorial contraction, and the last codimension 1 face in the boundary of the pseudo-effective cone corresponds to a Mori fibre space structure on the output of the MMP.

Remark 4.20. In this section we chose to follow [HM13] and [BCHM10], who build on [Sho96, §6] for the construction of polyhedral chambers. In fact it seems to be Shokurov who coined the terminology "geography of ample models" to refer to this polyhedral decomposition.

However we should point out that another possible thread of references would be to use [CL12] and [CL13], on which rely both [Kal13] and [KKL16]. In these references the polyhedral decomposition is derived from [ELM+06], and is equivalent to our decomposition into faces.

Remark 4.21. Recall that a *fan* is a collection of rational strongly convex polyhedral cones, such that each face (of any dimension) of a cone is also part of the collection, and such that the intersection of two cones is a face of each.

Lemma 4.15 almost gives a fan structure on $\mathcal{C}$: the only missing point would be to check the property for non inner faces. We do not complete this study since we will not need it, but we let the interested reader check that the collection of all faces of big chambers indeed forms a fan, which coincides with the fan structure of [ELM+06, Theorem 4.1] and [KKL16, Theorem 3.2].

4.C. **Generation and relations in the Sarkisov programme.** The goal of this section is to prove Theorem 4.29, which will allow us to define the group homomorphisms of the main theorems. The main technical intermediate step is Proposition 4.25, which explains the relation between our notion of rank r fibration and the combinatorics of the non-big boundary of the cone $\mathcal{C}$ as given in [Kal13].

The proof of the following lemma can be extracted from [HM13, Lemma 4.1].

Lemma 4.22. *Let Z be a smooth variety, $\pi\colon Z \longrightarrow X$ a birational morphism, and $\eta\colon X \longrightarrow B$ a terminal Mori fibre space. There exist ample divisors Δ, Δ' on Z such that π is the ample model of $K_Z + \Delta$ and the semiample model of $K_Z + \Delta'$ and $\eta \circ \pi$ is the ample model of $K_Z + \Delta'$.*

As a consequence, we get the following fact, which is essentially [Kal13, Proposition 3.1(ii)].

Proposition 4.23. *Let $t \geqslant 2$ be an integer. For $i = 1, \dots, t$, let $\eta_i\colon X_i \longrightarrow B_i$ be a terminal Mori fibre space and let $\theta_i\colon X_i \dashrightarrow X_{i+1}$ be a birational map (here θ_t goes from X_t to $X_{t+1} := X_1$). We assume moreover that $\theta_t \circ \cdots \circ \theta_1 = \mathrm{id}_{X_1}$.*

There exists a smooth variety Z, together with birational morphisms $\pi_i\colon Z \longrightarrow X_i$, $i = 1, \dots, t$, and ample $\mathbf{Q}$-divisors $A_1, \dots, A_m$ on Z such that the following hold:

(1) *The divisors $A_1, \dots, A_m$ generate the $\mathbf{R}$-vector space $N^1(Z)$.*

(2) *For $i = 1, \dots, t$, the birational morphism π_i and the morphism $\eta_i \circ \pi_i$ are ample models of an element of*

$$\mathcal{C} = \left\{ a_0 K_Z + \sum_{i=1}^{m} a_i A_i \;\middle|\; a_0, \dots, a_m \geqslant 0 \right\} \cap \overline{\mathrm{Eff}}(Z).$$

(3) *For $i = 1, \dots, t$ we have $\theta_i \circ \pi_i = \pi_{i+1}$ (with $\pi_{t+1} := \pi_1$). We then have a commutative diagram as in Figure 4.*

Proof. We take a smooth common resolution Z of the birational maps θ_i. This gives birational morphisms $\pi_i\colon Z \longrightarrow X$ satisfying (3). For each $i = 1, \dots, t$, we apply Lemma 4.22 to the morphism Z/X_i. This gives ample $\mathbf{Q}$-divisors Δ_i and Δ'_i on Z such that π_i is the ample model of $K_Z + \Delta_i$, and $\eta_i \circ \pi_i$ is the ample model of $K_Z + \Delta'_i$. We then choose some large rational number $\xi_i > 0$ such that $K_Z + \xi_i \Delta_i$ and $K_Z + \xi_i \Delta'_i$ are ample, and then define $A_i = K_Z + \xi_i \Delta_i$ and $A_{t+i} = K_Z + \xi_i \Delta'_i$.

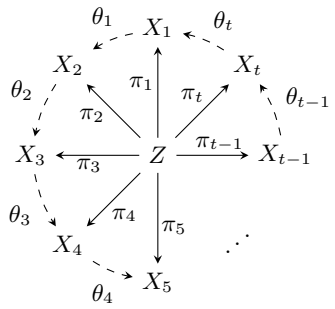


FIGURE 4. The commutative diagram in Proposition 4.23

This provides ample $\mathbf{Q}$-divisors $A_1, \dots, A_{2t}$ such that $K_Z + \Delta_i$ and $K_Z + \Delta_i'$ lie in

$$\left\{ a_0 K_Z + \sum_{i=1}^{2t} a_i A_i \;\middle|\; a_0, \dots, a_{2t} \geqslant 0 \right\} \cap \overline{\mathrm{Eff}}(Z)$$

for each $i \in \{1, \dots, 2t\}$, proving (2). Adding some additional ample divisors A_j, we can assume that $A_1, \dots, A_m$ generate the $\mathbf{R}$-vector space $N^1(Z)$, giving (1). □

In the following discussion (and until Corollary 4.28) we work with the setting given by Proposition 4.23, that is, the commutative diagram of Figure 4 and an associated choice of cone $\mathcal{C} \subset N^1(Z)$. Also recall that $\partial^+\mathcal{C} \subset \mathcal{C}$ is the subset of non-big divisors.

Lemma 4.24. *$\partial^+\mathcal{C}$ is the cone over a polyhedral complex homeomorphic to a disc or a sphere of dimension $\rho(Z) - 2$.*

Proof. Consider the auxiliary cone $\mathcal{C}'$ of classes of the form

$$\sum a_i A_i \text{ where } a_i \geqslant 0 \text{ for all } i.$$

In other words, $\mathcal{C}'$ is the cone over the convex hull of the A_i, and in particular $\mathcal{C}'$ is a closed subcone of the ample cone of Z. Let $\partial^+\mathcal{C}'$ be the points in the boundary of $\mathcal{C}'$ that are visible from the point K_Z. Formally:

$$\partial^+\mathcal{C}' = \{ D \in \mathcal{C}' \mid [D, K_Z] \cap \mathcal{C}' = \{D\} \}.$$

By an elementary convexity argument, this cone $\partial^+\mathcal{C}'$ is homeomorphic to the cone over a sphere or a disc of dimension $\rho(Z) - 2$, the first case occurring precisely if $-K_Z$ is in the interior of $\mathcal{C}'$. Then we have a continuous map

$$\begin{array}{rccc} \pi \colon & \partial^+\mathcal{C}' & \longrightarrow & \partial^+\mathcal{C} \\ & D & \longmapsto & \pi(D) \end{array}$$

that sends D to the intersection of the segment $[D, K_Z]$ with $\partial^+\mathcal{C}$. The intersection exists because $K_Z \notin \overline{\mathrm{Eff}}(Z)$, while $D \in \mathcal{C}$, and the intersection is unique by convexity of $\mathcal{C}$. The injectivity of π follows directly from the definition of $\partial^+\mathcal{C}'$, and π is also surjective, because by definition the cone $\mathcal{C}$ in contained in the cone over the convex hull of K_Z and the A_i, which is the same as the cone over the convex hull of K_Z and $\mathcal{C}'$. In conclusion π is a homeomorphism, as expected. □

Recall that the codimension of a face is taken relatively to the ambient space $N^1(Z)$, so in particular if $\mathcal{F}^k \subseteq \partial^+\mathcal{C}$ we have $k \geqslant 1$.

By Remark 4.18, a face $\mathcal{F}^1$ of codimension 1 in $\partial^+\mathcal{C}$ corresponds to a Mori fibre space, or equivalently a rank 1 fibration (Lemma 3.3). More generally, we now prove that inner codimension r faces in $\partial^+\mathcal{C}$ correspond to rank r fibrations.

Proposition 4.25. *Let $\mathcal{F}^r \subseteq \partial^+\mathcal{C}$ be an inner codimension r face. By Lemma 4.15, we can write $\mathcal{F}^r = \overline{\mathcal{A}_j} \cap \overline{\mathcal{A}_i}$ with $\mathcal{A}_j$ a chamber of maximal dimension and $\mathcal{A}_i \subseteq \partial^+\mathcal{C}$ the Mori chamber containing the interior of $\mathcal{F}^r$. Then*

(1) *The associated morphism $\varphi_{ji}\colon Z_j \longrightarrow Z_i$ is a rank r fibration.*

(2) *If $\mathcal{F}^s \subseteq \partial^+\mathcal{C}$ is an inner codimension s face and $\mathcal{F}^r \subseteq \mathcal{F}^s$, then the rank r fibration associated to $\mathcal{F}^r$ from (1) factorises through the rank s fibration associated to $\mathcal{F}^s$.*

Proof. (1) We check the assertions of Definition 3.1:

(RF2). By Lemma 4.15(*iii*), $\varphi_{ji}\colon Z_j \longrightarrow Z_i$ is a morphism with relative Picard rank equal to r, and $\dim Z_i < \dim Z_j$ because $\mathcal{A}_i \subseteq \partial^+\mathcal{C}$.

(RF4). This is Proposition 4.10(6).

(RF5). To show that $-K_{Z_j}$ is φ_{ji}-big, we take $D \in \overline{\mathcal{A}_j} \cap \mathcal{A}_i$. By Proposition 4.10, we have $D = K_Z + \Delta$ for some ample divisor Δ, and $\varphi_{j*}D \in \mathrm{Nef}(Z_j)$ is φ_{ji}-trivial. By Lemma 4.6(3) φ_j is a semiample model of any element of $\mathcal{A}_j$. So φ_j is a birational contraction and $\varphi_{j*}K_Z = K_{Z_j}$, which we rewrite as $-K_{Z_j} = \varphi_{j*}\Delta - \varphi_{j*}D$. Since Δ is ample and φ_j is birational, the divisor $\varphi_{j*}\Delta$ is big , which means we can write it as a sum of an ample and an effective divisor. So $-K_{Z_j}$ is the sum of a φ_{ji}-ample and an effective divisor and hence is φ_{ji}-big by Lemma 2.1.

(RF1). We prove that Z_j/Z_i is a Mori dream space:

(MD1) and (MD2). By Proposition 4.10, Z_j is $\mathbf{Q}$-factorial terminal, Z_i has rational singularities and $\dim Z_j > \dim Z_i$. A general fibre of φ_{ji} has rational singularities by Lemma 2.15. By Remark 4.13 we can assume that (Z, Δ), and also $(Z_j, \varphi_{j*}\Delta)$, are klt pairs. By Proposition 4.10(5) the divisor $K_{Z_j} + \varphi_{j*}\Delta = \varphi_{j*}D$ is φ_{ji}-trivial. We have just seen that $-K_{Z_j}$ is φ_{ji}-big. Then it follows from Lemma 2.16(1) that a general fibre of φ_{ji} is rationally connected.

(MD3). We show that the nef cone $\mathrm{Nef}(Z_j/Z_i)$ is generated by finitely many semiample divisors.

We take $D_j \in \mathrm{Nef}(Z_j)$ and set $D = \varphi_j^* D_j \in N^1(Z)$. We choose $D' \in \mathring{\mathcal{F}}^r \subseteq \mathcal{A}_i \cap \overline{\mathcal{A}_j}$. By Remark 4.14, for $t \gg 0$ we have $D + tD' \in \mathcal{C}$. By Lemma 4.11(1)&(2), we have $D + tD' \in \overline{\mathcal{A}_j}$. Since $\varphi_{j*}D'$ is φ_{ji}-trivial by Proposition 4.10(5)(*i*), we get that $\varphi_{j*}(D + tD') = D_j + t\varphi_{j*}D'$ is equivalent to D_j in $\mathrm{Nef}(Z_j/Z_i)$. Hence, any class in $\mathrm{Nef}(Z_j/Z_i)$ can be represented by a divisor in $\varphi_{j*}\overline{\mathcal{A}_j}$. We conclude that $\mathrm{Nef}(Z_j/Z_i)$ is generated by finitely many divisors of the form $\varphi_{j*}(K_Z + \Delta)$, where $K_Z + \Delta$ runs over the vertices of a polytope generating the cone $\overline{\mathcal{A}_j}$, and the $\varphi_{j*}(K_Z + \Delta)$ are semiample by Proposition 4.10(4).

(MD4). Let $D_j \in \mathrm{IntMov}(Z_j)$, in particular D_j is big. Set $D = \varphi_j^* D_j$ and pick $D' \in \mathring{\mathcal{F}}^r \subseteq \mathcal{A}_i \cap \overline{\mathcal{A}_j}$. By Remark 4.14, for $t \gg 0$ we have $\hat{D} := D + tD' \in \mathcal{C}$.

Replacing D by an arbitrary close class in $\mathcal{C}$ we can assume that $\hat{D} = D + tD' \in \mathcal{A}_k$ where $\mathcal{A}_k$ is of maximal dimension. We also replace D_j by $\varphi_{j*}D$, which is a small perturbation of the initial class hence still in $\mathrm{IntMov}(Z_j)$. We keep the same notation for simplicity. (Observe that after perturbation we lose the property $D = \varphi_j^* D_j$, but we will not need it). By finiteness of the chamber decomposition,

$\mathcal{A}_k$ does not depend on the choice of the large real t, which also implies $D' \in \overline{\mathcal{A}_k} \cap \mathcal{A}_i$. So we have $\mathcal{F}_{ji} \subseteq \mathcal{F}_{ki}$, hence a similar inclusion for the vector subspaces spanned by these faces. By Lemma 4.15(*iii*) this implies that all divisors contracted by φ_j are also contracted by φ_k, hence $f_k := \varphi_k \circ \varphi_j^{-1} \colon Z_j \dashrightarrow Z_k$ is a birational contraction.

As above D_j and $\hat{D}_j := D_j + t\varphi_{j*}D'$ represent the same class in $N^1(Z_j/Z_i)$. Moreover by Lemma 4.6(2) we have $\varphi_{j*}D' = \varphi_{ji}^* D_i$, and the pull-back of an ample divisor being movable we have $\varphi_{j*}D' \in \overline{\mathrm{Mov}}(Z_j)$. So we have $\hat{D}_j \in \mathrm{IntMov}(Z_j)$, and $\varphi_{j*}\hat{D} = \hat{D}_j$ with $\hat{D} \in \mathcal{A}_k$.

We now check that the birational contraction $f_k \colon Z_j \dashrightarrow Z_k$ is the ample model of $\hat{D}_j$. Since $\hat{D} \in \mathcal{A}_k$, we already have $\hat{D}_k := f_{k*}\hat{D}_j = \varphi_{k*}\hat{D} \in \mathrm{Ample}(Z_k)$. Let $p \colon W \longrightarrow Z$, $q_j \colon W \longrightarrow Z_j$ and $q_k \colon W \longrightarrow Z_k$ be common resolutions of φ_j and φ_k and

$$E := p^*\hat{D} - q_j^*\hat{D}_j, \qquad\qquad F := p^*\hat{D} - q_k^*\hat{D}_k$$

the q_j-exceptional and q_k-exceptional divisors respectively. By Proposition 4.10(4) we have $F \geqslant 0$. (On the other hand E might not be effective). We write $F = F_j + R$ where $F_j \geqslant 0$ is q_j-exceptional and $R \geqslant 0$ is q_j-effective. Then we have

$$\begin{aligned} E - F_j &= q_k^*\hat{D}_k - q_j^*\hat{D}_j + R \\ &= \mathrm{nef} - q_j\text{-trivial} + q_j\text{-effective} \\ &= q_j\text{-nef} + q_j\text{-effective}. \end{aligned}$$

The Negativity Lemma 2.2 gives $F_j - E \geqslant 0$, hence also $q_j^*\hat{D}_j - q_k^*\hat{D}_k = F - E = R + F_j - E \geqslant 0$, which by Lemma 4.6(3) achieves the proof that f_k is the ample model of $\hat{D}_j$.

Since $\hat{D}_j \in \mathrm{IntMov}(Z_j)$, its ample model f_k is a pseudo-isomorphism. Finally $\hat{D}_j \in f_k^*(\mathrm{Ample}(Z_k/Z_i))$ where Z_k is $\mathbf{Q}$-factorial, and by taking closures we obtain

$$\overline{\mathrm{Mov}}(Z_j/Z_i) \subseteq \bigcup f_l^*(\mathrm{Nef}(Z_l/Z_i))$$

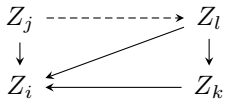
for some finite collection of pseudo-isomorphisms $f_l \colon Z_j \dashrightarrow Z_l$ over Z_i to $\mathbf{Q}$-factorial varieties.

For the other inclusion, we note that for any pseudo-isomorphism $f_l \colon Z_j \dashrightarrow Z_l$ over Z_i, we have $f_l^* \,\mathrm{Ample}(Z_l/Z_i) \subset \mathrm{Mov}(Z_l/Z_i)$ and the claim follows by taking closures.

(RF3). Let $D_j \in N^1(Z_j)$ be a divisor. We now show that the output of any D_j-MMP from Z_j over Z_i can be obtained by running a K_Z-MMP from Z. Let $D' \in \mathcal{F}^r \subseteq \overline{\mathcal{A}_j}$. Then by Proposition 4.10(4), φ_j is a semiample model of D', φ_i is its ample model, and by Lemma 4.6(2) $\varphi_{j*}D' = \varphi_{ji}^* H_i$ for some ample divisor H_i on Z_i. To run a D_j-MMP from Z_j over Z_i, we pick $H_j \in \mathrm{Ample}(Z_j)$ and consider all pseudo-effective convex combinations $D_t := \varepsilon(tD_j + (1-t)H_j) + \varphi_{ij}^* H_i$ for some $1 \gg \varepsilon > 0$. The set of the $\varphi_j^* D_t$ is a segment in a small neighborhood of D' inside $\mathcal{C}$. Therefore, any intermediate variety in this D_j-MMP over Z_i can be obtained by running a K_Z-MMP from Z. In particular the output of this MMP has the form $\operatorname{Proj} H^0(Z_j, D_{t_0}) = \operatorname{Proj} H^0(Z, \varphi_j^* D_{t_0})$ for some $t_0 \in (0,1)$, and by Proposition 4.10(3), this is a $\mathbf{Q}$-factorial and terminal variety, as expected.

(2) (Analogous to [LZ17, Proposition 3.10(2)]): Let $\mathcal{A}_i, \mathcal{A}_k \subseteq \partial^+\mathcal{C}$ be the chambers containing the interior of $\mathcal{F}^r$, $\mathcal{F}^s$ respectively. By Lemma 4.15(3) there exist maximal chambers $\mathcal{A}_j$ and $\mathcal{A}_l$ such that $\mathcal{F}^r = \overline{\mathcal{A}_j} \cap \overline{\mathcal{A}_i}$ and $\mathcal{F}^s = \overline{\mathcal{A}_l} \cap \overline{\mathcal{A}_k}$. Since

moreover $\mathcal{F}^r \subseteq \mathcal{F}^s$ implies that $\overline{\mathcal{A}_l} \cap \mathcal{A}_i \neq \emptyset$, by Proposition 4.10(5) we have a commutative diagram induced by the maps from Z:



We want to prove that the birational map $Z_j \dashrightarrow Z_l$ is a birational contraction.

Let $D \in \mathring{\mathcal{F}}^r \subseteq \mathcal{A}_i$. There exists an ample class $\Delta \in \mathcal{C}$ and $t_1 > 0$ such that $D = (1-t_1)\Delta + t_1 K_Z$. For $t_1 > t_0 > 0$ sufficiently close to t_0, any chamber of maximal dimension $\mathcal{A}_{j_0}$ such that $(1-t_0)\Delta + t_0 K_Z \in \mathcal{A}_{j_0}$ satisfies $\mathcal{F}^r \subset \overline{\mathcal{A}_{j_0}}$. Now there exists a small perturbation Δ' of Δ such that the segment $[\Delta', K_Z]$ meets successively a chamber $\mathcal{A}_{j_0}$ and then the chamber $\mathcal{A}_l$. Indeed, $t_1 > t_0$ and the ordering is preserved under a small perturbation. Up to replacing j by this j_0, by Remark 4.19 this segment corresponds to a K_Z-MMP with scaling of Δ', and provides the expected birational contraction from Z_j to Z_l. □

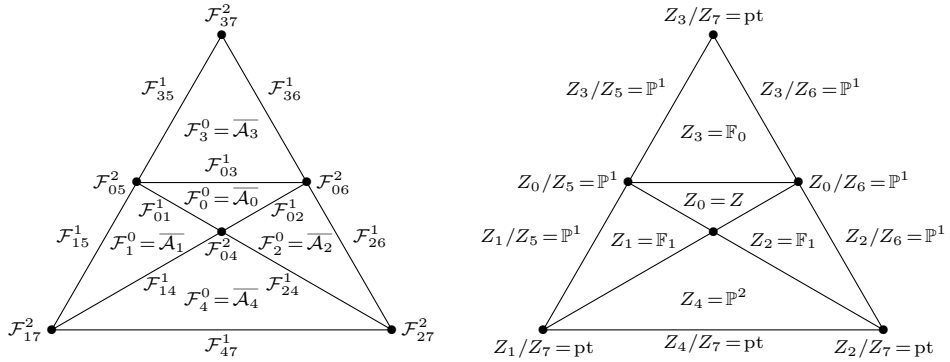


FIGURE 5. rank r fibrations in Example 4.17.

Example 4.26. On Figure 5 we label the boundary faces from Example 4.17 with their corresponding rank r fibration, as given by Proposition 4.25 ($r = 1$ or 2 here). We also indicate the images of ample models corresponding to chambers of maximal dimension.

Corollary 4.27. *If the intersection $\mathcal{F}^1_i \cap \mathcal{F}^1_j$ of non-big codimension 1 faces has codimension 2, then there is a Sarkisov link between the corresponding Mori fibre spaces.*

Proof. By Proposition 4.25 there is a rank 2 fibration corresponding to the codimension 2 face $\mathcal{F}^2 := \mathcal{F}^1_i \cap \mathcal{F}^1_j$ that factorises through the rank 1 fibrations associated to $\mathcal{F}^1_i$ and $\mathcal{F}^1_j$. This is exactly the definition of a Sarkisov link (Definition 3.8). □

Corollary 4.28. *Let $\mathcal{F}^3$ be a face in $\partial^+\mathcal{C}$ of codimension 3 and T/B be the associated rank 3 fibration, as given in Proposition 4.25. Then the elementary relation associated to T/B corresponds to the finite collection of codimension 1 faces $\mathcal{F}^1_1, \dots, \mathcal{F}^1_s$ containing $\mathcal{F}^3$, and ordered such that $\mathcal{F}^1_j$ and $\mathcal{F}^1_{j+1}$ share a codimension 2 face for all j (where indexes are taken modulo s).*

Proof. This is just a rephrasing of Proposition 4.3, using Proposition 4.25 to associate a rank 1 or 2 fibration dominated by T/B to each codimension 1 or 2 face containing $\mathcal{F}^3$, and using Corollary 4.27 to associate a Sarkisov link to each pair of codimension 1 faces sharing a common codimension 2 face. □

Let X/B a Mori fibre space. We denote by $\mathrm{BirMori}(X)$ the groupoid of birational maps between Mori fibre spaces birational to X. The group of birational selfmaps $\mathrm{Bir}(X)$ is a subgroupoid of $\mathrm{BirMori}(X)$. The motivation for introducing the notion of elementary relation is the following result. The first part is a reformulation of [HM13, Theorem 1.1]. The second part is strongly inspired by [Kal13, Theorem 1.3], observe however that our notion of elementary relation is more restrictive.

In the statement we use the formalism of presentations by generators and relations for groupoids. This is very similar to the more familiar setting of groups: we have natural notions of a free groupoid, and of a normal subgroupoid generated by a set of elements. We refer to [Bro06, §8.2 and 8.3] for details.

Theorem 4.29. *Let X/B be a terminal Mori fibre space.*

(1) *The groupoid* $\mathrm{BirMori}(X)$ *is generated by Sarkisov links and automorphisms.*

(2) *Any relation between Sarkisov links in* $\mathrm{BirMori}(X)$ *is generated by elementary relations.*

Proof. (1) is the main result of [HM13]. The idea of the proof is to take Z a resolution of a given birational map $\varphi\colon X_1/B_1 \dashrightarrow X_2/B_2$, and to consider the cone $\mathcal{C}$ with a choice of ample divisors as given by Proposition 4.23 (applied with $t=2$, $\theta_1=\varphi$, $\theta_2=\varphi^{-1}$). Then one takes a general 2-dimensional affine slice of $\mathcal{C}$ that passes through the codimension 1 faces associated to X_1/B_1 and X_2/B_2. The intersection of this slice with $\partial^+\mathcal{C}$ is a polygonal path corresponding to successive pairwise neighbour codimension 1 faces, and by Corollary 4.27 this gives a factorisation of φ into Sarkisov links.

(2). The proof is essentially the same as in [LZ17, Proposition 3.15], we repeat the argument for the convenience of the reader.

Let

$$X_0/B_0 \overset{\chi_1}{\dashrightarrow} X_1/B_1 \overset{\chi_2}{\dashrightarrow} \cdots \overset{\chi_t}{\dashrightarrow} X_t/B_t$$

be a relation between t Sarkisov links, meaning that $\chi_t \circ \cdots \circ \chi_1$ is the identity on $X_0 = X_t$. We take a smooth resolution Z dominating all the X_i/B_i, and consider the cone $\mathcal{C} \subset N^1(Z)$ constructed from a choice of ample divisors as in Proposition 4.23. We may assume $\rho(Z) \geqslant 4$ (otherwise we simply blow-up some points on Z), so that by Lemma 4.24 the non-big boundary $\partial^+\mathcal{C}$ is a cone over a polyhedral complex $\mathcal{S}$ homeomorphic to a disc or a sphere of dimension $\rho(Z)-2 \geqslant 2$. In particular, the section $\mathcal{S}$ is simply connected. Now we construct a 2-dimensional simplicial complex $\mathcal{B}$ embedded in $\mathcal{S}$ as follows. Vertices are the barycenters $p(\mathcal{F}^k)$ of codimension k faces $\mathcal{F}^k$ for $k=1,2$ or 3. We call k the type of the vertex. We put an edge between $p(\mathcal{F}^j)$ and $p(\mathcal{F}^k)$ if $\mathcal{F}^j$ is a proper face of $\mathcal{F}^k$, and a 2-simplex for each sequence $\mathcal{F}^3 \subset \mathcal{F}^2 \subset \mathcal{F}^1$. The complex $\mathcal{B}$ is homeomorphic to the barycentric subdivision of the 2-skeleton of the dual cell complex of $\mathcal{S}$. It follows that $\mathcal{B}$ is simply connected (recall that the 2-skeleton of a simply connected complex is again simply connected, see e.g. [Hat02, Corollary 4.12]). Then we restrict to the subcomplex $\mathcal{I} \subseteq \mathcal{B}$ corresponding to inner faces, which are the ones that intersect the relative interior of $\mathcal{S}$. The simplicial complex $\mathcal{I}$ is a deformation retract of the interior of $\mathcal{B}$, so $\mathcal{I}$ again is simply connected. By Proposition 4.25 we can associate

a rank r fibration to each vertex of type r, and two vertices are connected by an edge if and only if the corresponding fibrations factorise through each other. By Corollary 4.28, around each vertex of type 3 there is a unique disc whose boundary loop encodes an elementary relation. The complex $\mathcal{I}$ is a deformation retract of $\mathcal{B}$ (hence also simply connected) and its 2-dimensional components are the union of these discs. The initial relation corresponds to a loop in $\mathcal{I}$ that only passes through vertices of types 1 and 2. We can realise the homotopy of this loop to the constant loop inside the simply connected complex $\mathcal{I}$ by using these elementary relations, and this translates as a factorisation of the initial relation as a product of conjugates of elementary relations. □

The whole construction leading to the previous theorem can be made in a relative setting, that is, where all involved varieties admit a morphism to a fixed base variety B. In fact the paper [BCHM10] on which relies [HM13] is written with this level of generality. In the particular case where the base B has dimension $n-1$, we obtain the following statement, slightly more precise than Theorem 4.29(1).

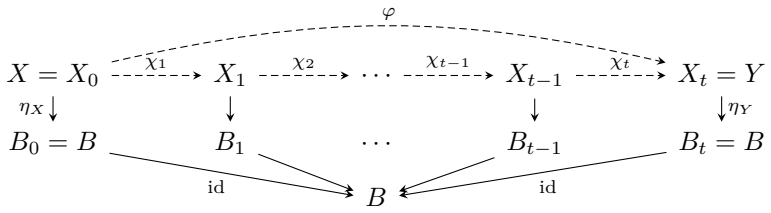


FIGURE 6. The diagram of Lemma 4.30

Lemma 4.30. *Let $\eta_X\colon X \longrightarrow B$ and $\eta_Y\colon Y \longrightarrow B$ be two conic bundles over the same base. Then any birational map $\varphi\colon X \dashrightarrow Y$ over B decomposes into a sequence of Sarkisov links of conic bundles over B. More precisely, we have a commutative diagram as in Figure 6, such that for each $i = 1, \dots, t$, B_i/B is a birational morphism, X_i/B_i is a conic bundle and χ_i is a Sarkisov link.*

4.D. **Examples of elementary relations.** In this section we give examples of elementary relations, mostly in dimension $n = 3$.

Example 4.31. Let X be a Fano variety with $\mathbf{Q}$-factorial terminal singularities and Picard rank 3. Then X/pt is a rank 3 fibration (Example 3.2(1)), hence there is an associated elementary relation. In the case where X is smooth of dimension 3, these relations were studied systematically by Kaloghiros, using a classification result by Mori-Mukai: see [Kal13, Example 4.9 and Figures 3,4 & 5]. With respect to the setting of §4.C, in these examples we have $Z = X$, $N^1(Z) \simeq \mathbf{R}^3$ and $\partial^+\mathcal{C}$ is the cone over a complex homeomorphic to a circle, which encodes the elementary relation. Observe that the simple 2-dimensional Example 4.26 also belongs to this family of examples.

Example 4.32. Let $L \cup L' \subset \mathbb{P}^3$ be two secant lines, and P the plane containing them. Let $X \longrightarrow \mathbb{P}^3$ be the blow-up of L with exceptional divisor E, let $\ell \subset E$ be the fibre intersecting the strict transform of L', and let $T \longrightarrow X$ be the blow-up of L', with exceptional divisor E'.

From T we can flop ℓ to get a 3-fold T', which is obtained by the same two blow-ups in the reverse order: first the blow-up $X' \longrightarrow \mathbb{P}^3$ of $L' \subset \mathbb{P}^3$ and then the blow-up $T' \longrightarrow X'$ of (the strict transform of) L on X'.

From T or T' one can contract the strict transform of P onto a smooth point, obtaining two 3-folds Y and Y' also related by the flop of ℓ.

The elementary relation associated to the rank 3 fibration T/pt (or equivalently to T'/pt), is depicted on Figure 7. There are five links in the relation, where χ_1 has type I, χ_2 and χ_4 have type II, χ_3 has type IV, and χ_5 has type III.

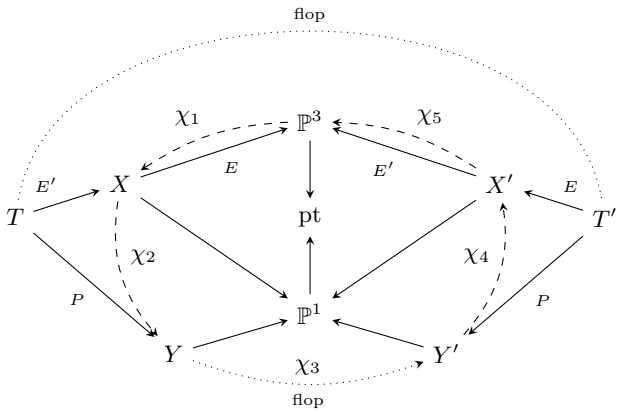


FIGURE 7. The elementary relation from Example 4.32.

Example 4.33. Consider the blow-up $\mathbb{F}_1 \longrightarrow \mathbb{P}^2$ of a point, with exceptional curve $\Gamma \subset \mathbb{F}_1$. In $\mathbb{P}^1 \times \mathbb{F}_1$, write $D = \mathbb{P}^1 \times \Gamma$, and $C = \{0\} \times \Gamma$. Let T be the blow-up of C, with exceptional divisor E. Then $T/\mathbb{P}^2$ is a rank 3 fibration, and we now describe the associated elementary relation (see Figure 8). We let the reader verify the following assertions (since all varieties are toric, one can for instance use the associated fans).

First the two-rays game $T/\mathbb{F}_1$ gives a link of type II

$$\chi_1 \colon \mathbb{P}^1 \times \mathbb{F}_1 \dashrightarrow \mathbb{P}^1 \boxtimes \mathbb{F}_1,$$

where $\mathbb{P}^1 \boxtimes \mathbb{F}_1$ denotes a Mori fibre space over $\mathbb{F}_1$ that is a non-trivial but locally trivial $\mathbb{P}^1$-bundle. The link χ_1 involves the pair $D \cup E$ of divisors of type II for $T/\mathbb{F}_1$.

The divisor D on T can be contracted in two ways to a curve $\mathbb{P}^1$, that is, T dominates a flop between $\mathbb{P}^1 \boxtimes \mathbb{F}_1$ and another variety X. This variety X admits a divisorial contraction to $\mathbb{P}^1 \times \mathbb{P}^2$, with exceptional divisor the strict transform of E, which here is a divisor of type I for $X/\mathbb{P}^2$. This corresponds to a link of type III

$$\chi_2 \colon \mathbb{P}^1 \boxtimes \mathbb{F}_1 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^2.$$

Finally the two-rays game $\mathbb{P}^1 \times \mathbb{F}_1/\mathbb{P}^2$, which factorises via $\mathbb{F}_1$ and $\mathbb{P}^1 \times \mathbb{P}^2$, gives a link of type I

$$\chi_3 \colon \mathbb{P}^1 \times \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{F}_1.$$

In conclusion we get an elementary relation $\chi_3 \circ \chi_2 \circ \chi_1 = \mathrm{id}$.

In contrast with Lemma 3.17, observe that D and E are divisors of type II for $T/\mathbb{F}_1$, but divisors of type I for $T/\mathbb{P}^2$.

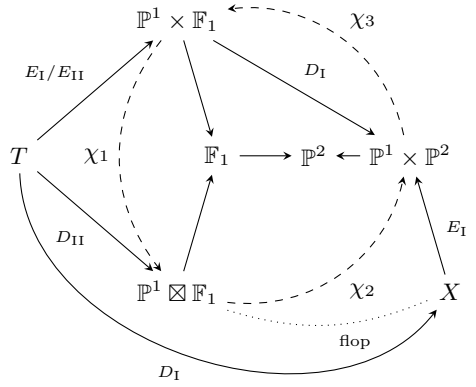


FIGURE 8. The elementary relation from Example 4.33. We indicate the type of contracted divisors in index.

Example 4.34 (Example 3.15 over $B = \mathbb{P}^2$)**.** Consider $\mathbb{P}^1 \times \mathbb{P}^2$, and let $\Gamma \subset \mathbb{P}^2$ be a line, $D \simeq \mathbb{P}^1 \times \Gamma$ the pull-back of Γ in $\mathbb{P}^1 \times \mathbb{P}^2$, $\Gamma' = \{t\} \times \Gamma \subset D$ a section and $p \in D \smallsetminus \Gamma'$ a point. Let $T \longrightarrow \mathbb{P}^1 \times \mathbb{P}^2$ be the blow-up of Γ' and p, with respective exceptional divisors D' and E, and denote again D the strict transform of $\mathbb{P}^1 \times \Gamma$ in T. Then the induced morphism $\eta\colon T \longrightarrow \mathbb{P}^2$ is a rank 3 fibration that gives rise to the relation on Figure 9.

The figure was computed using toric fans in $\mathbf{Z}^3$, starting from the standard fan of $\mathbb{P}^1 \times \mathbb{P}^2$ with primitive vectors $(1,0,0)$, $(0,1,0)$, $(-1,-1,0)$, $(0,0,1)$, $(0,0,-1)$, and with the following choices

$$D : (1,0,0), \qquad D' : (1,0,1), \qquad E : (1,1,-1).$$

The varieties T' and W' both have one terminal singularity, all other varieties are smooth. There are two distinct Francia flips from T', which are $T' \dashrightarrow T$ and $T' \dashrightarrow T''$. Observe also that the link χ_1 is exactly Example 3.11(2).

Example 4.35. The article [AZ17] contains a beautiful example of an elementary relation involving five Sarkisov links. In Figure 10 we reproduce the diagram from [AZ17, §5.2], and we refer to their paper for a detailed description of the varieties. The Sarkisov links χ_1 and χ_3 have type II, χ_2 has type I, χ_4 has type IV and χ_5 type III. The relation is associated to the rank 3 fibration Z_1'/pt, or equivalently to Z_2'/pt. In fact other equivalent choices of varieties of Picard rank 3 are omitted from the picture (dominating respectively Y_1', X_3', X_1' and X_1''). The morphisms from $Z, \bar{Z}$ and $\tilde{Z}$ to $\mathbb{P}^1$ are fibrations in cubic surfaces. Observe that the top rows of the Sarkisov diagrams display non trivial pseudo-isomorphisms, involving flips and flops. Note that each pseudo-isomorphism labeled "n flops" really corresponds to a

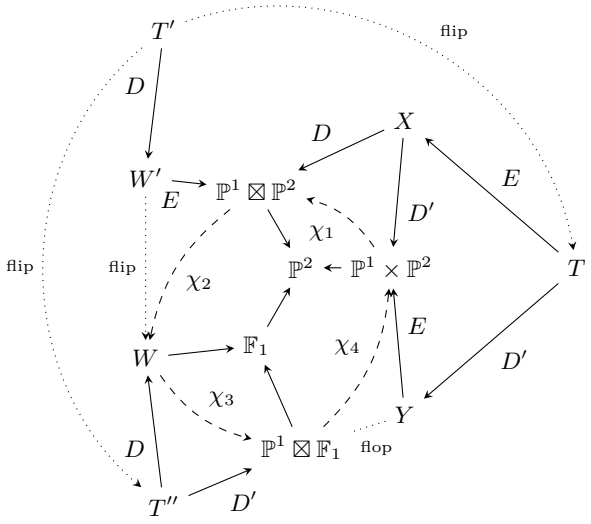


FIGURE 9. Elementary relation from Example 4.34.

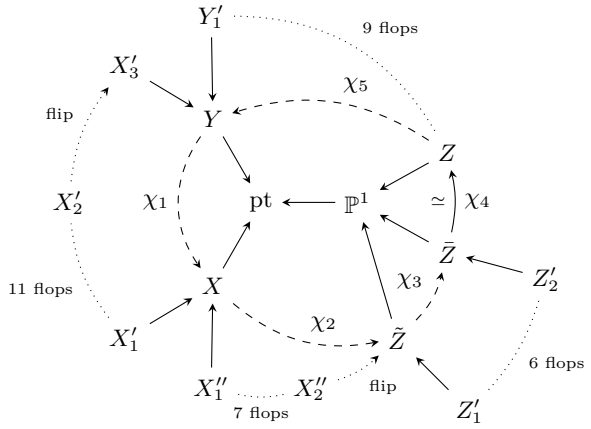


FIGURE 10. Elementary relation from Example 4.35.

single flop with n components (which by definition are all numerically proportional), in accordance with Remark 3.10.

5. Elementary relations involving Sarkisov links of conic bundles of type II

This section is devoted to the study of elementary relations involving Sarkisov links of conic bundles of type II that are complicated enough, meaning their covering gonality is large. We give some restriction on such relations that will allow us to

prove Theorem D. Firstly in Proposition 5.3 we cover the case of relations over a base of dimension $\leqslant n-2$, where n is the dimension of the Mori fibre spaces, using the BAB conjecture and working with Sarkisov links of large enough covering gonality. Secondly, the case of relations over a base of dimension $n-1$ is handled in Proposition 5.5, using only the assumption that the covering gonality is >1.

5.A. **A consequence of the BAB conjecture.** The following is a consequence of the BAB conjecture, which was recently established in arbitrary dimension by C. Birkar.

Proposition 5.1. *Let n be an integer, and let $\mathcal{Q}$ be the set of weak Fano terminal varieties of dimension n. There are integers $d, l, m \geqslant 1$, depending only on n, such that for each $X \in \mathcal{Q}$ the following hold:*

(1) $\dim(H^0(-mK_X)) \leqslant l$;

(2) *The linear system $|-mK_X|$ is base-point free;*

(3) *The morphism $\varphi\colon X \xrightarrow{|-mK_X|} \mathbb{P}^{\dim(H^0(-mK_X))-1}$ is birational onto its image and contracts only curves $C \subseteq X$ with $C \cdot K_X = 0$;*

(4) $\deg \varphi(X) \leqslant d$.

Proof. By [Bir16b, Theorem 1.1], varieties in $\mathcal{Q}$ form a bounded family (here we use the observation that for a given $X \in \mathcal{Q}$, the pair $(X, \emptyset)$ is ε-lc for any $0 < \varepsilon < 1$). In particular, by [Bir16a, Lemma 2.24], the Cartier index of such varieties is uniformly bounded. Then [Kol93, Theorem 1.1] gives the existence of $m = m(n)$ such that $|-mK_X|$ is base-point free for each $X \in \mathcal{Q}$. By [Bir16a, Theorem 1.2], we can increase m if needed, and assume that the associated morphism

$$\varphi\colon X \xrightarrow{|-mK_X|} \mathbb{P}^{\dim(H^0(-mK_X))-1}$$

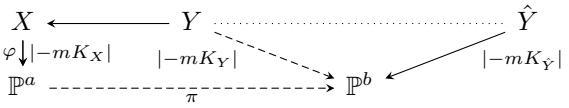
is birational onto its image. As it is a morphism, this implies that it contracts only curves $C \subseteq X$ with $C \cdot K_X = 0$. Finally, since $\mathcal{Q}$ is a bounded family, the two integers $\dim(H^0(-mK_X))$ and $\deg \varphi(X)$ are bounded. □

Corollary 5.2. *Let $\pi\colon Y \longrightarrow X$ be the blow-up of a reduced but not necessarily irreducible codimension 2 subvariety $\Gamma \subset X$, $Y \dashrightarrow \hat{Y}$ a pseudo-isomorphism, and assume that both X and $\hat{Y}$ are weak Fano terminal varieties of dimension $n \geqslant 3$, whose loci covered by curves with trivial intersection against the canonical divisor has codimension at least 2. Let φ be the birational morphism associated to the linear system $|-mK_X|$, with m given by Proposition 5.1, and assume that Γ is not contained in the exceptional locus $\mathrm{Ex}(\varphi)$. Then through any point of $\Gamma \smallsetminus \mathrm{Ex}(\varphi)$ there is an irreducible curve $C \subseteq \Gamma$ with $\mathrm{gon}(C) \leqslant d$, where d is the integer from Proposition 5.1.*

Proof. We choose the integers $d, l, m \geqslant 1$ associated to the dimension n in Proposition 5.1. We write $a = \dim(H^0(-mK_X)) - 1$ and $b = \dim(H^0(-mK_Y)) - 1$. Using the pseudo-isomorphism $Y \dashrightarrow \hat{Y}$, we also have $b = \dim(H^0(-mK_{\hat{Y}})) - 1$. By Proposition 5.1 the morphisms given by the linear systems $|-mK_X|$ and $|-mK_{\hat{Y}}|$ are birational onto their images and are moreover pseudo-isomorphisms onto their images, because of the assumption that the locus covered by curves with non-positive intersection against the canonical divisor has codimension at least 2.

Since $Y \longrightarrow X$ is the blow-up of Γ, each effective divisor equivalent to $-mK_Y$ is the strict transform of an effective divisor equivalent to $-mK_X$ passing through Γ

(with some multiplicity). In particular, we have $b \leqslant a$ and obtain a commutative diagram



where π is a linear projection away from a linear subspace $\mathcal{L} \simeq \mathbb{P}^r$ of $\mathbb{P}^a$ containing the image of Γ. Recall that we write $\varphi\colon X \longrightarrow \mathbb{P}^a$ the morphism given by $|-mK_X|$. The variety $\varphi(X) \subseteq \mathbb{P}^a$ has dimension n and degree $\leqslant d$ (Proposition 5.1), and is not contained in a hyperplane section. Since by assumption $\Gamma \subsetneq \mathrm{Ex}(\varphi)$, we get that φ induces a birational morphism from Γ to $\varphi(\Gamma)$.

We now prove that there is no (irreducible) variety $S \subseteq \varphi(X) \cap \mathcal{L}$ of dimension $n-1$ (recall that $\varphi(\Gamma) \subseteq \varphi(X) \cap \mathcal{L}$ has dimension $n-2$). Indeed, otherwise the strict transform of S on X would be a variety $S_X \subset X$ birational to S, so its strict transform in $\hat{Y}$, and in $\mathbb{P}^b$ is again birational to S (as the birational map from Y to its image in $\mathbb{P}^b$ is a pseudo-isomorphism). The linear system of the rational map $X \dashrightarrow \mathbb{P}^b$ is obtained from the linear system associated to $X \dashrightarrow \mathbb{P}^a$ by taking the subsystem associated to hyperplanes through $\mathcal{L}$. Hence, if $S \subseteq \mathcal{L}$, then every element of the linear system $|-mK_Y|$ contains the strict transform S_Y of S in $\hat{Y}$. This is impossible, as $|-mK_{\hat{Y}}|$ is base-point free (Proposition 5.1).

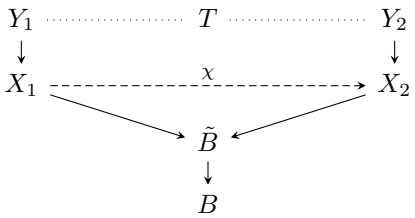
Now, the fact that $\varphi(X) \cap \mathcal{L} \subseteq \mathbb{P}^a$ does not contain any variety of dimension $\geqslant n-1$ implies, by Bézout Theorem, that all its irreducible components of dimension $n-2$ have degree $\leqslant d$. Therefore, each of the irreducible components of $\varphi(\Gamma)$ (birational to Γ) has degree $\leqslant d$.

We are now able to finish the proof, by showing that through any point $q \in \Gamma \smallsetminus \mathrm{Ex}(\varphi)$ there is an irreducible curve $C \subseteq \Gamma$ with $\mathrm{gon}(C) \leqslant d$. Since $\Gamma \longrightarrow \varphi(\Gamma)$ is a local isomorphism at q, it suffices to take a general linear projection from $\mathbb{P}^a$ to a linear subspace of dimension $n-2$, and to take C equal to the preimage of a line through the image of $\varphi(q)$. □

Proposition 5.3. *For each dimension $n \geqslant 3$, there exists an integer $d_n \geqslant 1$ depending only on n such that the following holds. If χ is a Sarkisov link of conic bundles of type* II *that arises in an elementary relation induced by rank* 3 *fibration T/B with* $\dim(T) = n$ *and* $\dim(B) \leqslant n-2$, *then* $\mathrm{cov.\,gon}(\chi) \leqslant \max\{d_n, 8\,\mathrm{conn.\,gon}(T)\}$.

Proof. We choose $d_n \geqslant 8$ to be bigger than the integers d given by Proposition 5.1 for the dimensions $3, \dots, n$, and prove the result for this choice of d_n.

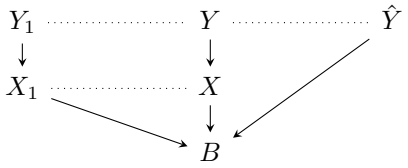
The Sarkisov link χ, which is dominated by T/B by assumption, has the form



where X_1, X_2, Y_1, Y_2 have dimension n and $\tilde{B}$ has dimension $n-1$. Since $\dim B \leqslant n-2$, we have $\rho(\tilde{B}/B) \geqslant 1$, and on the other hand $\rho(Y_i/B) \leqslant 3$, for $i = 1, 2$, which implies that $\rho(\tilde{B}/B) = 1$, and that the birational contractions $T \dashrightarrow Y_1$, $T \dashrightarrow Y_2$ are pseudo-isomorphisms. Moreover, $Y_1 \longrightarrow X_1$ contracts a divisor E onto a variety

$\Gamma_1 \subset X_1$ of dimension $n-2$, birational to its image $\tilde{\Gamma} \subset \tilde{B}$ via the morphism $X_1 \longrightarrow \tilde{B}$ (Lemma 3.23). We need to check that $\mathrm{cov.\,gon}(\Gamma_1) = \mathrm{cov.\,gon}(\tilde{\Gamma}) \leqslant d_n$, where d_n is chosen as above. We may then assume that $\mathrm{cov.\,gon}(\tilde{\Gamma}) > 1$.

Now, $\tilde{B}/B$ is a klt Mori fibre space by Lemma 3.13 and X_1/B is a rank 2 fibration by Lemma 3.4(1). By Lemma 3.5, the rank 2 fibration X_1/B is pseudo-isomorphic, via a sequence of log-flips over B, to another rank 2 fibration X/B such that $-K_X$ is relatively nef and big over B. We then use Lemma 2.18 to obtain a sequence of log-flips $Y_1 \dashrightarrow Y$ over B such that the induced map $Y \longrightarrow X$ is a divisorial contraction. By Lemma 3.5 again, we get a sequence of log-flips over B from Y/B to another rank 3 fibration $\hat{Y}/B$ such that $-K_{\hat{Y}}$ is relatively nef and big over B.



As $\mathrm{cov.\,gon}(\tilde{\Gamma}) > 1$, by Lemma 2.16(3) the codimension 2 subvariety $\Gamma_1 \subset X_1$ is not contained in the base-locus of the pseudo-isomorphism $X_1 \dashrightarrow X$. So the image $\Gamma \subset X$ of Γ_1 is birational to Γ_1, and it suffices to show that $\mathrm{cov.\,gon}(\Gamma) \leqslant d_n$. Observe that $Y \longrightarrow X$ is the blow-up of Γ (Lemma 2.14).

We take a general point $p \in B$, and consider the fibres over p in X, Y and $\hat{Y}$ respectively, that we denote by X_p, Y_p and $\hat{Y}_p$, and which are varieties of dimension

$$n_0 = n - \dim B \in \{2, \dots, n\}.$$

By Corollary 3.6 the two varieties X_p and $\hat{Y}_p$ are weak Fano terminal varieties. Moreover, Y_p and $\hat{Y}_p$ are pseudo-isomorphic, as $Y \dashrightarrow \hat{Y}$ is a sequence of log-flips over B.

Observe that $\tilde{\Gamma} \subset \tilde{B}$ is a hypersurface and that $\tilde{\Gamma} \longrightarrow B$ is surjective. Indeed, otherwise $\tilde{\Gamma}$ would be the preimage of a divisor on B, and we would have $\mathrm{cov.\,gon}(\tilde{\Gamma}) = 1$, as the preimage of each point of $\tilde{B} \longrightarrow B$ is covered by rational curves (Lemma 3.13), in contradiction with our assumption. This implies that the morphism $\Gamma \longrightarrow B$ induced by the restriction of X/B is again surjective.

We then denote by $\Gamma_p \subset X_p$ the codimension 2 subscheme $\Gamma_p = \Gamma \cap X_p$, which is the fibre of $\Gamma \longrightarrow B$ over p, and which is not necessarily irreducible. Observe that $Y_p \longrightarrow X_p$ is the blow-up of Γ_p, as $Y \longrightarrow X$ is the blow-up of Γ (as explained before) and because the fibre over p is transverse to Γ (Lemma 3.23(4)).

Suppose first that $n_0 = 2$, which corresponds to $\dim(\Gamma) = \dim(B)$. In this case, X_p and $Y_p \simeq \hat{Y}_p$ are smooth del Pezzo surfaces, because by Corollary 3.6 the locus covered by curves trivial against the canonical divisor has codimension 2, hence is empty in the case $n_0 = 2$. Moreover Γ_p is a disjoint union of r points, where r is the degree of the field extension $\mathbf{C}(B) \subseteq \mathbf{C}(\Gamma_1)$. As Y_p is obtained from X_p by blowing-up Γ_p, the degree of the field extension is at most 8, which implies that $\mathrm{cov.\,gon}(\Gamma) \leqslant 8 \cdot \mathrm{cov.\,gon}(B) \leqslant 8\,\mathrm{conn.\,gon}(T)$ (Lemma 2.25).

We now consider the case $n_0 \geqslant 3$, which implies that Γ_p has dimension $n_0 - 2 \geqslant 1$. We apply Corollary 5.2 to the blow-up $Y_p \longrightarrow X_p$ of Γ_p and the pseudo-isomorphism $Y_p \dashrightarrow \hat{Y}_p$. The fact that the loci on X_p or $\hat{Y}_p$ covered by curves with trivial intersection against the canonical divisor has codimension at least 2 follows from Corollary 3.6. We obtain that for a general p, $\Gamma_p \setminus \mathrm{Ex}(\varphi)$ is covered by curves

of gonality at most d_n. In conclusion, we have found an open set $U = \Gamma \smallsetminus \mathrm{Ex}(\varphi) \subseteq \Gamma$ covered by curves of gonality at most d_n, as expected. □

Remark 5.4. It is not clear to us whether Proposition 5.3 could also hold for a link χ of type II between arbitrary Mori fibre spaces.

For instance in the case of threefolds, if χ is a link of type II between del Pezzo fibrations that starts with the blow-up a curve of genus g contained in one fibre, we suspect that g cannot be arbitrary large but we are not aware of any bound in the literature.

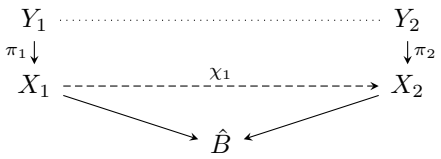
5.B. **Some elementary relations of length 4.**

Proposition 5.5. *Let χ_1 be a Sarkisov link of conic bundles of type* II *with* $\mathrm{cov.gon}(\chi_1) > 1$. *Let T/B be a rank* 3 *fibration with* $\dim B = \dim T - 1$, *which factorises through the Sarkisov link χ_1. Then, the elementary relation associated to T/B has the form*

$$\chi_4 \circ \chi_3 \circ \chi_2 \circ \chi_1 = \mathrm{id},$$

where χ_3 is a Sarkisov link of conic bundles of type II *that is equivalent to χ_1.*

Proof. The Sarkisov link χ_1 is given by a diagram



where X_1, X_2, Y_1, Y_2 are varieties of dimension n, and $\dim \hat{B} = n - 1$. Denote by $E_1 \subset Y_1$ and $E_2 \subset Y_2$ the respective exceptional divisors of the divisorial contractions π_1 and π_2. We denote again by $E_1, E_2 \subset T$ the strict transforms of these divisors, under the birational contractions $T \dashrightarrow Y_1$ and $T \dashrightarrow Y_2$. Then by Lemma 3.23(2), $E_1 \cup E_2$ is a pair of divisors of type II for $Y_1/\hat{B}$, hence also for T/B by Lemma 3.17. By Proposition 3.16(5), we are in one of the following mutually exclusive three cases:

(1) B is $\mathbf{Q}$-factorial, and there exists a divisor G of type I for T/B.
(2) B is not $\mathbf{Q}$-factorial.
(3) B is $\mathbf{Q}$-factorial, and there exists another pair $F_1 \cup F_2$ of divisors of type II for T/B.

We denote $\{X_i/B_i\}$ the finite collection of all rank 1 fibrations dominated by T/B. In each case we are going to show that this collection has cardinal 4.

Suppose first that (1) holds. By Proposition 3.16(1)(4) and Lemma 3.18, we can obtain such an X_i/B_i by a birational contraction contracting one the following four sets of divisors: $\{E_1\}$, $\{E_2\}$, $\{E_1, G\}$ or $\{E_2, G\}$. Moreover X_i/B_i is determined up to isomorphism by such a choice of contracted divisors:

• If $T \dashrightarrow X_i$ contracts $\{E_1, G\}$ or $\{E_2, G\}$, then $\rho(X_i) = \rho(T) - 2$ which implies $\rho(B_i/B) = 0$, that is, $B_i \longrightarrow B$ is an isomorphism. Then X_i is uniquely determined by Corollary 2.20.

• If $T \dashrightarrow X_i$ contracts $\{E_1\}$ or $\{E_2\}$, then $\rho(B_i/B) = 1$, and $B_i \dashrightarrow B$ is a birational contraction contracting the image of the divisor G. Then such a B_i is uniquely determined by Corollary 2.21.

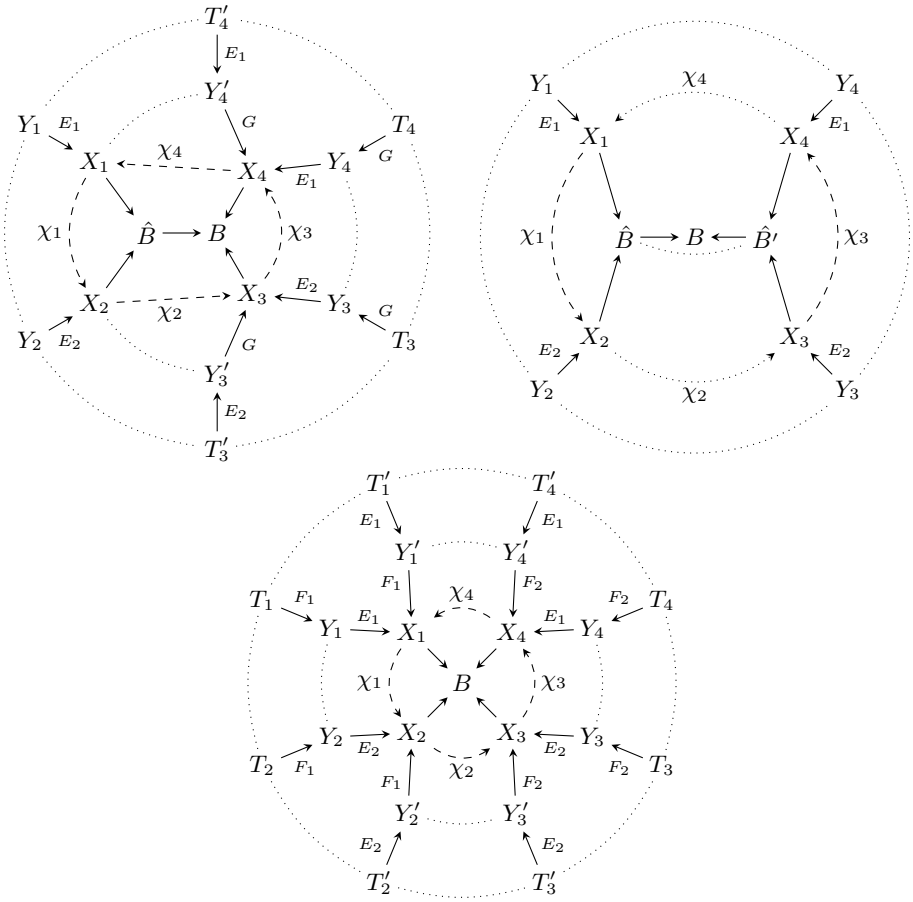


FIGURE 11. The elementary relation associated to T/B in cases (1), (2) and (3) of the proof of Proposition 5.5. Varieties are organized in circles according to their Picard rank over B.

In conclusion the relation given by Proposition 4.3 has the form

$$\chi_4 \circ \chi_3 \circ \chi_2 \circ \chi_1 = \mathrm{id},$$

and more precisely, up to a cyclic permutation exchanging the role of χ_1 and χ_3, we have a commutative diagram as in Figure 11, top left, where χ_2 and χ_4 have respectively type III and I, and χ_1 and χ_3 are equivalent Sarkisov links of type II.

Now consider Case (2). As $\hat{B}$ is $\mathbf{Q}$-factorial (Proposition 2.11), we have $\hat{B} \neq B$, hence $\rho(\hat{B}/B) = 1$ and the morphism $\hat{B} \longrightarrow B$ is a small contraction. By uniqueness of log-flip, there are exactly two small contractions from a $\mathbf{Q}$-factorial variety to B. Denote $\hat{B}' \longrightarrow B$ the other one. Then for each X_i/B_i, we have $B_i \simeq \hat{B}$ or $\hat{B}'$, and $\rho(X_i/B) = 2$. Hence the birational contraction $T \dashrightarrow X_i$ contracts exactly one divisor, which must be E_1 or E_2. Again this gives four possibilities. The actual existence of $X_3/\hat{B}'$ and $X_4/\hat{B}'$ arises from the two-rays games X_1/B and X_2/B.

We get a relation as in Figure 11, top right, with χ_1, χ_3 of type II and χ_2, χ_4 of type IV.

Finally consider Case (3). Then by Proposition 3.16(1)(4), each birational contraction $T \dashrightarrow X_i$ contracts one divisor among $E_1 \cup E_2$, and another one among $F_1 \cup F_2$. Again this gives four possibilities. In each case $\rho(B_i/B) = 0$ hence B_i is isomorphic to B, and then Corollary 2.20 says that X_i is determined up to isomorphism by such a choice. We obtain a relation with four links of type II, as on Figure 11, bottom. □

Remark 5.6. Example 4.33 illustrates why the assumption on the covering gonality is necessary in Proposition 5.5.

5.C. **Proof of Theorem D.** In order to prove Theorem D, we use the generators and relations of $\mathrm{BirMori}(X)$ which are given in Theorem 4.29. The key results are then Propositions 5.3 and 5.5.

Proof of Theorem D. We choose the integer d associated to the dimension n by Proposition 5.3, and set $M = \max\{d, 8\,\mathrm{conn.\,gon}(X)\}$. By Theorem 4.29(1), the groupoid $\mathrm{BirMori}(X)$ is generated by Sarkisov links and automorphisms of Mori fibre spaces. By Theorem 4.29(2), the relations are generated by elementary relations, so it suffices to show that every elementary relation is sent to the neutral element in the group

$$\underset{C \in \mathsf{CB}(X)}{*} \left(\underset{\mathsf{M}(C)}{\oplus} \mathbf{Z}/2 \right).$$

Let $\chi_t \circ \cdots \circ \chi_1 = \mathrm{id}$ be an elementary relation, coming from a rank 3 fibration T/B. We may assume that one of the χ_i is a Sarkisov link of conic bundles of type II with $\mathrm{cov.\,gon}(\chi_i) > M$, otherwise the relation is sent onto the neutral element as all χ_i are sent to the neutral element. We may moreover conjugate the relation and assume that χ_1 is a Sarkisov link of conic bundles of type II with $\mathrm{cov.\,gon}(\chi_1) > M$. By Proposition 5.3, we have $\dim(B) = n - 1$. Then, Proposition 5.5 implies that $t = 4$ and that χ_1 and χ_3 are equivalent Sarkisov links of conic bundles of type II. Applying the same argument to the relation $\chi_1 \circ \chi_4 \circ \chi_3 \circ \chi_2 = \mathrm{id}$ we either find that both χ_2 and χ_4 are sent to the neutral element or are equivalent Sarkisov links of conic bundles of type II (again by Proposition 5.5). Moreover, all the conic bundles involved in this relation are equivalent. This proves the existence of the groupoid homomorphism.

The fact that it restricts to a group homomorphism from $\mathrm{Bir}(X)$ is immediate, and the fact that it restricts as a group homomorphism

$$\mathrm{Bir}(X/B) \longrightarrow \underset{\mathsf{M}(X/B)}{\oplus} \mathbf{Z}/2$$

follows from Lemma 4.30. □

6. Image of the group homomorphism given by Theorem D

In this section, we study the image of $\mathrm{Bir}(X)$ under the group homomorphism given by Theorem D, and more precisely the image of $\mathrm{Bir}(X/B) \longrightarrow \oplus_{\mathsf{M}(X/B)} \mathbf{Z}/2$ for some conic bundles X/B. To simplify the notation, we will identify an equivalence class of marked conic bundles in $\mathsf{M}(X/B)$ with the associated generator of $\mathbf{Z}/2$. We can then speak about sums of elements of $\mathsf{M}(X/B)$, which we see in $\oplus_{\mathsf{M}(X/B)} \mathbf{Z}/2$, twice the same class being equal to zero.

6.A. **A criterion.**

Definition 6.1. Let $(X/B, \Gamma)$ be a marked conic bundle, and $\varphi\colon X/B \dashrightarrow Y/B$ a birational map over B between conic bundles. For a general point $p \in \Gamma$, and an irreducible curve $C \subseteq B$ transverse to Γ at p, let $b \in \mathbf{N}$ be the number of base-points of the birational surface map $\eta_X^{-1}(C) \dashrightarrow \eta_Y^{-1}(C)$ induced by φ that are equal or infinitely near to a point of the fibre of p. We call the class $\bar{b} \in \mathbf{Z}/2$ *the parity of* φ *along* Γ.

The following lemma shows that this definition does not depend on the choice of p or C. We shall use it to compute the image of the group homomorphism of Theorem D by studying locally a birational map near a hypersurface Γ of the base.

Lemma 6.2. *Let $\eta_X\colon X \longrightarrow B$ and $\eta_Y\colon Y \longrightarrow B$ be two conic bundles, $\varphi\colon X \dashrightarrow Y$ a birational map over B, and $\Gamma \subset B$ an irreducible hypersurface not contained in the discriminant locus of X/B.*

For any decomposition $\varphi = \chi_t \circ \cdots \circ \chi_1$ as in Lemma 4.30, the parity of φ along Γ is equal to the parity of the number of indexes $i \in \{1, \dots, t\}$ such that χ_i is a Sarkisov link of type II *whose marking $\Gamma_i \subset B_i$ is sent to Γ via B_i/B.*

Proof. Fix a decomposition $\varphi = \chi_t \circ \cdots \circ \chi_1$ as in Lemma 4.30, a general point $p \in \Gamma$ and an irreducible curve $C \subseteq B$ transverse to Γ at p. In particular p is a smooth point of both Γ and C. For $i = 0, \dots, t$, we denote by $\eta_i\colon X_i \longrightarrow B$ the morphism given by the composition $X_i \longrightarrow B_i \longrightarrow B$.

It suffices to prove, for $i = 0, \dots, t$, that the following holds:

(a) The morphism $\eta_i^{-1}(C) \longrightarrow C$ has general fibre $\mathbb{P}^1$, and the fibre over p is $\mathbb{P}^1$ (this means that Γ is not in the discriminant locus).

(b) If $i \geqslant 1$, then $\chi_i \circ \cdots \circ \chi_1$ induces a birational map between surfaces over C

$$\eta_0^{-1}(C) = \eta_X^{-1}(C) \dashrightarrow \eta_i^{-1}(C)$$

and the number of base-points that are equal or infinitely near to a point of the fibre of p has the same parity as the number of integers $j \in \{1, \dots, i\}$ such that χ_j is a Sarkisov link of type II with marking $\Gamma_j \subset B_j$, sent to Γ via B_j/B.

We proceed by induction on i. If $i = 0$, (a) follows from the assumption that Γ is not contained in the discriminant locus of X/B, and (b) is clear.

For $i \geqslant 1$, the birational map χ_i induces a birational map over C

$$\theta_i\colon \eta_{i-1}^{-1}(C) \dashrightarrow \eta_i^{-1}(C).$$

If χ_i is a Sarkisov link of type II with marking $\Gamma_i \subset B_i$, sent to Γ via B_i/B, it follows from the description of χ_i given in Lemma 3.23 that the restriction θ_i is the composition of the blow-up of a point on the fibre of p, the contraction of the strict transform of the fibre and of a birational map that is an isomorphism over an open subset of C that contains the fibre of p. This achieves the proof of (a) and (b) in this case, using the induction hypothesis.

If χ_i is a Sarkisov link of type II with a marking not sent to Γ or if χ_i is a Sarkisov link of type I or III, then the restriction θ_i of χ_i is an isomorphism over an open subset of C that contains the fibre of p. This follows again from Lemma 3.23 if the Sarkisov link is of type II and from Corollary 3.20 if it is of type I or III. As before, this gives the result by applying the induction hypothesis. □

Corollary 6.3. *Let X/B be a conic bundle, where $\dim(X) \geqslant 3$, and $\varphi \in \mathrm{Bir}(X/B)$. The image of φ under the group homomorphism*

$$\mathrm{Bir}(X/B) \longrightarrow \bigoplus_{\mathsf{M}(X/B)} \mathbf{Z}/2$$

of Theorem D is equal to the sum of equivalence classes of marked conic bundles $(X/B, \Gamma)$ with $\mathrm{cov.\,gon}(\Gamma) > \max\{d, 8\,\mathrm{conn.\,gon}(X)\}$ such that the parity of φ along Γ is odd.

Proof. Set $M = \max\{d, 8\,\mathrm{conn.\,gon}(X)\}$. Using Lemma 4.30, we decompose φ as $\varphi = \chi_t \circ \cdots \circ \chi_1$ where each χ_i is a Sarkisov link of conic bundles from X_{i-1}/B_{i-1} to X_i/B_i. Denote by $J \subseteq \{1, \ldots, t\}$ the subset of indexes i such that the Sarkisov link χ_i is of type II and satisfies $\mathrm{cov.\,gon}(\chi_i) > M$. By definition of the group homomorphism

$$\mathrm{Bir}(X/B) \longrightarrow \bigoplus_{\mathsf{M}(X/B)} \mathbf{Z}/2$$

of Theorem D, the image of φ is the sum of the equivalence classes of marked conic bundles of χ_i where i runs over J. For each $i \in J$, the marked conic bundle of χ_i is equal to $(X_i/B_i, \hat{\Gamma}_i)$ for some irreducible hypersurface $\hat{\Gamma}_i \subset B_i$ with $\mathrm{cov.\,gon}(\hat{\Gamma}_i) > M$. Hence, $(X_i/B_i, \hat{\Gamma}_i)$ is equivalent to $(X/B, \Gamma_i)$, where $\Gamma_i \subset B$ is the image of $\hat{\Gamma}_i \subset B_i$ via B_i/B. This implies that the image of φ is the sum of the classes of $(X/B, \Gamma_i)$, where i runs over J.

By Lemma 6.2, this sum is equal to the sum of equivalence classes of marked conic bundles $(X/B, \Gamma)$ with $\mathrm{cov.\,gon}(\Gamma) > M$ and such that the parity of φ along Γ is odd. □

6.B. **The case of trivial conic bundles and the proof of Theorem A.** Given a variety B, let $X = \mathbb{P}^1 \times B$, and X/B the second projection. The group $\mathrm{Bir}(X/B)$ is canonically isomorphic to $\mathrm{PGL}_2(\mathbf{C}(B))$, via the action

$$\begin{array}{ccc} \mathrm{PGL}_2(\mathbf{C}(B)) \times X & \dashrightarrow & X \\ \left(\begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}, ([u:v], t)\right) & \dashmapsto & ([a(t)u + b(t)v : c(t)u + d(t)v], t). \end{array}$$

For $B = \mathbb{P}^{n-1}$, the group $\mathrm{Bir}(X/B)$ corresponds, via a birational map $X \dashrightarrow \mathbb{P}^n$ sending the fibres of X/B to lines through a point $p \in \mathbb{P}^n$, to the subgroup of the Jonquières group associated to p consisting of birational maps of $\mathbb{P}^n$ that preserves a general line through p (in general a Jonquières element permutes such lines). Hence, $\mathrm{Bir}(X/B)$ corresponds to the factor $\mathrm{PGL}_2(\mathbf{C}(x_2, \ldots, x_n))$ of the group $\mathrm{PGL}_2(\mathbf{C}(x_2, \ldots, x_n)) \rtimes \mathrm{Bir}(\mathbb{P}^{n-1}) \subseteq \mathrm{Bir}(\mathbb{P}^n)$ described in §1.C.

For B general, we obtain many different varieties $X = \mathbb{P}^1 \times B$. It can also be that X is rational even if B is not, but then the conic bundle X/B is not equivalent to the trivial one $\mathbb{P}^n \times \mathbb{P}^1/\mathbb{P}^n$.

Lemma 6.4. *Any surjective group homomorphism $\tau\colon \mathrm{PGL}_2(\mathbf{C}(B)) \twoheadrightarrow G$ that is not an isomorphism factorises through the quotient*

$$\mathrm{PGL}_2(\mathbf{C}(B))/\mathrm{PSL}_2(\mathbf{C}(B)) \simeq \mathbf{C}(B)^*/(\mathbf{C}(B)^*)^2,$$

where the isomorphism corresponds to the determinant. In particular the target group G is abelian of exponent 2.

Proof. There exists a non trivial element $M \in \operatorname{Ker}\tau$ by assumption. Since the group $\mathrm{PGL}_2(\mathbf{C}(B))$ has trivial centre, we can find $N \in \mathrm{PGL}_2(\mathbf{C}(B))$ that does not commute with M. Then $\mathrm{id} \neq MNM^{-1}N^{-1} \in \mathrm{PSL}_2(\mathbf{C}(B)) \cap \operatorname{Ker}\tau$, and since $\mathrm{PSL}_2(\mathbf{C}(B))$ is a simple group we get $\mathrm{PSL}_2(\mathbf{C}(B)) \subseteq \operatorname{Ker}\tau$, which gives the result. □

Write $\mathrm{div}\colon \mathbf{C}(B)^* \longrightarrow \mathrm{Div}(B)$ the classical group homomorphism that sends a rational function onto its divisor of poles and zeros, and whose image is the group of principal divisors on B. Denoting by $\mathcal{P}_B$ the set of prime divisors on B, the group homomorphism div naturally gives a group homomorphism

$$\mathrm{PGL}_2(\mathbf{C}(B))/\mathrm{PSL}_2(\mathbf{C}(B)) \simeq \mathbf{C}(B)^*/(\mathbf{C}(B)^*)^2 \longrightarrow \bigoplus_{\mathcal{P}_B} \mathbf{Z}/2.$$

We project onto the sum of prime divisors with large enough covering gonality and identify the ones which are equivalent up to a birational map of B. This identification corresponds to taking orbits of the action of $\mathrm{Aut}_{\mathbf{C}}(\mathbf{C}(B))$ on $\mathbf{C}(B)$. The following lemma shows that the resulting group homomorphism extends from $\mathrm{Bir}(X/B)$ to $\mathrm{Bir}(X)$, and coincides with the group homomorphism from Theorem D.

Observe that for each $M \in \mathrm{PGL}_2(\mathbf{C}(B))$, we can speak about the parity of the multiplicity of $\det(M) \in \mathbf{C}(B)^*/(\mathbf{C}(B)^*)^2$ as pole or zero along an irreducible hypersurface $\Gamma \subset B$, as the multiplicity of an element of $(\mathbf{C}(B)^*)^2$ is always even.

Lemma 6.5. *Let B be a smooth variety of dimension at least 2, $X = \mathbb{P}^1 \times B$, and let $\varphi_M \in \mathrm{Bir}(X/B) \simeq \mathrm{PGL}_2(\mathbf{C}(B))$ be the birational map*

$$\varphi_M\colon ([u:v],t) \dashrightarrow ([a(t)u + b(t)v : c(t)u + d(t)v], t),$$

where

$$M = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \in \mathrm{PGL}_2(\mathbf{C}(B)).$$

The image of φ_M under the group homomorphism

$$\mathrm{Bir}(X/B) \longrightarrow \bigoplus_{\mathsf{M}(X/B)} \mathbf{Z}/2$$

of Theorem D is equal to the sum of the equivalence classes of marked conic bundles $(X/B, \Gamma)$ such that $\Gamma \subset B$ is a irreducible hypersurfaces of B with $\mathrm{cov.gon}(\Gamma) > \max\{d, 8\,\mathrm{conn.gon}(X)\}$ and such that the multiplicity of $\det(M)$ along Γ is odd.

Proof. We first observe that the image of $\mathrm{PSL}_2(\mathbf{C}(B)) \subseteq \mathrm{PGL}_2(\mathbf{C}(B)) \simeq \mathrm{Bir}(X/B)$ under the group homomorphism

$$\mathrm{Bir}(X/B) \longrightarrow \bigoplus_{\mathsf{M}(X/B)} \mathbf{Z}/2$$

is trivial, since $\mathrm{PSL}_2(\mathbf{C}(B))$ is simple and not abelian. Hence, the image of an element $\varphi \in \mathrm{Bir}(X/B) \simeq \mathrm{PGL}_2(\mathbf{C}(B))$ is uniquely determined by its determinant $\delta \in \mathbf{C}(B)^*/(\mathbf{C}(B)^*)^2$ (Lemma 6.4), and is the same as the image of the dilatation

$$\psi_\delta\colon ([u:v],t) \longmapsto ([\delta(t)u : v], t).$$

So we only need to prove the result for M equal to such a dilatation.

We denote as before by $\mathcal{P}_B$ the set of prime divisors on B. For $\delta \in \mathbf{C}(B)^*$ and $\Gamma \in \mathcal{P}_B$, we denote by $m_\delta(\Gamma) \in \mathbf{Z}$ the multiplicity of δ along Γ, so that

$$\mathrm{div}(\delta) = \sum_{\Gamma \in \mathcal{P}_B} m_\delta(\Gamma)\,\Gamma.$$

We also denote by $P_\delta(\Gamma) \in \{0,1\}$ the parity of ψ_δ along Γ as defined in Definition 6.1 and Lemma 6.2. The image of the dilatation $\psi_\delta \in \mathrm{Bir}(X/B)$ under the group homomorphism

$$\mathrm{Bir}(X/B) \longrightarrow \bigoplus_{\mathsf{M}(X/B)} \mathbf{Z}/2$$

is equal to the sum of equivalence classes of marked conic bundles $(X/B, \Gamma)$ such that $\Gamma \subset B$ is an irreducible hypersurface with

$$\mathrm{cov.\,gon}(\Gamma) > \max\{d, 8\,\mathrm{conn.\,gon}(X)\}$$

and such that $P_\delta(\Gamma)$ is odd (Corollary 6.3). To prove the result, it suffices to show that $P_\delta(\Gamma)$ and $m_\delta(\Gamma)$ have the same parity. For all $\delta, \delta' \in \mathbf{C}(B)^*$, we have

$$m_\delta(\Gamma) + m_{\delta'}(\Gamma) = m_{\delta\cdot\delta'}(\Gamma) \quad \text{and} \quad P_\delta(\Gamma) + P_{\delta'}(\Gamma) \equiv P_{\delta\cdot\delta'}(\Gamma) \pmod 2.$$

Indeed, the first equality follows from the definition of the multiplicity and the second follows from Lemma 6.2, since $\psi_\delta \circ \psi_{\delta'} = \psi_{\delta\cdot\delta'}$. The local ring $\mathcal{O}_\Gamma(B)$ being a DVR, the group $\mathbf{C}(B)^*$ is generated by elements $\delta \in \mathbf{C}(B)^*$ with $m_\delta(\Gamma) = 0$, and by one single element δ_0 which satisfies $m_{\delta_0}(\Gamma) = 1$. It therefore suffices to consider the case where $m_\delta(\Gamma) \in \{0,1\}$.

We take a general point $p \in \Gamma$, an irreducible curve $C \subseteq B$ transverse to Γ at p, and compute the number of base-points of the birational map $\theta\colon \mathbb{P}^1 \times C \dashrightarrow \mathbb{P}^1 \times C$ given by $([u:v],t) \longmapsto ([\delta(t)u:v],t)$ that are equal or infinitely near to a point of the fibre of p. If $m_\delta(\Gamma) = 0$, then δ is well defined on p, so the birational map θ induces an isomorphism $\mathbb{P}^1 \times \{p\} \longrightarrow \mathbb{P}^1 \times \{p\}$, which implies that $P_\delta(\Gamma) = 0$. If $m_\delta(\Gamma) = 1$, then δ has a zero of multiplicity one at p, so θ has exactly one base-point on $\mathbb{P}^1 \times \{p\}$, namely $([1:0],p)$. The composition of θ with the blow-up of $Z \longrightarrow \mathbb{P}^1 \times C$ of $([1:0],p)$ yields a birational map $Z \dashrightarrow \mathbb{P}^1 \times C$ with no more base-point on the exceptional divisor, as the multiplicity of both δ and v/u at the point is 1, so $P_\delta(\Gamma) = 1$. This achieves the proof. □

We can now give the proof of Theorem A.

Proof of Theorem A. We denote by $\mathrm{Dil}_{\mathbf{k}}$ the subgroup of birational dilatations

$$\begin{aligned}\mathrm{Dil}_{\mathbf{k}} &= \{(x_1,\dots,x_n) \dashmapsto (x_1\alpha(x_2,\cdots,x_n),x_2,\dots,x_n) \mid \alpha \in \mathbf{k}(x_2,\dots,x_n)^*\}\\ &\subseteq \mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^n) \simeq \mathrm{Aut}_{\mathbf{k}}(\mathbf{k}(x_1,\dots,x_n)).\end{aligned}$$

We denote $B = \mathbb{P}^{n-1}$ and use the birational map (defined over $\mathbf{k}$)

$$\begin{array}{ccc} X = \mathbb{P}^1 \times B & \dashrightarrow & \mathbb{P}^n \\ ([u:1],[t_1,\dots,t_{n-1}:1]) & \dashmapsto & [1:u:t_1:\cdots:t_{n-1}]\end{array}$$

that conjugates $\mathrm{Bir}(X)$ to $\mathrm{Bir}(\mathbb{P}^n)$, sending elements of the form

$$\{([u:v],t) \dashmapsto ([\alpha(t)u:v],t) \mid \alpha \in \mathbf{C}(B)^*\}$$

onto elements locally given by $(x_1,\dots,x_n) \dashmapsto (x_1\alpha(x_2,\cdots,x_n),x_2,\dots,x_n)$.

Now we pick a large enough integer D and consider the set $\mathcal{H}_D$ of degree D irreducible hypersurfaces in $\mathbb{P}^{n-1}$. For each element $\Gamma \in \mathcal{H}_D$, we consider an irreducible polynomial $P \in \mathbf{k}[x_0,\dots,x_n]$ of degree D defining the hypersurface Γ, choose $\alpha = P/x_0^D \in \mathbf{k}(\mathbb{P}^{n-1})$ and associate to Γ the element $\varphi_\alpha \in \mathrm{Bir}(X/B)$ given by

$$\varphi_\alpha\colon ([u:v],t) \dashmapsto ([\alpha(t)u:v],t).$$

By Lemma 6.5, the image of φ_α under the group homomorphism

$$\mathrm{Bir}(X/B) \longrightarrow \bigoplus_{\mathsf{M}(X/B)} \mathbf{Z}/2$$

of Theorem D is the unique marked conic bundle $(X/B, \Gamma)$ (as the hypersurface $\Gamma_0 \subset B$ given by $x_0 = 0$ satisfies $\mathrm{cov.gon}(\Gamma_0) = 1$). It remains to observe that we have enough elements in $\mathcal{H}_D$, up to birational maps of $\mathbb{P}^{n-1}$, namely as much as in the field $\mathbf{k}$. Indeed, if we take two general hypersurfaces $\Gamma_1, \Gamma_2 \subset \mathbb{P}^{n-1}$ of degree $\geqslant n+1$, then every birational map $\Gamma_1 \dashrightarrow \Gamma_2$ extends to a linear automorphism of $\mathbb{P}^{n-1}$; this can be shown by taking the suitable Veronese embedding of $\mathbb{P}^{n-1}$ such that the canonical divisors of Γ_1 and Γ_2 become hyperplane sections. The dimension of $\mathrm{PGL}_n(\mathbf{k})$ being bounded, for a large enough degree D we obtain a quotient of $\mathcal{H}_D$ by $\mathrm{PGL}_n(\mathbf{k})$ which has positive dimension, hence which has the same cardinality as the ground field $\mathbf{k}$. This quotient can be taken as the indexing set I in the statement of Theorem A. □

Remark 6.6. (1) As all birational dilatations in Theorem A belong to the Jonquières subgroup of elements preserving a pencil of lines, the restriction of the group homomorphism $\mathrm{Bir}(\mathbb{P}^n) \longrightarrow \oplus_I \mathbf{Z}/2$ to the Jonquières subgroup also is surjective. We will need other conic bundle structures on rational varieties to obtain Theorem C.

(2) The proof of Theorem A uses Lemma 6.5 in the case where $B = \mathbb{P}^{n-1}$. For a general basis B we can prove along the same lines that the image of the subgroup of $\mathrm{Bir}(X/B)$ given by

$$\{([u:v],t) \dashmapsto ([\delta(t)u:v],t) \mid \delta \in \mathbf{C}(B)^*\}$$

under the group homomorphism $\mathrm{Bir}(X/B) \longrightarrow \oplus_{\mathsf{M}(X/B)} \mathbf{Z}/2$ of Theorem D is infinite. We omit the proof here, as it is similar to the case of $B = \mathbb{P}^{n-1}$, and moreover we will prove a more general result in Proposition 6.9.

6.C. **The case of non-trivial conic bundles and the proof of Theorem B.** Recall that given a smooth conic $C \subset \mathbb{P}^2$ and a point $p \in \mathbb{P}^2 \smallsetminus C$, there is an involution $\iota(p, C) \in \mathrm{Bir}(\mathbb{P}^2)$ that preserves the conic C. It is defined on each general line L through p as the involution that fixes p and exchange the two intersection points $L \cap C$. We say that $\iota(p, C)$ is the *involution induced by the projection from* p. We now use this construction in family to produce interesting involutions on some conic bundles.

Lemma 6.7. *Let B be a smooth variety, $\hat{\eta}\colon P \longrightarrow B$ a locally trivial $\mathbb{P}^2$-bundle, and $X \subset P$ a closed subvariety such that the restriction of $\hat{\eta}$ is a conic bundle $\eta\colon X \longrightarrow B$. Let $s\colon B \dashrightarrow P$ be a rational section (i.e. a rational map, birational to its image, such that $\hat{\eta} \circ s = \mathrm{id}_B$), whose image is not contained in X. Let $\iota \in \mathrm{Bir}(X/B)$ be the birational involution whose restriction to a general fibre $\eta^{-1}(b)$ is the involution induced by the projection from $s(b)$. Let $\Gamma \subset B$ be an irreducible hypersurface not contained in the discriminant locus of η.*

Then the parity of ι along Γ (in the sense of Definition 6.1) is the parity of the multiplicity of $F(s)$ along Γ, where F is the local equation of X in P.

Proof. We choose a dense open subset $U \subseteq B$ which intersects Γ and trivialises the $\mathbb{P}^2$-bundle, and view X locally inside of $\mathbb{P}^2 \times U$. It is given by $F \in \mathbf{C}(B)[x, y, z]$, homogeneous of degree 2 in x, y, z. The fibre of $\eta\colon X \longrightarrow B$ over a general point of

Γ (respectively of B) is a smooth conic. The section s corresponds to $[\alpha : \beta : \gamma]$, where $\alpha, \beta, \gamma \in \mathbf{C}(B)$ are not all zero and are uniquely determined by s, up to multiplication by an element of $\mathbf{C}(B)^*$.

The evaluation $F(\alpha, \beta, \gamma) \in \mathbf{C}(B)$ at s is uniquely determined by s up to multiplication by the square of an element of $\mathbf{C}(B)^*$. The parity of the multiplicity of $F(\alpha, \beta, \gamma)$ along Γ is then well defined. The statement of the lemma consists in showing that this parity is equal to the parity of ι along Γ.

By multiplying by a suitable power of a local equation of Γ, we can choose that neither α, β, γ has a pole along Γ and that not all three vanish on Γ. Then, the restriction of α, β, γ gives an element $(\bar{\alpha}, \bar{\beta}, \bar{\gamma}) \in \mathbf{C}(\Gamma)^3 \smallsetminus \{0\}$. There exists a matrix in $\mathrm{GL}_3(\mathbf{C}(\Gamma))$ that sends $(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ to $(1, 0, 0)$. By extending this matrix as an element of $\mathrm{GL}_3(\mathbf{C}(B))$, we can assume that $(\alpha, \beta, \gamma) = (1, 0, 0)$.

We then write the equation of X as

$$F = ax^2 + bxy + cxz + dy^2 + eyz + fz^2$$

where $a, b, c, d, e, f \in \mathbf{C}(B)$ have no pole along Γ and are not all simultaneously zero on Γ, and obtain that

$$F(\alpha, \beta, \gamma) = a.$$

With these coordinates, the involution $\iota \in \mathrm{Bir}(P/B)$ is given by

$$\iota \colon [x : y : z] \longmapsto [-(x + \tfrac{b}{a}y + \tfrac{c}{a}z) : y : z].$$

If a does not vanish on Γ, then ι is an isomorphism on a general point of Γ, and the multiplicity of ι at Γ is equal to zero. This achieves the proof in this case.

Suppose that a is zero on Γ. It implies that either b or c is not zero on Γ. We denote by $E \subset P$ the preimage of Γ. It is an irreducible hypersurface and the general fibre of E/Γ is isomorphic to $\mathbb{P}^1$ because Γ is not contained in the discriminant of X/B. The variety E is contracted by ι onto the section of $E \longrightarrow \Gamma$ given by $[x : y : z] = [1 : 0 : 0]$. We denote by $\hat{\iota}, \nu \in \mathrm{Bir}(P/B)$ the birational maps locally given by

$$\hat{\iota} \colon [x : y : z] \longmapsto [-(x + by + cz) : y : z] \quad \text{and} \quad \nu \colon [x : y : z] \longmapsto [ax : y : z],$$

and observe that $\iota = \nu^{-1} \circ \hat{\iota} \circ \nu$. Moreover, the variety $\hat{X} = \nu(X) \subset P$ is locally given by

$$\hat{F} = x^2 + bxy + cxz + a(dy^2 + eyz + fz^2).$$

In particular, the preimage of Γ in $\hat{X}$ is the union of the hypersurfaces $E_1, E_2 \subset \hat{X}$, locally given by $x = 0$ and $x + by + cz = 0$ respectively. As b and c are not both zero on Γ, we have $E_1 \neq E_2$.

For $i = 1, 2$, a general fibre of $E_i \longrightarrow \Gamma$, for $i = 1, 2$, is isomorphic to $\mathbb{P}^1$. Moreover, the restriction of ν induces a birational map $E \dashrightarrow E_1$ over Γ, corresponding on a general fibre to the projection from a smooth conic to a line, via a point of the conic. The birational involution $\hat{\iota}$ induces a birational transformation of $\hat{X}$ whose restriction gives a birational map $E_1 \dashrightarrow E_2$ over Γ. We denote by $r \geqslant 1$ the multiplicity of a along Γ. As Γ is an irreducible hypersurface of B, the local ring $\mathcal{O}_{B,\Gamma}$ of rational functions of B defined on an open subset of Γ is a DVR. We can find $a_1, \ldots, a_r \in \mathbf{C}(B)$ of multiplicity 1 along Γ such that $a = a_1 \cdot a_2 \cdots a_r$. This allows us to write ν^{-1} as $\nu^{-1} = \nu_r \circ \cdots \circ \nu_1$, where ν_i is given by $[x : y : z] \longmapsto [x : a_i y : a_i z]$. We write $X_1 = \nu_1(\hat{X})$ and write $X_i = \nu_i(X_{i-1})$ for $i = 2, \ldots, r$. In particular,

$X_r = X$. The equation of X_i is given by

$$F_i = a_1 \cdots a_i x^2 + bxy + cxz + a_{i+1} \cdots a_r (dy^2 + eyz + fz^2).$$

For $i = 1, \dots, r-1$, the preimage of Γ in X_i is the union of $E_{1,i}$ and $E_{i,2}$ given by $x = 0$ and $by + cz = 0$. Writing $E_1 = E_{1,0}$ and $E_2 = E_{2,0}$, we obtain that ν_i contracts $E_{2,i-1}$ onto a rational section of Γ contained in $E_{2,i} \smallsetminus E_{1,i}$, for $i = 1, \dots, r$.

To compute the parity of ι along Γ, we denote by $C \subseteq \Gamma$ a general irreducible curve passing through a general point $p \in \Gamma$, and look at the birational map obtained by the restriction of ι to the preimage of C. Recalling that $\iota = \nu^{-1} \circ (\hat{\iota} \circ \nu)$, observe that the first map $\hat{\iota} \circ \nu$ is a local isomorphism at the point p, while the map ν^{-1} corresponds to a sequence of r elementary links. The parity of ι along Γ is then the class of r in $\mathbf{Z}/2$ as desired. □

Definition 6.8. We say that a conic bundle X/B is a *decomposable conic bundle* if X and B are smooth, and if we have closed embeddings $B \hookrightarrow \mathbb{P}^m$ and $X \hookrightarrow P$ where P is a $\mathbb{P}^2$-bundle over $\mathbb{P}^m$, which is the projection of a decomposable vector bundle of rank 3. We moreover ask that the morphism X/B comes from the restriction of the $\mathbb{P}^2$-bundle $P \longrightarrow \mathbb{P}^m$ and that $X \subset P$ is locally given by equations of degree 2 in the $\mathbb{P}^2$-bundle.

Proposition 6.9. *For each decomposable conic bundle $\eta\colon X \longrightarrow B$ with $\dim B \geqslant 2$, there are infinitely many involutions in $\mathrm{Bir}(X/B)$ which have distinct images via the group homomorphism $\mathrm{Bir}(X/B) \longrightarrow \oplus_{\mathsf{M}(X/B)} \mathbf{Z}/2$ of Theorem D. In particular, the image is infinite.*

Proof. We can see B as a closed subset $B \subseteq \mathbb{P}^m$, and obtain that $X \subset P$, where $\hat{\eta}\colon P \longrightarrow \mathbb{P}^m$ is the projectivisation of a rank 3 vector bundle. We can thus write $P = \mathbb{P}(\mathcal{O}_{\mathbb{P}^m} \oplus \mathcal{O}_{\mathbb{P}^m}(a) \oplus \mathcal{O}_{\mathbb{P}^m}(b))$ for some $a, b \geqslant 0$ (up to twisting and exchanging the factors). We view P as the quotient of $(\mathbb{A}^3 \smallsetminus \{0\}) \times (\mathbb{A}^{m+1} \smallsetminus \{0\})$ by $(\mathbb{G}_m)^2$ via

$$((\lambda, \mu), (x_0, x_1, x_2, y_0, \dots, y_m)) \longmapsto (\lambda x_0, \lambda \mu^{-a} x_1, \lambda \mu^{-b} x_2, \mu y_0, \dots, \mu y_m)$$

and denote by $[x_0 \colon x_1 \colon x_2 \,;\, y_0 \colon \dots \colon y_m]$ the class of $(x_0, x_1, x_2, y_0, \dots, y_m)$ (see [AO18, Definition 2.3, Remark 2.4] for more details).

Then X is equal to the preimage of B cut by the zero locus of an irreducible polynomial $G \in \mathbf{C}[x_0, x_1, x_2, y_0, \dots, y_m]$, that has degree 2 in x_0, x_1, x_2 (and suitable degree in $y_0, \dots, y_m$ so that the polynomial is homogeneous for the above action). For each integer $d \geqslant 1$, and for general homogeneous polynomials

$$u_0, v_0 \in \mathbf{C}[y_0, \dots, y_m]_d, \quad u_1, v_1 \in \mathbf{C}[y_0, \dots, y_m]_{d+a}, \quad u_2, v_2 \in \mathbf{C}[y_0, \dots, y_m]_{d+b},$$

(the subscript corresponding to the degree), the closed subvariety $\hat{\Gamma} \subset X$ of codimension 2 given by

$$\hat{\Gamma} = \left\{ ([x_0 : x_1 : x_2 \,;\, y_0 : \dots : y_m]) \in X \subseteq P \;\middle|\; \sum_{i=0}^{2} x_i u_i = \sum_{i=0}^{2} x_i v_i = 0 \right\}$$

is smooth, by Bertini theorem.

We now prove that the projection $X \longrightarrow B$ induces a birational morphism from $\hat{\Gamma}$ to its image $\Gamma \subset B$, an irreducible hypersurface of B. Solving the linear system $\sum_{i=0}^{2} x_i u_i = \sum_{i=0}^{2} x_i v_i = 0$ in x_0, x_1, x_2, we obtain that the preimage of $[y_0 : \dots : y_m]$ is $[u_1 v_2 - u_2 v_1 : u_2 v_0 - u_0 v_2 : u_0 v_1 - u_1 v_0 \,;\, y_0 : \dots : y_m]$, so the

projection induces a birational morphism from $\hat{\Gamma}$ to the hypersurface $\Gamma \subset B$ given by the polynomial $G(P_0, P_1, P_2, y_0, \ldots, y_m)$, where $P_0, P_1, P_2 \in \mathbf{C}[y_0, \ldots, y_m]$ are the polynomials $P_0 = u_1v_2 - u_2v_1$, $P_1 = u_2v_0 - u_0v_2$ and $P_2 = u_0v_1 - u_1v_0$.

We now show that the covering gonality $\mathrm{cov.\,gon}(\hat{\Gamma}) = \mathrm{cov.\,gon}(\Gamma)$ is large if d is large enough. We denote by $H_i, F_j \subset P$ the hypersurfaces given respectively by $x_i = 0$ and $y_j = 0$, and obtain that $\mathrm{Pic}(P) = \mathbf{Z}H_i \oplus \mathbf{Z}F_j$ for all $i \in \{0, 1, 2\}, j \in \{0, \ldots, m\}$. The class of all F_j is the same and denoted by F and $H_0 \sim H_1 + aF \sim H_2 + bF$. Note that $\hat{\Gamma}$ is a complete intersection in $\hat{\eta}^{-1}(B) \subseteq P$ of 3 hypersurfaces equivalent to $H_0+dF, H_0+dF, 2H_0+d_0F$ for some $d_0 \in \mathbf{Z}$ (depending on the equation of X). The canonical divisor of P being equivalent to $-H_0 - H_1 - H_2 - F_0 - \ldots - F_m = -3H_0 - (m + 1 - a - b)F$, we obtain by adjunction that the canonical divisor of $\hat{\Gamma}$ is the restriction to $\hat{\Gamma}$ of a divisor of P equivalent to $H_0 + (2d + d_0 - m - 1 + a + b)F$. The morphism associated to F is simply the projection $\hat{\Gamma} \longrightarrow \mathbb{P}^m$, which is birational onto its image. By Lemma 2.30(2)-(3), the divisor pF satisfies BVA_p, for each integer $p \geqslant 0$, and thus $K_{\hat{\Gamma}}$ satisfies BVA_p for $p = 2d+d_0-m-1+a+b \geqslant 0$ if d is large enough, by Lemma 2.30(1). This implies that $\mathrm{cov.\,gon}(\hat{\Gamma}) \geqslant p + 2$ by Theorem 2.29. By choosing d large enough we obtain that $\mathrm{cov.\,gon}(\Gamma) = \mathrm{cov.\,gon}(\hat{\Gamma})$ is large.

We now use the construction in Lemma 6.7 of the involution $\iota \in \mathrm{Bir}(X/B)$ associated with the $\mathbb{P}^2$-bundle P/B and the rational section $s\colon B \dashrightarrow P$ given by

$$[y_0 : \ldots : y_m] \mapsto [u_1v_2 - u_2v_1 : u_2v_0 - u_0v_2 : u_0v_1 - u_1v_0 \,;\, y_0 : \ldots : y_m].$$

By Lemma 6.7, the parity of ι along Γ is one and the parity of ι along any other irreducible hypersurface of B is zero (as Γ is the zero locus of $G(s)$ by construction). For a large integer d, the image of ι under the group homomorphism

$$\mathrm{Bir}(X/B) \longrightarrow \bigoplus_{\mathsf{M}(X/B)} \mathbf{Z}/2$$

of Theorem D is the equivalence class of $(X/B, \Gamma)$. Taking larger and larger d, we obtain infinitely many involutions in the image of the group homomorphisms, which are distinct and thus generate a group isomorphic to an infinite direct sum of $\mathbf{Z}/2$, as the covering gonality of the hypersurfaces goes to infinity with d. □

Proof of Theorem B. We use the group homomorphism

$$\mathrm{Bir}(X) \longrightarrow \mathop{\ast}_{C \in \mathsf{CB}(X)} \left(\bigoplus_{\mathsf{M}(C)} \mathbf{Z}/2 \right)$$

of Theorem D. By assumption, X/B is a decomposable conic bundle (in the sense of Definition 6.8). By Proposition 6.9, the image of $\mathrm{Bir}(X/B)$ contains a group isomorphic to an infinite direct sum of $\mathbf{Z}/2$.

To finish the proof of Theorem B, we take a subfield $\mathbf{k} \subseteq \mathbf{C}$ over which X, B and η are defined, and check that the involutions in $\mathrm{Bir}(X/B)$ that are used to provide the large image are defined over $\mathbf{k}$. Firstly, the involutions provided by Lemma 6.7 are defined over $\mathbf{k}$ as soon as the rational section $s\colon B \dashrightarrow P$ is. Secondly, the construction of Proposition 6.9 works for general polynomials in $\mathbf{C}[y_0, \ldots, y_m]$ of some fixed degrees.

Since a dense open subset of an affine space $\mathbb{A}^n_{\mathbf{C}}$ contains infinitely many $\mathbf{k}$-points for each subfield $\mathbf{k} \subseteq \mathbf{C}$ (follows from the fact that the $\mathbf{Q}$-points of $\mathbb{A}^n$ are dense), we can assume that the polynomials, and thus the section, are defined over $\mathbf{k}$. □

7. Non-equivalent conic bundles

In this section, we construct infinitely many non-equivalent conic bundles on $\mathbb{P}^n$, showing that the set $\mathsf{CB}(\mathbb{P}^n)$ is infinite for $n \geqslant 3$ (by contrast, observe that $\mathsf{CB}(\mathbb{P}^2)$ consists of one element). This allows us to prove Theorems E and C.

7.A. **Studying the discriminant locus.** The main result of this section is Proposition 7.10. We prove in particular that if two standard conic bundles $X_1/\mathbb{P}^2$ and $X_2/\mathbb{P}^2$ are ramified over smooth curves Δ_1 and Δ_2 such that the conic bundles $(X_1 \times \mathbb{P}^n)/(\mathbb{P}^2 \times \mathbb{P}^n)$ and $(X_1 \times \mathbb{P}^n)/(\mathbb{P}^2 \times \mathbb{P}^n)$ are equivalent, then there exist surjective morphisms $\Delta_1 \dashrightarrow \Delta_2$ and $\Delta_2 \dashrightarrow \Delta_1$.

The following notion is called an *embedded conic* in [Sar82, page 358].

Definition 7.1. Let V be a smooth quasi-projective variety. An *embedded conic fibration* is a projective morphism $\eta\colon X \longrightarrow V$ that is the restriction of a locally trivial $\mathbb{P}^2$-bundle $\hat{\eta}\colon P \longrightarrow V$, and such that $X \subset P$ is a closed subvariety, given locally by an equation of degree 2. Precisely, for each $p \in V$, there exists an affine open subset $U \subseteq V$ containing p such that $\hat{\eta}^{-1}(U)$ is isomorphic to $U \times \mathbb{P}^2$, and the image of $\eta^{-1}(U) \subset U \times \mathbb{P}^2$ is a closed subvariety, irreducible over $\mathbf{C}(U)$, and defined by a polynomial $F \in \mathbf{C}[U][x, y, z]$ homogeneous of degree 2 in the coordinates x, y, z.

Remark 7.2. Let $\eta\colon X \longrightarrow V$ be a flat projective morphism between smooth quasi-projective varieties, with generic fibre an irreducible conic. Then, η is an embedded conic fibration in a natural way. This is done by taking the locally trivial $\mathbb{P}^2$-bundle $P = \mathbb{P}(\eta_*(\omega_X^{-1}))$ over V, where ω_X is the canonical line bundle of X (see [Sar82, §1.5]).

The following definition is equivalent to the one of [Sar82, Definition 1.4].

Definition 7.3. A *standard conic bundle* is a morphism $\eta\colon X \longrightarrow B$ which is a conic bundle (in the sense of Definition 2.13), and which is moreover *flat* with X and B smooth. This implies that η is also an *embedded conic fibration* in the $\mathbb{P}^2$-bundle $\mathbb{P}(\eta_*(\omega_X^{-1})) \longrightarrow B$ (see Remark 7.2).

Remark 7.4. Let us make some comparisons between the above definitions.

An *embedded conic fibration* (Definition 7.1) over a projective base is not necessarily a *conic bundle* (Definition 2.13), as the relative Picard rank can be > 1. Conversely, a conic bundle X/B is not necessarily an embedded conic fibration, but it is one if the conic bundle is *standard* (Definition 7.3) (as explained just above) or *decomposable* (Definition 6.8).

Moreover, a decomposable conic bundle is not always standard, as some fibres can be equal to $\mathbb{P}^2$. It is not clear to us if there exist standard conic bundles which are not decomposable.

Definition 7.5. Let V be a smooth quasi-projective variety and $\eta\colon X \longrightarrow V$ a flat embedded conic fibration.

For each irreducible closed subset $\Gamma \subseteq V$, we define the *multiplicity of the discriminant of η along* Γ as follows. We take an open subset $U \subseteq V$ that intersects Γ and such that $\eta^{-1}(U)$ is a closed subset of $U \times \mathbb{P}^2$, of degree 2, and consider a symmetric matrix $M \in \mathrm{Mat}_{3\times 3}(\mathbf{C}(U))$ that defines the equation of $\eta^{-1}(U)$. We choose M such that all coefficients of M are contained in the local ring $\mathcal{O}_\Gamma(U) \subset \mathbf{C}(U)$ of rational functions defined on a general point of Γ, and such that the residue matrix

$\overline{M} \in \mathrm{Mat}_{3\times 3}(\mathbf{C}(\Gamma))$ is not zero. This is possible as the morphism is flat, and defines M uniquely, up to multiplication by an invertible element of $\mathcal{O}_\Gamma(U)$.

Now we define the multiplicity of the discriminant of η along Γ to be the least integer $m \geqslant 0$ such that the determinant lies in $\mathfrak{m}_\Gamma(U)^m$, where $\mathfrak{m}_\Gamma(U)$ is the maximal ideal of $\mathcal{O}_\Gamma(U)$, kernel of the ring homomorphism $\mathcal{O}_\Gamma(U) \twoheadrightarrow \mathbf{C}(\Gamma)$.

We define the *discriminant divisor of* η to be $\sum m_D D$, where the sum runs over all irreducible hypersurfaces $D \subset V$ and where $m_D \in \mathbf{N}$ is the the multiplicity of the discriminant of η along D as defined above.

Remark 7.6. If $\eta\colon X \longrightarrow V$ is moreover a conic bundle, the definition of the discriminant given in Definition 7.5 is compatible with the definition of discriminant locus given in Definition 2.13: the discriminant locus is the reduced part of the discriminant divisor of η. Moreover, if η is a standard conic bundle, the discriminant divisor is reduced [Sar82, Corollary 1.9]. The multiplicity of the discriminant divisor along irreducible hypersurfaces of V is always 0 or 1 in this case. We will however not only consider hypersurfaces but also closed subvarieties of lower dimension.

Using the local description of the matrix that defines η as a flat embedded conic fibration, one can prove the following:

Proposition 7.7 ([Sar82, Proposition 1.8])**.** *Let V be a smooth quasi-projective variety, let $\eta\colon X \longrightarrow V$ be a flat embedded conic fibration, such that X is smooth. The discriminant divisor Δ of η has the following properties: for each point $p \in V$, the fibre $f_p = \eta^{-1}(p)$ is given as follows:*

$$f_p \text{ is } \begin{cases} \text{a smooth conic} \\ \text{the union of two distinct lines} \\ \text{a double line} \end{cases} \iff p \text{ is } \begin{cases} \text{not on } \Delta \\ \text{a smooth point of } \Delta \\ \text{a singular point of } \Delta. \end{cases}$$

We shall need the following folklore result:

Lemma 7.8. *Let V be a smooth quasi-projective variety and let $\eta_1\colon X_1 \longrightarrow V$ and $\eta_2\colon X_2 \longrightarrow V$ be two flat embedded conic fibrations. Let $\psi\colon X_1 \dashrightarrow X_2$ be a birational map over V. Let $\Gamma \subseteq V$ be a closed irreducible subvariety such that $\eta_1^{-1}(\Gamma)$ is not contained in the base-locus of ψ, that the preimage $\eta_2^{-1}(\Gamma)$ is irreducible and that a general fibre of $\eta_2^{-1}(\Gamma) \longrightarrow \Gamma$ is the union of two distinct lines. We moreover assume that the multiplicity of the discriminant of η_2 along Γ is 1. Then, one of the following holds:*

(1) *Every fibre of $\eta_1^{-1}(\Gamma) \longrightarrow \Gamma$ is a double line (non-reduced fibre).*
(2) *The preimage $\eta_1^{-1}(\Gamma)$ is irreducible and a general fibre of $\eta_1^{-1}(\Gamma) \longrightarrow \Gamma$ is the union of two distinct lines.*

Proof. Replacing V by an open subset that intersects Γ, we can assume that X_1 and X_2 are closed subvarieties of $V \times \mathbb{P}^2$ given by a polynomial of degree 2 in the coordinates of $\mathbb{P}^2$. We denote by $\mathcal{O}_\Gamma(V) \subset \mathbf{C}(V)$ the subring of rational functions that are defined on a general point of Γ and consider the surjective residue homomorphism $\mathcal{O}_\Gamma(V) \twoheadrightarrow \mathbf{C}(\Gamma)$. The quadratic equations of X_1 and X_2 correspond to symmetric matrices $M_1, M_2 \in \mathrm{Mat}_{3\times 3}(\mathbf{C}(V))$, defined up to scalar multiplication. Since both η_1 and η_2 are flat, we can choose $M_1, M_2 \in \mathrm{Mat}_{3\times 3}(\mathcal{O}_\Gamma(V))$ such that the residue matrices $\overline{M}_1, \overline{M}_2 \in \mathrm{Mat}_{3\times 3}(\mathbf{C}(\Gamma))$ are not zero.

The fact that $\eta_2^{-1}(\Gamma)$ is irreducible and that a general fibre of $\eta_2^{-1}(\Gamma) \longrightarrow \Gamma$ is the union of two distinct lines is equivalent to asking that the quadratic form

associated to M_2 corresponds to a singular irreducible conic over the field $\mathbf{C}(\Gamma)$. It then corresponds to the union of two lines defined over an extension of degree 2 of $\mathbf{C}(\Gamma)$, which intersect into a point defined over $\mathbf{C}(\Gamma)$. After a change of coordinates on $X_2 \subset V \times \mathbb{P}^2$, applying an element of $\mathrm{PGL}_3(\mathbf{C}(V))$ which restricts to an element of $\mathrm{PGL}_3(\mathbf{C}(\Gamma))$, we can assume that the point is $[0:0:1]$ and completing the square we assume that the restriction is given by $F = ax^2 + by^2$ for some $a, b \in \mathbf{C}(\Gamma)^*$, where $-\frac{a}{b} \in \mathbf{C}(\Gamma)^*$ is not a square. This corresponds to saying that $\overline{M}_2$ is equal to the diagonal matrix $\mathrm{diag}(a, b, 0)$.

The birational map ψ is given by

$$\left(v, \begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) \dashmapsto \left(v, A(v) \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$$

for some $A \in \mathrm{GL}_3(\mathbf{C}(V))$. This implies that M_1 and ${}^tA \cdot M_2 \cdot A$ are collinear in $\mathrm{Mat}_{3\times 3}(\mathbf{C}(V))$.

As $\eta_1^{-1}(\Gamma)$ is not contained in the base-locus of ψ, we can assume that $A \in \mathrm{Mat}_{3\times 3}(\mathcal{O}_\Gamma(V))$ is such that its residue $\bar{A} \in \mathrm{Mat}_{3\times 3}(\mathbf{C}(\Gamma))$ is not zero. We can moreover choose an element $S \in \mathrm{GL}_3(\mathcal{O}_\Gamma(V))$, with residue $\bar{S} \in \mathrm{GL}_3(\mathbf{C}(\Gamma))$, and replace A with $A \cdot S$. This corresponds to a coordinate change of $\mathbb{P}^2 \times V$ at the source, which only affects X_1 and not X_2. We can then reduce to the following possibilities for $\bar{A}$, according to the rank of the 2×3 matrix obtained from the first two rows of $\bar{A}$:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \mu_1 & \mu_2 & \mu_3 \end{pmatrix}, \qquad \begin{pmatrix} \alpha & 0 & 0 \\ \beta & 0 & 0 \\ \mu_1 & \mu_2 & \mu_3 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mu_1 & \mu_2 & \mu_3 \end{pmatrix},$$

where $\alpha, \beta, \mu_1, \mu_2, \mu_3 \in \mathbf{C}(\Gamma)$ and $(\alpha, \beta) \neq (0, 0)$.

In the first case, ${}^t\bar{A} \cdot \overline{M}_2 \cdot \bar{A} = \overline{M}_2$, so $\eta_2^{-1}(\Gamma)$ has the same properties as $\eta_1^{-1}(\Gamma)$, which gives (2).

The second case gives ${}^t\bar{A} \cdot \overline{M}_2 \cdot \bar{A} = \mathrm{diag}(\alpha^2 a + \beta^2 b, 0, 0)$. As $(\alpha, \beta) \neq (0, 0)$ and $-\frac{a}{b} \in \mathbf{C}(\Gamma)^*$ is not a square, we have $\alpha^2 a + \beta^2 b \neq 0$. The quadratic form associated to this matrix is then a double line, and we obtain (1).

It remains to study the last case, which yields ${}^t\bar{A} \cdot \overline{M}_2 \cdot \bar{A} = 0$. This means that all coefficents of ${}^tA \cdot M_2 \cdot A$ belong to the maximal ideal $\mathfrak{m} = \mathfrak{m}_\Gamma(V)$ of $\mathcal{O}_\Gamma(V)$, kernel of the residue homomorphism $\mathcal{O}_\Gamma(V) \twoheadrightarrow \mathbf{C}(\Gamma)$. Applying S as before, we can assume that $\mu_1 = 1$, $\mu_2 = \mu_3 = 0$, since the rank of $\bar{A}$ is 1. We write $M_2 = \mathrm{diag}(a, b, 0) + (\nu_{i,j})_{1 \leqslant i,j \leqslant 3}$ where $\nu_{i,j} \in \mathfrak{m}$ for all i, j, and obtain $\det(M_2) \equiv a \cdot b \cdot \nu_{3,3} \pmod{\mathfrak{m}^2}$. As the multiplicity of the discriminant of η_2 along Γ is 1, this implies that $\nu_{3,3} \in \mathfrak{m} \smallsetminus \mathfrak{m}^2$. We compute ${}^tA \cdot M_2 \cdot A \equiv \mathrm{diag}(\nu_{3,3}, 0, 0) \pmod{\mathfrak{m}^2}$. The quadratic form associated to this matrix is a double line, so again we obtain (1). □

We give two examples to illustrate the need for all the assumptions in Lemma 7.8:

Example 7.9. We work over the affine plane $V = \mathbb{A}^2$ and consider

$$X_1 = \{([x:y:z], (u,v)) \in \mathbb{P}^2 \times \mathbb{A}^2 \mid x^2 v + y^2 - z^2 = 0\},$$
$$X_2 = \{([x:y:z], (u,v)) \in \mathbb{P}^2 \times \mathbb{A}^2 \mid x^2 v + y^2 - u^2 z^2 = 0\},$$
$$X_2' = \{([x:y:z], (u,v)) \in \mathbb{P}^2 \times \mathbb{A}^2 \mid x^2 uv + y^2 - z^2 = 0\}.$$

The projection onto the second factor gives three flat embedded conic fibrations $\eta_1 \colon X_1 \longrightarrow \mathbb{A}^2$, $\eta_2 \colon X_2 \longrightarrow \mathbb{A}^2$, $\eta_2' \colon X_2' \longrightarrow \mathbb{A}^2$, with discriminant divisors being respectively given by $v = 0$, $u^2 v = 0$ and $uv = 0$. The birational maps of $\mathbb{P}^2 \times \mathbb{A}^2$

given by $([x:y:z],(u,v)) \longmapsto ([xu:yu:z],(u,v))$ and $([x:y:z],(u,v)) \longmapsto ([2x:(u+1)y+(u-1)z:(u-1)y+(u+1)z],(u,v))$ provide two birational maps $\psi\colon X_1 \dashrightarrow X_2$ and $\psi'\colon X_1 \dashrightarrow X_2'$ over $\mathbb{A}^2$.

Choosing $\Gamma \subset \mathbb{A}^2$ to be the line $\{u=0\}$, the result of Lemma 7.8 does not hold for ψ and for ψ', because a general fibre of $\eta_1^{-1}(\Gamma) \longrightarrow \Gamma$ is a smooth conic. In both cases, exactly one hypothesis is not satisfied. Namely, the multiplicity of the discriminant of η_2 along Γ is 2 instead of 1, and the surface $\eta_2'^{-1}(\Gamma)$ is not irreducible.

The idea of the proof of the following statement was given to us by C. Böhnig and H.-C. Graf von Bothmer.

Proposition 7.10. *Let B be a smooth surface, and for $i=1,2$, let $\eta_i\colon X_i \longrightarrow B$ be a standard conic bundle with discriminant a smooth irreducible curve $\Delta_i \subset B$. Assume that there exists a commutative diagram*

$$\begin{array}{ccc} X_1 \times Y & \overset{\psi}{\dashrightarrow} & X_2 \times Y \\ {\scriptstyle \eta_1\times \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \eta_2 \times \mathrm{id}} \\ B \times Y & \overset{\theta}{\dashrightarrow} & B \times Y \end{array}$$

where Y is smooth and ψ, θ are birational.

Then, for a general point $p \in Y$, the image of $\Delta_1 \times \{p\}$ is contained in $\Delta_2 \times Y$ and the morphism $\Delta_1 \longrightarrow \Delta_2$ obtained by composing

$$\Delta_1 \xrightarrow{\sim} \Delta_1 \times \{p\} \overset{\theta}{\dashrightarrow} \Delta_2 \times Y \xrightarrow{\mathrm{pr}_1} \Delta_2$$

is surjective (here $\mathrm{pr}_1\colon \Delta_2 \times Y \longrightarrow \Delta_2$ is the first projection).

Proof. For $i=1,2$, the discriminant divisor of η_i is reduced [Sar82, Corollary 1.9], so consists of Δ_i. As Δ_i is smooth, $\eta_i^{-1}(p)$ is the union of two distinct lines for each $p \in \Delta_i$ (Proposition 7.7). Since $\rho(X_i/B_i)=1$, the preimage $\eta_i^{-1}(\Delta_i)$ is irreducible. The morphism $(X_i \times Y)/(B \times Y)$ is a standard conic bundle whose discriminant divisor is reduced, consisting of the smooth hypersurface $\Delta_i \times Y \subset B \times Y$.

We choose a dense open subset $U \subseteq B \times Y$ on which θ is defined and whose complement is of codimension 2 (since $B \times Y$ is smooth). In particular, $U \cap (\Delta_1 \times Y)$ is not empty, so $U \cap (\Delta_1 \times \{p\}) \neq \emptyset$ for a general point $p \in Y$. After restricting the open subset, we can moreover assume that $\eta_1^{-1}(U)$ is a closed subset of $U \times \mathbb{P}^2$, given by the quadratic form induced by a matrix $M_1 \in \mathrm{GL}_3(\mathbf{C}(U))$. The coefficients of the matrix can moreover be chosen in $\mathbf{C}(B) \subseteq \mathbf{C}(B \times Y) = \mathbf{C}(U)$, as the equation of $X_1 \times Y$ in $\mathbb{P}^2 \times Y$ is locally the equation of X_1 in $\mathbb{P}^2$, independent of Y.

We define $C \subset B \times Y$ to be image of $\Delta_1 \times \{p\}$ by θ, which is a point or an irreducible curve, as Δ_1 is an irreducible curve. The aim is now to show that $C \subseteq \Delta_2 \times Y$ and that $\mathrm{pr}_1(C) = \Delta_2$. We choose an open subset $V \subseteq B \times Y$ intersecting C such that $\eta_2^{-1}(V)$ is contained in $\mathbb{P}^2 \times V$ and is given by the quadratic form given by a symmetric matrix $M_2 \in \mathrm{Mat}_{3\times 3}(\mathbf{C}(V))$. The morphism η_2 being flat, we can choose the coefficients of M_2 to be defined on C and such that the residue matrix in $\overline{M}_2 \in \mathrm{Mat}_{3\times 3}(\mathbf{C}(C))$ is not zero. The birational map ψ is locally given by

$$\begin{array}{ccc} U \times \mathbb{P}^2 & \dashrightarrow & V \times \mathbb{P}^2 \\ \left(u, \begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) & \dashmapsto & \left(\theta(u), A(u) \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) \end{array}$$

for some $A \in \mathrm{GL}_3(\mathbf{C}(U))$. The explicit form of the map ψ gives

$$\lambda \cdot M_1 = {}^tA \cdot \theta^*(M_2) \cdot A$$

where $\lambda \in \mathbf{C}(U)^*$ is a scalar and $\theta^*(M_2)$ is the matrix obtained from M_2 by applying to its coefficients the field isomorphism $\theta^* \colon \mathbf{C}(V) \longrightarrow \mathbf{C}(U)$. As the rational map θ induces a dominant rational map $\Delta_1 \times \{p\} \dashrightarrow C$, we have a field homomorphism $\mathbf{C}(C) \longrightarrow \mathbf{C}(\Delta_1 \times \{p\}) \simeq \mathbf{C}(\Delta_1)$, that we denote by $\bar{\theta}^*$. It induces a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_C(V) & \xrightarrow{\theta^*} & \mathcal{O}_{\Delta_1 \times \{p\}}(U) \\ \downarrow & & \downarrow \\ \mathbf{C}(C) & \xrightarrow{\bar{\theta}^*} & \mathbf{C}(\Delta_1 \times \{p\}) \ \simeq \ \mathbf{C}(\Delta_1). \end{array}$$

We denote by $X' \subset U \times \mathbb{P}^2$ the subvariety given by the quadratic form associated to the matrix $\theta^*(M_2)$. We observe that the coefficients of $\theta^*(M_2)$ are defined over $\Delta_1 \times \{p\}$ and that the residue gives a matrix $\overline{\theta^*(M_2)} \in \mathrm{Mat}_{3\times 3}(\mathbf{C}(\Delta_1))$ which is obtained by applying the field homomorphism $\bar{\theta}^*$ to the entries of $\overline{M}_2 \in \mathrm{Mat}_{3\times 3}(C)$. The morphism $\mathrm{pr}_1 \colon X' \longrightarrow U$ is then an embedded conic fibration, which is flat after maybe reducing the open subset U (but still having $U \cap (\Delta_1 \times \{p\}) \neq \emptyset$).

We can apply Lemma 7.8 to the birational map $X' \dashrightarrow X$ given by

$$\left(u, \begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) \dashmapsto \left(u, A(u)^{-1} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$$

and to $\Gamma = \Delta_1 \times \{p\}$. Indeed, $(\eta_1 \times \mathrm{id})^{-1}(\Delta_1 \times \{p\})$ is irreducible as $\eta_1^{-1}(\Delta_1)$ is irreducible, and every fibre of $(\eta_1 \times \mathrm{id})^{-1}(\Delta_1 \times \{p\}) \longrightarrow \Delta_1 \times \{p\}$ is the union of two distinct lines, as the same holds for $\eta_1^{-1}(\Delta_1) \longrightarrow \Delta_1$ by Proposition 7.7. Lemma 7.8 gives two possibilities for the matrix $\bar{\theta}^*(M_2) \in \mathrm{Mat}_{3\times 3}(\mathbf{C}(\Delta_1))$: either it is of rank 1 (case (1)) or it is of rank 2, corresponding to a singular irreducible conic (case (2)). This gives the same two possibilities for $\overline{M}_2 \in \mathrm{Mat}_{3\times 3}(C)$ as $\bar{\theta}^*$ is a field homomorphism. As the rank of M_2 is smaller than 3, the variety C is in the discriminant of $(X_2 \times Y)/(B \times Y)$ and is thus contained in $\Delta_2 \times Y$ as desired. It remains to see that C is not contained in $\{q\} \times Y$ for some point q. Indeed, the preimage $(\eta_2 \times \mathrm{id})^{-1}(\{q\} \times Y)$ is isomorphic to $\eta_2^{-1}(\{q\}) \times Y$, which is not irreducible, as $\eta_2^{-1}(\{q\})$ is the union of two lines (again by Proposition 7.7), but which is reduced. □

7.B. **Conic bundles associated to smooth cubic curves.** The principal result in this section is Proposition 7.15, which provides a family of conic bundles that we shall use in the next section to prove Theorem E.

Lemma 7.11. *For each $p = [\alpha : \beta] \in \mathbb{P}^1$, the set*

$$\mathcal{S}_p = \{[x_0 : x_1 : x_2] \in \mathbb{P}^2 \mid \alpha x_0^2 + \beta x_1 x_2 = \alpha x_1^2 + \beta x_0 x_2 = \alpha x_2^2 + \beta x_0 x_1 = 0\}$$

consists of three points if $\alpha(\alpha^3 + \beta^3) = 0$ and is empty otherwise.

Proof. Since $\mathcal{S}_{[0:1]} = \{[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]\}$ and $\mathcal{S}_{[1:0]} = \emptyset$, we may assume that $\alpha \in \mathbf{C}^*$ and $\beta = 1$. If $[x_0 : x_1 : x_2] \in \mathcal{S}_p$, then $\alpha(x_0^3 - x_1^3) = x_0(\alpha x_0^2 + x_1 x_2) - x_1(\alpha x_1^2 + x_0 x_2) = 0$. The equations being symmetric, we get $x_0^3 = x_1^3 = x_2^3$. In particular $x_0 x_1 x_2 \neq 0$, so the three equations are equivalent to

$$\alpha = -\frac{x_1 x_2}{x_0^2} = -\frac{x_0 x_1}{x_2^2} = -\frac{x_0 x_2}{x_1^2},$$

which implies that $\alpha^3 = -1$. For the three possible values of α, we observe that $\mathcal{S}_{[\alpha:1]} = \{[1 : x_1 : -\alpha/x_1] \mid x_1^3 = 1\}$ consists of three points. □

Lemma 7.12. *For each $\xi \in \mathbf{C}$ such that $\xi^3 \neq -\frac{1}{8}$, the hypersurface $X_\xi \subset \mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(2,1)$ given by*

$$X_\xi = \left\{ ([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) \in \mathbb{P}^2 \times \mathbb{P}^2 \;\middle|\; \sum_{i=0}^{2} (x_i^2 + 2\xi \frac{x_0 x_1 x_2}{x_i}) y_i = 0 \right\}$$

is smooth, irreducible, rational over $\mathbf{Q}(\xi)$, and satisfies $\rho(X_\xi) = 2$. The second projection gives a standard conic bundle $X_\xi/\mathbb{P}^2$. The discriminant curve $\Delta_\xi \subset \mathbb{P}^2$ is given by

$$-\xi^2(y_0^3 + y_1^3 + y_2^3) + (2\xi^3 + 1) y_0 y_1 y_2 = 0$$

and is the union of three lines if $\xi = 0$ or if $\xi^3 = 1$, and is a smooth cubic otherwise.

Proof. To show that X_ξ is smooth, irreducible, rational over $\mathbf{Q}(\xi)$ and that $\rho(X_\xi) = 2$, it suffices to show that the first projection $X_\xi \longrightarrow \mathbb{P}^2$ is a (Zariski locally trivial) $\mathbb{P}^1$-bundle. This amounts to showing that the coefficients of the linear polynomial in the variables y_i defining X_ξ are never zero, i.e. that for each $[x_0 : x_1 : x_2] \in \mathbb{P}^2$ we cannot have $x_0^2 + 2\xi x_1 x_2 = x_1^2 + 2\xi x_0 x_2 = x_2^2 + 2\xi x_0 x_1 = 0$. This follows from Lemma 7.11 and from the hypothesis $\xi^3 \neq -\frac{1}{8}$.

The equation of X_ξ is given by

$$(x_0\, x_1\, x_2) \cdot M \cdot \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = 0 \text{ with } M = \begin{pmatrix} y_0 & \xi y_2 & \xi y_1 \\ \xi y_2 & y_1 & \xi y_0 \\ \xi y_1 & \xi y_0 & y_2 \end{pmatrix} \in \mathrm{Mat}_{3\times 3}(\mathbf{C}[y_0, y_1, y_2]).$$

The polynomial $\det(M)$ is equal to

$$\det(M) = \lambda(y_0^3 + y_1^3 + y_2^3) + \mu y_0 y_1 y_2, \text{ with } \lambda = -\xi^2 \text{ and } \mu = 2\xi^3 + 1.$$

In particular, the fibres of the second projection $X_\xi/\mathbb{P}^2$ are all conics (the coefficients of x_i^2 is y_i so not all coefficients can be zero) and a general one is irreducible. As the threefold X_ξ is smooth, irreducible and satisfies $\rho(X_\xi) = 2$, the morphism $X_\xi/\mathbb{P}^2$ is a standard conic bundle. Its discriminant is given by the zero locus of $\det(M)$, which is a polynomial of degree 3 which has the classical Hesse Form. The discriminant corresponds to a smooth cubic if $\lambda(27\lambda^3 + \mu^3) \neq 0$, and to the union of three lines in general position otherwise. To prove this classical fact, we compute the partial derivatives of $\det(M)$, which are $(3\lambda y_0^2 + \mu y_1 y_2, 3\lambda y_1^2 + \mu y_0 y_2, 3\lambda y_2^2 + \mu y_0 y_1)$. By Lemma 7.11, this has no zeroes in $\mathbb{P}^2$ if $\lambda(27\lambda^3 + \mu^3) \neq 0$ and has three zeroes otherwise. It remains to observe that $27\lambda^3 + \mu^3 = (8\xi^3 + 1)(\xi^3 - 1)^2$. □

Remark 7.13. Let $\mathbf{k}$ be a subfield of $\mathbf{C}$ and $\xi \in \mathbf{k}$. Then the curve Δ_ξ of Lemma 7.12 is defined over $\mathbf{k}$ and has a $\mathbf{k}$-rational point, namely the inflexion point $[0 : 1 : -1]$. When $\mathbf{k} = \mathbf{C}$, one can prove that all elliptic curves are obtained in this way; for smaller fields this does not seem to be true. We will however show that there are enough such curves.

We thank P. Habegger for helpful discussions concerning the next lemma.

Lemma 7.14. *Let $\mathbf{k} \subseteq \mathbf{C}$ be a subfield.*

(1) *For each* $\xi \in \mathbf{k}$*, with* $\xi^3 \notin \{0, -\frac{1}{8}, 1\}$*, we denote (as in Lemma 7.12) by* Δ_ξ *the smooth cubic curve defined over* $\mathbf{k}$ *given by*

$$-\xi^2(y_0^3 + y_1^3 + y_2^3) + (2\xi^3 + 1)y_0y_1y_2 = 0.$$

The j*-invariant of* Δ_ξ *is equal to*

$$\left(\frac{16\xi^{12} + 464\xi^9 + 240\xi^6 + 8\xi^3 + 1}{\xi^2(8\xi^9 - 15\xi^6 + 6\xi^3 + 1)}\right)^3.$$

(2) *There is a subset* $J \subseteq \mathbf{k}$ *having the same cardinality as* $\mathbf{k}$ *such that for all* $\xi, \xi' \in J$*, the following are equivalent:*

(*i*) *There exist surjective morphisms* $\Delta_\xi \dashrightarrow \Delta_{\xi'}$ *and* $\Delta_{\xi'} \dashrightarrow \Delta_\xi$ *defined over* $\mathbf{C}$*;*

(*ii*) $\xi = \xi'$*.*

Proof. (1). By Lemma 7.12, Δ_ξ is a smooth cubic curve if $\xi^3 \notin \{0, -\frac{1}{8}, 1\}$. We choose the inflexion point $[0 : 1 : -1] \in \Delta_\xi$ to be the origin, make a coordinate change so that the inflexion line is the line at infinity, and thusly obtain a Weierstrass form. Then we compute the j-invariant as in [Sil09, III.1 page 42]; this is tedious but straightforward. This can also be done using the formulas from [AD09, page 240].

(2). Let $\xi, \xi' \in \mathbf{k}$ be such that $\xi^3, (\xi')^3 \notin \{0, -\frac{1}{8}, 1\}$. We see the curves Δ_ξ and $\Delta_{\xi'}$ as elliptic curves defined over $\mathbf{k}$ with origin $O = [0 : 1 : -1]$. Suppose that there is a surjective morphism $\varphi\colon \Delta_\xi \dashrightarrow \Delta_{\xi'}$ defined over $\mathbf{C}$. It sends the origin of Δ_ξ onto a $\mathbf{C}$-rational point of $\Delta_{\xi'}$. Applying a translation at the target, we can assume that $\varphi(O) = O$, which means that φ is an isogeny, and that Δ_ξ and $\Delta_{\xi'}$ are isogenous over $\mathbf{C}$ (see [Sil09, Definition, §III.4 page 66]).

We now choose a sequence $p_1, p_2, \dots$ of increasing prime numbers such that for each $i \geqslant 2$, the prime number p_i does not appear in the denominator of the j-invariant of $\Delta_{p_{i'}}$ for each $i' < i$. For each $i \geqslant 1$, the j-invariant of Δ_{p_i} is an element of $\mathbf{Q}$ having a denominator divisible by p_i (follows from (1)), so Δ_{p_i} does not have potential good reduction modulo p_i but this does not hold for $\Delta_{p_{i'}}$ for $i' > i$, which then has potential good reduction modulo p_i [Sil09, Proposition 5.5, §VII.5, page 197]. This implies that there is no isogeny $\Delta_{p_i} \longrightarrow \Delta_{p_{i'}}$ defined over any number field K and where one curve has good reduction and the other has bad reduction [Sil09, Corollary 7.2, §VII.7, page 202], and thus no isogeny defined over $\mathbf{C}$ [MW90, Lemma 6.1]. If $\mathbf{k}$ is countable, this achieves the proof of (2).

It remains to consider the case where $\mathbf{k}$ is an uncountable subfield of $\mathbf{C}$. The set of j-invariants of curves Δ_ξ, where $\xi \in \mathbf{k}$ is such that $\xi^3 \notin \{0, -\frac{1}{8}, 1\}$, is then uncountable too.

We denote by $\Omega \subseteq \mathbf{C}^2$ the set consisting of pairs $(j_1, j_2) \in \bar{\mathbf{Q}}^2$ such that the curves of j-invariants j_1 and j_2 are isogenous. The set Ω is a countable union of algebraic curves defined over $\mathbf{Q}$, given by the zero set of the so-called modular transformation polynomials (see [Lan87, 5§3] and in particular [Lan87, Theorem 5, Chapter 5§3, page 59]). Moreover, these curves are irreducible and invariant under the exchanges of variables $(x, y) \longmapsto (y, x)$ [Lan87, Theorem 3, Chapter 5§3, page 55], so are not vertical or horizontal lines in $\mathbf{C}^2$.

We write $S = \{\xi \in \mathbf{k} \mid \xi^3 \notin \{0, -\frac{1}{8}, 1\}\}$. Then, by the previous paragraph, for each element $\xi \in S$ the curve Δ_ξ is isogeneous (over $\mathbf{C}$) to only countably many isomorphism classes of $\Delta_{\xi'}$ with $\xi' \in \mathbf{k}$. Putting an equivalence relation on

S saying that two elements are equivalent if the curves are isogeneous over $\mathbf{C}$ (see [Sil09, III.6, Theorem 6.1(a)]), we obtain that each equivalence class is countable, so the set of equivalence classes has the cardinality of S, or equivalently of $\mathbf{k}$. This achieves the proof. □

Proposition 7.15. *Let $\mathbf{k}$ be a subfield of $\mathbf{C}$. For each $n \geqslant 3$, there is a set J having the cardinality of $\mathbf{k}$ indexing decomposable conic bundles X_i/B_i defined over $\mathbf{k}$, where X_i, B_i are smooth varieties rational over $\mathbf{k}$, and such that two conic bundles X_i/B_i and X_j/B_j are equivalent (over $\mathbf{C}$) if and only if $i = j$.*

Proof. We choose the set $J \subseteq \mathbf{k}$ of Lemma 7.14(2), and consider, for each $\xi \in J$, the hypersurface $X_\xi \subset \mathbb{P}^2 \times \mathbb{P}^2$ of Lemma 7.12, which is given by

$$X_\xi = \left\{ ([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) \in \mathbb{P}^2 \times \mathbb{P}^2 \;\middle|\; \sum_{i=0}^{2} (x_i^2 + 2\xi \frac{x_0 x_1 x_2}{x_i}) y_i = 0 \right\}$$

By Lemma 7.12, the second projection gives a standard conic bundle $X_\xi \longrightarrow \mathbb{P}^2$ whose discriminant curve $\Delta_\xi \subset \mathbb{P}^2$ is given by $-\xi^2(y_0^3 + y_1^3 + y_2^3) + (2\xi^3 + 1)y_0 y_1 y_2$. Note that $(X_\xi \times \mathbb{P}^{n-3})/(\mathbb{P}^2 \times \mathbb{P}^{n-3})$ (or simply $X_\xi/\mathbb{P}^2$ if $n = 3$) is a decomposable conic bundle defined over $\mathbf{k}$, as it is embedded in the trivial $\mathbb{P}^2$-bundle $(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^{n-3})/(\mathbb{P}^2 \times \mathbb{P}^{n-3})$ by construction. Moreover, $X_\xi \times \mathbb{P}^{n-3}$ is birational to $\mathbb{P}^n$ over $\mathbf{k}$ (Lemma 7.12). By Proposition 7.10, two conic bundles $(X_\xi \times \mathbb{P}^{n-3})/(\mathbb{P}^2 \times \mathbb{P}^{n-3})$ and $(X_{\xi'} \times \mathbb{P}^{n-3})/(\mathbb{P}^2 \times \mathbb{P}^{n-3})$ are equivalent only if there exist surjective morphisms $\Delta_\xi \twoheadrightarrow \Delta_{\xi'}$ and $\Delta_{\xi'} \twoheadrightarrow \Delta_\xi$. This is only possible if $\xi = \xi'$, by Lemma 7.14(2). □

7.C. **Proofs of Theorems E and C.**

Proof of Theorem E. By Theorem D, we have a group homomorphism and a groupoid homomorphism

$$\begin{array}{ccc} \mathrm{Bir}(\mathbb{P}^n) & \longrightarrow & *_{C \in \mathsf{CB}(\mathbb{P}^n)} \left(\oplus_{\mathsf{M}(C)} \mathbf{Z}/2 \right) \\ \cup\!| & \nearrow & \\ \mathrm{BirMori}(\mathbb{P}^n) & & \end{array}$$

For each subfield $\mathbf{k} \subseteq \mathbf{C}$, we can embed $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^n)$ into $\mathrm{Bir}_{\mathbf{C}}(\mathbb{P}^n)$ and look at the image in $*_{C \in \mathsf{CB}(\mathbb{P}^n)} \left(\oplus_{\mathsf{M}(C)} \mathbf{Z}/2 \right)$. We consider the set of decomposable conic bundles X_i/B_i defined over $\mathbf{k}$ indexed by J of Proposition 7.15, which give pairwise distinct elements of $C_i \in \mathsf{CB}(\mathbb{P}^n)$, and associate to these birational maps $\psi_i \colon X_i \dashrightarrow \mathbb{P}^n$ defined over $\mathbf{k}$. For each $i \in J$, there is an involution $\iota_i \in \psi_i \, \mathrm{Bir}_{\mathbf{k}}(X_i/B_i) \psi_i^{-1} \subseteq \mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^n)$ whose image in $\oplus_{\mathsf{M}(C_i)} \mathbf{Z}/2$ is not trivial by Proposition 6.9. One can thus take a projection $\oplus_{\mathsf{M}(C_i)} \mathbf{Z}/2 \longrightarrow \mathbf{Z}/2$ such that the image of ι_i is non-trivial. We obtain a surjective group homomorphism from $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^n)$ to $*_{i \in J} \mathbf{Z}/2$ where J has the cardinality of $\mathbf{k}$ and such that each involution $\iota_i \in \mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^n)$ is sent onto the generator indexed by i. There is thus a section of this surjective group homomorphism. □

Remark 7.16. As Proposition 7.15 gives an infinite image, the above proof naturally gives a surjective homomorphism to the group $*_J(\oplus_{\mathbf{Z}} \mathbf{Z}/2)$, but since there is an abstract surjective homomorphism from $*_J \mathbf{Z}/2$ to this group, we chose not to mention the direct sum in the statement of the theorem.

Moreover, with the alternative form the existence of a section would be far less clear. Indeed, $(\mathbf{Z}/2)^3$ does not embed in $\mathrm{Bir}(X/B)$ and $(\mathbf{Z}/2)^7$ does not embed

in $\mathrm{Bir}(X)$, for X rationally connected of dimension 3 [Pro11, Pro14], so it seems probable that $\oplus_{\mathbf{Z}} \mathbf{Z}/2$ does not embed in $\mathrm{Bir}(X)$ for any variety X.

Proof of Theorem C. We consider a subfield $\mathbf{k}$ of $\mathbf{C}$, an integer $n \geqslant 3$, and a subset $S \subset \mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^n)$ of cardinality smaller than the one of $\mathbf{k}$. We want to construct a surjective homomorphism $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^n) \twoheadrightarrow \mathbf{Z}/2$ such that the group G generated by $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^n)$, by all Jonquières elements and by S is contained in the kernel. We use the group homomorphism

$$\tau\colon \mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^n) \longrightarrow \mathop{\ast}_{J} \mathbf{Z}/2$$

given by Theorem E. Each $j \in J$ corresponds to a conic bundle X_j/B_j. The group $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^n)$ is in the kernel of τ. The group of Jonquières elements is conjugated to the subgroup $J \subset \mathrm{Bir}(\mathbb{P}^1 \times \mathbb{P}^{n-1})$ consisting of elements sending a general fibre of $\mathbb{P}^1 \times \mathbb{P}^{n-1}/\mathbb{P}^{n-1}$ onto another one. The action on the base yields an exact sequence

$$1 \longrightarrow \mathrm{Bir}(\mathbb{P}^1 \times \mathbb{P}^{n-1}/\mathbb{P}^{n-1}) \longrightarrow J \longrightarrow \mathrm{Bir}(\mathbb{P}^{n-1}) \longrightarrow 1.$$

This gives $J = \mathrm{Bir}(\mathbb{P}^1 \times \mathbb{P}^{n-1}/\mathbb{P}^{n-1}) \rtimes J'$, where $J' \subset J$ is the group isomorphic to $\mathrm{Bir}(\mathbb{P}^{n-1})$ that acts on $\mathbb{P}^1 \times \mathbb{P}^{n-1}$ with trivial action on the first factor. We can assume that $\mathbb{P}^1 \times \mathbb{P}^{n-1}/\mathbb{P}^{n-1} = X_{j_0}/B_{j_0}$ for some $j_0 \in J$. The image of $\mathrm{Bir}(\mathbb{P}^1 \times \mathbb{P}^{n-1}/\mathbb{P}^{n-1})$ by τ is contained in the group $\mathbf{Z}/2$ indexed by j_0. Now observe that $J' \subset \mathrm{Ker}\,\tau$. Indeed, we first decompose an element of $J' \simeq \mathrm{Bir}(\mathbb{P}^{n-1})$ as a product of Sarkisov links between Mori fibre spaces $Y_i \longrightarrow S_i$, where Y_i has dimension $n-1$, and observe that taking the product with $\mathbb{P}^1$ gives Sarkisov links between the Mori fibre spaces $Y_i \times \mathbb{P}^1 \longrightarrow S_i \times \mathbb{P}^1$ of dimension n. Each of the Sarkisov links of type II arising in such decomposition has covering gonality 1, as $\mathrm{cov.\,gon}(\Gamma \times \mathbb{P}^1) = 1$ for each variety Γ.

We consider the group homomorphism $\hat{\tau}\colon \mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^n) \longrightarrow \ast_{J\smallsetminus\{j_0\}} \mathbf{Z}/2$ obtained by composing τ with the projection $\ast_J \mathbf{Z}/2 \longrightarrow \ast_{J\smallsetminus\{j_0\}} \mathbf{Z}/2$ obtained by forgetting the factor indexed by j_0.

The image by $\hat{\tau}$ of all Jonquières elements is trivial, hence the group $\hat{\tau}(G)$ has at most the cardinality of S, which by assumption is strictly smaller than the cardinality of J. We construct the expected morphism by projecting from $\hat{\tau}(\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^n))$ onto a factor $\mathbf{Z}/2$ which is not in the image of G. □

8. Complements

8.A. **Quotients and SQ-universality.** A direct consequence of Theorem E is that we have a lot of quotients of $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^n)$ for $n \geqslant 3$.

Firstly, we can have quite small quotients (which is not the case for $\mathrm{Bir}_{\mathbf{C}}(\mathbb{P}^2)$ which has no non-trivial countable quotient, as mentioned before):

Corollary 8.1. *For each $n \geqslant 3$, each subfield $\mathbf{k} \subseteq \mathbf{C}$, and each integer $m \geqslant 1$, there are (abstract) surjective group homomorphisms from $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^n)$ to the dihedral group D_{2m} of order $2m$ and the symmetric group Sym_m. In particular, there is a normal subgroup of $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^n)$ of index r for each even integer $r > 1$.*

Proof. Follows from Theorem E and the fact that D_{2m} and Sym_m are generated by involutions. □

Secondly, we get much larger quotients:

Corollary 8.2. *For any $n \geqslant 3$, any subfield $\mathbf{k} \subseteq \mathbf{C}$ and any integer $m \geqslant 1$, there are (abstract) surjective group homomorphisms*

$$\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^n) \twoheadrightarrow \mathrm{SL}_m(\mathbf{k}), \qquad \mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^n) \twoheadrightarrow \mathrm{Bir}_{\overline{\mathbf{Q}}}(\mathbb{P}^2).$$

Proof. We observe that $\mathrm{SL}_m(\mathbf{k})$ has the cardinality of $\mathbf{k}$ and that $\mathrm{Bir}_{\overline{\mathbf{Q}}}(\mathbb{P}^2)$ is countable. Hence, both groups have at most the cardinality of $\mathbf{k}$. Both groups are generated by involutions: for $\mathrm{Bir}_{\overline{\mathbf{Q}}}(\mathbb{P}^2)$ this is by the Noether-Castelnuovo Theorem which says that $\mathrm{Bir}_{\overline{\mathbf{Q}}}(\mathbb{P}^2)$ is generated by the standard quadric involution and by $\mathrm{Aut}_{\overline{\mathbf{Q}}}(\mathbb{P}^3) \simeq \mathrm{PGL}_3(\overline{\mathbf{Q}}) = \mathrm{PSL}_3(\overline{\mathbf{Q}})$, and thus is generated by involutions. Hence, the two groups are quotients of $*_J \mathbf{Z}/2$. The result then follows from Theorem E. □

Similarly, over $\mathbf{C}$ we get:

Corollary 8.3. *For any $n \geqslant 3$, there exists a surjective group homomorphism*

$$\mathrm{Bir}_{\mathbf{C}}(\mathbb{P}^n) \twoheadrightarrow \mathrm{Bir}_{\mathbf{C}}(\mathbb{P}^2).$$

Recall that a group G is *SQ-universal* if any countable group embeds in a quotient of G. The free group $\mathbf{Z} * \mathbf{Z}$ was an early example of SQ-universal group. More generally any nontrivial free product $G_1 * G_2$ distinct from $\mathbf{Z}/2 * \mathbf{Z}/2$ is SQ-universal, see [Sch73, Theorem 3]. From a modern point of view, this also follows from [MO15], by looking at the action of any loxodromic isometry on the associated Bass-Serre tree. In particular, taking $G_1 = \mathbf{Z}/2 * \mathbf{Z}/2$ and $G_2 = \mathbf{Z}/2$, we get that $\mathbf{Z}/2 * \mathbf{Z}/2 * \mathbf{Z}/2$ is SQ-universal.

Corollary 8.4. *For any field $\mathbf{k} \subseteq \mathbf{C}$ and any $n \geqslant 3$, the Cremona group $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^n)$ admits a surjective morphism to the SQ-universal group $\mathbf{Z}/2 * \mathbf{Z}/2 * \mathbf{Z}/2$. In particular, $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^n)$ also is SQ-universal.*

Proof. Follows from Theorem E and from the fact that $\mathbf{Z}/2 * \mathbf{Z}/2 * \mathbf{Z}/2$ is SQ-universal. □

8.B. **Hopfian property.** Recall that a group G is *hopfian* is every surjective group homomorphism $G \twoheadrightarrow G$ is an isomorphism. It was proven in [Dés07] that the group $\mathrm{Bir}_{\mathbf{C}}(\mathbb{P}^2)$ is hopfian. An open question, asked by I. Dolgachev (see [Dés17]), is whether the Cremona group $\mathrm{Bir}_{\mathbf{C}}(\mathbb{P}^n)$ is generated by involutions for each n, the answer being yes in dimension 2 and open in dimension $\geqslant 3$. Theorem E relates these two notions and shows that we cannot generalise both results at the same time (being hopfian and generated by involutions) to higher dimension.

Corollary 8.5. *For each $n \geqslant 3$ and each subfield $\mathbf{k} \subseteq \mathbf{C}$, the group $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^n)$ is not hopfian if it is generated by involutions.*

Proof. Follows from Theorem E, as the group homomorphisms provided by Theorem E is not injective, and because $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^n)$ has the same cardinality as $\mathbf{k}$ (the set of all polynomials of degree n with coefficients in $\mathbf{k}$ has the same cardinality as $\mathbf{k}$). □

8.C. **More general fields.** Every field isomorphism $\mathbf{k} \xrightarrow{\sim} \mathbf{k}'$ naturally induces an isomorphism $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^n) \xrightarrow{\sim} \mathrm{Bir}_{\mathbf{k}'}(\mathbb{P}^n)$. More generally, it associates to each variety and each rational map defined over $\mathbf{k}$, a variety and a rational map defined over $\mathbf{k}'$. It then induces an isomorphism between the group of birational maps defined

over $\mathbf{k}$ and $\mathbf{k}'$ of the varieties obtained. This implies that the five Theorems A-E also hold for each ground field which is abstractly isomorphic to a subfield of $\mathbf{C}$. This includes any field of rational functions of any algebraic variety defined over a subfield of $\mathbf{C}$ as these fields have characteristic zero and cardinality smaller or equal than the one of $\mathbf{C}$.

8.D. **Amalgamated product structure.** We work over the field $\mathbf{C}$. In the next result, an element of $\mathsf{CB}(X)$ is said to be decomposable if it is the class of a decomposable conic bundle (in the sense of Definition 6.8).

Theorem 8.6. *For each integer $n \geqslant 3$, and let X/B be a conic bundle, where X is a terminal variety of dimension n. We denote by ρ the group homomorphism*

$$\rho\colon \mathrm{Bir}(X) \longrightarrow \mathop{\ast}_{C\in\mathsf{CB}(X)} \left(\mathop{\oplus}_{\mathsf{M}(C)} \mathbf{Z}/2 \right)$$

given by Theorem D. For each $C \in \mathsf{CB}(X)$ we fix a choice of representative X_C/B_C, and we denote $G_C = \rho^{-1}(\rho(\mathrm{Bir}(X_C/B_C))) \subseteq \mathrm{Bir}(X)$. Then, the following hold:

(1) *For all $C \neq C'$ in $\mathsf{CB}(X)$, the group $A = G_C \cap G_{C'}$ contains $\ker\rho$ and does not depend on the choice of C, C';*

(2) *The group $\mathrm{Bir}(X)$ is the free product of the groups G_C, $C \in \mathsf{CB}(X)$, amalgamated over their common intersection A:*

$$\mathrm{Bir}(X) = \mathop{\ast}_{A} G_C.$$

(3) *For each decomposable $C \in \mathsf{CB}(X)$ we have $A \subsetneq G_C$. Moreover, the free product of (2) is non-trivial (i.e. $A \subsetneq G_C \subsetneq \mathrm{Bir}(X)$ for each C) as soon as $\mathsf{CB}(X)$ contains two distinct decomposable elements. This is for instance the case when X is rational, as $\mathsf{CB}(X)$ then contains uncountably many decomposable elements.*

Proof. (1). For each $C \in \mathsf{CB}(X)$, we denote by $H_C = \left(\oplus_{\mathsf{M}(C)} \mathbf{Z}/2\right)$ the factor indexed by C in the free product $\ast_{C\in\mathsf{CB}(X)} \left(\oplus_{\mathsf{M}(C)} \mathbf{Z}/2\right) = \ast_{C\in\mathsf{CB}(X)} H_C$. By definition of the group homomorphism, for each $C \in \mathsf{CB}(X)$ we have $\rho(\mathrm{Bir}(X_C/B_C)) \subseteq H_C$. As H_C is a $\mathbb{F}_2$-vector space with basis $\mathsf{M}(C)$ and $\rho(\mathrm{Bir}(X_C/B_C))$ is a linear subspace, there exists a projection $H_C \longrightarrow \rho(\mathrm{Bir}(X_C/B_C))$. We then denote by

$$\rho'\colon \mathrm{Bir}(X) \longrightarrow \mathop{\ast}_{C\in\mathsf{CB}(X)} \rho(\mathrm{Bir}(X_C/B_C))$$

the group homomorphism induced for each C by the projection $H_C \longrightarrow \rho(\mathrm{Bir}(X_C/B_C))$. By definition of the free product, we obtain $H_C \cap H_{C'} = \mathrm{id}$ for all $C \neq C'$. This implies that $G_C \cap G_{C'} = \ker\rho' \supseteq \ker\rho$.

(2). We first observe that by construction the groups G_C generate the group $\mathrm{Bir}(X)$. The fact that $\mathrm{Bir}(X) = \ast_A G_C$ corresponds to saying that all relations in $\mathrm{Bir}(X)$ lie in the groups G_C. This follows from the group homomorphism ρ to a free product, where no relation between the groups H_C exists.

(3). The fact that $A \subsetneq G_C$ for each decomposable C follows from Proposition 6.9. Hence, the free product of (2) is non-trivial if there are least two C corresponding to decomposable conic bundles. If X is rational, then we moreover have uncountably many such elements by Proposition 7.15. □

In Theorem 8.6, one could be tempted to say that $A = \ker\rho$, but this is not clear. Indeed, it could be that some elements of $\oplus_{\mathsf{M}(C)} \mathbf{Z}/2$ are in the image of $\mathrm{Bir}(X)$ but not in the image of $\mathrm{Bir}(X/B)$.

8.E. **Cubic varieties.** Here again we work over $\mathbf{C}$. We recall the following result, which allows to apply Theorem B to any smooth cubic hypersurface of dimension $\geqslant 3$:

Lemma 8.7. *Let $n \geqslant 4$ and let $\ell \subset X \subset \mathbb{P}^n$ be a line on a smooth cubic hypersurface. We denote by $\hat{X}$ and P the respective blow-ups of X and $\mathbb{P}^n$ along ℓ. Then, the projection away from ℓ gives rise to a decomposable conic bundle and a decomposable $\mathbb{P}^2$-bundle*

$$\hat{X} \subset P = \mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-2}} \oplus \mathcal{O}_{\mathbb{P}^{n-2}} \oplus \mathcal{O}_{\mathbb{P}^2}(1)) \longrightarrow \mathbb{P}^{n-2}.$$

Moreover, the discriminant of the conic bundle is a hypersurface of degree 5.

Proof. We take coordinates $[y_0 : y_1 : \cdots : y_{n-2} : u : v]$ on $\mathbb{P}^n$ and assume that $\ell \subset \mathbb{P}^n$ is the line given by $y_0 = y_1 = \cdots = y_{n-2} = 0$. The equation of X is then given by

$$Au^2 + 2Buv + Cv^2 + 2Du + 2Ev + F = 0$$

where $A, B, C, D, E, F \in \mathbf{C}[y_0, \ldots, y_{n-2}]$ are homogeneous polynomials of degree $1, 1, 1, 2, 2, 3$ respectively.

As in the proof of Proposition 6.9, we view $P = \mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-2}} \oplus \mathcal{O}_{\mathbb{P}^{n-2}} \oplus \mathcal{O}_{\mathbb{P}^{n-2}}(1))$ as the quotient of $(\mathbb{A}^2 \smallsetminus \{0\}) \times (\mathbb{A}^{n-1} \smallsetminus \{0\})$ by $(\mathbb{G}_m)^2$ via

$$((\lambda, \mu), (x_0, x_1, x_2, y_0, y_1, \cdots, y_{n-2})) \longmapsto (\lambda x_0, \lambda x_1, \lambda\mu^{-1} x_2, \mu y_0, \cdots, \mu y_{n-2})$$

and denote by $[x_0 \colon x_1 \colon x_2 \,;\, y_0 \colon \cdots \colon y_{n-2}] \in P$ the class of $(x_0, x_1, x_2, y_0, \cdots, y_{n-2})$. The birational morphism

$$\begin{array}{ccc} P & \longrightarrow & \mathbb{P}^n \\ [x_0 \colon x_1 \colon x_2 \,;\, y_0 \colon y_1 \colon y_2 \colon \cdots \colon y_{n-2}] & \longmapsto & [x_2 y_0 \colon \cdots \colon x_2 y_{n-2} : x_0 : x_1] \end{array}$$

is the blow-up of ℓ, so $\hat{X}$ is given by

$$Ax_0^2 + 2Bx_0x_1 + Cx_1^2 + 2Dx_2x_0 + 2Ex_2x_1 + Fx_2^2 = 0,$$

which is then a conic bundle over $\mathbb{P}^2$. The discriminant of the curve gives a hypersurface $\Delta \subset \mathbb{P}^2$ of degree 5, given by the determinant of $\left(\begin{smallmatrix} A & B & D \\ B & C & E \\ D & E & F \end{smallmatrix}\right)$. □

Corollary 8.8. *For each $n \geqslant 4$ and each smooth cubic hypersurface $X \subset \mathbb{P}^n$, there exists a surjective group homomorphism $\mathrm{Bir}(X) \twoheadrightarrow \oplus_{\mathbf{Z}} \mathbf{Z}/2$*

Proof. Follows from the application of Theorem B to the conic bundle associated to blow-up a line of X (Lemma 8.7). □

Every smooth cubic threefold $X \subset \mathbb{P}^4$ is not rational, and moreover two such cubics are birational if and only if they are projectively equivalent, i.e. equal up to an element of $\mathrm{Aut}(\mathbb{P}^4) = \mathrm{PGL}_5(\mathbf{C})$ [CG72]. We moreover get the following:

Proposition 8.9. *Let $X \subset \mathbb{P}^4$ be a general smooth cubic hypersurface. We have a surjective group homomorphism $\mathrm{Bir}(X) \twoheadrightarrow *_J \mathbf{Z}/2$, where J has the cardinality of $\mathbf{C}$.*

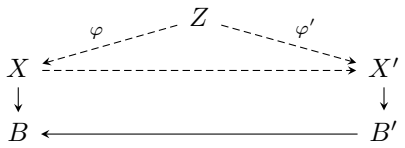
Proof. The map of Lemma 8.7 associates to each smooth cubic threefold X and each line $\ell \subset X$ a quintic curve $\Delta \subset \mathbb{P}^2$ and also a theta-characteristic; this induces a birational map between the pairs (ℓ, X) of lines on smooth cubic threefolds, up to $\mathrm{PGL}_5(\mathbf{C})$, and the pairs (θ, Δ), where $\Delta \subset \mathbb{P}^2$ is a smooth quintic and θ is a

theta-characteristic, again up to $\mathrm{PGL}_3(\mathbf{C})$ [CMF05, Theorem 4.1 and Proposition 4.2].

In particular, taking a general smooth cubic hypersurface $X \subset \mathbb{P}^4$ and varying the lines $\ell \subset X$ (which form a 2-dimensional family), we obtain a family J of dimension 2 of smooth quintics $\Delta \subset \mathbb{P}^2$, not pairwise equivalent modulo $\mathrm{PGL}_3(\mathbf{C})$. This yields conic bundles that are not pairwise equivalent, parametrised by a complex algebraic variety of dimension 2. Applying the group homomorphism of Theorem D and projecting on the corresponding factors provides a surjective group homomorphism $\mathrm{Bir}(X) \twoheadrightarrow *_J \mathbf{Z}/2$, similarly as in the proof of Theorem E. □

8.F. **Fibrations graph.** We explain how to get a natural graph structure from the set of rank r fibrations, similarly as in [LZ17].

Let Z be a variety birational to a Mori fibre space. We construct a sequence of nested graphs $\mathcal{G}_n$, $n \geqslant 1$, as follows. The set of vertices of $\mathcal{G}_n$ are rank r fibrations X/B, for any $r \leqslant n$, with a choice of a birational map $\varphi\colon Z \dashrightarrow X$, and modulo Z-equivalence (Definition 4.1). We denote $(X/B, \varphi)$ such an equivalence class. We put an oriented edge from $(X/B, \varphi)$ to $(X'/B', \varphi')$ if $\rho(X'/B') = \rho(X/B) - 1$ and the birational maps from Z induce a factorisation of X/B through X'/B', that is, if there is a morphism $B' \longrightarrow B$ and a birational contraction $X \dashrightarrow X'$ such that the following diagram commutes



We call the graph $\mathcal{G} := \bigcup_n \mathcal{G}_n$ the *fibrations graph* associated with Z. The group $\mathrm{Bir}(Z)$ naturally acts on each graph $\mathcal{G}_n$, and so also on $\mathcal{G}$, by precomposition :

$$g \cdot (X/B, \varphi) := (X/B, \varphi \circ g^{-1}).$$

The fact that Sarkisov links generate $\mathrm{BirMori}(Z)$ is equivalent to the fact that $\mathcal{G}_2$ is a connected graph. Lemma 4.2 implies that $\mathcal{G}_3$ is the 1-skeleton of a square complex, where each square has one vertex of rank 3, one vertex of rank 1 and two vertices of rank 2. The fact that elementary relations generate all relations in $\mathrm{BirMori}(Z)$ is equivalent to the fact that this square complex is simply connected.

It is not clear to us if for $n \geqslant 4$ the graph $\mathcal{G}_n$ is still the 1-skeleton of a cube complex.

References

[AO18] H. Ahmadinezhad & T. Okada. Stable rationality of higher dimensional conic bundles. *Épijournal Geom. Algébrique*, 2:Art. 5, 13, 2018. 5, 66

[AZ16] H. Ahmadinezhad & F. Zucconi. Mori dream spaces and birational rigidity of Fano 3-folds. *Adv. Math.*, 292:410–445, 2016. 33

[AZ17] H. Ahmadinezhad & F. Zucconi. Circle of Sarkisov links on a Fano 3-fold. *Proc. Edinb. Math. Soc. (2)*, 60(1):1–16, 2017. 52

[Alb02] M. Alberich-Carramiñana. *Geometry of the plane Cremona maps*, volume 1769 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2002. 2

[AD09] M. Artebani & I. Dolgachev. The Hesse pencil of plane cubic curves. *Enseign. Math. (2)*, 55(3-4):235–273, 2009. 74

[ADHL15] I. Arzhantsev, U. Derenthal, J. Hausen & A. Laface. *Cox rings*, volume 144 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2015. 13

[BDE+17] F. Bastianelli, P. De Poi, L. Ein, R. Lazarsfeld & B. Ullery. Measures of irrationality for hypersurfaces of large degree. *Compos. Math.*, 153(11):2368–2393, 2017. 19, 21

[Bir16a] C. Birkar. Anti-pluricanonical systems on Fano varieties. *Preprint arXiv:1603.05765*, 2016. 7, 54

[Bir16b] C. Birkar. Singularities of linear systems and boundedness of Fano varieties. *Preprint arXiv:1609.05543*, 2016. 7, 54

[BCHM10] C. Birkar, P. Cascini, C. D. Hacon & J. McKernan. Existence of minimal models for varieties of log general type. *J. Am. Math. Soc.*, 23(2):405–468, 2010. 9, 11, 12, 13, 14, 35, 36, 43, 50

[Bla10] J. Blanc. Groupes de Cremona, connexité et simplicité. *Ann. Sci. Éc. Norm. Supér. (4)*, 43(2):357–364, 2010. 3

[BFT17] J. Blanc, A. Fanelli & R. Terpereau. Automorphisms of $\mathbb{P}^1$-bundles over rational surfaces. *Preprint arXiv:1707.01462*, 2017. 3

[BFT18] J. Blanc, A. Fanelli & R. Terpereau. Maximal connected algebraic subgroups of the Cremona group in three variables. *manuscript in preparation*, 2018. 6

[BH15] J. Blanc & I. Hedén. The group of Cremona transformations generated by linear maps and the standard involution. *Ann. Inst. Fourier (Grenoble)*, 65(6):2641–2680, 2015. 6

[BZ18] J. Blanc & S. Zimmermann. Topological simplicity of the Cremona groups. *Amer. J. Math.*, 140(5):1297–1309, 2018. 3

[Bro06] R. Brown. *Topology and groupoids*. BookSurge, LLC, Charleston, SC, 2006. Third edition of *Elements of modern topology* [McGraw-Hill, New York, 1968],. 49

[Can14] S. Cantat. Morphisms between Cremona groups, and characterization of rational varieties. *Compos. Math.*, 150(7):1107–1124, 2014. 3

[CL13] S. Cantat & S. Lamy. Normal subgroups in the Cremona group. *Acta Math.*, 210(1):31–94, 2013. With an appendix by Yves de Cornulier. 3

[CX18] S. Cantat & J. Xie. Algebraic actions of discrete groups: The p-adic method. *Acta Math.*, to appear, 2018. 3

[CMF05] S. Casalaina-Martin & R. Friedman. Cubic threefolds and abelian varieties of dimension five. *J. Algebraic Geom.*, 14(2):295–326, 2005. 80

[CL12] P. Cascini & V. Lazić. New outlook on the minimal model program, I. *Duke Math. J.*, 161(12):2415–2467, 2012. 44

[Cas01] G. Castelnuovo. Le trasformazioni generatrici del gruppo cremoniano nel piano. *Atti della R. Accad. delle Scienze di Torino*, (36):861–874, 1901. 2

[CG72] C. H. Clemens & P. A. Griffiths. The intermediate Jacobian of the cubic threefold. *Ann. of Math. (2)*, 95:281–356, 1972. 79

[Cor95] A. Corti. Factoring birational maps of threefolds after Sarkisov. *J. Algebraic Geom.*, 4(2):223–254, 1995. 18

[Cor00] A. Corti. Singularities of linear systems and 3-fold birational geometry. In *Explicit birational geometry of 3-folds*, volume 281 of *London Math. Soc. Lecture Note Ser.*, pages 259–312. Cambridge Univ. Press, Cambridge, 2000. 17

[CL13] A. Corti & V. Lazić. New outlook on the minimal model program, II. *Math. Ann.*, 356(2):617–633, 2013. 44

[CPR00] A. Corti, A. Pukhlikov & M. Reid. Fano 3-fold hypersurfaces. In *Explicit birational geometry of 3-folds*, pages 175–258. Cambridge: Cambridge University Press, 2000. 25

[DGO17] F. Dahmani, V. Guirardel & D. Osin. Hyperbolically embedded subgroups and rotating families in groups acting on hyperbolic spaces. *Mem. Amer. Math. Soc.*, 245(1156), 2017. 3

[Dan74] V. I. Danilov. Non-simplicity of the group of unimodular automorphisms of an affine plane. *Mat. Zametki*, 15:289–293, 1974. 3

[Dem70] M. Demazure. Sous-groupes algébriques de rang maximum du groupe de Cremona. *Ann. Sci. École Norm. Sup. (4)*, 3:507–588, 1970. 2

[Dés07] J. Déserti. Le groupe de Cremona est hopfien. *C. R. Math. Acad. Sci. Paris*, 344(3):153–156, 2007. 77

[Dés17] J. Déserti. Cremona maps and involutions. *Preprint arXiv:1708.01569*, 2017. 77

[ELM+06] L. Ein, R. Lazarsfeld, M. Mustaţă, M. Nakamaye & M. Popa. Asymptotic invariants of base loci. *Ann. Inst. Fourier (Grenoble)*, 56(6):1701–1734, 2006. 44

[Enr95] F. Enriques. *Conferenze di Geometria: fundamenti di una geometria iperspaziale*. Bologna, 1895. 3, 5

[Fuj99] O. Fujino. Applications of Kawamata's positivity theorem. *Proc. Japan Acad. Ser. A Math. Sci.*, 75(6):75–79, 1999. 15

[Fuj15] O. Fujino. Some remarks on the minimal model program for log canonical pairs. *J. Math. Sci. Univ. Tokyo*, 22(1):149–192, 2015. 15

[FL10] J.-P. Furter & S. Lamy. Normal subgroup generated by a plane polynomial automorphism. *Transform. Groups*, 15(3):577–610, 2010. 3

[GK17] F. Gounelas & A. Kouvidakis. Measures of irrationality of the Fano surface of a cubic threefold. *Preprint arXiv:1707.00853v3*, 2017. 20, 21

[HM07] C. D. Hacon & J. Mckernan. On Shokurov's rational connectedness conjecture. *Duke Math. J.*, 138(1):119–136, 2007. 16

[HM13] C. D. Hacon & J. McKernan. The Sarkisov program. *J. Algebraic Geom.*, 22(2):389–405, 2013. 5, 7, 9, 32, 35, 37, 43, 44, 49, 50

[Har77] R. Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52. 12, 16

[Hat02] A. Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002. 49

[HK00] Y. Hu & S. Keel. Mori dream spaces and GIT. *Michigan Math. J.*, 48:331–348, 2000. 14

[Hud27] H. P. Hudson. *Cremona transformations in plane and space*. Cambridge, University Press, 1927. 5

[Isk91] V. A. Iskovskikh. Generators of the two-dimensional Cremona group over a nonclosed field. *Trudy Mat. Inst. Steklov.*, 200:157–170, 1991. 5

[Isk96] V. A. Iskovskikh. Factorization of birational mappings of rational surfaces from the point of view of Mori theory. *Uspekhi Mat. Nauk*, 51(4(310)):3–72, 1996. 5

[IKT93] V. A. Iskovskikh, F. K. Kabdykairov & S. L. Tregub. Relations in a two-dimensional Cremona group over a perfect field. *Izv. Ross. Akad. Nauk Ser. Mat.*, 57(3):3–69, 1993. Translation in Russian Acad. Sci. Izv. Math. 42 (1994), no. 3, 427–478. 4

[Isk87] V. A. Iskovskikh. Cremona group. In *Encyclopedia of Mathematics*. 1987. 2, 3

[Kal13] A.-S. Kaloghiros. Relations in the Sarkisov program. *Compos. Math.*, 149(10):1685–1709, 2013. 4, 7, 9, 42, 44, 49, 50

[KKL16] A.-S. Kaloghiros, A. Küronya & V. Lazić. Finite generation and geography of models. In *Minimal models and extremal rays (Kyoto, 2011)*, volume 70 of *Adv. Stud. Pure Math.*, pages 215–245. 2016. 9, 11, 12, 13, 14, 36, 44

[Kat87] S. Katz. The cubo-cubic transformation of $\mathbf{P}^3$ is very special. *Math. Z.*, 195(2):255–257, 1987. 26

[Kol86] J. Kollár. Higher direct images of dualizing sheaves. I. *Ann. of Math. (2)*, 123(1):11–42, 1986. 12

[Kol93] J. Kollár. Effective base point freeness. *Math. Ann.*, 296(4):595–605, 1993. 24, 54

[Kol96] J. Kollár. *Rational curves on algebraic varieties*, volume 32. Springer-Verlag, Berlin, 1996. 11, 12, 16

[Kol97] J. Kollár. Singularities of pairs. In *Algebraic geometry—Santa Cruz 1995*, volume 62 of *Proc. Sympos. Pure Math.*, pages 221–287. Amer. Math. Soc., Providence, RI, 1997. 16

[Kol17] J. Kollár. Conic bundles that are not birational to numerical Calabi-Yau pairs. *Épijournal Geom. Algébrique*, 1:Art. 1, 14, 2017. 5

[KM92] J. Kollár & S. Mori. Classification of three-dimensional flips. *J. Amer. Math. Soc.*, 5(3):533–703, 1992. 12

[KM98] J. Kollár & S. Mori. *Birational geometry of algebraic varieties*, volume 134 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original. 9, 10, 14, 15, 16

[Kra96] H. Kraft. Challenging problems on affine n-space. *Astérisque*, (237):Exp. No. 802, 5, 295–317, 1996. Séminaire Bourbaki, Vol. 1994/95. 6

[LZ17] S. Lamy & S. Zimmermann. Signature morphisms from the Cremona group over a non-closed field. *Preprint arXiv:1707.07955, to appear in JEMS*, 2017. 4, 33, 47, 49, 80

[Lan87] S. Lang. *Elliptic functions*, volume 112 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1987. With an appendix by J. Tate. 74

[Laz04] R. Lazarsfeld. *Positivity in algebraic geometry. I*. Springer-Verlag, Berlin, 2004. Classical setting: line bundles and linear series. 9

[Lon16] A. Lonjou. Non simplicité du groupe de Cremona sur tout corps. *Ann. Inst. Fourier (Grenoble)*, 66(5):2021–2046, 2016. 3

[MW90] D. W. Masser & G. Wüstholz. Estimating isogenies on elliptic curves. *Invent. Math.*, 100(1):1–24, 1990. 74

[Mat02] K. Matsuki. *Introduction to the Mori program*. Universitext. Springer-Verlag, New York, 2002. 11, 15

[MO15] A. Minasyan & D. Osin. Acylindrical hyperbolicity of groups acting on trees. *Math. Ann.*, 362(3-4):1055–1105, 2015. 77

[MM85] S. Mori & S. Mukai. Classification of Fano 3-folds with $B_2 \geqslant 2$. I. In *Algebraic and topological theories (Kinosaki, 1984)*, pages 496–545. Kinokuniya, Tokyo, 1985. 27

[Nak04] N. Nakayama. *Zariski-decomposition and abundance*, volume 14 of *MSJ Memoirs*. Mathematical Society of Japan, Tokyo, 2004. 10

[Pan99] I. Pan. Une remarque sur la génération du groupe de Cremona. *Bol. Soc. Brasil. Mat. (N.S.)*, 30(1):95–98, 1999. 5

[PS15] I. Pan & A. Simis. Cremona maps of de Jonquières type. *Canad. J. Math.*, 67(4):923–941, 2015. 5

[Poo07] B. Poonen. Gonality of modular curves in characteristic p. *Math. Res. Lett.*, 14(4):691–701, 2007. 20

[Pro11] Y. Prokhorov. p-elementary subgroups of the Cremona group of rank 3. In *Classification of algebraic varieties*, EMS Ser. Congr. Rep., pages 327–338. Eur. Math. Soc., Zürich, 2011. 3, 76

[Pro12] Y. Prokhorov. Simple finite subgroups of the Cremona group of rank 3. *J. Algebraic Geom.*, 21(3):563–600, 2012. 3

[Pro14] Y. Prokhorov. 2-elementary subgroups of the space Cremona group. In *Automorphisms in birational and affine geometry*, volume 79 of *Springer Proc. Math. Stat.*, pages 215–229. 2014. 3, 76

[PS16] Y. Prokhorov & C. Shramov. Jordan property for Cremona groups. *Amer. J. Math.*, 138(2):403–418, 2016. 3

[Sar82] V. G. Sarkisov. On conic bundle structures. *Izv. Akad. Nauk SSSR Ser. Mat.*, 46(2):371–408, 432, 1982. 68, 69, 71

[Sch73] P. E. Schupp. A survey of SQ-universality. pages 183–188. Lecture Notes in Math., Vol. 319, 1973. 77

[SU04] I. P. Shestakov & U. U. Umirbaev. The tame and the wild automorphisms of polynomial rings in three variables. *J. Amer. Math. Soc.*, 17(1):197–227, 2004. 6

[Sho96] V. V. Shokurov. 3-fold log models. *J. Math. Sci.*, 81(3):2667–2699, 1996. Algebraic geometry, 4. 43

[Sil09] J. H. Silverman. *The arithmetic of elliptic curves*, volume 106 of *Graduate Texts in Mathematics*. Springer, Dordrecht, second edition, 2009. 74, 75

[Ume85] H. Umemura. On the maximal connected algebraic subgroups of the Cremona group. II. In *Algebraic groups and related topics (Kyoto/Nagoya, 1983)*, volume 6 of *Adv. Stud. Pure Math.*, pages 349–436. North-Holland, Amsterdam, 1985. 3

[Zim18] S. Zimmermann. The Abelianization of the real Cremona group. *Duke Math. J.*, 167(2):211–267, 2018. 3

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