

# QUOTIENTS OF GROUPS OF BIRATIONAL TRANSFORMATIONS OF CUBIC DEL PEZZO FIBRATIONS

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ABSTRACT. We prove that the group of birational transformations of a Del Pezzo fibration of degree 3 over a curve is not simple, by giving a surjective group homomorphism to a free product of infinitely many groups of order 2. As a consequence we also obtain that the Cremona group of rank 3 is not generated by birational maps preserving a rational fibration. Besides, the subgroup of  $\text{Bir}(\mathbb{P}^3)$  generated by all connected algebraic subgroups is a proper normal subgroup.

## 1. INTRODUCTION

**1.1. Non-simplicity of the Cremona groups.** Let  $X$  be an algebraic variety over a field  $\mathbb{k}$ . We denote by  $\text{Bir}_{\mathbb{k}}(X)$  its group of birational automorphisms. During the last few decades, there have been numerous papers concerning various properties of these groups. A case of particular interest is  $X = \mathbb{P}^n$ , when the group  $\text{Bir}_{\mathbb{k}}(\mathbb{P}^n)$  is called the *Cremona group* of rank  $n$ . Rank one Cremona group over  $\mathbb{k}$  is just  $\text{PGL}_2(\mathbb{k})$ . However, already for  $n = 2$  we get a much more complicated object, which is very far from being a (finite dimensional) linear algebraic group. These days, we know quite a lot about the group  $\text{Bir}_{\mathbb{k}}(\mathbb{P}^2)$ , especially when the base field  $\mathbb{k}$  is algebraically closed of characteristic zero. In particular, we know the presentation of this group in terms of generators and relations [Giz82, IKT93, UZ19], description of finite subgroups [DI09] or more generally finitely generated subgroups and other different group-theoretic properties: see [Can13, Dés19] for some overviews and references. Here we only want to emphasize that most questions about Cremona groups remain widely open in dimension greater than 2.

The aim of this paper is to get some insights about the structure of  $\text{Bir}(X)$  for  $X$  a three-dimensional complex algebraic variety, and in particular to address a question about the simplicity of such a group. For Cremona groups, this question goes back to F. Enriques, and it remained unsolved for more than 100 years. The non-simplicity of  $\text{Bir}_{\mathbb{C}}(\mathbb{P}^2)$  as an abstract group was first proven by S. Cantat and S. Lamy in [CL13] using an action of  $\text{Bir}_{\mathbb{C}}(\mathbb{P}^2)$  on an infinite-dimensional hyperbolic space. Later this result was generalized by A. Lonjou to an arbitrary base field [Lon16].

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Another approach to non-simplicity of  $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ , based on explicit presentation of  $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$  by generators and relations, was discovered by S. Zimmermann in [Zim18]. The question of the non-simplicity of higher rank Cremona groups remained open until the recent preprint [BLZ19], where the so-called *Sarkisov program* and recent achievements in the Borisov-Alexeev-Borisov conjecture [Bir16a] allowed the authors to prove the following result:

**Theorem 1.1.** *Let  $B \subseteq \mathbb{P}^m$  be a smooth projective complex variety,  $P \rightarrow \mathbb{P}^m$  a decomposable  $\mathbb{P}^2$ -bundle (projectivisation of a decomposable rank 3 vector bundle) and  $X \subset P$  a smooth closed subvariety such that the projection to  $\mathbb{P}^m$  gives a conic bundle  $\eta: X \rightarrow B$ . Then there exists a group homomorphism*

$$\mathrm{Bir}(X) \rightarrow \bigoplus_{\mathbb{Z}} \mathbb{Z}/2$$

which is surjective in restriction to  $\mathrm{Bir}(X/B) = \{\phi \in \mathrm{Bir}(X) \mid \eta \circ \phi = \eta\}$ .

Moreover, if there exists a subfield  $\mathbb{k} \subseteq \mathbb{C}$  over which  $X, B$  and  $\eta$  are defined, the image of elements of  $\mathrm{Bir}(X/B)$  defined over  $\mathbb{k}$  is also infinite.

Theorem 1.1 applies to any product  $X = \mathbb{P}^1 \times B$ , to smooth cubic hypersurfaces  $X \subseteq \mathbb{P}^n$  and to many other non-rational varieties of dimension  $n \geq 3$ . Clearly, it also includes the case of  $X = \mathbb{P}^1 \times \mathbb{P}^{n-1}$  which is birational to  $\mathbb{P}^n$ , hence proving that Cremona groups  $\mathrm{Bir}(\mathbb{P}^n)$  are not simple. For a version of this result in dimension 2 over a non-closed field, see [Sch19].

**1.2. Are there simple “large” groups of birational automorphisms?** In view of the previous discussion, it seems very natural to ask the following

**Question (S. Cantat).** Is there a complex algebraic variety  $X$  such that the group  $\mathrm{Bir}(X)$  is “infinite-dimensional” and is a simple (abstract) group?

Let us now prove that a potential counterexample to this question must have dimension at least 3. Firstly, the curve case is trivial. Let  $X$  be an algebraic surface. We may assume that  $X$  is smooth projective and minimal. If  $X$  is of general type, then Matsumura’s theorem [Mat63] says that  $\mathrm{Bir}(X)$  is finite. If  $X$  is birational to  $\mathbb{P}^2$ , then  $\mathrm{Bir}(X)$  is isomorphic to the Cremona group of rank 2, which is not simple. According to Enriques-Kodaira classification, in the remaining cases either  $X$  is a non-rational ruled surface, or  $K_X$  is nef. In the latter case  $\mathrm{Bir}(X) = \mathrm{Aut}(X)$  and by Matsusaka’s theorem [Mat58] the group  $\mathrm{Aut}(X)$  has a structure of a locally algebraic group with finite or countably many components. Finally, if  $X$  is ruled over a non-rational curve  $B$ , then

$$\mathrm{Bir}(X) \simeq \mathrm{PGL}_2(\mathbb{C}(B)) \rtimes \mathrm{Aut}(B)$$

which is of infinite dimension, but is not simple. Indeed, the subgroup  $\mathrm{PSL}_2(\mathbb{C}(B)) \subsetneq \mathrm{PGL}_2(\mathbb{C}(B))$  is a proper normal subgroup of  $\mathrm{Bir}(X)$  (if  $\mathrm{Aut}(B)$  is not trivial, one can of course also consider  $\mathrm{PGL}_2(\mathbb{C}(B))$ ).

**1.3. Threefolds and Mori fibrations.** Let  $X$  be a rationally connected three-dimensional algebraic variety over an algebraically closed field of characteristic zero. Run the Minimal Model Program on  $X$ . Recall that during this program we stay in the category of projective normal varieties with at worst terminal  $\mathbb{Q}$ -factorial singularities. Since  $X$  is rationally connected, on the final step we get a *Fano-Mori fibration*  $f : \tilde{X} \rightarrow Z$ , which means that  $\dim Z < \dim X$ ,  $Z$  is normal,  $f$  has connected fibers, the anticanonical Weil divisor  $-K_{\tilde{X}}$  is ample over  $Z$ , and  $\rho(\tilde{X}/Z) = 1$ . Then we have one of the following three possibilities:

- (1)  $Z$  is a rational surface and a general fiber of  $f$  is a conic;
- (2)  $Z \simeq \mathbb{P}^1$  and a general fiber of  $f$  is a smooth Del Pezzo surface;
- (3)  $Z$  is a point and  $\tilde{X}$  is a  $\mathbb{Q}$ -Fano threefold of Picard rank 1.

We see that Theorem 1.1 covers the first case of this trichotomy (or at least most of them), so it is natural to proceed with the cases of Del Pezzo fibrations and Fano threefolds. Moreover, every Del Pezzo fibration of degree  $\geq 4$  is birational to a conic bundle (see e.g. [BCDP19, Lemma 2.5]), so it is natural to study the next cases, which correspond to Del Pezzo fibrations of degrees 3, 2 or 1.

Our goal is to study the first case which is not covered by Theorem 1.1 — the case of Del Pezzo fibrations of degree 3. Our main result is the following statement:

**Theorem A.** *For each complex Del Pezzo fibration  $\pi : X \rightarrow B$  of degree 3 over a curve  $B$ , there exists a group homomorphism*

$$\mathrm{Bir}(X) \twoheadrightarrow \ast_{\mathbb{N}} \mathbb{Z}/2$$

*whose restriction to  $\mathrm{Bir}(X/B) = \{\varphi \in \mathrm{Bir}(X) \mid \pi\varphi = \pi\}$  is surjective.*

As the following example shows, there are cubic Del Pezzo fibrations which are not birational to conic bundles and such that not all birational transformations preserve the cubic Del Pezzo fibration. Theorem A shows then that  $\mathrm{Bir}(V_2)$  is not simple, a fact that does not follow from [BLZ19].

**Example 1.2** ([Sob02]). Let  $V_m \subset \mathbb{P}^1 \times \mathbb{P}^3$  be a general divisor of bidegree  $(m, 3)$ . Note that the variety  $V_1$  is rational: the restriction of the projection  $\mathbb{P}^1 \times \mathbb{P}^3 \rightarrow \mathbb{P}^3$  on  $V_1$  gives a birational morphism  $V_1 \rightarrow \mathbb{P}^3$ .

Consider the variety  $V_2$ . Let  $[u_0 : u_1]$  and  $[x_0 : x_1 : x_2 : x_3]$  be the homogeneous coordinates on the factors. Then  $V$  is given by the equation

$$A(x_0, x_1, x_2, x_3)u_0^2 + B(x_0, x_1, x_2, x_3)u_0u_1 + C(x_0, x_1, x_2, x_3)u_1^2 = 0,$$

where  $A$ ,  $B$  and  $C$  are polynomials of degree 3. The system  $A = B = C = 0$  has 27 distinct solutions  $q_i \in \mathbb{P}^3$ ,  $i = 1, \dots, 27$ , which give 27 lines  $L_i = \mathbb{P}^1 \times q_i \subset V_2$ . The Galois involution  $\tau$  of the double cover  $V_2 \rightarrow \mathbb{P}^3$  is well defined outside  $L_i$ , so  $\tau \in \text{Bir}(V_2)$ . One can show that  $V_2$  is not birationally rigid. The involution  $\tau$  corresponds to the flop of the 27 lines mentioned above, giving another structure of Mori fiber space on  $V_2$  and  $\tau$  is a Sarkisov link of type IV (see §2.3). In [Sob02] it is shown that  $V_2$  is not birational to a conic bundle and that

$$\text{Bir}(V_2) = \langle \tau \rangle * \text{Bir}(V_2/\mathbb{P}^1).$$

Note that Theorem A also applies to the group  $\text{Bir}_{\mathbb{C}}(\mathbb{P}^3)$ , and gives a proof of the non-simplicity of  $\text{Bir}_{\mathbb{C}}(\mathbb{P}^3)$ , alternative to the one of [BLZ19] (using in fact the four first sections of [BLZ19], so essentially the half of the paper). Even if the quotients that we obtain are isomorphic to the ones of [BLZ19], the normal subgroups of  $\text{Bir}_{\mathbb{C}}(\mathbb{P}^3)$  that we construct are quite different. For instance, every birational map of  $\mathbb{P}^3$  that preserves a conic fibration is in the kernel of our group homomorphisms. We then obtain the following result on generators of  $\text{Bir}_{\mathbb{C}}(\mathbb{P}^3)$ :

**Theorem B.** *There is a surjective group homomorphism*

$$\rho: \text{Bir}_{\mathbb{C}}(\mathbb{P}^3) \rightarrow \underset{\mathbb{N}}{*} \mathbb{Z}/2$$

having the following property:

Denoting by  $G \subset \text{Bir}_{\mathbb{C}}(\mathbb{P}^3)$  the group generated by all birational maps  $\varphi \in \text{Bir}_{\mathbb{C}}(\mathbb{P}^3)$  preserving a fibration  $\pi: \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$  whose general fibers are rational (which means that  $\pi\varphi = \pi$ ), we have  $\text{Aut}_{\mathbb{C}}(\mathbb{P}^3) \subsetneq G \subsetneq \text{Ker } \rho$ . In particular, we have a strict inclusion  $G \subsetneq \text{Bir}_{\mathbb{C}}(\mathbb{P}^3)$ .

Theorem B generalises the three-dimensional version of [BLZ19, Theorem C], which proves that  $\text{Bir}_{\mathbb{C}}(\mathbb{P}^3)$  is not generated by birational maps preserving a linear fibration  $\mathbb{P}^3 \dashrightarrow \mathbb{P}^2$  (this corresponds to the Tame problem for  $\text{Bir}_{\mathbb{C}}(\mathbb{P}^3)$ , see [BLZ19, §1.C] for more details on this question). We also obtain the following natural proper normal subgroup of  $\text{Bir}_{\mathbb{C}}(\mathbb{P}^3)$  (for a definition and characterisations of algebraic subgroups of  $\text{Bir}_{\mathbb{C}}(\mathbb{P}^n)$ , see for instance [BF13]):

**Theorem C.** *Let  $n \geq 1$ . The subgroup of  $\text{Bir}_{\mathbb{C}}(\mathbb{P}^n)$  generated by all connected algebraic subgroups of  $\text{Bir}_{\mathbb{C}}(\mathbb{P}^n)$  is a normal subgroup  $H_n \subset \text{Bir}_{\mathbb{C}}(\mathbb{P}^n)$ , that contains  $\text{Aut}_{\mathbb{C}}(\mathbb{P}^n) \cong \text{PGL}_{n+1}(\mathbb{C})$ . The group  $H_n$  is equal to  $\text{Bir}_{\mathbb{C}}(\mathbb{P}^n)$  if  $n \in \{1, 2\}$  and is a strict normal subgroup of  $\text{Bir}_{\mathbb{C}}(\mathbb{P}^n)$  if  $n = 3$ .*

**1.4. The strategy of the proof and a more detailed result.** Let us briefly describe how one can obtain the group homomorphisms of Theorems A and B. According to the Sarkisov program, any birational map between two Mori fiber spaces is a composition of birational maps, called Sarkisov links. Moreover, the relations between Sarkisov links are generated by so-called *elementary relations* ([BLZ19, Theorem 4.29], that we reproduce here in Theorem 2.14). We follow the strategy of [BLZ19], which studies for a Mori fiber space  $X$  the *groupoid*  $\text{BirMori}(X)$  of all birational maps between Mori fiber spaces birational to  $X$  and we construct as in [BLZ19] a groupoid morphism from  $\text{BirMori}(X)$  to a free product of  $\mathbb{Z}/2\mathbb{Z}$  whose restriction to  $\text{Bir}(X)$  gives the desired group homomorphisms of Theorems A and B.

The groupoid morphisms constructed in [BLZ19] send every every Sarkisov link in  $\text{BirMori}(X)$  onto the identity, except some special Sarkisov links between conic bundles which are complicated enough. We follow the same approach, but focus on Sarkisov links between Del Pezzo fibrations. The ones we are interested in are called *Bertini links* in the sequel (see Definition 2.6). We put equivalence classes on such links (Definition 2.8) and associate to every Bertini link  $\chi$  an integer number  $\mathbf{g}(\chi)$ , the *genus* of  $\chi$  (see Definition 2.9), which measures the “complexity” of this link and only depends on the equivalence class. We then associate to each Mori fiber space  $X$  the set  $\mathcal{B}_X$  of all equivalence classes of Bertini links between varieties birational to  $X$  and denote by  $\mathcal{B}_X^{\geq g} \subset \mathcal{B}_X$  the subset of equivalence classes of Bertini links having genus  $\geq g$ .

The advantage of working only with Bertini links of “high complexity” is that they occur only in very simple elementary relations. Altogether, in Section 3.3 it allows us to prove the following statement:

**Theorem D.** *There is an integer  $g \geq 0$  such that for each complex Del Pezzo fibration  $X \rightarrow B$  over a curve, there exists a group homomorphism*

$$\text{Bir}(X) \rightarrow \ast_{\mathcal{B}_X^{\geq g}} \mathbb{Z}/2$$

*which is the restriction of a groupoid homomorphism  $\text{BirMori}(X) \rightarrow \ast_{\mathcal{B}_X^{\geq g}} \mathbb{Z}/2$  that sends every Sarkisov link  $\chi$  of Bertini type with  $\mathbf{g}(\chi) \geq g$  to the generator of  $\mathbb{Z}/2$  indexed by equivalence class of  $\chi$ , and all other Sarkisov links and all automorphisms of Mori fiber spaces birational to  $X$  onto the trivial element.*

Theorem A will follow from Theorem D. Note that in Theorem D there is no restriction on the degree of the del Pezzo fibration  $X/B$ , but we only prove that the image is non-trivial for del Pezzo fibrations of degree 3. This is done in Section 4, where we show that for a del Pezzo fibration  $X/B$  of degree 3, the image by the group homomorphism of Theorem D of a natural birational

involution associated with a 2-section of  $X/B$  of genus  $g' \geq g$  (where  $g$  is as in Theorem D) is the generator of one of the  $\mathbb{Z}/2$ , indexed by a Sarkisov link of genus  $g'$ . It then suffices to show that the genus of 2-sections in  $X/B$  is not bounded (a fact already proven in [BCDP19]) to show that the image of  $\text{Bir}(X)$  under the group homomorphism surjects onto a free product of  $\mathbb{Z}/2$  indexed by an infinite set.

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## 2. PRELIMINARIES ON THE SARKISOV PROGRAM

**2.1. Notation and conventions.** Throughout the paper, we stick to the following agreements:

- All the varieties and maps are defined over  $\mathbb{C}$ , if not stated otherwise;
- The plain arrows ( $\longrightarrow$ ) denote morphisms; the dashed arrows ( $\dashrightarrow$ ) denote rational maps; the dotted arrows ( $\cdots\rightarrow$ ) denote pseudo-isomorphisms (isomorphism outside a codimension 2 subsets).
- We write  $X \simeq Y$  if two varieties are isomorphic, and  $X \approx Y$  if they are pseudo-isomorphic.
- A *birational contraction* is a birational map such that the inverse does not contract any divisor.

Below we very briefly recall some standard notions of the Minimal Model Program (MMP in the sequel) which are used in in this paper. Our main references are [KM98] and [BCHM10]; see also [BLZ19, §2].

**2.2. Rank  $r$  fibrations.** At the heart of this paper, as well as in [BLZ19], lies the notion of *rank  $r$  fibration*. It puts together the notions of terminal Mori fiber space, of Sarkisov links and of elementary relations between Sarkisov links, for  $r = 1, 2$  and  $3$  respectively.

**Definition 2.1.** Let  $r \geq 1$  be an integer. A morphism  $\eta: X \rightarrow B$  is a *rank  $r$  fibration* if the following conditions are satisfied:

- (1)  $X/B$  is a Mori dream space;
- (2)  $\dim X > \dim B \geq 0$  and  $\rho(X/B) = r$ ;
- (3)  $X$  is  $\mathbb{Q}$ -factorial and terminal, and for any divisor  $D$  on  $X$ , the output of any  $D$ -MMP from  $X$  over  $B$  is still  $\mathbb{Q}$ -factorial and terminal;
- (4)  $B$  has klt singularities.
- (5)  $-K_X$  is  $\eta$ -big.

**Remark 2.2.** For the definition of a Mori dream space we refer to [BLZ19, Definition 2.3], which is a relative version of [HK00, Definition 1.10]. Standard examples include toric varieties and Fano varieties (with  $B = \text{pt}$ ). For us, the reason to work with Mori dream spaces is that they behave well with respect to the MMP. Namely, if  $X/B$  is a Mori dream space, then for any divisor  $D$  one can run a  $D$ -MMP on  $X$  over  $B$ , and there exists only finitely many possible outputs of such a process (see [HK00, Proposition 1.11] or [KKL16, Theorem 5.4]).

Further, we say that a rank  $r$  fibration  $X/B$  *factorises through* a rank  $r'$  fibration  $X'/B'$ , or that  $X'/B'$  *is dominated by*  $X/B$ , if the fibrations  $X/B$  and  $X'/B'$  fit in a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{\hspace{10em}} & & & B \\ & \searrow \text{dashed} & X' & \xrightarrow{\hspace{2em}} & B' & \xrightarrow{\hspace{2em}} & B \end{array}$$

where  $X \dashrightarrow X'$  is a birational contraction, and  $B' \rightarrow B$  is a morphism with connected fibers. Note that  $r \geq r'$ .

**Proposition 2.3** ([BLZ19, Lemma 3.3]). *Let  $\eta : X \rightarrow B$  be a surjective morphism between normal varieties. Then  $X/B$  is a rank 1 fibration if and only if  $X/B$  is a terminal Mori fiber space.*

**Remark 2.4.** In threefold case, there are three possible cases for a Mori fiber space  $X/B$ :

- (1) If  $\dim(B) = 0$ , then  $X$  is a Fano variety with  $\rho(X) = 1$ ;
- (2) If  $\dim(B) = 1$ , then  $X$  is a Del Pezzo fibration over a curve  $B$ ;
- (3) If  $\dim(B) = 2$ , then  $X$  is a conic bundle over the surface  $B$ .

In the following lemma we gather some properties of rank  $r$  fibrations which we will need later.

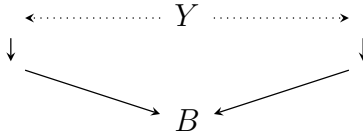
**Lemma 2.5.** *Let  $r \geq 1$  and let  $\eta : X \rightarrow B$  be a rank  $r$  fibration. Then, the following hold:*

- (1) *If  $X'$  is obtained from  $X$  by making a log-flip (resp. a divisorial contraction) over  $B$ , then  $X'/B$  is a rank  $r$  fibration (resp. a rank  $(r - 1)$ -fibration).*
- (2) *For a general point  $b \in B$  the fiber  $X_b = \eta^{-1}(b)$  is pseudo-isomorphic to a weak Fano terminal variety, and the curves in  $X_b$  which are non-positive against  $K_{X_b}$  cover a subset of codimension at least 2 in  $X_b$ .*

*Proof.* (1) is [BLZ19, Lemma 3.4(1)] and (2) is [BLZ19, Corollary 3.6]. □

**2.3. Sarkisov links.** The notion of a rank 2 fibration corresponds to the notion of Sarkisov link. Namely, a 2-ray game allows to show [BLZ19, Lemma 3.7] that a rank 2 fibration  $Y/B$  factorises

through exactly two rank 1 fibrations  $X_1/B_1, X_2/B_2$  (up to isomorphisms), which both fit into a diagram



where the top dotted arrows are sequences of log-flips, and the other four arrows are morphisms of relative Picard rank 1. In this situation we say that the birational map  $\chi : X_1 \dashrightarrow X_2$  is a *Sarkisov link*. In the diagram above, there are two possibilities for the sequence of two morphisms on each side of the diagram: either the first arrow is a Mori fiber space, or it is a divisorial contraction and then the second arrow is a Mori fiber space. Thus, we have 4 possibilities, which correspond to the classical definition of *Sarkisov links of type I, II, III and IV*, as illustrated on Figure 1.

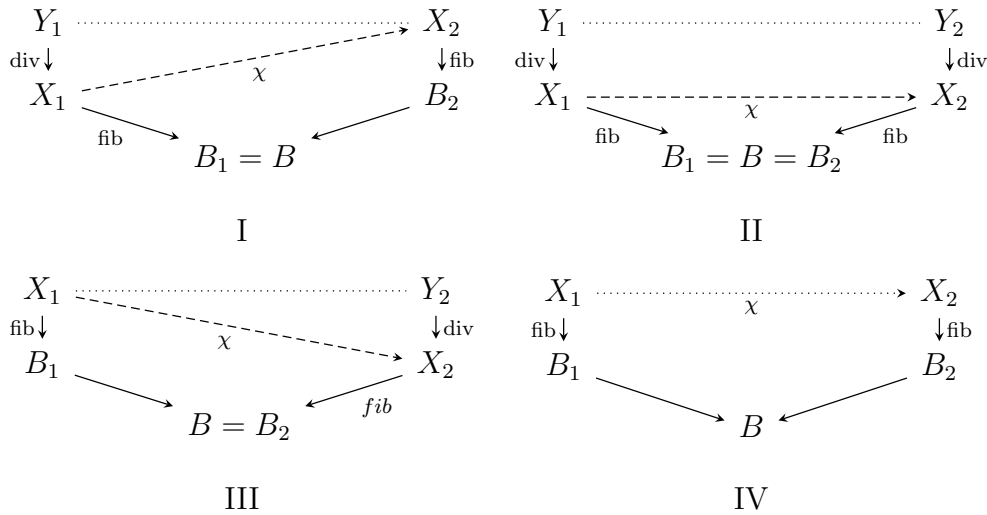


FIGURE 1. The four types of Sarkisov links.

The dashed arrow are pseudo-isomorphisms, the plain arrows are morphisms of relative Picard rank 1, and the arrows labeled by “div” are divisorial contractions.

**2.4. Crucial example: Bertini links.** In this paper, one particular family of Sarkisov links will be especially important for us. Those are the only links which will be sent to non-trivial elements by the group homomorphism of Theorem D. Let us now define this kind of links.

**Definition 2.6.** We say that a Sarkisov link  $\chi : X_1 \dashrightarrow X_2$  is of *Bertini type* (or  $\chi$  is just a *Bertini link*) if

- (1)  $\chi$  has type II with  $B$  being a curve and  $X_1, X_2$  being of dimension 3;



(2)  $\eta_1$  is a blow-up of a curve  $\Gamma_1 \subset X_1$ , so that the generic fiber of  $Y_1/B$  is a Del Pezzo surface of degree 1;

**Remark 2.7.** The definition implies that the birational map between the generic fibers of  $X_1/B$  and  $X_2/B$  induced by  $\chi$  is equal, up to an isomorphism of the target, to a birational involution that lifts to the Bertini involution on the generic fiber of  $Y_1/B$ , and implies that  $X_1/B$  and  $X_2/B$  are Del Pezzo fibrations of the same degree  $d \geq 2$  (follows from [Isk96, Theorem 2.6]). This motivates the terminology. Moreover, this implies that  $\eta_2$  is the blow-up of a curve  $\Gamma_2$  which is birational to  $\Gamma_1$ .

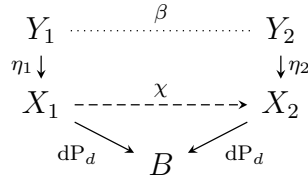
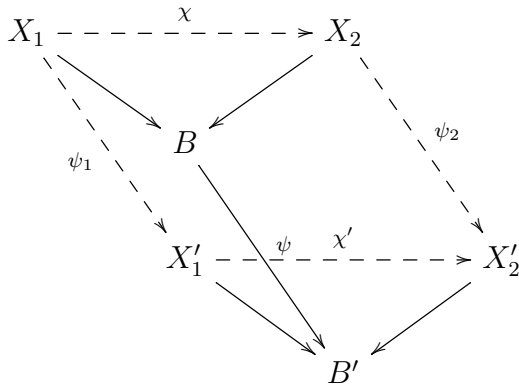


FIGURE 2. A Bertini link

**Definition 2.8.** We say that two Bertini links  $\chi : X_1 \dashrightarrow X_2$  and  $\chi' : X'_1 \dashrightarrow X'_2$  over  $B$  and  $B'$  respectively are *equivalent*, if there exists a commutative diagram



where for each  $i \in \{1, 2\}$ ,  $\psi_i$  is a birational map inducing an isomorphism between the generic fibers of the Del Pezzo fibrations  $X_i/B$  and  $X'_i/B'$ , and where  $\psi : B \rightarrow B'$  is an isomorphism. The equivalence class of  $\chi$  will be denoted  $[\chi]$ .

For each Mori fiber space  $X/B$ , we denote by  $\mathcal{B}_X$  the set of equivalence classes of Bertini links  $\chi : X_1 \dashrightarrow X_2$ , where  $X_1$  and  $X_2$  are birational to  $X$ .

**Definition 2.9.** Let  $\chi$  be a Bertini link. The *genus*  $\mathbf{g}(\chi)$  of  $\chi$  is the geometric genus of the curve  $\Gamma_1$  blow up by  $\eta_1$ :

$$\mathbf{g}(\chi) = \mathbf{g}(\Gamma_1).$$

**Remark 2.10.** Equivalent Bertini links have the same genus. One can thus define  $\mathcal{B}_X^{\geq g} \subset \mathcal{B}_X$  to be the subset of equivalence classes of Bertini type links  $\chi$  such that  $\mathbf{g}(\chi) \geq g$ .

Moreover, the curves  $\Gamma_1$  and  $\Gamma_2$  associated to a Bertini link  $\chi$  are birational (see Remark 2.7), so we have  $\mathbf{g}(\chi) = \mathbf{g}(\chi^{-1})$  for each Bertini link  $\chi$ .

**2.5. Elementary relations.** Finally, the notion of rank 3 fibration recovers the notion of an *elementary relation* between Sarkisov links, introduced in [LZ17] and [BLZ19].

**Definition 2.11.** Let  $X/B$  and  $X'/B'$  be two rank  $r$  fibrations, and  $T \dashrightarrow X, T \dashrightarrow X'$  two birational maps from the same variety  $T$ . We say that  $X/B$  and  $X'/B'$  are  *$T$ -equivalent* (the birational maps being implicit) if there exist a pseudo-isomorphism  $X \dashrightarrow X'$  and an isomorphism  $B \xrightarrow{\sim} B'$  such that all these maps fit in a commutative diagram:

$$\begin{array}{ccc} & T & \\ \swarrow \text{---} & & \searrow \text{---} \\ X & \text{---} & X' \\ \downarrow & & \downarrow \\ B & \xrightarrow{\sim} & B' \end{array}$$

As every rank 3 fibration  $T/B$  is a Mori dream space, there are only finitely many rank 1 and rank 2 fibrations dominated by  $T/B$  (see Remark 2.2). Moreover, if  $T/B$  factorises through a rank 1 fibration  $X/B'$ , then up to  $T$ -equivalence there are exactly two rank 2 fibrations which factorise through  $X/B'$  and dominated by  $T/B$  [BLZ19, Lemma 4.2]. From these two facts one can deduce the following statement:

**Proposition 2.12** ([BLZ19, Proposition 4.3]). *Let  $T/B$  be a rank 3 fibration. Then there are only finitely many Sarkisov links  $\chi_i$  dominated by  $T/B$ , and they fit in a relation*

$$\chi_t \circ \cdots \circ \chi_1 = \text{id}.$$

**Definition 2.13.** In the situation of Proposition 2.12, we say that

$$\chi_t \circ \cdots \circ \chi_1 = \text{id}$$

is an *elementary relation* between Sarkisov links, coming from the rank 3 fibration  $T/B$ . Observe that the elementary relation is uniquely defined by  $T/B$ , up to taking the inverse, cyclic permutations and insertion of isomorphisms.

Now we are ready to state a keynote result, which allows us to study the structure of birational automorphism groups. Its first part is basically the Sarkisov program itself, following [HM13]. The second part is inspired by [Kal13, Theorem 1.3] and proven in [BLZ19].

Let  $X/B$  a Mori fiber space. We denote by  $\text{BirMori}(X)$  the *groupoid*<sup>1</sup> of birational maps between Mori fiber spaces birational to  $X$ . Note that  $\text{Bir}(X)$  is a subgroupoid of  $\text{BirMori}(X)$ .

**Theorem 2.14** ([BLZ19, Theorem 4.29]). *Let  $X/B$  be a terminal Mori fiber space.*

- (1) *The groupoid  $\text{BirMori}(X)$  is generated by Sarkisov links and automorphisms.*
- (2) *Any relation between Sarkisov links in  $\text{BirMori}(X)$  is generated by elementary relations.*

### 3. ELEMENTARY RELATIONS INVOLVING BERTINI LINKS

Let  $\chi_t \circ \dots \circ \chi_1 = \text{id}$  be an elementary relation between Sarkisov links, coming from a rank 3 fibration  $T/B$ . Assume that one of the  $\chi_i$  is a Sarkisov link  $\chi_i: X_i \dashrightarrow X_{i+1}$  of type II over a curve  $B_i$ . Depending on whether  $B$  is a point or a curve, there are only two possible ways in which  $T/B$  can factorize through  $\chi_i$ , see Figure 3.

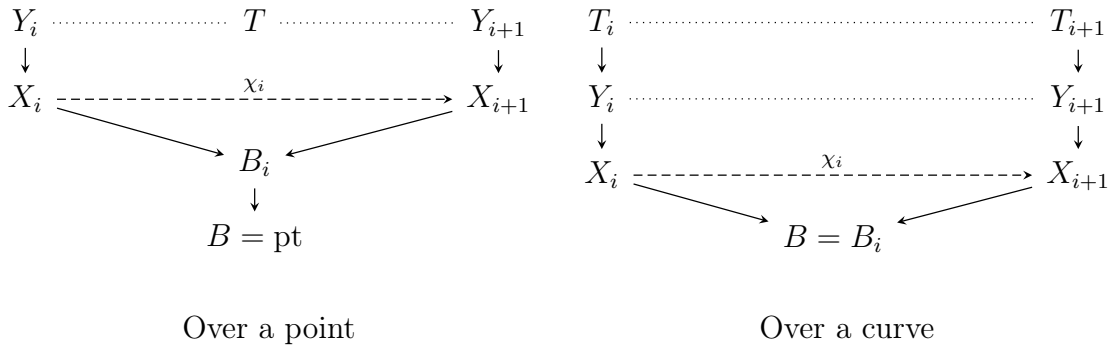


FIGURE 3. Two types of relations

We call them *relation over a point* and *relation over a curve* respectively. The goal of the next two paragraphs is to show that Bertini links of large genus do not occur in non-trivial relations over a point, and occur only in very simple relations over a curve.

**3.1. Relations over a point.** In this paragraph we prove that Bertini links of large genus do not occur in non-trivial elementary relations over a point (Proposition 3.3).

The following statement is a consequence of the Borisov-Alexeev-Borisov conjecture, established in [KMMT00, Kaw92] for dimension  $\leq 3$  and in [Bir16b] for any dimension.

<sup>1</sup>For the formalism of groupoids and of generators and relations in groupoids, we refer to [Bro06, §8.2 and 8.3].

**Proposition 3.1** ([BLZ19, Proposition 5.1]). *Let  $n$  be an integer, and let  $\mathcal{F}_n$  be the set of weak Fano terminal varieties of dimension  $n$ . There are integers  $d, l, m \geq 1$ , depending only on  $n$ , such that for each  $X \in \mathcal{F}_n$  the following hold:*

- (1)  $h^0(-mK_X) \leq l$ ;
- (2) The linear system  $|-mK_X|$  is base-point free;
- (3) The morphism

$$\phi_{|-mK_X|}: X \rightarrow \mathbb{P}^{h^0(-mK_X)-1}$$

is birational onto its image and contracts only curves  $C \subseteq X$  with  $C \cdot K_X = 0$ ;

- (4)  $\deg \phi_{|-mK_X|}(X) \leq d$ .

**Corollary 3.2.** *There is an integer  $N \geq 0$  such that the following holds. Let  $X$  be a terminal threefold, and  $\eta: Y \rightarrow X$  be the blow-up of a reduced not necessarily irreducible curve  $\Gamma \subset X$ . Assume that both  $X$  and  $Y$  are pseudo-isomorphic to weak Fano terminal threefolds having a small anticanonical morphism. Then the geometric genus of  $\Gamma$  satisfies  $\mathbf{g}(\Gamma) \leq N$ .*

*Proof.* We have the following commutative diagram:

$$\begin{array}{ccccc}
 \widehat{X} & \xleftarrow{\dots\dots\dots} & X & \xleftarrow{\eta} & Y & \xrightarrow{\dots\dots\dots} & \widehat{Y} \\
 \downarrow \phi_{|-mK_{\widehat{X}}|} & & \downarrow \psi & & \downarrow \phi_{|-mK_{\widehat{Y}}|} & & \downarrow \\
 \mathbb{P}^a & \supset & \Gamma & & \mathbb{P}^b & & \\
 & & \downarrow & & & & \\
 & & \widetilde{\Gamma} & & & & 
 \end{array}$$

where  $\widehat{X}$  and  $\widehat{Y}$  are weak terminal Fano threefolds, and the positive integers  $m$  and  $a = h^0(-mK_X) - 1 = h^0(-mK_{\widehat{X}}) - 1$ ,  $b = h^0(-mK_Y) - 1 = h^0(mK_{\widehat{Y}}) - 1 \leq l$  are given by Proposition 3.1. By hypothesis, the morphism  $\phi_{|-mK_{\widehat{X}}|}$  is small, so  $\text{Exc}(\psi)$  is a closed subset of  $X$  of dimension  $\leq 1$ , and each irreducible curve contained in  $\text{Exc}(\psi)$  has genus 0. As we may assume that  $\mathbf{g}(\Gamma) > 0$ , we obtain  $\Gamma \not\subseteq \text{Exc}(\psi)$ . Set  $\widetilde{\Gamma} = \psi(\Gamma)$ .

Since  $\eta_*(mK_Y) = mK_X$ , we obtain that  $b \leq a$  and the induced birational map  $\pi: \mathbb{P}^a \dashrightarrow \mathbb{P}^b$  is a linear projection away from a linear subspace  $L \subset \mathbb{P}^a$ . The variety  $\widetilde{X} = \psi(X) \subset \mathbb{P}^a$  is a threefold of degree  $\leq d$  by Proposition 3.1, and is not contained in a hyperplane section. As in [BLZ19, Corollary 5.2], one can show that there is no irreducible surfaces  $S \subset \widetilde{X} \cap L$ . Then Bézout Theorem implies that all irreducible components of  $\widetilde{X} \cap L$  of dimension 1 have degree  $\leq d$ . Thus,  $\deg(\widetilde{\Gamma}) \leq d$ .

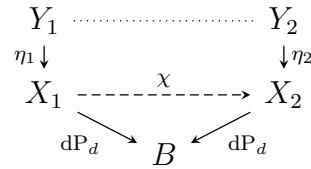
By Castelnuovo’s theorem (see e.g. [Har81]), it implies that  $\mathbf{g}(\tilde{\Gamma})$  is bounded by some constant  $N$ , depending only on  $a$  and  $d$ , so the same holds for the geometric genus of  $\tilde{\Gamma}$ , which is equal to the geometric genus of  $\Gamma$ .  $\square$

**Proposition 3.3.** *There exists some positive integer  $g$  such that no Bertini link  $\chi$  with  $\mathbf{g}(\chi) \geq g$  occurs in a non-trivial elementary relation over a point.*

*Proof.* From Lemma 2.5 we know that  $Y_1/\text{pt}$  and  $X_1/\text{pt}$  are rank 3 and rank 2 fibrations respectively. Moreover, they both pseudo-isomorphic to weak Fano terminal varieties. It remains to apply Corollary 3.2 and take  $g = N + 1$ .  $\square$

**3.2. Relations over a curve.** The main result of this paragraph is Proposition 3.6 which shows that if a Bertini link occurs in some elementary relation over a curve, then this relation has a very simple form. Before proving this, we need some lemmas.

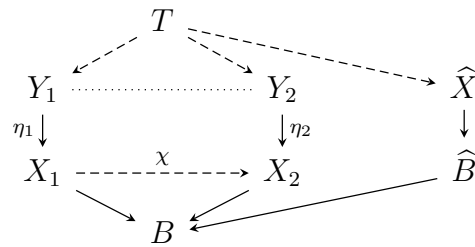
**Lemma 3.4.** *Let us take a Bertini link*



dominated by a rank 3 fibration  $T/B$ , where  $B$  is a curve, and let  $\hat{X}/\hat{B}$  be a rank 1 fibration (or equivalently a Mori fiber space) such that  $T/B$  factorizes through  $\hat{X}/\hat{B}$  (see the picture below). Then the following hold:

- (1) *The associated morphism  $\hat{B} \rightarrow B$  is an isomorphism;*
- (2) *The birational contraction  $T \dashrightarrow Y_1$  contracts a divisor onto a point or a curve contained in one fiber of  $Y_1/B$ , and the generic fibers of  $T/B$  and  $Y_1/B$  are isomorphic Del Pezzo surfaces of degree 1 and Picard rank 2;*

(3) *Exactly one of the two birational maps  $X_1 \dashrightarrow \hat{X}$  and  $X_2 \dashrightarrow \hat{X}$  induces an isomorphism between the generic fibers of  $X_i/B_i \simeq X_i/B$  and of  $\hat{X}/B$ .*



*Proof.* Let  $\epsilon$  be the generic point of  $B$  and let  $(X_1)_\epsilon, (X_2)_\epsilon, \hat{X}_\epsilon, (Y_1)_\epsilon, (Y_2)_\epsilon$  and  $T_\epsilon$  be the generic fibers of  $X_1/B, X_2/B, \hat{X}/B, Y_1/B, Y_2/B$  and  $T/B$  respectively. As  $\chi$  is a Bertini link, the

surfaces  $(Y_1)_\epsilon$  and  $(Y_2)_\epsilon$  are isomorphic Del Pezzo surfaces of degree 1 and Picard rank 2, and the contractions of the two extremal rays give two morphisms  $(Y_1)_\epsilon \rightarrow (X_1)_\epsilon$  and  $(Y_2)_\epsilon \rightarrow (X_2)_\epsilon$  induced by  $\eta_1$  and  $\eta_2$ .

By Lemma 2.5 a general fiber of  $T \rightarrow B$  is pseudo-isomorphic, and thus isomorphic, to a weak Del Pezzo surface. As  $(Y_1)_\epsilon$  is a Del Pezzo surface of degree 1, the birational contraction  $T \dashrightarrow Y_1$  contracts a divisor onto a point or a curve contained in one fiber of  $Y_1/B$ . In particular,  $T_\epsilon$  is isomorphic to  $(Y_1)_\epsilon$ . This proves (2). As  $T/B$  factorises through  $\widehat{X}/\widehat{B}$ , we have a commutative diagram

$$\begin{array}{ccc}
 T & \xrightarrow{\pi} & B \\
 \searrow \varphi & & \nearrow \sigma \\
 \widehat{X} & \xrightarrow{\widehat{\pi}} & \widehat{B}
 \end{array}$$

where  $\varphi$  is a birational contraction, and  $\sigma$  is a morphism with connected fibers. As  $\dim \widehat{B} \leq 2$ , the generic fiber  $\widehat{B}_\epsilon$  of  $\widehat{B}/B$  is either a curve or a point, and we get morphisms of generic fibers

$$T_\epsilon \xrightarrow{f} \widehat{X}_\epsilon \xrightarrow{g} \widehat{B}_\epsilon.$$

The contractions of the two extremal rays of  $T_\epsilon$  give  $(X_1)_\epsilon$  and  $(X_2)_\epsilon$ , and both are Del Pezzo surfaces of Picard rank 1, so there is no morphism to a curve, and  $\widehat{B}_\epsilon$  is a point, which means that  $\widehat{B} \rightarrow B$  is an isomorphism (as  $B$  and  $\widehat{B}$  are normal), proving (1). As  $\widehat{X}/\widehat{B}$  is a rank 1 fibration, the Picard rank of  $\widehat{X}_\epsilon$  is 1, so  $f$  corresponds to one of the contractions of the two extremal rays of  $T_\epsilon$ . This gives (3).  $\square$

We also need the following easy lemma.

**Lemma 3.5** ([BLZ19, Lemma 2.22]). *Let  $T \rightarrow Y$  and  $Y \rightarrow X$  be two divisorial contractions between  $\mathbb{Q}$ -factorial varieties, with respective exceptional divisors  $E$  and  $F$ . Assume that there exists a morphism  $X \rightarrow B$  such that  $T/B$  is a Mori dream space. Then there exist two others  $\mathbb{Q}$ -factorial varieties  $T'$  and  $Y'$ , with a pseudo-isomorphism  $T \dashrightarrow T'$  and birational contractions  $T' \rightarrow Y' \rightarrow X$ , with respective exceptional divisors the strict transforms of  $F$  and  $E$ , such that the following diagram commutes:*

$$\begin{array}{ccc}
 T & \dashrightarrow & T' \\
 E \downarrow & & \downarrow F \\
 Y & & Y' \\
 & \searrow F & \swarrow E \\
 & & X
 \end{array}$$

We can now prove the main statement of this paragraph.

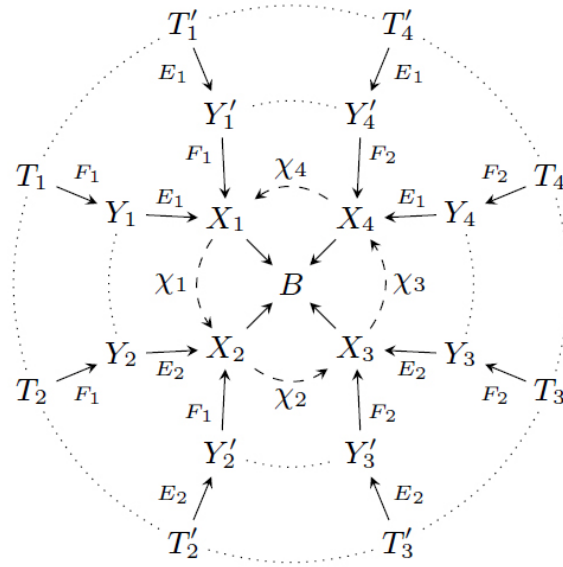
**Proposition 3.6.** *Let  $\chi_1 : X_1/B \dashrightarrow X_2/B$  be a Bertini link appearing in some non-trivial elementary relation over a curve. Then this relation has the form*

$$\chi_4 \circ \chi_3 \circ \chi_2 \circ \chi_1 = \text{id},$$

where  $\chi_1$  and  $\chi_3$  are Bertini links, equivalent via  $\chi_1$  and  $\chi_2$ .

*Proof.* Let  $T/B$  be a rank 3 fibration corresponding to an elementary relation  $\chi_n \circ \dots \circ \chi_1 = \text{id}$ . Denote by  $\{X_i/B_i\}_{i=1}^n$  the finite collection of rank 1 fibrations dominated by  $T/B$  and corresponding to Sarkisov links  $\chi_i : X_i \dashrightarrow X_{i+1}$ . We are going to show that  $n = 4$ .

First, Lemma 3.4(1) shows that the morphism  $B_i \rightarrow B$  is an isomorphism for each  $i$ . In particular, all  $\chi_i$  are Sarkisov links of type II over  $B$ . We will show that the links are given as in the diagram below, where the varieties associated with  $\chi_i$  are arranged in circles according to their Picard number and where the morphisms are labelled by divisors which those morphisms contract.



We start with the link  $\chi_1$  and try to recover the whole relation. Let  $E_1 \subset Y_1$  and  $E_2 \subset Y_2$  be exceptional divisors of the divisorial contractions  $Y_1 \rightarrow X_1$  and  $Y_2 \rightarrow X_2$  respectively. We again denote by  $E_1, E_2 \subset T$  the strict transforms of these divisors on  $T$ . We denote by  $F_1$  the divisor contracted by the divisorial contraction  $T_1 \rightarrow Y_1$ . By Lemma 3.4(2), the generic fibers of  $T_1/B$  and  $Y_1/B$  are isomorphic, so the image of  $F_1$  is a point or a curve contained in a fiber of  $Y_1/B$ .

The same then holds for  $T_2 \rightarrow Y_2$ , and we obtain the following part of the above diagram.

$$\begin{array}{ccc}
 T_1 & \cdots & T_2 \\
 F_1 \downarrow & & \downarrow F_1 \\
 Y_1 & \cdots & Y_2 \\
 E_1 \downarrow & & \downarrow E_2 \\
 X_1 & \xrightarrow{\chi_1} & X_2 \\
 & \searrow & \swarrow \\
 & B & 
 \end{array}$$

Denote by  $F_2 \subset Y_1$  the fiber containing the image of  $F_1$ . As before, we again write  $F_2$  for the strict transforms of  $F_2$  in the threefolds  $X_i, Y_i, T_i$ .

For each integer  $i$ , the rank 3 fibration  $T_i/B$  factorises through  $X_i/B_i = X_i/B$ . We obtain the following key facts:

(1) The morphism  $T_i \rightarrow X_i$  contracts *exactly one* divisor among  $F_1$  and  $F_2$ . Indeed, each fiber of our Del Pezzo fibration  $X_i \rightarrow B$  is an *irreducible* surface.

(2) The morphism  $T_i \rightarrow X_i$  contracts *exactly one* divisor among  $E_1$  and  $E_2$ . This is because exactly one of the two birational maps  $X_1 \dashrightarrow X_i$  and  $X_2 \dashrightarrow X_i$  induces an isomorphism between the generic fibers of  $X_1/B$  or  $X_2/B$  with  $X_i/B$  (Lemma 3.4(3)).

We use these to finish the proof. Let us consider then next link  $\chi_2$  that starts from  $X_2$ . Its resolution is given by

$$\begin{array}{ccc}
 T'_2 & \cdots & T'_3 \\
 \downarrow & & \downarrow \\
 Y'_2 & \cdots & Y'_3 \\
 \downarrow & & \downarrow \\
 X_2 & \xrightarrow{\chi_2} & X_3 \\
 & \searrow & \swarrow \\
 & B & 
 \end{array}$$

As  $T_2 \approx T'_2$  over  $X_2$  and the birational map  $\chi_2 \circ \chi_1$  should not be an isomorphism, the morphism  $Y'_2 \rightarrow X_2$  must contract  $F_1$ , and  $T'_2 \rightarrow Y'_2$  must contract  $E_2$  (this exists by Lemma 3.5). As  $T_2 \approx T'_3$  and  $Y'_2 \approx Y'_3$ , the morphism  $T'_3 \rightarrow Y'_3$  contracts  $E_2$ . Then (2) implies that  $Y'_3 \rightarrow X_3$  contracts  $F_2$ . Proceeding this game further, we recover all labels and prove  $n = 4$ .

It remains to notice that the links  $\chi_2$  and  $\chi_4$  induce isomorphisms on the generic fibers of the corresponding Del Pezzo fibrations. By definition, it means that  $\chi_1$  is equivalent to  $\chi_3$ .  $\square$

**3.3. Proof of Theorem D.** We are now ready to prove Theorem D. We use the presentation of the groupoid BirMori given in Theorem 2.14 and the description of elementary relations involving Bertini links, obtained in Propositions 3.3 and 3.6.



*Proof of Theorem D.* Let  $X$  be a Del Pezzo fibration of degree 3. By Theorem 2.14(1), the groupoid  $\text{BirMori}(X)$  is generated by Sarkisov links and automorphisms of Mori fiber spaces. By Theorem 2.14(2), the relations are generated by elementary relations. Fix a positive integer  $g \in \mathbb{Z}_{>0}$  given by Proposition 3.3. Consider the map

$$\Phi : \text{BirMori}(X) \longrightarrow \underset{\mathcal{B}_X^{\geq g}}{*} \mathbb{Z}/2$$

which sends each Sarkisov link  $\chi$  of Bertini type with  $\mathbf{g}(\chi) \geq g$  to the generator of  $\mathbb{Z}/2$  indexed by  $[\chi]$ , and all other Sarkisov links and automorphisms of Mori fiber spaces are sent to zero. To check that  $\Phi$  is a well-defined groupoid homomorphism, we need to show that every elementary relation is sent to the neutral element.

Let  $\chi_n \circ \cdots \circ \chi_1 = \text{id}$  be a non-trivial elementary relation between the Sarkisov links. We may assume that it involves a Bertini link  $\chi_1$  with  $\mathbf{g}(\chi_1) \geq g$ , otherwise the relation is sent to the neutral element automatically. By Proposition 3.3, our relation is a relation over a curve. Then Proposition 3.6 shows that  $n = 4$ . Moreover,  $\chi_1$  and  $\chi_3$  are equivalent links of Bertini type, while  $\chi_2$  and  $\chi_4$  are not of Bertini type. Thus, our elementary relation is sent to the neutral element. This proves the existence of the groupoid homomorphism. The restriction on  $\text{Bir}(X)$  gives a group-theoretic homomorphism.  $\square$

#### 4. IMAGE OF THE GROUP HOMOMORPHISM GIVEN BY THEOREM D

In this section, we prove that the image of the group homomorphism given by Theorem D is large when  $X$  is a cubic Del Pezzo fibration. Recall that a Bertini involution on a cubic surface  $F \subset \mathbb{P}^3$  associated with two general points  $p, q \in F$  is given by  $\pi\iota\pi^{-1} \in \text{Bir}(X)$ , where  $\pi : \widehat{F} \rightarrow F$  is the blow-up of the two points  $p, q$  and  $\iota \in \text{Aut}(\widehat{F})$  is the Bertini involution of the Del Pezzo surface  $\widehat{F}$  of degree 1, given by the double covering  $\widehat{F} \xrightarrow{|-2K_{\widehat{F}}|} V \subset \mathbb{P}^3$  onto a quadric cone  $V \subset \mathbb{P}^3$ . The following result does this in family on a cubic Del Pezzo fibration.

**Proposition 4.1** ([BCDP19, Proposition 5.3]). *Let  $X/B$  be a Del Pezzo fibration of degree 3. For each positive integer  $m > 0$  there exists a smooth 2-section  $\Gamma \subset X$  with genus  $\mathbf{g}(\Gamma) \geq m$ , and a birational involution  $\iota_\Gamma$  which acts on a general fibre  $F$  of  $X/B$  as the Bertini involution of the cubic surface  $F$ , centered at the two points of  $F \cap \Gamma$ .*

*Sketch of proof.* We may assume that  $X$  is brought to a standard model. Take a section  $Z \subset X$ , and let  $\eta : Y \rightarrow X$  be its blow-up (note that  $Z$  exists by [Kol96, IV, Theorem 6.8]). Then  $Y$  is a Del Pezzo fibration of degree 2. Denote by  $F_X$  a general fiber of  $X \rightarrow \mathbb{P}^1$ , and by  $F_Y$  a general

fiber of  $Y \rightarrow \mathbb{P}^1$ . For some large integer  $n$ , consider the divisor

$$D = -K_Y + nF_Y.$$

One can show that  $D$  is base-point-free, for  $n$  big enough. Now take two general elements  $D_1, D_2 \in |D|$ , and consider the curve  $\Delta = D_1 \cap D_2$ . Then  $Q$  is a smooth curve contained in the smooth locus of  $Y$ .

As  $D_1, D_2$  correspond to the strict transforms of two hyperplanes in  $\mathbb{P}^3$  through the point  $Z \cap F_X$ , we get that  $F_Y \cap \Delta$  consists of 2 points, i.e.  $\Delta$  is a 2-section. Now let  $\Gamma = \eta(\Delta)$ , which is a 2-section of  $X/B$ . Straightforward calculations show that

$$\deg(-K_\Gamma) = 4n - K_Y^3,$$

which proves that the genus of  $\Gamma$  is bigger than  $m$  if one chooses  $n$  large enough. The birational involution  $\iota_\Gamma \in \text{Bir}(X)$  is then obtained by applying to a general fibre  $F$  of  $X/B$  the Bertini involution of the cubic surface  $F$ , centered at the two points of  $F \cap \Gamma$ .  $\square$

To deduce Theorem A and Theorem B from Theorem D, we will show that the involutions  $\iota_\Gamma$  of Proposition 4.1 are sent, via the group homomorphism of Theorem D, onto infinitely many different generators of  $\ast_{\mathcal{B}_X^{\geq g}} \mathbb{Z}/2$ .

**Lemma 4.2.** *Let  $X/B$  and  $Y/B$  be two Mori fiber spaces over the same base, and  $\varphi : X \dashrightarrow Y$  be a birational map over  $B$ . There exists a decomposition of  $\varphi$  into a product of Sarkisov links*

$$\varphi = \chi_1 \circ \cdots \circ \chi_r,$$

where  $\chi_i : X_i/B_i \dashrightarrow X_{i+1}/B_{i+1}$  and the following hold:

- (1) If  $\dim(B) = 2$ , one can assume that  $\dim(B_i) = 2$  for each  $i$ ;
- (2) If  $X/B$  is a Del Pezzo fibration of degree  $d \leq 3$  over a curve  $B$ , then one can choose each  $\chi_i$  to be a Sarkisov link of type II, each  $B_i$  equal to  $B$ , and all  $X_i/B_i$  to be Del Pezzo fibrations of the same degree  $d$  (here we set  $X = X_1, Y = X_{r+1}$ );

*Proof.* We first note that the proof of [HM13, Theorem 1.1] works in relative settings, i.e. the map  $\varphi$  can be decomposed into a product of Sarkisov links  $\chi_i$  over the fixed base  $B$ . This means that all intermediate varieties admit morphisms to that  $B$ , as in the figure below.

$$\begin{array}{ccccccc}
 X = X_1 & \xrightarrow{\chi_1} & X_2 & \xrightarrow{\chi_2} & \cdots & \xrightarrow{\chi_{r-1}} & X_r & \xrightarrow{\chi_r} & X_{r+1} = Y \\
 \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
 B & & B_2 & & \cdots & & B_r & & B \\
 & & \searrow & & & & \swarrow & & \swarrow \\
 & & & & & & & & B
 \end{array}$$

If  $\dim B = 2$ , then all  $B_i$  are surfaces, proving (1). To prove (2), we suppose that  $X/B$  is a Del Pezzo fibration of degree  $d \leq 3$ . We denote by  $(X_i)_\epsilon$  the generic fibre of  $X_i/B$ , for each  $i$ , and obtain a birational map  $(X_1)_\epsilon \dashrightarrow (X_i)_\epsilon$  for each  $i$ . As  $(X_1)_\epsilon$  is a Del Pezzo surface of degree  $d \leq 3$  with Picard rank equal to 1, the same holds for  $(X_i)_\epsilon$  (follows from the classification of Sarkisov links between two-dimensional Mori fibrations over a perfect field given in [Isk96, Theorem 2.6]). This implies that the generic fibre  $(B_i)_\epsilon$  of  $B_i \rightarrow B$  is a point and thus  $B_i \rightarrow B$  is an isomorphism. The Sarkisov link  $\chi_i$  is then of type II. This achieves the proof of (2).  $\square$

**Lemma 4.3.** *Let  $g \geq 0$  be an integer as in Theorem D. Let  $X/B$  be a cubic Del Pezzo fibration over a curve  $B$ , and  $\iota_\Gamma \in \text{Bir}(X)$  be the birational involution associated with a smooth 2-section  $\Gamma$  of  $X/B$  with  $\mathbf{g}(\Gamma) \geq g$ , as in Proposition 4.1. Then, the image of  $\iota_\Gamma$  under the group homomorphism*

$$\text{Bir}(X) \rightarrow \underset{\mathcal{B}_X^{\geq g}}{*} \mathbb{Z}/2$$

*of Theorem D is the generator of a group  $\mathbb{Z}/2$  indexed by the equivalence class of a Bertini link of genus  $\mathbf{g}(\Gamma)$ .*

*Proof.* By Lemma 4.2(2), the map  $\varphi$  can be decomposed into the product of Sarkisov links  $\chi_1, \dots, \chi_r$  between cubic Del Pezzo fibrations over  $B$ . The decomposition given in the proof of [HM13, Theorem 1.3] comes from birational contractions  $\pi_i : Z \dashrightarrow X_i$ , where  $Z$  is the resolution of indeterminacy points of  $\varphi$ .

$$\begin{array}{ccccccc}
 & & Z & & & & \\
 & \swarrow \pi_1 & & \searrow \pi_{r+1} & & & \\
 X = X_1 & \xrightarrow{\chi_1} & X_2 & \xrightarrow{\chi_2} & \cdots & \xrightarrow{\chi_{r-1}} & X_r & \xrightarrow{\chi_r} & X_{r+1} = X \\
 & & \swarrow \pi_2 & & \searrow \pi_r & & & & \\
 & & & & & & & & B
 \end{array}$$

Denote by  $S_1 \subset X_1$  and  $S_{r+1} \subset X_{r+1}$  the divisors contracted by  $\varphi$  and  $\varphi^{-1}$  respectively. Further, let  $E_i \subset Z$  be  $\pi_i$ -exceptional divisor over  $S_i$ , where  $i \in \{1, r+1\}$ . Note that the generic fiber

of  $Z/B$  is a Del Pezzo surface of degree 1 and Picard rank 2 over  $\mathbb{C}(B)$ . Thus each  $\pi_i$  contracts precisely one of  $E_1$  and  $E_{r+1}$ , and possibly some “vertical” divisors contained in the fibers of  $Z/B$ . If  $\pi_i$  and  $\pi_{i+1}$  contract the same divisor  $E_j$ , the Sarkisov link  $\chi_i$  induces an isomorphism between the generic fibers of  $X_i/B$  and  $X_{i+1}/B$ . If the contracted divisors are different, the corresponding link is a Bertini link. Since the first and the last morphisms (i.e.  $\pi_1$  and  $\pi_{r+1}$ ) contract different divisors, the number of Bertini links must be odd. Moreover, all Bertini links have the same genus  $\mathbf{g}(\Gamma)$ .  $\square$

We now can complete the proof of Theorem A.

*Proof of Theorem A.* Let  $X/B$  be a cubic Del Pezzo fibration and let us write

$$\mathrm{Bir}(X/B) = \{\varphi \in \mathrm{Bir}(X) \mid \pi\varphi = \pi\}.$$

By Theorem D, there exists an integer  $g \geq 0$  and a group homomorphism

$$\rho: \mathrm{Bir}(X) \longrightarrow \underset{\mathcal{B}_X^{\geq g}}{*} \mathbb{Z}/2$$

which sends every Sarkisov link  $\chi$  of Bertini type with  $\mathbf{g}(\chi) \geq g$  to the generator indexed by equivalence class of  $\chi$ , and all other Sarkisov links and all automorphisms of Mori fiber spaces birational to  $X$  onto the trivial element. For each integer  $g' \geq g$ , Proposition 4.1 gives a birational involution  $\iota_{\Gamma} \in \mathrm{Bir}(X/B)$  associated to a curve of genus  $\mathbf{g}(\Gamma) \geq g'$ , and whose image by  $\rho$  is the generator of a Sarkisov link of genus  $\mathbf{g}(\Gamma)$  (Lemma 4.3). We obtain then infinitely many distinct involutions  $\iota_{\Gamma} \in \mathrm{Bir}(X)$  that are sent onto infinitely many distinct generators of  $\underset{\mathcal{B}_X^{\geq g}}{*} \mathbb{Z}/2$ . Projecting on the corresponding factors gives a group homomorphism  $\mathrm{Bir}(X) \rightarrow \underset{\mathbb{N}}{*} \mathbb{Z}/2$  whose restriction to  $\mathrm{Bir}(X/B)$  is surjective.  $\square$

**4.1. Proof of Theorems B and C.** We now use the group homomorphisms constructed in Theorems A and D and the relative Sarkisov program to prove Theorem B and C. We work as before over the field of complex numbers, and thus write  $\mathrm{Aut}(\mathbb{P}^n) = \mathrm{Aut}_{\mathbb{C}}(\mathbb{P}^n)$  and  $\mathrm{Bir}(\mathbb{P}^n) = \mathrm{Bir}_{\mathbb{C}}(\mathbb{P}^n)$ .

*Proof of Theorem B.* Let  $X/B$  be a Del Pezzo fibration of degree 3 such that  $X$  is rational (one can take e.g.  $V_1$  from Example 1.2). Fixing a birational map  $\psi: X \dashrightarrow \mathbb{P}^3$  gives an isomorphism  $\mathrm{Bir}(\mathbb{P}^3) \simeq \mathrm{Bir}(X)$ . We then obtain an induced homomorphism

$$\rho: \mathrm{Bir}(\mathbb{P}^3) \simeq \mathrm{Bir}(X) \longrightarrow \underset{\mathbb{N}}{*} \mathbb{Z}/2$$

given by Theorem A. By construction, it is obtained by composing the group homomorphism  $\hat{\rho}: \mathrm{Bir}(X) \rightarrow \underset{\mathcal{B}_X^{\geq g}}{*} \mathbb{Z}/2$  of Theorem D with a projection  $\underset{\mathcal{B}_X^{\geq g}}{*} \mathbb{Z}/2 \rightarrow \underset{\mathbb{N}}{*} \mathbb{Z}/2$ . Let  $\varphi \in \mathrm{Bir}(\mathbb{P}^3)$  be a

birational map preserving a rational fibration  $\pi: \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$  (which means  $\pi\varphi = \pi$ ). We want to show that  $\varphi \in \text{Ker } \rho$ , which corresponds to ask that  $\varphi' = \psi^{-1}\varphi\psi$  is in the kernel of  $\hat{\rho}$ .

Writing  $\pi' = \pi\psi$ , the element  $\varphi' \in \text{Bir}(X)$  satisfies  $\pi'\varphi' = \pi'$ . Consider the following diagram:

$$\begin{array}{ccc}
 Z & \overset{\text{MMP}/\mathbb{P}^2}{\dashrightarrow} & Z' \\
 \sigma \downarrow & \searrow \eta_Z & \downarrow \eta_{Z'} \\
 X & \overset{\pi'}{\dashrightarrow} & \mathbb{P}^2
 \end{array}$$

where  $\sigma$  is a resolution of indeterminacy points of  $\pi'$ , and  $Z \dashrightarrow Z'$  is a Minimal Model Program over  $\mathbb{P}^2$ . As the general fibres of  $Z \rightarrow \mathbb{P}^2$  are rational curves, the morphism  $Z' \rightarrow \mathbb{P}^2$  is a Mori fibre space which is a conic bundle. Denote by  $\mu: X \dashrightarrow Z'$  the induced birational map and put  $\varphi'' = \mu\varphi'\mu^{-1}$ . Then  $\varphi'' \in \text{Bir}(Z'/\mathbb{P}^2)$ . By Lemma 4.2(1), the map  $\varphi''$  can be decomposed into the product of Sarkisov links (of conic bundles over surfaces dominating  $\mathbb{P}^2$ ) of type II and thus into a product of Sarkisov links with no Bertini links. This implies that  $\psi^{-1}\varphi\psi$  is in the kernel of the group homomorphism  $\text{Bir}(X) \rightarrow \ast_{\mathcal{D}_X^{\geq g}} \mathbb{Z}/2$  of  $\rho$ , as desired.  $\square$

**Lemma 4.4.** *Every algebraic subgroup of  $\text{Bir}(\mathbb{P}^3)$  that is isomorphic to  $\mathbb{G}_a$  or  $\mathbb{G}_m$  is conjugate, in  $\text{Bir}(\mathbb{P}^3)$ , to a subgroup of  $\text{Aut}(\mathbb{P}^3)$ .*

*Proof.* In the case of  $\mathbb{G}_m$ , the result follows from [Bia66, Corollary 2]. For the additive case, we do as in the proof of [BL15, Proposition 4.1]: one may find a birational map  $\mathbb{P}^3 \dashrightarrow X$ , where  $X$  is a smooth variety, that conjugates the group to a subgroup of  $\text{Aut}(X)$  (using a result of Weil, see [Wei55] or [Kra18]).

Then there exists an open set  $U \subset X$  where the action is a translation, which corresponds to the existence of an equivariant isomorphism  $U \cong \mathbb{G}_a \times V$ , where the action on  $V$  is trivial and the action on  $\mathbb{G}_a$  is the translation (follows from the work of Rosenlicht [Ros56]). As  $V$  is unirational and of dimension 2, it is rational. So we can assume that  $V = \mathbb{A}^2$  and then obtain a linear action on  $\mathbb{A}^3$ , that we can extend to  $\mathbb{P}^3$ .  $\square$

*Proof of Theorem C.* As a conjugate of a connected algebraic subgroup of  $\text{Bir}(\mathbb{P}^n)$  is again a connected algebraic subgroup of  $\text{Bir}(\mathbb{P}^n)$ , the subgroup of  $\text{Bir}(\mathbb{P}^n)$  generated by all connected algebraic subgroups of  $\text{Bir}(\mathbb{P}^n)$  is a normal subgroup  $H_n \subset \text{Bir}(\mathbb{P}^n)$ . It contains the connected algebraic subgroup  $\text{Aut}(\mathbb{P}^n) \cong \text{PGL}_{n+1}(\mathbb{C})$ . If  $n = 1$ , we have  $H_1 = \text{Bir}(\mathbb{P}^1)$  as  $\text{Bir}(\mathbb{P}^1) = \text{Aut}(\mathbb{P}^1)$ . If  $n = 2$ , the Noether-Castelnuovo theorem [Cas01] implies that  $\text{Bir}(\mathbb{P}^2)$  is generated by  $\text{Aut}(\mathbb{P}^2)$  and by the standard quadratic transformation  $\sigma: [x : y : z] \dashrightarrow [yz : xz : x]$ . As  $\sigma$  corresponds locally to

$(x, y) \mapsto (x^{-1}, y^{-1})$ , conjugate to  $(x, y) \mapsto (-x, -y)$  via

$$(x, y) \mapsto \left( \frac{x+1}{x-1}, \frac{y+1}{y-1} \right),$$

we find  $\sigma \in H$ . This shows that  $H_2 = \text{Bir}(\mathbb{P}^2)$ .

It remains to prove the main part of Theorem C, i.e. to prove that  $H_3 \neq \text{Bir}(\mathbb{P}^3)$ . To do this, it suffices to show that  $H_3$  is generated by elements  $\varphi \in H_3 \subset \text{Bir}(\mathbb{P}^3)$  that are conjugate in  $\text{Bir}(\mathbb{P}^3)$  to elements of  $\text{Aut}(\mathbb{P}^3)$  and to apply Theorem B. Every connected algebraic subgroup  $G \subset \text{Bir}(\mathbb{P}^3)$  is a linear algebraic group [BF13, Remark 2.21] and is thus generated by its algebraic subgroups isomorphic to  $\mathbb{G}_m$  or to  $\mathbb{G}_a$ . The proof ends by applying Lemma 4.4.  $\square$

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