DYNAMICAL DEGREES OF (PSEUDO)-AUTOMORPHISMS FIXING CUBIC HYPERSURFACES

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Abstract. We give a way to construct group of pseudo-automorphisms of rational varieties of any dimension that fix pointwise the image of a cubic hypersurface of $\mathbb{P}^n$. These group are free products of involutions, and most of their elements have dynamical degree > 1. Moreover, the Picard group of the varieties obtained is not big, if the dimension is at least 3.

We also answer a question of E. Bedford on the existence of birational maps of the plane that cannot be lifted to automorphisms of dynamical degree > 1, even if we compose them with an automorphism of the plane.

1. Introduction

A birational map $\varphi\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ (or a Cremona transformation) is a rational map given by

$$(x_0 : \cdots : x_n) \dashrightarrow (P_0(x_0, \ldots, x_n) : \cdots : P_n(x_0, \ldots, x_n)),$$

where all P_i are homogeneous polynomials of the same degree, which admits an inverse of the same type. Choosing all P_i without common component, the degree $\deg(\varphi)$ of φ is by definition the degree of the polynomials P_i, or equivalently the degree of the pull-back of hyperplanes of $\mathbb{P}^n$ by φ.

The (*first*) *dynamical degree* of φ is the number

$$\lim_{n\to\infty} (\deg(\varphi^n))^{1/n},$$

which always exists, since $\deg(\varphi^{a+b}) \leq \deg(\varphi^a) \cdot \deg(\varphi^b)$ for any $a, b \geq 0$. It is moreover invariant under conjugation.

There is a sequence of articles which provide families of examples of birational maps of $\mathbb{P}^2$ with dynamical degree > 1, lifting to automorphisms of a smooth rational surface obtained by blowing-up a finite number of points (among them, see [BedKim06], [BedDil06], [McM07], [BedKim09], [BedKim10], [Dil11], [DésGri10]....) The general way of producing examples is to start by a simple birational map (quadratic involution, automorphism of the affine plane,...), and to compose it with a linear automorphism of $\mathbb{P}^2$ to impose that the base-points of the inverse "come back" to the base-points of the map after a certain number of iteration of the map.

This approach was generalised in dimension 3, in [BedKim11], to provide pseudo-automorphisms of projective 3-folds of dynamical degree > 1, starting from a special family of quadratics elements of $\mathrm{Bir}(\mathbb{P}^3)$. Recall that a pseudo-automorphism of X is a birational self-map $\varphi \in \mathrm{Bir}(X)$ such that φ and φ^{-1} do not contract any codimension 1 set; it is the same as automorphism for smooth projective surfaces.

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Other examples of pseudo-automorphisms of dynamical degree > 1 of rational projective varieties of any dimension were given in [PerZha11], using actions of Weyl groups on blow-ups of $\mathbb{P}^n$ at a finite number of points.

In this article, we give another way of constructing examples, which also works in any dimension. This produces large groups of pseudo-automorphisms, where almost all elements have dynamical degree > 1. Moreover, the rank of the Picard group of the varieties obtained does not need to be very large, contrary to what happens in dimension 2, or to the examples of [BedKim11] and [PerZha11].

We recall the following construction, defined in [Giz94, page 42, Example 3] over the name of R_p.

Definition 1.1. *Let $Q \subset \mathbb{P}^n$ be a cubic hypersurface, and let $p \in Q$ be a smooth point. We define an involution $\sigma_{p,Q} \in \mathrm{Bir}(\mathbb{P}^n)$ which fixes pointwise Q by the following: if L is a general line of $\mathbb{P}^n$ passing through p, we have $\sigma_{p,Q}(L) = L$ and the restriction of $\sigma_{p,Q}$ to L is the involution that fixes $(L \cap Q)\backslash\{p\}$.*

From the geometric definition, we can easily get an algebraic definition by polynomials (see [Giz94] or Section 2). We will show that any $\sigma_{p,Q}$ lifts to an automorphism of a smooth variety obtained by blowing-up codimension 2 subsets of $\mathbb{P}^n$. Taking a finite number of points on the same cubic gives a huge group of pseudo-automorphisms of a rational n-fold:

Theorem 1. *Let $Q \subset \mathbb{P}^n$ be a cubic hypersurface, and let $p_1, \dots, p_k \in Q$ be distinct smooth points. For $i = 1, \dots, k$, we write $\Gamma_i = \{x \in Q \mid$ the line through x and q_i is tangent to Q at $x\}$.*

Denote by $\pi\colon X \to \mathbb{P}^n$ the following birational morphism: it first blows-up all points $p_1, \dots, p_k$, then blows-up the strict transform of Γ_1, then the strict transform of Γ_2 and so on until blowing-up the strict transform of Γ_k.

If $p_i \notin \Gamma_j$ for any $i \neq j$, then $\sigma_{p_1,Q}, \dots, \sigma_{p_k,Q}$ lift to pseudo-automorphisms $\hat{\sigma}_1, \dots, \hat{\sigma}_k$ of X, that generate a free product

$$G = \star_{i=1}^{k} < \hat{\sigma}_i >,$$

having the following properties:

1. *Any element of G of finite order is conjugate to $\hat{\sigma}_i$ for some i (and has dynamical degree 1);*
2. *Any element conjugate to $(\hat{\sigma}_i\hat{\sigma}_j)^m$ for $i \neq j$ and $m \geq 1$ is of infinite order, and its dynamical degree is equal to 1 if and only if $d = 3$;*
3. *Any other element has dynamical degree > 1.*
4. *Each element of G fixes pointwise the lift of the cubic Q on X.*

Corollary 1.2. *For any $n \geq 3$, there exist a rational smooth n-fold X with $\mathrm{rk}\ \mathrm{Pic}(X) = 7$ admitting a group of pseudo-automorphisms G isomorphic to the free group with 2 generators, such that all elements of $G\backslash\{1\}$ have dynamical degree > 1. Moreover, G fixes pointwise an hypersurface isomorphic to a general smooth cubic of $\mathbb{P}^n$.*

Remark 1.3. *Any pseudo-automorphism of a smooth projective surface X with $\mathrm{rk}\ \mathrm{Pic}(X) \leq 10$ has dynamical degree 1. All previously known examples of smooth rational varieties admitting pseudo-automorphisms with dynamical degree > 1 had Picard rank bigger than 10.*

Section 2 is devoted to the proof of Theorem 1 and of its corollary.

Question 1.4. *What is the minimal rank of a smooth projective rational 3-fold admitting an automorphism of dynamical degree >1?*

Restricting to dimension $n=2$, Theorem 1 gives the existence of group of automorphisms of rational surfaces with many elements of dynamical degree >1. The rank of the Picard group is however quite large, at least 16. This is because the varieties Γ_i are in fact union of 4 distinct points.

As we said above, the usual way to construct automorphisms of projective rational surfaces with dynamical degree >1 is to take a birational map of small degree, and then to compose it with an automorphism so that all base-points of the inverse are sent after some iterations onto base-points of the map. This approach gives rise to the following question of Eric Bedford, also stated and studied by Julie Déserti and Julien Grivaux in [DésGri10]:

Question 1.5. *Does there exist a birational map of the projective plane φ of degree >1 such that for all $\tau\in \mathrm{Aut}(\mathbb{P}^2)$ the map $\tau\varphi$ is not birationally conjugate to an automorphism of dynamical degree >1?*

Here, by "conjugate to an automorphism", we mean the existence of a birational map $\nu\colon \mathbb{P}^2 \dashrightarrow X$, where X is a projective smooth surface, such that $\nu(\tau\varphi)\nu^{-1}\in \mathrm{Aut}(X)$.

It is know that if φ is a birational map of degree 2, there exists an automorphism τ such that $\tau\varphi$ is conjugate to an automorphism of dynamical degree >1. We recall this fact in Section 3. The same was also proved by different authors for some special maps φ of degree 3. Using the involutions $\sigma_{p,Q}$, we prove in Section 3 that the same holds for a *general* map of degree 3.

The possible map φ of Question 1.5 cannot thus be a general cubic transformation, or a transformation of degree 2. Section 4 is devoted to the proof of the following result, showing that the map can be of degree 6, and answering the question of Bedford, Déserti and Grivaux:

Theorem 2. *Let $\chi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the birational map given by*

$$\chi\colon (x:y:z) \dashrightarrow (xz^5+(yz^2+x^3)^2 : yz^5+x^3z^3 : z^6).$$

For any automorphism $\tau\in\mathrm{Aut}(\mathbb{P}^2)$, the birational map $\tau\chi\in\mathrm{Bir}(\mathbb{P}^2)$ is not conjugate to an automorphism of a smooth projective rational surface.

Question 1.6. *Does there exists a transformation of degree <6 having the above property?*

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2. The maps $\sigma_{p,Q}$, their lifts and the groups generated by these

Let us describe algebraically the map $\sigma_{p,Q}$ introduced in Definition 1.1.

For this, choose homogeneous coordinates $(x_1:x_2:\cdots:x_n:y)$ on $\mathbb{P}^n$ and assume, up to a change of coordinates, that p is equal to $(0:\cdots:0:1)$. The equation of Q is thus $y^2P_1+yP_2+P_3$, where $P_1,P_2,P_3\in\mathbb{C}[x_1,\ldots,x_n]$ are homogeneous of degree $1,2,3$. The involution $\sigma_{p,Q}$ sends a point $(x_1:x_2:\cdots:x_n:y)$ onto

$$(-x_1(P_2+2yP_1):\cdots:-x_n(P_2+2yP_1):P_2y+2P_3).$$

The point p is a base-point of multiplicity 2. The subscheme $\Gamma_p \subset Q \subset \mathbb{P}^n$ of codimension 2 given by $P_2 + 2yP_1 = 0$ and $yP_2 + 2P_3 = 0$ is also contained in the base-locus, and $\sigma_{p,Q}$ is defined on $\mathbb{P}^n \backslash \Gamma_p$.

The cone $V_p \subset \mathbb{P}^n$ given by $(P_2)^2 - 4P_3P_1$ is contracted onto Γ_p by $\sigma_{p,Q}$, and the hypersurface given by $P_2 + 2yP_1$ is contracted onto the point p.

Note that Γ_p is also given by the intersection of Q with the hypersurface of equation $P_2 + 2yP_1 = 0$, or with the cone V_p, and corresponds to the points $q \in Q$ such that the line passing through p and q is tangent to Q at q, as defined in the introduction.

Proposition 2.1. *Denote by $\pi_p \colon X_p \to \mathbb{P}^n$ the blow-up of p, by $\pi_\Gamma \colon X \to X_p$ the blow-up of the strict transform of Γ_p and write $\pi = \pi_p \circ \pi_\Gamma \colon X \to \mathbb{P}^n$. The lift $\hat{\sigma}_p = \pi^{-1}\sigma\pi$ of $\sigma_{p,Q}$ is an automorphism of X.*

Writing $H \subset \mathrm{Pic}(X)$ the pull-back of an hyperplane of $\mathbb{P}^n$ by π, $E \subset \mathrm{Pic}(X)$ the pull-back of $\pi_p^{-1}(p)$ by π_Γ and F the exceptional divisor of π_Γ, (H, E, F) is a basis of a sub-$\mathbb{Z}$-module of $\mathrm{Pic}(X)$ invariant by $\hat{\sigma}_p$ (this sub-module is equal to $\mathrm{Pic}(X)$ if and only if Γ_p is irreducible). Moreover, the action of $\hat{\sigma}_p$ relative to this basis is

$$\begin{bmatrix} 3 & 2 & 4 \\ -2 & -1 & -4 \\ -1 & -1 & -1 \end{bmatrix}.$$

Proof. We can view X_p in $\mathbb{P}^n \times \mathbb{P}^{n-1}$ as

$$X_p = \Big\{ \big((x_1 : x_2 : \cdots : x_n : y), (z_1 : \cdots : z_n)\big) \;\Big|\; x_i z_j = x_j z_i \text{ for } 1 \leq i, j \leq n \Big\},$$

where $\pi_p \colon X_p \to \mathbb{P}^n$ is given by the projection on the first factor. The variety X_p is covered by open subsets $U_1, \ldots, U_n$, where U_i is the set where $z_i \neq 0$.

Each U_i is isomorphic to $\mathbb{A}^{n-1} \times \mathbb{P}^1$. For $i = 1$, the isomorphism is given by:

$$\begin{array}{ccc} \mathbb{A}^{n-1} \times \mathbb{P}^1 & \stackrel{\simeq}{\longrightarrow} & U_1 \\ \big((t_2, \ldots, t_n), (\alpha : \beta)\big) & \longmapsto & \big((\alpha : \alpha t_2 : \cdots : \alpha t_n : \beta), (1 : t_2 : \cdots : t_n)\big). \end{array}$$

The lift of σ preserves U_1 and restricts to the following birational map

$$\big((t_1, \ldots, t_n), (\alpha : \beta)\big) \dashrightarrow \big((t_1, \ldots, t_n), (-(\alpha R_2 + 2\beta R_1) : \beta R_2 + 2\alpha R_3)\big),$$

where $R_i = P_i(1, t_2, \ldots, t_n)$. On this chart, Γ_p is given by $\alpha R_2 + 2\beta R_1 = 0$ and $R_2\beta + 2\alpha R_3 = 0$. Blowing-up the corresponding ideal, we obtain the variety $W_1 \subset \mathbb{A}^n \times \mathbb{P}^1 \times \mathbb{P}^1$ given by

$$W_1 = \big\{((t_1, \ldots, t_n), (\alpha : \beta), (u : v)) \mid u(R_2\beta + 2\alpha R_3) = -v(\alpha R_2 + 2\beta R_1)\big\},$$

and the blow-up is given by the projection $W_1 \to \mathbb{A}^n \times \mathbb{P}^1$ on the first two factors. The map $\hat{\sigma}$ corresponds in these coordinates to

$$((t_1, \ldots, t_n), (\alpha : \beta), (u : v)) \mapsto ((t_1, \ldots, t_n), (u : v), (\alpha : \beta)),$$

and is thus an automorphism (the same calculation holds on the other charts).

Since E is exchanged with the strict transform of the hypersurface of equation $P_2 + 2yP_1$, which has degree 2, passes through p with multiplicity 1, and also through a general point of Γ_p, E is sent onto $2H - E - F$. Moreover, H is exchanged with hyperplanes of degree 3 having multiplicity 2 at p and multiplicity one at a general point of Γ_p. This shows that H is exchanged with $3H - 2E - F$. The fact that $\hat{\sigma}_p$ is an involution gives the last column of the matrix, i.e. that F is exchanged with

$4H-4E-F$, which corresponds to the cone $(P_2)^2+4P_1P_3=0$, which has degree 4, passes through p with multiplicity 4, and through Γ_p with multiplicity 1. □

We now generalise the construction by taking many points on the same cubic hypersurface.

Proposition 2.2. *Let $Q \subset \mathbb{P}^n$ be a cubic hypersurface, and let $p_1, \dots, p_k \in Q$ be distinct smooth points. For $i = 1, \dots, k$ we write σ_i the map $\sigma_{p_i,Q}$ given in Definition 1.1, and by $\Gamma_i = \Gamma_{p_i} \subset \mathbb{P}^n$ the codimension 2 subset associated. We assume that for $p_i \notin \Gamma_j$ for any $i \neq j$.*

Denote by $\pi\colon X \to \mathbb{P}^n$ the following birational morphism: it first blows-up all points $p_1, \dots, p_k$, then blows-up the strict transform of Γ_1, then the strict transform of Γ_2 and so on until blowing-up the strict transform of Γ_k.

Then, $\sigma_1, \dots, \sigma_k$ lift to pseudo-automorphisms $\hat{\sigma}_1, \dots, \hat{\sigma}_k$ of X.

Proof. Applying Proposition 2.1, σ_i lifts to an automorphism of a variety obtained by blowing-up first p_i and then the strict transform of Γ_i. Since $p_j \notin \Gamma_i$ for $j \neq i$, all these points correspond to points of the strict transform of Q, and are thus fixed by the lift of σ_i. We can thus blow-up all the points p_j with $j \neq i$ and σ_i again lifts to an automorphism. The strict transforms of the lifts of Γ_j for $j \neq i$ are again contained in the strict transform of W and thus fixed pointwise by the automorphism. We blow-up all these schemes and lift σ_i to an automorphism of a variety X_i obtained.

Note that $X_1 = X$, but that in general X_i is not the same as X_j for $i \neq j$, since the varieties Γ_i and Γ_j intersect: the choice in the order of the sets blown-up is important. The maps $X_j \dashrightarrow X_i$ induced by the blow-ups are pseudo-isomorphisms, they are isomorphisms outside the pull-back of the sets where Γ_i and Γ_j intersect, which have codimension ≥ 2. This implies that $\hat{\sigma_1} \in \mathrm{Aut}(X_1)$ and that all others $\hat{\sigma_i}$ are pseudo-automorphisms of X. □

The following proposition describes the group generated by these pseudo-automorphisms, and the dynamical properties of its elements. It yields – with Proposition 2.2 – the proof of Theorem 1.

Proposition 2.3. *Let $\hat{\sigma}_1, \dots, \hat{\sigma}_k \in \mathrm{Bir}(X)$ be pseudo-automorphisms as in Proposition 2.2. These element generate a free product*

$$G = \star_{i=1}^{k} < \hat{\sigma}_i >,$$

and we have the following description of elements of G:

1. *Any element of finite order is conjugate to a $\hat{\sigma}_i$ and has dynamical degree 1;*
2. *Any element conjugate to $(\hat{\sigma}_i\hat{\sigma}_j)^m$ for $i \neq j$ and $m \geq 1$ is of infinite order, and its dynamical degree is equal to 1;*
3. *Any other element has dynamical degree > 1.*

Proof. Let $H \in \mathrm{Pic}(X)$ be the pull-back of an hyperplane of $\mathbb{P}^n$, denote by $E_1, \dots, E_k \in \mathrm{Pic}(X)$ the total pull-back of the divisors obtained by blowing-up the p_i, and by $F_1, \dots, F_k \in \mathrm{Pic}(X)$ the exceptional divisors associated to $\Gamma_1, \dots, \Gamma_k$, using the same notation as before. The actions of $\hat{\sigma_1},\dots,\hat{\sigma_k}$ on $\mathrm{Pic}(X)$ are given by Proposition 2.1:

$$\begin{array}{rcllll} \hat{\sigma_i}(H) &=& 3H & -2E_i & -F_i, \\ \hat{\sigma_i}(E_i) &=& 2H & -E_i & -F_i, \\ \hat{\sigma_i}(F_i) &=& 4H & -4E_i & -F_i, \\ \hat{\sigma_i}(E_j) &=& E_j & \text{for } i\neq j, \\ \hat{\sigma_i}(F_j) &=& F_j & \text{for } i\neq j. \end{array}$$

Writing $\nu_i=\hat{\sigma_i}(H)-H=2H-2E_i-F_i$ we get

$$\begin{array}{rcl} \hat{\sigma_i}(H) &=& H+\nu_i, \\ \hat{\sigma_i}(\nu_i) &=& -\nu_i, \\ \hat{\sigma_i}(\nu_j) &=& \nu_j+2\nu_i \text{ for } i\neq j. \end{array} \tag{1}$$

Let us choose any element $\varphi=\sigma_{a_r}\dots\sigma_{a_1}$, where $a_1,\dots,a_r\in\{1,\dots,k\}$, $a_i\neq a_{i+1}$ for $i=1,\dots,r-1$. By induction on r, we prove that $\varphi(H)=H+\sum\limits_{i=1}^{k}\alpha_i v_i$, satisfying the following properties

- (i) $\alpha_1,\dots,\alpha_k$ are non-negative integers;
- (ii) $\alpha_{a_r}>\alpha_i$ for $i\neq a_r$;
- (iii) if $r>1$, then $\alpha_{a_{r-1}}>\alpha_i$ for $i\notin\{a_r,a_{r-1}\}$;
- (iv) $\sum_{i=1}^k\alpha_i\geq(\frac{5}{3})^t$, where $t=\#\{i\mid i\geq 3, a_i\neq a_{i-2}\}$.

When $r=1$, the result is obvious since $\varphi(H)=H+\nu_{a_1}$. We assume the result true for $r-1$ and prove it for r. We have $\varphi(H)=\sigma_{a_r}(V)$, where $V=\sigma_{a_{r-1}}\circ\dots\circ\sigma_1(H)=H+\sum\limits_{i=1}^{k}\beta_i v_i$ and all β_i satisfy the properties above. In particular, $\sum\limits_{i\neq a_r}\beta_i\geq\beta_{a_{r-1}}>\beta_{a_r}$. Applying (1), we get

$$\begin{array}{rcl} \alpha_i &=& \beta_i \text{ for } i\neq a_r, \\ \alpha_{a_r} &=& 1-\beta_{a_r}+2\sum\limits_{i\neq a_r}\beta_i>\sum\limits_{i\neq a_r}\beta_i\geq\beta_{a_{r-1}}>\beta_{a_r}, \end{array}$$

which proves the first three assertions. To prove (iv), we compute

$$\sum_{i=1}^{k}\alpha_i=1-4\beta_{a_r}+3\sum_{i=1}^{k}\beta_i.$$

We always have $\sum\limits_{i=1}^{k}\alpha_i>\sum\limits_{i=1}^{k}\beta_i$. It suffices thus to prove that if $r\geq 3$ and $a_r\neq a_{r-2}$, then $\sum\limits_{i=1}^{k}\alpha_i\geq\frac{5}{3}\sum\limits_{i=1}^{k}\beta_i$.

The fact that $r\geq 3$ and $a_r\neq a_{r_2}$ gives $\beta_{a_r}<\beta_{a_{r-1}}$ and $\beta_{a_r}<\beta_{a_{r-2}}$, and thus implies

$$\begin{array}{rcl} \sum\limits_{i=1}^{k}\alpha_i-\frac{5}{3}\sum\limits_{i=1}^{k}\beta_i &=& 1-4\beta_{a_r}+(3-\frac{5}{3})\sum\limits_{i=1}^{k}\beta_i, \\ &=& 1-\frac{8}{3}\beta_{a_r}+\frac{4}{3}\sum\limits_{i\neq a_r}\beta_i, \end{array}$$

which is positive since $2\beta_{a_r}<\beta_{a_{r-1}}+\beta_{a_{r-2}}\leq\sum\limits_{i\neq a_r}\beta_i$.

Now that $(i)-(iv)$ have been proved, we show how they imply the result. First, assertions (i) and (ii) show that G is the free product of the groups $<\sigma_i>\cong\mathbb{Z}/2\mathbb{Z}$.

Second, any non-trival element of the group is conjugated to $\varphi = \sigma_{a_r} \dots \sigma_{a_1}$, where $a_1, \dots, a_r \in \{1, \dots, k\}$, $a_i \neq a_{i+1}$ for $i = 1, \dots, r-1$ and $a_r \neq a_1$.

The element φ has finite order if and only if $r = 1$. If $r > 1$, we compute its dynamical degree by computing $\deg(\varphi^n)$ for $n \in \mathbb{N}$. The degree here is the degree as a birational map of $\mathbb{P}^3$, which is the degree of the system $\pi(\varphi^{-n}(H))$. Since each ν_i corresponds to a divisor of degree 2, we get

$$\deg(\varphi^n) = 1 + 2\sum_{i=1}^{k} \alpha_i \text{ if } \varphi^{-n}(H) = H + \sum_{i=1}^{k} \alpha_k \nu_k.$$

The assertions above imply that if the set $\{a_1, \dots, a_r\}$ has at least three elements, $\deg(\varphi^n) \geq (\frac{5}{3})^n$, so the dynamical degree of φ is strictly bigger than 1. The only case where the dynamical degree 1 could be one is when $\varphi = (\hat{\sigma}_i \hat{\sigma}_j)^m$ for $i \neq j$ and $m \geq 1$. It remains to prove that in this case, the dynamical degree is 1; and we only have to consider the case $m = 1$. The submodule of $\mathrm{Pic}(X)$ generated by H, ν_i, ν_j is invariant by φ, and the action relative to this basis is

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 3 & -2 \\ 1 & 2 & -1 \end{pmatrix},$$

which has only one eigenvalue, equal to 1. This achieves the proof. □

Remark 2.4. *Note that the dynamical degree of any element of the free group G generated above is easy to compute.*

(i) As we observed in the above proof, the dynamical degree of $\sigma_i \cdot \sigma_j$, for $i \neq j$, is the biggest eigenvalue of

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 3 & -2 \\ 1 & 2 & -1 \end{pmatrix},$$

whose characteristic polynomial is $(x-1)^3$. This dynamical degree is thus 1.

(ii) We can do a similar calculation with $\sigma_i \cdot \sigma_j \cdot \sigma_k$ where i, j, k are pairwise distinct. The dynamical degree is the the highest real eigenvalue of

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 2 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 2 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 9 & 15 & 10 & -6 \\ 3 & 6 & 3 & -2 \\ 1 & 2 & 2 & -1 \end{pmatrix},$$

which is $9 + 4\sqrt{5} \sim 17.944272$.

(iii) All other dynamical degrees can be computed in the same way.

Remark 2.5. *With the descriptions above, it is easy to take explicit cubic hypersurfaces, for example smooth ones, and to compute explicitly the locus to blow-up and the involutions.*

Now that Theorem 1 is proved, we finish the section with the proof of its corollary.

proof of Corollary 1.2. In any dimension $n \geq 3$, we take a smooth cubic hypersurface $Q \subset \mathbb{P}^n$, and choose three distinct general points p_1, p_2, p_3 such that the line through two of them intersects the cubic into another point. These points in Q satisfy then the conditions of Theorem 1, and yields pseudo-automorphisms $\hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3$

of the variety X obtained by blowing-up p_1, p_2, p_3 and the varieties $\Gamma_1, \Gamma_2, \Gamma_3$ associated. These latter being irreducible, the rank of $\mathrm{Pic}(X)$ is exactly 7 (a fact which is false in dimension 2).

Because $<\hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3>$ is the free product $\star_{i=3}^{k} <\hat{\sigma}_i>$, the group generated by $\alpha = \hat{\sigma}_1\hat{\sigma}_2\hat{\sigma}_3$ and $\beta = \hat{\sigma}_2\hat{\sigma}_1\hat{\sigma}_2\hat{\sigma}_3\hat{\sigma}_2$ is the free group over two generators. Moreover, none of the non-trivial elements of the group is conjugate to an element of length < 3, so each element has dynamical degree > 1. □

3. The involutions on $\mathbb{P}^2$ and the blow-up

In this section, we deal with dimension 2.

3.1. **Degree 2.** We will say that two birational maps φ, φ' are *projectively equivalent* if $\varphi = \alpha\varphi'\beta$ for some $\alpha, \beta \in \mathrm{Aut}(\mathbb{P}^2)$. In Question 1.5, we can only study equivalence classes, since $\alpha\varphi\beta$ is conjugate to $\varphi(\beta\alpha^{-1})$. We can also replace φ with φ^{-1}.

There are three equivalence classes of birational maps of $\mathbb{P}^2$ of degree 2. Each such map has three base-points of multiplicity 1, which are not collinear, and the classes correspond to

(i) three points p_1, p_2, p_3 that belong to $\mathbb{P}^2$ as proper points;
(ii) two points p_1, p_2 that belong to $\mathbb{P}^2$ as proper points, the point p_3 is infinitely near to p_1;
(iii) one point p_1 is a proper point of $\mathbb{P}^2$, p_2 is infinitely near to p_1 and p_3 is infinitely near to p_2

There are many known examples of quadratic maps of type (i) that are conjugate to automorphisms of projective surfaces and have dynamical degree > 1. See for example [BedKim06] or [BedKim09]. For examples of type (iii), see [BedKim10]. In fact, in [Dil11], all possible types of quadratic maps preserving cubics and being conjugate to to automorphisms of projective surfaces and have dynamical degree > 1 are constructed. They depend on an orbit data $[n_1, n_2, n_3]$, which provides all types (i), (ii), (iii) depending on the number of n_i equal.

All these examples yield the following:

Lemma 3.1. *If φ is a birational map of $\mathbb{P}^2$ of degree 2, there exists an automorphism $\tau \in \mathrm{Aut}(\mathbb{P}^2)$ such that $\tau\varphi$ is conjugate to an automorphism of a smooth projective rational surface with dynamical degree > 1.*

3.2. **Degree 3.** It is also possible to describe equivalence classes of elements of $\mathrm{Bir}(\mathbb{P}^2)$ of degree 3. There are in fact 32 algebraic families, corresponding to the type of base-points (if some are collinear, or if some infinitely near,...), or equivalently to the curves contracted (see [CerDes08, Table of page 176]). The family of biggest dimension (dimension 2) consists of cubic maps φ having five proper base-points, no 3 being collinear. All others have dimension < 1.

We will prove that for a general cubic map $\varphi \in \mathrm{Bir}(\mathbb{P}^2)$, there exists $\tau \in \mathrm{Aut}(\mathbb{P}^2)$ such that $\tau\varphi$ is conjugate to an automorphism of positive entropy. To do this, we will use involutions $\sigma_{p,Q}$ associated to a point p of a smooth cubic Γ (see Definition 1.1). Recall that $\sigma_{p,Q}$ preserves a general line L passing through p and restricts on it to the unique involution that fixes the two points $(L \cap \Gamma)\backslash\{p\}$.

The base-points of $\sigma_{p,Q}$ are described by the following lemma:

Lemma 3.2. [Bla08, Proposition 12] *Let $C \subset \mathbb{P}^2$ be a smooth cubic curve, let $p \in C$ and let $\sigma_{p,Q}$ be the element defined in Definition* 1.1. *The following occur:*

1. *The degree of $\sigma_{p,Q}$ is* 3, *and ${\sigma_{p,Q}}^2 = 1$, i.e. $\sigma_{p,Q}$ is a cubic involution.*
2. *The base-points of $\sigma_{p,Q}$ are the point p – which has multiplicity* 2 *– and the four points p_1, p_2, p_3, p_4 such that the line passing through p and p_i is tangent at p_i to C.*
3. *If p is not an inflexion point of C, all the points $p_1, ..., p_4$ belong to $\mathbb{P}^2$. Otherwise, only three of them belong to $\mathbb{P}^2$, and the fourth is the point in the blow-up of p that corresponds to the tangent of C at p.* □

Since $\sigma_{p,Q}$ is an involution, the blow-up $\pi\colon X \to \mathbb{P}^2$ of its five base-points conjugates it to an automorphism of X. We now describe the action of this automorphism on the Picard group of X.

Lemma 3.3. *Let $C \subset \mathbb{P}^2$ be a smooth cubic curve, let $p \in C$ and let $\sigma_{p,Q}$ be the element defined in Definition* 1.1. *Let p_1, p_2, p_3, p_4 be the base-points of $\sigma_{p,Q}$ of multiplicity one* (*see Proposition* 3.2)*, and let $\pi\colon X \to \mathbb{P}^2$ be the blow-up of the five base-points.*

Denote by $L \subset \mathrm{Pic}(X)$ the pull-back of a general line of $\mathbb{P}^2$, by E_i the divisor corresponding to the point p_i, and by E the divisor corresponding to p, which is the total pull-back on X of the curve contracted on p (*if p is an inflexion point, E corresponds to a reducible curve*). *The set $(L, E, E_1, \dots, E_4)$ is an orthogonal basis of* $\mathrm{Pic}(X)$*; the element have self-intersection* $(1, -1, -1, -1, -1)$ *and the action of* $\hat{\sigma} = \pi^{-1}\sigma_{p,Q}\pi \in \mathrm{Aut}(X)$ *on the Picard group is*

$$\begin{bmatrix} 3 & 2 & 1 & 1 & 1 & 1 \\ -2 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 & -1 & 0 \\ -1 & -1 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Proof. Only the action of $\hat{\sigma}$ is not clear. By Lemma 3.2, the map $\sigma_{p,Q}$ is a cubic involution, and its base-points are p with multiplicity 2, and $p_1, \dots, p_4$ with multiplicity 1. This implies that $\hat{\sigma}(L) = 3L - 2E - E_1 - E_2 - E_3 - E_4$. Because $\sigma_{p,Q}$ preserve the pencil of lines passing through p, we have $\hat{\sigma}(L - E) = L - E$. The lift of this pencil on X gives a conic bundle $X \to \mathbb{P}^1$, with four singular fibres, each one being the union of E_i and $L - E - E_i$ for $i = 1, \dots, 4$. This implies that the set $\{E_i, L - E - E_i\}$ is invariant for $i = 1, \dots, 4$. Computing the intersection with L and $\hat{\sigma}(L)$ shows that $\hat{\sigma}(E_i) = L - E - E_i$, for $i = 1, \dots, 4$. This achieves the proof. □

Proposition 3.4. *Let $C \subset \mathbb{P}^2$ be a smooth cubic curve, let $p \in C$ and let $\sigma_{p,Q}$ be the element defined in Definition* 1.1. *There exists an automorphism τ of $\mathbb{P}^2$, acting via a translation of order* 3 *on C, such that $\tau\sigma_{p,Q}$ is conjugate to an automorphism of a smooth projective rational surface Y, with dynamical degree* > 1.

Proof. Denote as above by $p_1, \dots, p_4$ the base-points of $\sigma_{p,Q}$ of multiplicity 1. Recall (Lemma 3.2) that p_1, p_2, p_3, p_4 are the points of C such that the tangent of C at p_i passes through p; if p is an inflexion point, one of the points is the point infinitely near to p_1 corresponding to the tangent direction of C.

We change coordinates on $\mathbb{P}^2$ and put the curve C into its Hessian form, which is the equation

$$x^3 + y^3 + z^3 + \lambda xyz = 0$$

for some $\lambda \in \mathbb{C}$ satisfying $\lambda^3 \neq -27$. Let $H \subset \mathbb{P}^2$ be the group generated by

$$(x : y : z) \mapsto (y : z : x),$$

$$(x : y : z) \mapsto (x : \omega y : \omega^2 z),$$

where ω is a 3-rd rood of unity. One directly sees that H is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^2$ and preserves C. Moreover, the action of any of the non-trivial elements of H on C is fixed-point-free. We obtain thus an isomorphism of H with the 3-torsion of the group of translations $C \subset \mathrm{Aut}(C)$.

Let us denote by $\pi\colon Y \to \mathbb{P}^2$ the blow-up of the orbit of $\{p, p_1, \dots, p_4\}$ by τ. If one of the p_i is infinitely near to p, then its orbit consists of points infinitely near to the orbit of p. As before, we denote by $L \subset \mathrm{Pic}(Y)$ the pull-back of a general line of $\mathbb{P}^2$, by E_i the divisor corresponding to the point p_i, and by E the divisor corresponding to p, which is the total pull-back on X of the curve contracted on p. The automorphism τ lifts to an automorphism $\hat{\tau} = \pi^{-1}\tau\pi \in \mathrm{Aut}(Y)$, which sends E_i onto the divisor corresponding to $\tau(p_i)$, and sends E onto the divisor corresponding to $\tau(p)$. The group $\mathrm{Pic}(X)$ is generated by L and by $\{\hat{\tau}^i(E), \hat{\tau}^i(E_1), \dots, \hat{\tau}^i(E_4)\}_{i=0}^2$. The birational involution $\sigma_{p,Q} \in \mathrm{Bir}(\mathbb{P}^2)$ lifts to an automorphism of the surface obtained by blowing-up $p, p_1, \dots, p_4$; because this one fixes each of the other points blown-up (which belong to the curve C), it lifts to an automorphism $\hat{\sigma}$ of Y. This shows that $\tau\sigma_{p,Q}$ is conjugate by π^{-1} to the automorphism $\hat{\tau}\hat{\sigma}$ of Y, and it suffices to show that its dynamical degree is > 1 to conclude. This amounts to find a real eigenvalue of the action of $\hat{\tau}\hat{\sigma}$ on $\mathrm{Pic}(Y) \otimes_{\mathbb{Z}} \mathbb{R}$ which is bigger than 1. The action of $\hat{\sigma}$ on $\mathrm{Pic}(Y)$ is given by Lemma 3.3, and the action of $\hat{\tau}$ fixes L and permutes the exceptional divisors according to the action of τ on the points.

Because τ does not fix any point of C, the divisors $E, \hat{\tau}(E), \hat{\tau}^2(E)$ are distinct. This is also true for the divisors $E_i, \hat{\tau}(E_i), \hat{\tau}^2(E_i)$ for $i = 1, \dots, 4$. Note that $\tau(p_i) = p_j$ is also impossible, because it would imply that τ sends the line tangent to C at p_i onto the line tangent to C at p_j, and thus τ would fix p. It remains to study two possible cases:

1) **There exists an $i \in \{1, \dots, 4\}$ such that $\tau(p) = p_i$ or $\tau^2(p) = p_i$.** Replacing τ by τ^2 and renumbering the p_i if needed, we can assume that $\tau(p) = p_1$. We see that $\tau(p_1) = \tau^2(p)$ is distinct from $\tau^i(p_j)$ for $j \geq 2$. The sub-$\mathbb{Z}$-module of $\mathrm{Pic}(Y)$ generated by

$$L, E, \hat{\tau}(E) = E_1, \hat{\tau}(E_1), \sum_{i=2}^{4} E_i, \sum_{i=2}^{4} \hat{\tau}(E_i), \sum_{i=2}^{4} \hat{\tau}^2(E_i)$$

is invariant, and the action of $\hat{\tau}$ and $\hat{\sigma}$, relatively to this basis, are given by

$$\begin{bmatrix} 3 & 2 & 1 & 0 & 3 & 0 & 0 \\ -2 & -1 & -1 & 0 & -3 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

The action of $\hat{\tau}\hat{\sigma}$ is thus the product of the two matrices, which is

$$\begin{bmatrix} 3 & 2 & 1 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -2 & -1 & -1 & 0 & -3 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

The characteristic polynomial is $x^7-2x^6+2x-1=(x-1)(x^6-x^5-x^4-x^3-x^2+1)$, whose real eigenvalues are $\lambda, 1, \lambda^{-1}$, where $\lambda \sim 1.946856$. This number is the dynamical degree of $\hat{\tau}\hat{\sigma}$ (and also of $\tau\sigma_{p,Q}$).

2) **For** $i = 1, \ldots, 4$, $p_i \notin \{\tau(p), \tau^2(p)\}$**.**

In this case, the sub-$\mathbb{Z}$-module of $\mathrm{Pic}(Y)$ generated by

$$L, 2E + \sum_{i=1}^{4} E_i, 2\hat{\tau}(E) + \sum_{i=1}^{4} \hat{\tau}(E_i), 2\hat{\tau}^2(E) + \sum_{i=1}^{4} \hat{\tau}^2(E_i)$$

is invariant, and the actions of $\hat{\sigma}$ and $\hat{\tau}$ are given by

$$\begin{bmatrix} 3 & 8 & 0 & 0 \\ -1 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The action of $\hat{\tau}\hat{\sigma}$ is thus the product of the two matrices, which is

$$\begin{bmatrix} 3 & 8 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The characteristic polynomial is $x^4 - 3x^3 + 3x - 1 = (x-1)(x+1)(x^2 - 3x + 1)$, whose real eigenvalues are $-1, 1, \frac{3\pm\sqrt{5}}{2}$. The dynamical degree is then $\frac{3+\sqrt{5}}{2} \sim 2.618034$. □

Corollary 3.5. *Let $\varphi \in \mathrm{Bir}(\mathbb{P}^2)$ be a birational map of degree 3.*

1. *If all base-points of φ and φ^{-1} are proper points of the plane, there exists an automorphism $\tau \in \mathrm{Aut}(\mathbb{P}^2)$ such that $\tau\varphi$ is conjugate to an automorphism of a smooth projective rational surface with dynamical degree > 1.*
2. *In the algebraic set of birational maps of $\mathbb{P}^2$ of degree 3, the set of maps having this property is a dense subset with complement of codimension 1.*

Proof. (1) We can replace φ with $\alpha\varphi\beta$, where $\alpha, \beta \in \mathrm{Aut}(\mathbb{P}^2)$. In particular, we can assume that the base-point of multiplicity 2 of both φ and φ^{-1} is $p = (1:0:0)$ and that a general line passing through this point is invariant by α. The other base-points of φ are p_2, p_3, p_4, p_5. Note that no 2 of the p_i are collinear with p, because otherwise the linear system of φ (being cubics of degree 3, with multiplicity 2 at p and multiplicity 1 at $p_1, \dots, p_5$) would be reducible. If three of the p_i are collinear, the line passing through these points would have self-intersection -2 on the blow-up of $p, p_1, \dots, p_5$, so the map φ^{-1} would have a base-point being infinitely near (to p). This implies that no 3 of the points $p, p_1, \dots, p_5$ are collinear. Choosing the good automorphism β, we can then assume that

$$p = (1:0:0),\ p_1 = (0:1:0),\ p_2 = (0:0:1),\ p_3 = (1:1:1),\ p_4 = (a:b:c),$$

for some $a, b, c \in \mathbb{C}^*$, no two being equal. We consider the birational cubic involution

$$\begin{aligned}\sigma\colon (x:y:z) \dashrightarrow \big(-ayz(\ &(c-b)x \ + \ (a-c)y \ + \ (b-a)z\ ) :\\ y(\ &a(c-b)yz + b(a-c)xz + c(b-a)xy\ ) :\\ z(\ &a(c-b)yz + b(a-c)xz + c(b-a)xy\ )\ \big),\end{aligned}$$

and observe that its base-points are $p_1, \dots, p_4$, and p with multiplicity 2. It preserves a general line passing through $p = (1:0:0)$. Moreover it fixes pointwise the smooth cubic curve $C \subset \mathbb{P}^2$ of equation

$$b(a-c)x^2z + c(b-a)x^2y + a(a-c)y^2z + a(b-a)yz^2 + 2a(c-b)xyz = 0.$$

In particular, the map σ is equal to the involution $\sigma_{p,Q}$ associated to $p \in C$, according to Definition 1.1. Because σ and φ have the same linear system (same degree, same base-points with same multiplicities), then φ is equal to $\sigma\gamma$ for some $\gamma \in \mathrm{Aut}(\mathbb{P}^2)$. Assertion (1) follows then from Proposition 3.4. Note that the existence of σ (which is uniquely determined by $p, p_1, \dots, p_4$) can also be seen more geometrically, by looking at the automorphism group of the del Pezzo surface of degree 4 obtained by blowing-up $p, p_1, \dots, p_4$, (see [Bla09b, Lemma 9.11]).

It remains to prove Assertion (2). Any cubic birational map φ of $\mathbb{P}^2$ has one base-point of multiplicity 2 and four base-points of multiplicity 1. And two maps with the same base-points differ only by the post-composition with an automorphism. The set of cubic birational maps is then parametrised by one point of $\mathbb{P}^2$, a set of four other points, that are on $\mathbb{P}^2$ or infinitely near, and one automorphism of $\mathbb{P}^2$. The biggest dimension is when all points are on $\mathbb{P}^2$ and no 3 are collinear, which is exactly the set where the map and its inverse have only proper base-points. It has the dimension of $(\mathbb{P}^2)^5 \times \mathrm{PGL}(3, \mathbb{C})$, which is 18. The set of all other maps has only dimension 17; it corresponds to the cases where 3 points are collinear or one point is infinitely near. □

Remark 3.6. *i*) *By Proposition* 3.4*, the same result holds for map projectively equivalent to* $\sigma_{p,Q}$ *where* p *is an inflexion point of a smooth cubic* Q*. These are the maps having* 4 *proper base-points* p, p_1, p_2, p_3 *of multiplicity* $2, 1, 1, 1$ *such that* p_1, p_2, p_3 *are collinear, and a point* p_5 *infinitely near to* p*.*

ii) *It also holds for other special cubics maps: some with two proper base-points* (*see* [BedDil06])*, with one proper base-point* (*see* [BedKim10])*, or some maps of degree* 3 *with exactly two proper base-points* (*see* [BedDil06] *and* [DésGri10])*.*

iii) If a birational map $\varphi \in \mathrm{Bir}(\mathbb{P}^2)$ of degree 3 has all its base-points which are proper but 3 are collinear, then it is projectively equivalent to

$$\psi_\lambda \colon (x : y : z) \dashrightarrow (yz(y - z + (\lambda - 1)x) : xy(z - \lambda y) : xz(z - \lambda y))$$

for some $\lambda \in \mathbb{C} \setminus \{0, 1\}$. In particular, φ^{-1} has only 4 proper base-points.

Question 3.7. *Does there exists a $\lambda \in \mathbb{C} \setminus \{0, 1\}$ such that for any $\tau \in \mathrm{Aut}(\mathbb{P}^2)$ the map $\tau\psi_\lambda$ is not conjugate to an automorphism of a projective surface (with dynamical degree > 1)?*

4. The example

4.1. **Actions on infinitely near points.** Before proving Theorem 2, we need some general tools.

Let X, Y be two smooth projective rational surfaces, and let $\psi \colon X \dashrightarrow Y$ be a birational map. If p is a point of X or a point infinitely near, which is not a base-point of ψ, we define a point $\psi^\bullet(p)$, which will also be a point of Y or a point infinitely near. For this, take a minimal resolution

$$\begin{array}{ccc} & Z & \\ {\scriptstyle \pi_1} \swarrow & & \searrow {\scriptstyle \pi_2} \\ X & \underset{\psi}{\dashrightarrow} & Y, \end{array}$$

where π_1, π_2 are sequences of blow-ups. Because p is not a base-point of ψ, it corresponds, via π_1, to a point of Z or infinitely near. Using π_2, we view this point on Y, again maybe infinitely near, and call it $\psi^\bullet(p)$. Observe that this point is not a base-point of ψ^{-1}, and that $(\psi^{-1})^\bullet(\psi^\bullet(p)) = p$.

Remark 4.1. *If p is not a base-point of $\phi \in \mathrm{Bir}(X)$ and $\phi(p)$ is not a base-point of $\psi \in \mathrm{Bir}(X)$, we have $(\psi\phi)^\bullet(p) = \psi^\bullet(\phi^\bullet(p))$. If p is a general point of X, then $\phi^\bullet(p) = \phi(p) \in X$.*

Example 4.2. *If $p = (1 : 0 : 0) \in \mathbb{P}^2$ and $\psi \in \mathrm{Bir}(\mathbb{P}^2)$ is the map $(x : y : z) \dashrightarrow (yz + x^2 : xz : z^2)$, the point $\psi^\bullet(p)$ is not equal to $p = \psi(p)$, but is infinitely near to it.*

The following easy lemma will be important for the proof of Theorem 2.

Lemma 4.3. *Let $\varphi \in \mathrm{Bir}(\mathbb{P}^2)$ be a birational map and let p be a point of $\mathbb{P}^2$ (or infinitely near). If there exists an integer $N \geq 0$ such that p is a base-point of φ^{-k} for any $k \geq N$ but is not a base-point of φ^k for any $k \geq N$, then φ is not conjugate to an automorphism of a smooth projective surface.*

Proof. We prove first that $(\varphi^k)^\bullet(p)$ and $(\varphi^l)^\bullet(p)$ are two distinct points of $\mathbb{P}^2$ (or infinitely near), for any $k, l \geq N$ with $k \neq l$. Otherwise, assuming that $l > k$, the equality $(\varphi^k)^\bullet(p) = (\varphi^l)^\bullet(p)$ implies that φ^{-l} is defined at $(\varphi^k)^\bullet(p)$ (because it is defined at $(\varphi^l)^\bullet(a)$), and that $(\varphi^{-l})^\bullet((\varphi^k)^\bullet(p)) = p$. In particular, φ^{k-l} is defined at p, and $(\varphi^{k-l})^\bullet(p) = p$, which means that $(\varphi^{(k-l)m})^\bullet(p) = p$ for any $m \geq 0$. This is incompatible with the fact that p is a base-point of φ^{-i} for any $i \geq N$.

The set $\{(\varphi^k)^\bullet(p)\}_{k=N}^\infty$ is thus an infinite set of points that belong to $\mathbb{P}^2$, as proper or infinitely near points. Suppose now that there exists a birational map $\alpha \colon \mathbb{P}^2 \dashrightarrow S$, where S is a smooth projective surface, that conjugates φ to an automorphism $\hat{\varphi} = \alpha\varphi\alpha^{-1}$ of S. The map α having only a finite number of base-points, there exists $M \geq N$ such that no one of the points $\{(\varphi^k)^\bullet(p)\}_{k=M}^{\infty}$ is a base-point of α. Writing $p_k = \alpha^\bullet((\varphi^k)^\bullet(p))$ for any $k \geq M$, we obtain a family of distinct points $\{p_k\}_{k=M}^{\infty}$ such that $\hat{\varphi}(p_k) = p_{k+1}$ for each $k \geq M$. Writing $p_{k-m} = \hat{\varphi}^{-m}(p_k)$ for any $m \geq 0$, we obtain an orbit $\{p_k\}_{k\in\mathbb{Z}}$ of the automorphism $\hat{\varphi}$, so that $p_k \neq p_l$ for $k \neq l$. Increasing maybe M, we can assume that p_k is not a base-point of α^{-1} for any $k \geq M$ and any $k \leq -M$. This implies that $(\varphi^M)^\bullet(p)$ is not a base-point of the map $\varphi^{-2M} = \alpha^{-1}\hat{\varphi}^{2M}\alpha$; indeed, $\alpha^\bullet((\varphi^M)^\bullet(p)) = p_m$, and $(\hat{\varphi}^{-2m})^\bullet(p_m) = p_{-m}$, which is not a base-point of α^{-1}.

We obtain a contradiction with the fact that p is a base-point of φ^M but not of φ^{-M}. □

The section is devoted to the proof of Theorem 2. We will always denote by $\chi \in \mathrm{Bir}(\mathbb{P}^2)$ the birational map

$$\chi\colon (x:y:z) \dashrightarrow (xz^5 + (yz^2+x^3)^2 : yz^5 + x^3z^3 : z^6)$$

which restricts on the affine plane where $z = 1$ to the automorphism

$$(x, y) \mapsto (x + (y + x^3)^2, y + x^3),$$

being the composition of $(x, y) \mapsto (x + y^2, y)$ with $(x, y) \mapsto (x, y + x^3)$.

4.2. **Basic description of the map χ.** The proof of Theorem 2 will rely on the study of the base-points of χ, and of its inverse, which is

$$\chi^{-1}\colon (x:y:z) \mapsto (xz^5 - y^2z^4 : yz^5 - (xz-y^2)^3 : z^6).$$

It will use two main properties: both χ and χ^{-1} have only one proper base-point, but the geometry of the base-points of the two maps are different (see below for more details). Note that many other examples can be constructed in the same way, the map χ is only the simplest one having the above properties.

4.3. **Base-points of χ.** As all birational maps of $\mathbb{P}^2$ which contract only one curve, χ has only one proper base-point, namely $p_1 = (0:1:0)$, and all its base-points are "in tower" (see [Bla09, Lemma 2.2]). This means that the 8 base-points of χ, that we denote by $p_1, \ldots, p_8$, are such that p_i is infinitely near to p_{i-1} for $i = 2, \ldots, 8$. We denote by $\pi\colon X \to \mathbb{P}^2$ the blow-up of the 8 base-points, and write $\mathcal{C} \subset X$ the strict transform of the line $C \subset \mathbb{P}^2$ of equation $z = 0$, which is the only curve of $\mathbb{P}^2$ contracted by χ. We denote by $\mathcal{E}_i \subset X$ the strict transform of the curve obtained by blowing-up p_i. The intersection form on X corresponds to the dual diagram of Figure 1 (this can be checked directly in local charts or by the decomposition of χ into two simple maps as above).

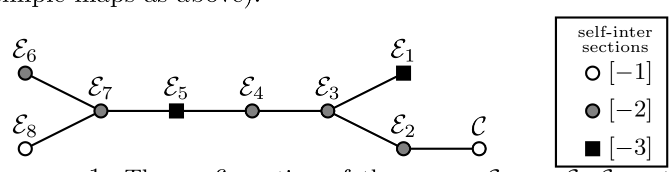


FIGURE 1. The configuration of the curves $\mathcal{E}_1, \ldots, \mathcal{E}_8, \mathcal{C}$ on the surface X. Two curves are connected by an edge if their intersection is positive (and here equal to 1). The self intersections correspond to the shape of the vertices.

Let us write $\varphi = \tau\chi$, where τ is an automorphism of $\mathbb{P}^2$. Because π is the blow-up of the base-points of χ, which are also the base-points of φ, the map $\eta = \varphi\pi$ is a birational morphism $X \to \mathbb{P}^2$, which is the blow-up of the base-points of φ^{-1}. In fact, Figure 2 is the minimal resolution of φ.

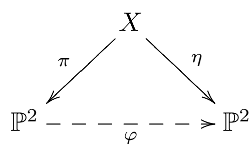


FIGURE 2. The minimal resolution of indeterminacies of φ.

The line $C \subset \mathbb{P}^2$ being the only curve of $\mathbb{P}^2$ contracted by φ, the morphism η contracts $\mathcal{C}$, and the union of 7 other irreducible curves, which are among the curves $\mathcal{E}_1, \ldots, \mathcal{E}_8$. The configuration of Figure 1 shows that η contracts the curves

$$\mathcal{C}, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4, \mathcal{E}_1, \mathcal{E}_5, \mathcal{E}_7, \mathcal{E}_6$$

following this order.

We study now the blow-up $\eta \colon X \to \mathbb{P}^2$ in the same way as we did for π. We denote by $q_1, \ldots, q_8$ the base-points of φ^{-1} (or equivalently the points blown-up by η), so that $q_1 = (1 : 0 : 0)$, and q_i is infinitely near to q_{i-1} for $i = 2, \ldots, 8$. We denote by $D \subset \mathbb{P}^2$ the line which is contracted by φ^{-1} (which is the image by τ of the line of equation $z = 0$), and write $\mathcal{D} \subset X$ the strict transform by η^{-1} of the curve D. We then denote by $\mathcal{F}_i$ the strict transform of the curve obtained by blowing-up q_i. Because of the order of the curves contracted by η, we get equalities between $\mathcal{C}, \mathcal{E}_1, \ldots, \mathcal{E}_8$ and $\mathcal{D}, \mathcal{F}_1, \ldots, \mathcal{F}_8$, according to Figure 3.

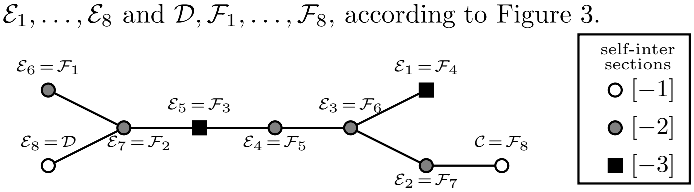


FIGURE 3. The configuration of the curves $\mathcal{E}_i, \mathcal{F}_i, \mathcal{C}, \mathcal{D}$ on the surface X.

In particular, we see that the configuration of the points $p_1, \ldots, p_8$ is not the same as the one of the points $q_1, \ldots, q_8$. Saying that a point a is *proximate to* a point b if a is infinitely near to b and belongs to the strict transform of the curve obtained by blowing-up b, the configuration of the points p_i and of the points q_i are given in Figure 4.

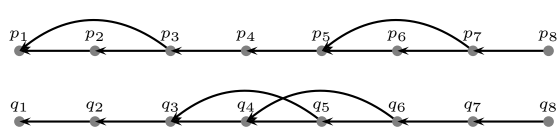


FIGURE 4. The configuration of the points $p_1, \ldots, p_8$ and of the points $q_1, \ldots, q_8$. An arrow corresponds to the relation "is proximate to".

4.4. **The proof of the theorem.**

proof of Theorem 2. We write as above $\varphi = \tau\chi$, where τ is an automorphism of $\mathbb{P}^2$, and recall that $p_1, \dots, p_8$ are the base-points of φ, and $q_1, \dots, q_8$ are the base-points of φ^{-1}. Our aim is to show that p_3 is a base-point of φ^i and not of φ^{-i}, for any $i > 0$. This will imply that φ is not conjugate to an automorphism of a smooth projective rational surface, by Lemma 4.3.

Denote by k the lowest positive integer such that p_1 is a base-points of φ^{-k}; if no such integer exists, we write $k = \infty$.

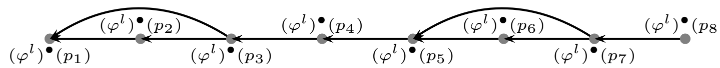
For any integer i such that $1 \leq i < k$, the point p_1 is not a base-point of φ^{-i}, hence the maps φ and φ^{-i} have no common base-point. This implies that the set of base-points of the map $\varphi^{i+1} = \varphi \circ \varphi^i$ is the union of the base-points of φ^i and of the points $(\varphi^{-i})^\bullet(p_j)$ for $j = 1, \dots, 8$. The map φ^{-i} being defined at p_1, the point $(\varphi^{-i})^\bullet(p_j)$ is proximate to the point $(\varphi^{-i})^\bullet(p_k)$ if and only if p_j is proximate to p_k.

Proceeding by induction on i, we obtain the following assertions:

(1) For any integer i with $1 \leq i \leq k$, the set of base-points of φ^i is equal to

$$\{(\varphi^{-m})^\bullet(p_j) \mid j = 1, \dots, 8, m = 0, \dots, i-1\}.$$

(2) For any integer l with $0 \leq -l < k$, the configuration of the points $\{(\varphi^l)^\bullet(p_j)\}_{j=1}^8$ is given by



In particular, p_3 is a base-point of φ^i for any i satisfying $1 \leq i \leq k$.

If $k = \infty$, this implies that p_3 is a base-point of φ^i for any $i > 0$, and by definition of k, the point p_1 is not a base-point of φ^{-i} for any $i > 0$, so neither is p_3. We can thus assume that k is a positive number.

Observe that q_1 is not a base-point of φ^i for $1 \leq i \leq k-1$. Indeed, otherwise q_1 would be equal to $(\varphi^{-m})^\bullet(p_j)$ for some m, j satisfying $0 \leq m \leq k-2$ and $1 \leq j \leq 8$. This would imply that p_j is a base-point of φ^{m+1}, which is impossible. We see thus that φ^{-1} has no common base-point with φ^i for $1 \leq i \leq k-1$. In particular, the set of common base-points of φ^{-1} and φ^k is equal to

$$B = \{(\varphi^{-(k-1)})^\bullet(p_j)\}_{j=1}^8 \cap \{q_j\}_{j=1}^8.$$

Because p_1 is a base-point of φ^{-k} and not of $\varphi^{-(k-1)}$, the point $(\varphi^{-(k-1)})^\bullet(p_1)$, which is a base-point of φ^k, is also a base-point of φ^{-1}. In particular, the set B is not empty. The configuration of the two sets of points $\{(\varphi^{-(k-1)})^\bullet(p_j)\}_{j=1}^8$ and $\{q_j\}_{j=1}^8$ implies that $q_1 = (\varphi^{-(k-1)})^\bullet(p_1)$. Moreover, either $B = \{q_1\}$ or $B = \{q_1, q_2\}$. Indeed, the point q_3 is proximate to q_1 but not to q_2, whereas $(\varphi^{-(k-1)})^\bullet(p_3)$ is proximate to $(\varphi^{-(k-1)})^\bullet(p_1)$ and $(\varphi^{-(k-1)})^\bullet(p_2)$.

The point $(\varphi^{-(k-1)})^\bullet(p_3)$ is therefore a point infinitely near to q_1, which corresponds, on the blow-up $\eta\colon X \to \mathbb{P}^2$, to a point that belong, as proper or infinitely near point, to one of the curves $\mathcal{F}_1$ or $\mathcal{F}_2$, equal respectively to $\mathcal{F}_7$, $\mathcal{F}_6$ (see Figure 3). Applying φ^{-1} to it corresponds to apply $\pi\eta^{-1}$, so $(\varphi^{-k})^\bullet(p_3)$ is a point that is infinitely near to p_6, and thus to p_3. Because p_3 is not a base-point of φ^{-i} for $1 \leq i \leq k$, the point p_3 is not a base-point of $\varphi^{-(k+i)}$ and $(\varphi^{-(k+i)})^\bullet(p_3)$ is infinitely near to $(\varphi^{-i})^\bullet(p_3)$. In particular, $(\varphi^{-2k})^\bullet(p_3)$ is infinitely near to $(\varphi^{-k})^\bullet(p_3)$, which is infinitely near to p_3. Continuing like this, we get the following assertion:

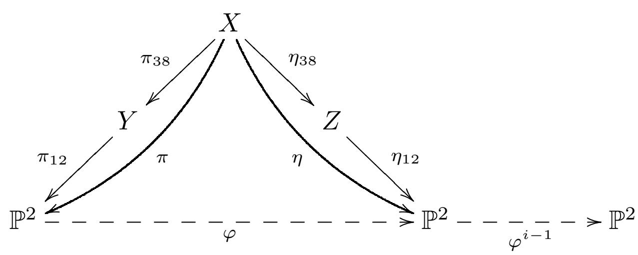
(2) For any $i \geq 1$, the point p_3 is not a base-point of φ^{-i}.

It remains to show that p_3 is a base-point of φ^i for each $i \geq 1$ to get the result. To do this, we will need the following assertion:

(3) For any $i \geq 1$, the point q_3 is not a base-point of φ^i.

Note that (3) could be proved in the same way as (2), reversing the order of φ and φ^{-1}. We quickly recall the way to deduce it. Note that q_3 is not a base-point of φ^i for $1 \leq i \leq k-1$, because q_1 is not a base-point of φ^i (see above). Since q_3 does not belong to B, which is the set of common base-points of φ^{-1} and φ^k, the point q_3 is not a base-point of φ^k. The point q_3 is infinitely near to $q_1 = (\varphi^{-(k-1)})^\bullet(p_1)$, in the second neighbourhood. The point $(\varphi^{k-1})^\bullet(q_3)$ is thus infinitely near to p_1. On the blow-up $\pi\colon X \to \mathbb{P}^2$, the point $(\varphi^{k-1})^\bullet(q_3)$ corresponds thus to a point that belongs, as a proper or infinitely point, to $\mathcal{E}_1$ or $\mathcal{E}_2$, equal respectively to $\mathcal{F}_4$ and $\mathcal{F}_7$. The point $(\varphi^k)^\bullet(q_3)$ is then a point infinitely near to q_4, and then to q_3. As before, the fact that q_3 is not a base-point of φ^i for $i = 1, \ldots, k$ and that $(\varphi^k)^\bullet(q_3)$ is infinitely near to q_3 implies that q_3 is not a base-point of φ^i for any $i \geq 0$, proving Assertion (3).

It remains to see that Assertion (3) implies that p_3 is a base-point of φ^i for any $i \geq 1$. For $i = 1$, this is obvious. For $i > 1$, we decompose φ^i into $\varphi^{i-1} \circ \varphi$, wedecompose $\pi\colon X \to \mathbb{P}^2$ into $\pi = \pi_{12} \circ \pi_{38}$, where $\pi_{12}\colon Y \to \mathbb{P}^2$ is the blow-up of p_1, p_2 and $\pi_{38}\colon X \to Y$ is the blow-up of $p_3, \ldots, p_8$, and do the same with η. This yields the following commutative diagram



Note that η_{38} contracts $\mathcal{F}_8, \ldots, \mathcal{F}_3$ onto the point $q_3 \in X_2$, which is not a base-point of $\varphi^{i-1} \circ \eta_{12}$. Let us take the system of conics of $\mathbb{P}^2$ passing through p_1, p_2, p_3 and denote by Λ the lift of this system on Y, which is a system of smooth curves passing through q_3 with movable tangents, having dimension 2. The strict transform on X of Λ is a system of curves intersecting $\mathcal{E}_3$ at a general movable point. The map η_{38} contracts the curves $\mathcal{C}, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4, \mathcal{E}_1, \mathcal{E}_5$. The curve $\mathcal{E}_3$ being contracted and being not the last one, the image of the system by η_{38} passes through q_3 with a fixed tangent (corresponding to the point q_4). The point q_3 being not a base-point of $\varphi^{i-1} \circ \eta_{12}$, the image of the system $\Lambda \subset Y$ by $\varphi^{i-1} \circ \eta \circ (\pi_{38})^{-1}$ has a fixed tangent at the point $(\varphi^{i-1} \circ \eta_{12})(q_3)$. This shows that p_3 is a base-point of $\varphi^{i-1} \circ \eta \circ (\pi_{38})^{-1}$, and thus of φ^i. □

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