DEGENERATION OF DYNAMICAL DEGREES IN FAMILIES OF MAPS

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Abstract. The *dynamical degree* of a dominant rational map $f : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ is the quantity $\delta(f) := \lim(\deg f^n)^{1/n}$. We study the variation of dynamical degrees in 1-parameter families of maps f_T. We make three conjectures concerning, respectively, the set of t such that: (1) $\delta(f_t) \le \delta(f_T) - \epsilon$; (2) $\delta(f_t) < \delta(f_T)$; (3) $\delta(f_t) < \delta(f_T)$ and $\delta(g_t) < \delta(g_T)$ for "independent" families of maps. We give a sufficient condition for our first conjecture to hold, prove that the condition is true for monomial maps, and provide evidence for our second and third conjectures by proving them for certain non-trivial families.

1. Introduction

Let $f : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ be a dominant rational map. A fundamental invariant attached to f is its (*first*) *dynamical degree*, which is the quantity

$$\delta(f) = \lim_{n\to\infty}\Bigl(\deg(f^n)\Bigr)^{1/n}.$$

(Using the fact that $\deg(f^{n+m}) \le \deg(f^n)\deg(f^m)$, the existence of the limit is an easy convexity argument; cf. [2, Proposition 9.6.4].) We recall that f is said to be *algebraically stable* if $\delta(f) = \deg(f)$, which in turn is equivalent to $\deg(f^n) = \delta(f)^n = (\deg f)^n$ for all $n \ge 1$.

In this paper we study the variation of dynamical degrees as f moves in a family. We consider a smooth irreducible quasi-projective curve $T/\mathbb{C}$ and a family

$$f_T : \mathbb{P}^N_T \dashrightarrow \mathbb{P}^N_T$$

of dominant rational maps, i.e., for every $t \in T(\mathbb{C})$, the specialization f_t is a dominant rational map. We start with three conjectures, followed by some brief remarks, and then we provide some evidence for our

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conjectures by proving that they are true for certain specific non-trivial families of maps.

Conjecture 1. *For all $\epsilon > 0$, the set*

$$\bigl\{t \in T(\mathbb{C}) : \delta(f_t) \le \delta(f_T) - \epsilon\bigr\}$$

is finite.

Conjecture 2. *If T and f_T are defined over $\bar{\mathbb{Q}}$, then the* exceptional set

$$\mathcal{E}(f_T) := \bigl\{t \in T(\bar{\mathbb{Q}}) : \delta(f_t) < \delta(f_T)\bigr\}$$

is a set of bounded height.

Conjecture 3. *Let $g_T : \mathbb{P}^N_T \dashrightarrow \mathbb{P}^N_T$ be another family of dominant rational maps, and let $\mathcal{E}(f_T)$ and $\mathcal{E}(g_T)$ be exceptional sets as defined in Conjecture 2. Then*[1]

$$\mathcal{E}(f_T) \cap \mathcal{E}(g_T) \text{ is infinite} \quad\Longrightarrow\quad \mathcal{E}(f_T) \mathbin{\triangle} \mathcal{E}(g_T) \text{ is finite.}$$

Conjecture 1 is inspired by Xie [7, Theorem 4.1], a special case of which implies that Conjecture 1 is true for families of birational maps of $\mathbb{P}^2$.[2] Our primary goal in this paper is to provide justification for Conjectures 2 and 3 by analyzing in depth an interesting three-parameter family of rational maps and showing that the conjectures are true for one-parameter subfamilies. The maps $f_{a,b,c} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ that we study are defined by

$$f_{a,b,c}\bigl([X,Y,Z]\bigr) = [XY, XY + aZ^2, bYZ + cZ^2]. \tag{1}$$

For $abc \ne 0$, we first show that $\delta(f_{a,b,c}) < 2$ if and only if there is a root of unity ξ with the property that $c^2 = (\xi + \xi^{-1})^2 ab$; cf. Theorem 10. Taking a, b, c to be polynomials in one variable, we use this criterion to prove Conjectures 2 and 3 for 1-parameter subfamilies of the family (1).

Theorem 4. *Let $f_{a,b,c} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the map (1).*

(a) (Corollary 11): *Let $a(T), b(T), c(T) \in \bar{\mathbb{Q}}[T]$ be non-zero polynomials satisfying $\delta(f_{a(T),b(T),c(T)}) = 2$. Then the exceptional set*

$$\mathcal{E}(f_{a(T),b(T),c(T)}) = \bigl\{t \in \bar{\mathbb{Q}} : \delta(f_{a(t),b(t),c(t)}) < 2\bigr\}$$

is a set of bounded height.

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[1]We recall that the *symmetric set difference* of two sets A and B is the set $A \mathbin{\triangle} B := (A \cup B) \smallsetminus (A \cap B)$, or alternatively $A \mathbin{\triangle} B := (A \smallsetminus B) \cup (B \smallsetminus A)$.

[2]In a private communication, Xie has indicated that the methods used in [6] should suffice to prove Conjecture 1 more generally for rational maps that are extensions to $\mathbb{P}^2$ of dominant polynomial endomorphisms $\mathbb{A}^2 \to \mathbb{A}^2$.

(b) (Theorem 13): *Let* $a_1(T), b_1(T), c_1(T), a_2(T), b_2(T), c_2(T) \in \bar{\mathbb{Q}}[T]$ *be non-zero polynomials such that*

$$\delta(f_{a_1(T),b_1(T),c_1(T)}) = 2 \quad \textit{and} \quad \delta(f_{a_2(T),b_2(T),c_2(T)}) = 2.$$

Then

$$\#\Big(\mathcal{E}(f_{a_1(T),b_1(T),c_1(T)}) \cap \mathcal{E}(f_{a_2(T),b_2(T),c_2(T)})\Big) = \infty$$
$$\implies \#\Big(\mathcal{E}(f_{a_1(T),b_1(T),c_1(T)}) \triangle \mathcal{E}(f_{a_2(T),b_2(T),c_2(T)})\Big) < \infty.$$

Remark 5. We observe that Conjectures 1 and 3 appear to be geometric, since they are stated over $\mathbb{C}$, while Conjecture 2 is clearly arithmetic in nature. This dichotomy is, however, somewhat misleading, since proofs of unlikely intersection statements such as Conjecture 3 invariably require a considerable amount of arithmetic. On the other hand, Conjecture 1 may well admit a geometric proof.

Remark 6. We note that Conjecture 3 should be only half the story. The other half would be a statement saying that if $\mathcal{E}(f_T) \cap \mathcal{E}(g_T)$ is infinite, then f_T and g_T are "geometrically dependent." We do not currently know how to formulate this precisely.

Remark 7. The conjectures, examples, and results in this paper were inspired by work of Xie [7]. In particular, he proves a beautiful theorem on the reduction modulo p of a birational map $f : \mathbb{P}^2_{\mathbb{Q}} \dashrightarrow \mathbb{P}^2_{\mathbb{Q}}$. In the context of "degeneration of dynamical degree in families," Xie's map f should be viewed as a family of maps over $T = \operatorname{Spec}\mathbb{Z}$, and the reduction $\tilde{f}_p : \mathbb{P}^2_{\mathbb{F}_p} \dashrightarrow \mathbb{P}^2_{\mathbb{F}_p}$ of f modulo p is the specialization of f to the fiber over p. Xie [7] proves that

$$\lim_{p\to\infty} \delta(\tilde{f}_p) = \delta(f).$$

One might suspect that in fact $\delta(\tilde{f}_p) = \delta(f)$ for all sufficiently large primes p, but Xie gives an intriguing example [7, Section 5] of a birational map

$$f : \mathbb{P}^2_{\mathbb{Q}} \longrightarrow \mathbb{P}^2_{\mathbb{Q}}, \quad f\big([X,Y,Z]\big) = [XY, XY - 2Z^2, YZ + 3Z^2]$$

having the property that there is a strict inequality $\delta(\tilde{f}_p) < \delta(f)$ for all primes p.

A fundamental inequality from [7] that Xie uses to study dynamical degrees in families says that there is an absolute constant $\gamma > 0$ such that

$$\delta(f) \ge \gamma \cdot \frac{\deg(f^2)}{\deg(f)} \quad \text{for all birational maps } f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2.$$

The crucial point here is that γ is independent of f, so for example, one can replace f by f^n without changing γ. We ask whether such estimates hold more generally.

Conjecture 8. *Let $N \ge 1$. There exists a constant $\gamma_N > 0$ such that for all dominant rational maps $f : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ we have*

$$\delta(f) \ge \gamma_N \cdot \min_{0 \le k < N} \frac{\deg(f^{k+1})}{\deg(f^k)}.$$

It is possible that Conjecture 8 is too optimistic, and we should instead take a minimum over $0 \le k \le \kappa(N)$ for some upper index that grows more rapidly with N, but we will at least prove that Conjecture 8 is true as stated for monomial maps. More precisely, we prove that if $f : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ is a dominant monomial map, then Conjecture 8 holds with $\gamma_N = (2^{1/N} - 1)/2N^2$; see Section 5.

Question 9. Conjecture 2 says that the set $\mathcal{E}(f_T)$ is a set of bounded height. In view of the many dynamical Galois equidistribution theorems proven in recent years, it is natural to ask if $\mathcal{E}(f_T)$ likewise admits such a description. Thus for each $t \in \mathcal{E}(f_T)$, let $t_1, \ldots, t_r$ be the distinct $\operatorname{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ conjugates of t, and define a measure

$$\mu_t := \frac{1}{r} \sum_{i=1}^{r} (\text{point mass at } t_i).$$

Then is it true that $\lim \mu_t$ converge (weakly) to a meaure on $\mathbb{P}^N(\mathbb{C})$ supported on $\mathcal{E}(f_T)$ as t ranges over points in $\mathcal{E}(f_T)$ with $h(t) \to \infty$?

We briefly summarize the contents of this article:

- §2 We study the geometry and algebraic stability of the family of maps $f_{a,b,c}$ defined by (1), and we prove Conjecture 2 for these families when a, b, c are polynomials of one variable.
- §3 We prove Conjecture 3 for a pair of families of maps f_{a_1,b_1,c_1} and f_{a_2,b_2,c_2}, where the a_i, b_i, c_i are again polynomials of one variable.
- §4 We sketch a proof (essentially due to Xie) that Conjecture 8 implies Conjecture 1, more generally over higher dimensional base varieties; see Theorem 17.
- §5 We prove Conjecture 8 for monomial maps.

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2. A Bounded Height Example

In this section we study a family of rational maps inspired by Xie's map [7, Section 5] described in Remark 7. We set the following notation.

Definition. Let $\mathcal{R}$ be an integral domain with field of fractions $\mathcal{K}$. For each triple $a, b, c \in \mathcal{R}$, let $f_{a,b,c} : \mathbb{P}^2_{\mathcal{R}} \to \mathbb{P}^2_{\mathcal{R}}$ be the rational map

$$f_{a,b,c}\bigl([X, Y, Z]\bigr) = [XY, XY + aZ^2, bYZ + cZ^2].$$

We also define the set of *exceptional triples* to be

$$\mathcal{Z}(\bar{\mathcal{K}}) = \left\{ (a, b, c) \in \mathbb{A}^3(\bar{\mathcal{K}}) : \begin{array}{c} \zeta c^2 + (\zeta + 1)^2 ab = 0 \text{ for some} \\ \text{root of unity } \zeta \in \bar{\mathcal{K}} \end{array} \right\}.$$

(We note that replacing ζ by ζ^2, we could alternatively define $\mathcal{Z}(\bar{\mathcal{K}})$ to be the set of triples satisfying $c^2 = (\zeta + \zeta^{-1})^2 ab$.)

Theorem 10. *Let $(a, b, c) \in \mathbb{A}^3(\bar{\mathcal{K}})$ with $abc \neq 0$. Then*

$$f_{a,b,c} \text{ is algebraically stable} \iff (a, b, c) \notin \mathcal{Z}(\bar{\mathcal{K}}).$$

Corollary 11. *Let $a(T), b(T), c(T) \in \bar{\mathbb{Q}}[T]$ be non-zero polynomials such that $f_{a(T),b(T),c(T)}$ is algebraically stable. Then*

$$\bigl\{ t \in \bar{\mathbb{Q}} : f_{a(t),b(t),c(t)} \text{ is not algebraically stable} \bigr\}$$

is a set of bounded height.

The key to proving Theorem 10 is an analysis of the geometry of the map $f_{a,b,c}$.

Proposition 12. *Let $a, b, c \in \mathcal{K}$ with $abc \neq 0$.*

(a) *The map $f_{a,b,c}$ is birational, and its indeterminacy locus is the set*

$$I(f_{a,b,c}) = \bigl\{ [0, 1, 0], [1, 0, 0] \bigr\}.$$

(b) *The critical locus of f is the set*

$$\operatorname{Crit}(f) = \{Y = 0\} \cup \{Z = 0\}.$$

(c) *Let $[\alpha, \beta, \gamma] \in \mathbb{P}^2(\bar{\mathcal{K}})$. Then the set $f^{-1}_{a,b,c}\bigl([\alpha, \beta, \gamma]\bigr)$ consists of a single point except in the following situations*:

$$f^{-1}_{a,b,c}\bigl([0, a, c]\bigr) = \{Y = 0\} \setminus \bigl\{[1, 0, 0]\bigr\},$$
$$f^{-1}_{a,b,c}\bigl([1, 1, 0]\bigr) = \{Z = 0\} \setminus \bigl\{[0, 1, 0], [1, 0, 0]\bigr\},$$
$$f^{-1}_{a,b,c}\bigl([1, 1, \gamma]\bigr) = \emptyset \textit{ if } \gamma \neq 0.$$

Proof. To ease notation, we let $f = f_{a,b,c}$.
(a) Let

$$g(X,Y,Z) = \bigl(ab^2X(Y-X), (cX-cY+aZ)^2, b(cX-cY+aZ)(Y-X)\bigr).$$

Then an easy calculation in affine coordinates gives

$$g\circ f(X,Y,Z) = a^2b^2YZ^2\cdot(X,Y,Z),$$

which shows that f is birational with f^{-1} induced by g; cf. [7, Section 5].[3] The indeterminacy locus of f is the set where

$$XY = XY + aZ^2 = bYZ + cZ^2 = 0.$$

We note that $XY = 0$ forces $aZ^2 = 0$, and hence $Z = 0$ under our assumption that $a \ne 0$. This gives two possible points in $I(f)$, namely $[0,1,0]$ and $[1,0,0]$, and it is clear that these points are in $I(f)$.
(b) The critical locus $\operatorname{Crit}(f)$ of f is the set where

$$\det\begin{pmatrix} Y & X & 0 \\ Y & X & 2aZ \\ 0 & bZ & bY+2cZ^2 \end{pmatrix} = -2abYZ^2 = 0.$$

(c) It is a standard fact that if $\Gamma\subset\mathbb{P}^2$ is a curve with the property that $f(\Gamma)$ is a point, then necessarily $\Gamma\subseteq\operatorname{Crit}(f)$; see for example [4, Lemma 23(c)]. We compute

$$\{Y=0\}\xrightarrow{f}[0,a,c] \quad\text{and}\quad \{Z=0\}\xrightarrow{f}[1,1,0]\in\operatorname{Fix}(f_{a,b,c}).$$

An easy calculation shows that

$$f^{-1}\bigl([0,a,c]\bigr)\subset\{Y=0\} \quad\text{and}\quad f^{-1}\bigl([1,1,0]\bigr)\subset\{Z=0\},$$

so the inverse images are the indicated sets with $I(f)$ removed.

It remains to determine for which points P the inverse image $f^{-1}(P)$ is empty. Since we do not actually need this result in the sequel, we leave the elementary, albeit slightly more involved, calculation to the interested reader. □

Proof of Theorem 10. To ease notation, we let $f = f_{a,b,c}$. The map f is not algebracally stable if and only if there is a curve $\Gamma\subset\mathbb{P}^2$ and an $N\ge 1$ such that $f^N(\Gamma)\subset I(f)$.

In particular, if f is not algebracally stable, then there is some $0\le n\le N-1$ such that $\dim f^n(\Gamma)=1$ and $\dim f^{n+1}(\Gamma)=0$. Proposition 12 tells us that the only curves that f collapses are the curves $Y=0$ and $Z=0$. Further,

$$f\bigl(\{Z=0\}\bigr) = [1,1,0]\in\operatorname{Fix}(f),$$

[3]As in [7], we can decompose $f = f_2\circ f_1$ with biratonal maps $f_1 = [XY, XY + bYZ, Z^2]$ and $f_2 = [X, X+aZ, -X+Y+cZ]$.

so if $f^n(\Gamma) = \{Z = 0\}$, then for all $k \ge 1$ we have $f^{n+k}(\Gamma) = [1, 1, 0] \notin I(f)$. So we have shown that f is not algebraically stable if and only if there is a curve Γ and integers $N > n \ge 0$ such that

$$f^n(\Gamma) = \{Y = 0\} \quad \text{and} \quad f^N(\Gamma) = [0, 1, 0]. \tag{2}$$

(We can't have $f^N(\Gamma)$ equal to the other point $[1, 0, 0]$ in $I(f)$, since $f\big(\{Y = 0\}\big) = [0, a, c]$ and $f\big(\{X = 0\}\big) = \{X = 0\}$, so once we get to a point with $X = 0$, applying f never gets back to a point with $X \ne 0$.)

We observe that (2) is true if and only if

$$\begin{aligned} [0, 1, 0] = f^N(\Gamma) &= (f^{N-n-1} \circ f \circ f^n)(\Gamma) \\ &= (f^{N-n-1} \circ f)\big(\{Y = 0\}\big) \\ &= f^{N-n-1}\big([0, a, c]\big). \end{aligned}$$

So we have proven that f is not algebraically stable if and only if there is an $n \ge 0$ such that the Z-coordinate of $f^n\big([0, a, c]\big)$ vanishes. We also note that $f\big([0, 0, 1]\big) = [0, a, c]$, so we may as well start at $[0, 0, 1]$. This prompts us to define polynomials $U_n, V_n \in \mathbb{Z}[a, b, c]$ by the formula

$$[0, U_n, V_n] = f^n\big([0, 0, 1]\big).$$

Then we have shown that

$$f_{a,b,c} \text{ is not algebraically stable} \iff V_n(a, b, c) = 0 \text{ for some } n \ge 1.$$

We now observe that $(V_n)_{n\ge 0}$ is a linear recurrence, at least until reaching a term that vanishes. Indeed, we have

$$\begin{aligned} [0, U_{n+1}, V_{n+1}] = f\big([0, U_n, V_n]\big) &= [0, aV_n^2, bU_nV_n + cV_n^2] \\ &= [0, aV_n, bU_n + cV_n]. \end{aligned}$$

Thus

$$\begin{pmatrix} U_{n+1} \\ V_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & a \\ b & c \end{pmatrix} \begin{pmatrix} U_n \\ V_n \end{pmatrix},$$

and repeated application together with the initial value $(U_0, V_0) = (0, 1)$ gives the matrix formula

$$\begin{pmatrix} U_n \\ V_n \end{pmatrix} = \begin{pmatrix} 0 & a \\ b & c \end{pmatrix}^n \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

Letting λ and $\bar\lambda$ be the eigenvalues of $\left(\begin{smallmatrix} 0 & a \\ b & c \end{smallmatrix}\right)$, i.e., the roots of

$$T^2 - cT - ab = 0,$$

an elementary linear algebra calculation yields

$$V_n = \frac{\lambda^{n+1} - \bar\lambda^{n+1}}{\lambda - \bar\lambda}.$$

(Unless $c^2 + 4ab = 0$, which we will deal with later.) So f is not algebraically stable if and only if there is some $n \ge 1$ such that $\lambda^n = \bar{\lambda}^n$. Writing λ and $\bar{\lambda}$ explicitly, we find that f is not algebraically stable if and only if (a, b, c) satisfies

$$c + \sqrt{c^2 + 4ab} = \zeta\bigl(c - \sqrt{c^2 + 4ab}\bigr) \quad \text{for some root of unity } \zeta \in \bar{\mathbb{Q}}.$$

A little algebra yields

$$\zeta c^2 + (\zeta + 1)^2 ab = 0,$$

which is the desired result.

It remains to deal with the case that $c^2 + 4ab = 0$, i.e., $\lambda = \bar{\lambda}$. But then an easy calculation shows that

$$V_n = (n + 1)(c/2)^n,$$

so V_n never vanishes under our assumption that $abc \ne 0$. □

Proof of Corollary 11. According to Theorem 10, the map $f_{a(t),b(t),c(t)}$ is not algebraically stable if and only if there is a root of unity $\zeta \in \bar{\mathbb{Q}}$ with the property that

$$\zeta c(t)^2 + (\zeta + 1)^2 a(t)b(t) = 0.$$

As ζ varies over roots of unity, the polynomials

$$\zeta c(T)^2 + (\zeta + 1)^2 a(T)b(T) \in \mathbb{Q}[T] \tag{3}$$

have bounded degree (depending on a, b, c) and have coefficients of bounded height. But the heights of the roots of a polynomial are easily bounded in terms of the degree and the heights of the coefficients; see for example [5, Theorem VIII.5.9]. Hence the roots of the polynomials (3) have height bounded independently of ζ. □

3. An Unlikely Intersection Example

Our goal in this section is to prove a non-trivial case of Conjecture 3 for the intersection of the exceptional sets of two maps.

Definition. Let $\mathcal{R}$ be an integral domain with field of fractions $\mathcal{K}$, and for non-zero $a, b, c \in \mathcal{R}$, let $f_{a,b,c} : \mathbb{P}^2_{\mathcal{R}} \to \mathbb{P}^2_{\mathcal{R}}$ denote the map

$$f_{a,b,c}\bigl([X, Y, Z]\bigr) = [XY, XY + aZ^2, bYZ + cZ^2]$$

that we already studied in Section 2. We define the *exceptional set of* $f_{a,b,c}$ to be the set of prime ideals

$$\mathcal{E}(f_{a,b,c}) = \bigl\{\mathfrak{p} \in \operatorname{Spec}(\mathcal{R}) : \delta(\tilde{f}_{a,b,c} \bmod \mathfrak{p}) < \delta(f_{a,b,c})\bigr\}.$$

Theorem 13. *Let $a_1, b_1, c_1, a_2, b_2, c_2 \in \bar{\mathbb{Q}}[T]$ be non-zero polynomials. For $i = 1, 2$, let*

$$f_{a_i,b_i,c_i}\big([X,Y,Z]\big) = [XY, XY + a_iZ^2, b_iYZ + c_iZ^2]$$

be the associated families of rational maps, and assume that they are algebraically stable as maps over the function field $\bar{\mathbb{Q}}(T)$. Then

$$\#\Big(\mathcal{E}(f_{a_1,b_1,c_1}) \cap \mathcal{E}(f_{a_2,b_2,c_2})\Big) = \infty \qquad \Longrightarrow \quad \#\Big(\mathcal{E}(f_{a_1,b_1,c_1}) \triangle \mathcal{E}(f_{a_2,b_2,c_2})\Big) < \infty.$$

Proof. We recall that Theorem 10 says that if $a, b, c \in \mathcal{R}$ are non-zero and if $f_{a,b,c}$ is algebraically stable, i.e., $\delta(f_{a,b,c}) = 2$, then

$$\mathcal{E}(f_{a,b,c}) = \left\{ \mathfrak{p} \in \operatorname{Spec}(\mathcal{R}) : \begin{array}{c} \zeta c^2 + (\zeta+1)^2 ab \equiv 0 \pmod{\mathfrak{p}\mathcal{R}[\zeta]} \\ \text{for some root of unity } \zeta \in \bar{\mathcal{K}} \end{array} \right\}. \tag{4}$$

We apply this with $\mathcal{R} = \bar{\mathbb{Q}}[T]$.

To ease notation, we let

$$F_1 = f_{a_1,b_1,c_1} \quad \text{and} \quad F_2 = f_{a_2,b_2,c_2}.$$

Then the assumption that $\#\big(\mathcal{E}(F_1) \cap \mathcal{E}(F_2)\big) = \infty$ and (4) imply that there are infinitely many triples $(\zeta_1, \zeta_2, t) \in \bar{\mathbb{Q}}^3$ with ζ_1 and ζ_2 roots of unity such that we have simultaneously

$$\zeta_1 c_1(t)^2 + (\zeta_1+1)^2 a_1(t) b_1(t) = 0,$$
$$\zeta_2 c_2(t)^2 + (\zeta_2+1)^2 a_2(t) b_2(t) = 0.$$

To further ease notation, for $i = 1, 2$ we define polynomials

$$B_i(U,T) := Uc_i(T)^2 + (U+1)^2 a_i(T) b_i(T) \in \bar{\mathbb{Q}}[U,T],$$

and then we know that there are infinitely many pairs of roots of unity (ζ_1, ζ_2) such that the polynomials

$$B_1(\zeta_1, T) \quad \text{and} \quad B_2(\zeta_2, T)$$

have a common root in $\bar{\mathbb{Q}}$. Equivalently, the resultant of $B_1(\zeta_1, T)$ and $B_2(\zeta_2, T)$ vanishes. Thus if we define

$$G(U_1, U_2) := \operatorname{Res}_T\big(B_1(U_1,T), B_2(U_2,T)\big) \in \bar{\mathbb{Q}}[U_1, U_2],$$

then we know that $G(U_1, U_2) = 0$ has infinitely many solutions (ζ_1, ζ_2) with roots of unity ζ_1 and ζ_2.

A famous result of Ihara, Serre, and Tate [3, Chapter 8, Theorem 6.1] says that a curve in $\mathbb{G}_m^2$ contains infinitely many torsion points if and only if it contains a torsion-point-translate of a subtorus of $\mathbb{G}_m^2$. Hence we can find a finite collection of pairs of integer $(m_1, n_1), \ldots, (m_r, n_r)$

not equal to $(0,0)$ such that every solution to $G(U_1,U_2)=0$ in roots of unity lies on one of the tori $U_1^{n_i}=U_2^{m_i}$. It suffices to concentrate on one pair (m,n) for which there are infinitely many such solutions. Equivalently, we fix integers $(m,n)\neq(0,0)$ such that $G(V^m,V^n)=0$ has infinitely many solutions with $V=\zeta$ a root of unity. This implies that that $G(V^m,V^n)=0$ identically in the Laurent ring $\bar{\mathbb{Q}}[V^{\pm 1}]$, i.e.,

$$\operatorname{Res}_T\bigl(B_1(V^m,T),B_2(V^n,T)\bigr)=0 \quad \text{in } \bar{\mathbb{Q}}[V^{\pm 1}]. \tag{5}$$

To ease our computations, we note that $B_i(U,T)=U^2B_i(U^{-1},T)$, so the four resultants

$$\operatorname{Res}_T\bigl(B_1(U_1^{\pm 1},T),B_2(U_2^{\pm 1},T)\bigr)$$

differ from one another by a factor of the form $U_1^iU_2^j$. In particular, if one of them vanishes for some non-zero value of (U_1,U_2), then they all vanish. Hence without loss of generality, we may assume in (5) that m and n are non-negative.

It follows from (5) that $B_1(V^m,T)$ and $B_2(V^n,T)$ have a common root in the field $\overline{\mathbb{Q}(V)}$, or equivalently, they have a non-trivial common factor in the ring $\bar{\mathbb{Q}}(V)[T]$. Clearing denominators and using the fact that $\bar{\mathbb{Q}}[V,T]$ is a UFD, we can find a greatest common divisor

$$C(V,T):=\gcd_{\bar{\mathbb{Q}}(V)[T]}\bigl(B_1(V^m,T),B_2(V^n,T)\bigr)\in\bar{\mathbb{Q}}[V,T],$$

and hence

$$\begin{aligned} B_1(V^m,T)&=C(V,T)D_1(V,T),\\ B_2(V^n,T)&=C(V,T)D_2(V,T), \end{aligned} \tag{6}$$

with non-zero polynomials $D_1(V,T),D_2(V,T)\in\bar{\mathbb{Q}}[V,T]$ such that D_1 and D_2 have no non-trivial common factor in $\bar{\mathbb{Q}}[V,T]$. Our assumptions imply that the set

$$\bigl\{t\in\bar{\mathbb{Q}}:C(\zeta,t)=0 \text{ for some root of unity } \zeta\in\bar{\mathbb{Q}}\bigr\}$$

is infinite, and hence we see that $\deg_V C(V,T)\geq 1$. Our goal now is to prove:

Claim 14. The symmetric set difference $\mathcal{E}(F_1)\mathbin{\triangle}\mathcal{E}(F_2)$ is finite.

We start with a definition.

Definition. For any polynomial $F(X,Y)\in\bar{\mathbb{Q}}[X,Y]$, we define

$$\mathcal{T}_X\bigl(F(X,Y)\bigr):=\bigl\{y\in\bar{\mathbb{Q}}:P(\zeta,y)=0 \text{ for some root of unity } \zeta\in\bar{\mathbb{Q}}\bigr\}.$$

We observe that for any integer $n\geq 1$ and any root of unity $\xi\in\bar{\mathbb{Q}}$, we have

$$\mathcal{T}_X\bigl(F(\xi X^n,Y)\bigr)=\mathcal{T}_X\bigl(F(X,Y)\bigr).$$

We have shown that there is a finite collection of pairs of integers (m,n) such that there is a factorization as described in (6) and such that $\mathcal{E}(F_1)\cup\mathcal{E}(F_2)$ is the union of a finite set and the sets

$$\mathcal{T}\bigl(B_1(V^m,T)\bigr)\cup\mathcal{T}\bigl(B_2(V^n,T)\bigr).$$

Hence in order to prove Claim 14, we are reduced to proving:

Claim 15. The set $\mathcal{T}\bigl(B_1(V^m,T)\bigr)\bigtriangleup\mathcal{T}\bigl(B_2(V^n,T)\bigr)$ is finite.

We stress that Claim 15 is not an immediate consequence of (6). Indeed, one might imagine there being many roots of unity ζ such that $D_1(\zeta,T)$ has a root t with $B_2(\zeta^n,t)\ne 0$, and every such t would then be in $\mathcal{T}\bigl(B_1(V^m,T)\bigr)\smallsetminus\mathcal{T}\bigl(B_2(V^n,T)\bigr)$. So we need to further exploit the specific form of the polynomials B_1 and B_2. The following rather strange lemma gives the information that we need.

Lemma 16. *Let $a(T),b(T)\in\bar{\mathbb{Q}}[T]$ be non-zero polynomials, and let*

$$B(U,T):=a(T)U+b(T)(U+1)^2\in\bar{\mathbb{Q}}[U,T].$$

Then one of the following is true:

(a) *There is a root of unity $\xi\in\bar{\mathbb{Q}}^*$ such that $B(\xi,T)=0$.*
(b) *For every integer $n\ge 1$ and every irreducible divisor $p(V,T)\in\bar{\mathbb{Q}}[V,T]$ of $B(V^n,T)$ with $\deg_V p\ge 1$, we have*

$$\mathcal{T}_U\bigl(B(U,T)\bigr)=\mathcal{T}_V\bigl(p(V,T)\bigr)\cup(\textit{finite set}).$$

Proof. We observe that for every $\zeta\in\boldsymbol{\mu}_n$, the polynomial $p(\zeta V,T)$ is also an irreducible divisor of $B(V^n,T)$. Let d be the largest divisor of n such that $p(V,T)\in\bar{\mathbb{Q}}[V^d,T]$. This means that there is an irreducible $q(U,T)\in\bar{\mathbb{Q}}[U,T]$ such that

$$p(V,T)=q(V^d,T).$$

It follows that $B(V^n,T)$ is divisible by

$$C(V,T):=\prod_{\zeta\in\boldsymbol{\mu}_n/\boldsymbol{\mu}_d}p(\zeta V,T)=\prod_{\zeta\in\boldsymbol{\mu}_n/\boldsymbol{\mu}_d}q(\zeta^dV^d,T),$$

since the factors in the product are non-associate irreducible polynomials. (The only way for two of them to be associate, and not equal, would be for $p(V,T)$ to be a monomial in V, which would contradict the fact that $B(V^n,T)$ is not divisible by V from our assumption that $b(T)\ne 0$.)

The fact that $C(V,T)\mid B(V^n,T)$ implies that

$$\begin{aligned}2n=\deg_V B(V^n,T)&\ge\deg_V C(V,T)\\&=\frac{n}{d}\deg_V p(V,T)=n\deg_U q(U,T)\ge n.\end{aligned}$$

Hence $\deg_U q(U,T) = 1$ or 2.

Case 1: $\boldsymbol{\deg_U q(U,T) = 2}$. In this case $B(V^n,T)$ and $C(V,T)$ both have V-degree equal to 2, so we find that

$$B(V^n,T) = D(T)C(V,T) = D(T) \prod_{\zeta\in\boldsymbol{\mu}_n/\boldsymbol{\mu}_d} p(\zeta V,T)$$

for some non-zero $D(T) \in \bar{\mathbb{Q}}[T]$. Hence

$$\begin{aligned}\mathcal{T}_U\bigl(B(U,T)\bigr) &= \mathcal{T}_V\bigl(B(V^n,T)\bigr)\\ &= \mathcal{T}_V\bigl(D(T)\bigr) \cup \bigcup_{\zeta\in\boldsymbol{\mu}_n/\boldsymbol{\mu}_d} \mathcal{T}_V\bigl(p(\zeta V,T)\bigr)\\ &= \underbrace{\mathcal{T}\bigl(D(T)\bigr)}_{\text{finite set}} \cup \mathcal{T}_U\bigl(p(U,T)\bigr).\end{aligned}$$

Case 2: $\boldsymbol{\deg_U q(U,T) = 1}$. In this case we have

$$\deg_V B(V^n,T) = 2n \quad \text{and} \quad \deg_V C(V,T) = n.$$

We are going to exploit the fact that the polynomial $B(U,T)$ satisfies the identity

$$U^2 B(U^{-1},T) = B(U,T),$$

i.e., $B(U,T)$ is a reciprocal polynomial in the U variable. Writing $B(V^n,T) = p(V,T)r(V,T)$ for some $r(V,T) \in \bar{\mathbb{Q}}[V,T]$, we find that

$$B(V^n,T) = V^{2n}B(V^{-n},T) = V^{\deg_V p}p(V^{-1},T)\cdot V^{\deg_V r}r(V^{-1},T),$$

and hence $V^d p(V^{-1},T)$ is also an irreducible divisor of $B(V^n,T)$. (We are using here the fact that $\deg_V p(V,T) = \deg_V q(V^d,T) = d$.) Hence either $V^d p(V^{-1},T)$ is an associate of one of the polynomials $p(\zeta V,T)$ dividing $C(V,T)$ that we already know divides $B(V^n,T)$, or else it is a new irreducible divisor.

Suppose first that

$$V^d p(V^{-1},T) = \gamma p(\zeta V,T) \quad \text{for some } \zeta\in\boldsymbol{\mu}_n \text{ and some } \gamma\in\bar{\mathbb{Q}}^*. \tag{7}$$

We write $q(U,T) = \alpha(T)U + \beta(T)$ and substitute into (7). This yields

$$\alpha(T) + \beta(T)V^d = \gamma\alpha(T)\zeta^d V^d + \gamma\beta(T).$$

Hence $\alpha(T) = \gamma\beta(T)$ and $\beta(T) = \gamma\alpha(T)\zeta^d$. These imply that

$$\alpha(T) = \gamma\beta(T) = \gamma^2\alpha(T)\zeta^d,$$

so $\gamma = \zeta^{-d/2}$ is a root of unity. Further, we see that

$$p(V,T) = q(V^d,T) = \alpha(T)V^d + \beta(T) = \beta(T)(\gamma V^d + 1),$$

and hence $p\bigl((-\gamma)^{1/d},T\bigr) = 0$. It follows that $B\bigl((-\gamma)^{1/d},T\bigr) = 0$, which concludes the proof of Lemma 16 in this case.

Finally, suppose that $V^d p(V^{-1},T)$ is a new irreducible divisor of $B(V^n,T)$. Then just as earlier, we obtain n/d such divisors by replacing V with ζV for $\zeta \in \boldsymbol{\mu}_n/\boldsymbol{\mu}_d$. Comparing V-degrees, this gives a complete factorization

$$B(V^n,T) = D(T) \prod_{\zeta\in\boldsymbol{\mu}_n/\boldsymbol{\mu}_d} p(\zeta V,T)\cdot(\zeta V)^d p(\zeta^{-1}V^{-1},T).$$

for some $D(T)\in\bar{\mathbb{Q}}[T]$. This allows us to compute

$$\begin{aligned}\mathcal{T}_U\bigl(B(U,T)\bigr) &= \mathcal{T}_V\bigl(B(V^n,T)\bigr)\\ &= \mathcal{T}_V\bigl(D(T)\bigr)\cup\bigcup_{\zeta\in\boldsymbol{\mu}_n/\boldsymbol{\mu}_d}\Bigl(\mathcal{T}_V\bigl(p(\zeta V,T)\bigr)\cup\mathcal{T}_V\bigl(V^d p(\zeta V^{-1},T)\bigr)\Bigr)\\ &= \underbrace{\mathcal{T}\bigl(D(T)\bigr)}_{\text{finite set}}\cup\mathcal{T}_U\bigl(p(V,T)\bigr).\end{aligned}$$

This completes the proof of Lemma 16. $\square$

We now have the tools needed to complete the proof of Claim 15 and with it, the proof of Theorem 13. We recall that we have factorizations (see (6))

$$B_1(V^m,T) = C(V,T)D_1(V,T) \text{ and } B_2(V^n,T) = C(V,T)D_2(V,T),$$

where $C(V,T)\in\bar{\mathbb{Q}}[V,T]$ satisfies $\deg_V C(V,T)\ge 1$. We choose an irreducible factor $p(V,T)\in\bar{\mathbb{Q}}[V,T]$ of $C(V,T)$ satisfying $\deg_V p(V,T)\ge 1$. Then $p(V,T)$ is an irreducible factor of both $B_1(V^m,T)$ and of $B_2(V^n,T)$. We know that

$$B_1(\xi,T)\ne 0 \quad\text{and}\quad B_2(\xi,T)\ne 0 \quad\text{for all roots of unity } \xi,$$

since otherwise the dynamical degree F_1 or F_2 over $\mathbb{Q}(T)$ would be strictly smaller than 2, contrary to our algebraic stability assumption. Using this fact and applying Lemma 16 twice, we find that

$$\begin{aligned}\mathcal{T}_V\bigl(B_1(V^m,T)\bigr) &= \mathcal{T}_V\bigl(p(V,T)\bigr)\cup(\text{finite set}),\\ \mathcal{T}_V\bigl(B_2(V^n,T)\bigr) &= \mathcal{T}_V\bigl(p(V,T)\bigr)\cup(\text{finite set}),\end{aligned}$$

and hence

$$\mathcal{T}_V\bigl(B_1(V^m,T)\bigr)\bigtriangleup\mathcal{T}_V\bigl(B_2(V^n,T)\bigr) = (\text{finite set}). \tag{8}$$

This completes the proof of Theorem 13. $\square$

4. Conjecture 8 Implies Conjecture 1

In this section we sketch the proof, essentially due to Xie [7, Theorem 4.1], that Conjecture 8 implies Conjecture 1. More generally, we show that Conjecture 8 implies a generalizataion of Conjecture 1 to families of arbitrary dimension.

Theorem 17. *Assume that Conjecture* 8 *is true for a given* $N \geq 1$. *Let*

$$f_T : \mathbb{P}^N_T \dashrightarrow \mathbb{P}^N_T$$

be a family of dominant rational maps over a smooth irreducible base variety T, *all defined over an algebraically closed field* K. *Then for all* $\epsilon > 0$, *the set*

$$\bigl\{t \in T(K) : \delta(f_t) \leq \delta(f_T) - \epsilon\bigr\}$$

is contained in a proper Zariski closed subset of T.

Proof. We first view $f = f_T$ as a rational map over the function field $\overline{K(T)}$. Then using the fact that $\gamma_N > 0$ and the definition of dynamical degree, we find that for any $k \geq 1$ we have

$$\begin{aligned}\lim_{n\to\infty}\left(\gamma_N\frac{\deg(f^{(k+1)n})}{\deg(f^{kn})}\right)^{1/n} &= \lim_{n\to\infty}\gamma_N^{1/n}\frac{\bigl(\deg(f^{(k+1)n})^{1/(k+1)n}\bigr)^{k+1}}{\bigl(\deg(f^{kn})^{1/kn}\bigr)^{k}}\\ &= \frac{\delta(f)^{k+1}}{\delta(f)^k} = \delta(f).\end{aligned}$$

In particular, we can find an $m = m(\epsilon, N)$ such that for all $0 \leq k < N$ we have

$$\left(\gamma_N \cdot \frac{\deg(f^{(k+1)m})}{\deg(f^{km})}\right)^{1/m} \geq \delta(f) - \epsilon. \tag{9}$$

We next observe that for any family $g : \mathbb{P}^N_T \dashrightarrow \mathbb{P}^N_T$ of dominant rational maps, the set

$$U(g) := \bigl\{t \in T(K) : \deg(g_t) = \deg(g)\bigr\}$$

is a non-empty Zariski open subset of T. We set

$$U_\epsilon := \bigcap_{k=0}^{N} U(f^{km}) \subset T(K),$$

where $m = m(\epsilon, N)$ is as in (9).

Finally, for $t \in U_\epsilon$ we compute

$$\begin{aligned}\delta(f_t) &= \delta(f_t^m)^{1/m} \quad \text{follows easily from definition of } \delta,\\ &\geq \left(\gamma_N \cdot \min_{0\leq k<N}\frac{\deg(f_t^{(k+1)m})}{\deg(f_t^{km})}\right)^{1/m} \text{Conjecture 8 applied to } f_t^m,\end{aligned}$$

$$= \left(\gamma_N \cdot \min_{0\le k<N} \frac{\deg(f^{(k+1)m})}{\deg(f^{km})}\right)^{1/m} \text{since } t \in U_\epsilon := \bigcap_{0\le k<N} U(f^{km}),$$
$$\ge \delta(f) - \epsilon \quad \text{from (9).}$$

This completes the proof of Theorem 17. □

5. A Dynamical Degree Estimate for Monomial Maps

As noted in the introduction, Xie [7] has shown that there is a constant $\gamma > 0$ such that

$$\delta(f) \ge \gamma \cdot \frac{\deg(f^2)}{\deg(f)} \quad \text{for all birational maps } f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2.$$

In this section we prove an analogous result for dominant monomial maps. We recall that a *monomial map* is an endomorphism of the torus $\mathbb{G}_m^N$, i.e., a map

$$\varphi_A : \mathbb{G}_m^N \longrightarrow \mathbb{G}_m^N$$

of the form

$$\varphi_A(X_1,\ldots,X_N) = \left(X_1^{a_{11}}X_2^{a_{12}}\cdots X_N^{a_{1N}},\ldots,X_1^{a_{N1}}X_2^{a_{N2}}\cdots X_N^{a_{NN}}\right),$$

where $A = (a_{ij}) \in \operatorname{Mat}_N(\mathbb{Z})$ is an N-by-N matrix with integer coefficients. The associated rational map $\varphi_A : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ is dominant if and only if $\det(A) \ne 0$.

Corollary 18. *Let* $A \in \operatorname{Mat}_N(\mathbb{Z})$ *be a matrix with* $\det(A) \ne 0$, *and let* $\varphi_A : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ *be the associated monomial map. Then*

$$\delta(\varphi_A) \ge \frac{2^{1/N}-1}{2N^2} \min_{0\le k\le N-1} \frac{\deg(\varphi_A^{k+1})}{\deg(\varphi_A^k)}.$$

Before starting the proof, we set some notation and quote a result due to Hasselblatt and Propp. For matrices $A \in \operatorname{Mat}_N(\mathbb{R})$ with real coefficients, we define

$$D(A) = \sum_{j=1}^N \max{}^+\{-a_{ij} : 1 \le i \le N\} + \max{}^+\left\{\sum_{j=1}^N a_{ij} : 1 \le i \le N\right\},$$

and we write

$$\|A\| = \max\bigl\{|a_{ij}| : 1 \le i,j \le N\bigr\},$$
$$\lambda(A) = \max\bigl\{|\alpha| : \alpha \in \mathbb{C} \text{ is an eigenvalue of } A\bigr\},$$

for the sup-norm and the spectral radius of the matrix A.

Proposition 19 (Hasselblatt–Propp)**.** *Let* $A \in \operatorname{Mat}_N(\mathbb{Z})$ *be a matrix satisfying* $\det(A) \ne 0$.

(a) *The degree of* φ_A *is given by* $\deg \varphi_A = D(A)$.
(b) *The dynamical degree of* φ_A *is given by* $\delta(\varphi_A) = \lambda(A)$.

Proof. For (a), see [1, Proposition 2.14], and for (b), see [1, Theorem 6.2]. □

We now verify that $D(A)$ induces a distance function on $\mathrm{Mat}_N(\mathbb{R})$ that is equivalent to the easier-to-deal-with sup norm.

Lemma 20. *Let* $A \in \mathrm{Mat}_N(\mathbb{R})$. *Then*

$$\frac{1}{2N} D(A) \le \|A\| \le N D(A).$$

Proof. From the definition of $D(A)$ we see that for every i, j we have

$$-a_{ij} \le D(A) \quad \text{and} \quad \sum_{k=1}^{N} a_{ik} \le D(A).$$

Hence every $a_{ij} \ge -D(A)$, while for every i, j we can estimate

$$a_{ij} = \sum_{k=1}^{N} a_{ik} - \sum_{\substack{k=1 \\ k \ne j}} a_{ik} \le D(A) + \sum_{\substack{k=1 \\ k \ne j}} D(A) = N D(A).$$

This proves that $-D(A) \le a_{ij} \le N D(A)$, so in particular $|a_{ij}| \le N D(A)$, so $\|A\| \le N D(A)$. And for the other direction, the triangle inequality applied to each term in the definition of $D(A)$ immediately gives $D(A) \le 2N\|A\|$. □

Proposition 21. *Let* $A \in \mathrm{Mat}_N(\mathbb{R})$ *be a matrix. Then*

$$\|A^{k+1}\| \le \frac{\lambda(A) \cdot \|A^k\|}{2^{1/N} - 1} \quad \textit{for some } 0 \le k \le N - 1.$$

Proof. Write the characteristic polynomial of A as

$$\det(xI - A) = \prod_{i=1}^{N} (x - \lambda_i) = \sum_{j=0}^{N} (-1)^j \sigma_j x^{N-j},$$

where σ_j is the j'th elementary symmetric polynomial of $\lambda_1, \ldots, \lambda_N$. We note for future reference that σ_j is a sum of $\binom{N}{j}$ monomials, each monomial being a product of j of the λ_i's, which combined with $\lambda(A) = \max |\lambda_j|$ gives the upper bound

$$|\sigma_j| \le \binom{N}{j} \lambda(A)^j. \tag{10}$$

We are going to use the Cayley-Hamilton theorem, which says that A satisfies its characteristic polynomial.

For notational convenience, we set $\epsilon := 2^{1/N} - 1$. We suppose that

$$\|A^{k+1}\| > \epsilon^{-1}\lambda(A)\|A^k\| \quad \text{for all } 0 \le k \le N-1 \tag{11}$$

and derive a contradiction. Iterating (11), we find that

$$\lambda(A)^{N-k}\|A^k\| < \epsilon^{N-k}\|A^N\| \quad \text{for all } 0 \le k \le N-1. \tag{12}$$

We use this to estimate

$$\begin{aligned}
\|A^N\| &= \left\| \sum_{j=1}^{N} (-1)^j \sigma_j A^{N-j} \right\| && \text{Cayley-Hamilton theorem,} \\
&\le \sum_{j=1}^{N} |\sigma_j| \|A^{N-j}\| && \text{triangle inequality,} \\
&\le \sum_{j=1}^{N} \binom{N}{j} \lambda(A)^j \|A^{N-j}\| && \text{from (10),} \\
&< \sum_{j=1}^{N} \binom{N}{j} \epsilon^j \|A^N\| && \text{from (12) with } k = N-j, \\
&= \big((1+\epsilon)^N - 1\big)\|A^N\| && \text{binomial theorem,} \\
&= \|A^N\| && \text{since } \epsilon := 2^{1/N} - 1.
\end{aligned}$$

This strict inequality is a contradiction, so (11) is false, which completes the proof of Proposition 21. □

Proof of Corollary 18. We choose some $0 \le k \le N-1$ such that Proposition 21 holds, and then we use Lemma 20 and Proposition 19 to estimate

$$\frac{1}{2^{1/N} - 1} \ge \frac{\|A^{k+1}\|}{\lambda(A) \cdot \|A^k\|} \ge \frac{(2N)^{-1} D(A^{k+1})}{\lambda(A) N D(A^k)} = \frac{\deg(\varphi_A^{k+1})}{2N^2 \delta(\varphi_A) \deg(\varphi_A^k)}.$$

This completes the proof of Corollary 18. □

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