# ON DEGREES OF BIRATIONAL MAPPINGS

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ABSTRACT. We prove that the degrees of the iterates $\deg(f^n)$ of a birational map satisfy $\liminf(\deg(f^n)) < +\infty$ if and only if the sequence $\deg(f^n)$ is bounded, and that the growth of $\deg(f^n)$ can not be arbitrarily slow, unless $\deg(f^n)$ is bounded.

## 1. DEGREE SEQUENCES

Let $\mathbf{k}$ be a field. Consider a projective variety $X$, a polarization $H$ of $X$ (given by hyperplane sections of $X$ in some embedding $X \subset \mathbb{P}^N$), and a birational transformation $f$ of $X$, all defined over the field $\mathbf{k}$. Let $k$ be the dimension of $X$. The **degree** of $f$ with respect to the polarization $H$ is the integer

$$\deg_H(f) = (f^*H) \cdot H^{k-1} \tag{1.1}$$

where $f^*H$ is the total transform of $H$, and $(f^*H) \cdot H^{k-1}$ is the intersection product of $f^*H$ with $k-1$ copies of $H$. The degree is a positive integer, which we shall simply denote by $\deg(f)$, even if it depends on $H$. When $f$ is a birational transformation of the projective space $\mathbb{P}^k$ and the polarization is given by $O_{\mathbb{P}^k}(1)$ (i.e. by hyperplanes $H \subset \mathbb{P}^k$), then $\deg(f)$ is the degree of the homogeneous polynomial formulas defining $f$ in homogeneous coordinates.

The degrees are submultiplicative, in the following sense:

$$\deg(f \circ g) \leq c_{X,H} \deg(f) \deg(g) \tag{1.2}$$

for some positive constant $c_{X,H}$ and for every pair of birational transformations. Also, if the polarization $H$ is changed into another polarization $H'$, there is a positive constant $c$ such that $\deg_H(f) \leq c \deg_{H'}(f)$ (see [7, 11, 13]).

The **degree sequence** of $f$ is the sequence $(\deg(f^n))_{n\geq 0}$; it plays an important role in the study of the dynamics and the geometry of $f$. There are infinitely, but only countably many degree sequences (see [14]); unfortunately, not much is known on these sequences when $\dim(X) \geq 3$. In this article, we obtain the following basic results.

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- The sequence $(\deg(f^n))_{n\geq 0}$ is bounded if and only if it is bounded along an infinite subsequence (see Theorems A and B in § 2 and 3).
- If the sequence $(\deg(f^n))_{n\geq 0}$ is unbounded, then its growth can not be arbitrarily small; for instance, $\max_{0\leq j\leq n}\deg(f^j)$ is asymptotically bounded from below by the inverse of the diagonal Ackermann function (see Theorem C in § 4 for an effective result).

We focus on birational transformations because a rational dominant transformation which is not birational has a topological degree $\delta > 1$, and this forces an exponential growth of the degrees: $1 < \delta^{1/k} \leq \lim_n(\deg(f^n)^{1/n})$ where $k = \dim(X)$ (see [7] and [4], pages 120–126).

## 2. Automorphisms of the affine space

We start with the simpler case of automorphisms of the affine space; the goal of this section is to introduce a $p$-adic method to study degree sequences.

**Theorem A (Urech).–** *Let $f$ be an automorphism of the affine space $\mathbb{A}^k_{\mathbf{k}}$. If* $\deg(f^n)$ *is bounded along an infinite subsequence, then it is bounded.*

In fact, Urech proves in [14] that $\max_{0\leq j\leq n}\deg(f^j)$ is bounded from below by $\alpha n^{1/k}$ for some constant $\alpha > 0$ when $(\deg(f^n))$ is unbounded. Here, we content ourselves with the simpler version stated in Theorem A.

2.1. **Urech's proof.** Assume $\deg(f^{n_i}) \leq B$ for some sequence $n_1 < n_2 < \ldots < n_\ell$ of positive integers. The iterates $f^{n_i}$ are in the vector space $\mathsf{End}_B(\mathbb{A}^k_{\mathbf{k}})$ of endomorphisms of $\mathbb{A}^k_{\mathbf{k}}$ of degree $\leq B$. This vector space has dimension

$$\dim(\mathsf{End}_B(\mathbb{A}^k_{\mathbf{k}})) = k\begin{pmatrix} k+B \\ B \end{pmatrix}. \tag{2.1}$$

Thus, if $\ell > \dim(\mathsf{End}_B(\mathbb{A}^k_{\mathbf{k}}))$ there is a non-trivial linear relation between the $f^{n_i}$ in the vector space $\mathsf{End}_B(\mathbb{A}^k_{\mathbf{k}})$, which we can write

$$f^n = \sum_{m=1}^{n-1} a_m f^m \tag{2.2}$$

for some integer $n \leq n_\ell$ and some coefficients $a_m \in \mathbf{k}$. Then, every iterate $f^N$ of $f$ with $N \geq n$ is a linear combination of the automorphisms $f^m$ with $m < n$, and so $\deg(f^N)$ is bounded from above by the maximum of the degrees of $f^m$ for $0 \leq m \leq n-1$. This shows that the sequence $(\deg(f^N))_{N\geq 0}$ is bounded.

2.2. **The $p$-adic argument.** Let us give a second proof when $\mathrm{char}(\mathbf{k}) = 0$, which will be generalized in § 3 to treat the case of birational transformations.

2.2.1. *Tate diffeomorphisms.* Let $p$ be a prime number. Let $K$ be a field of characteristic 0 which is complete with respect to an absolute value $|\cdot|$ satisfying $|p| = 1/p$; such an absolute value is automatically ultrametric (see [9], Ex. 2 and 3, Chap. I.2). Let $R = \{x \in K;\ |x| \leq 1\}$ be the valuation ring of $K$; in the vector space $K^k$, the unit **polydisk** is the subset $\mathsf{U} = R^k$.

Fix a positive integer $k$, and consider the ring $R[\mathbf{x}] = R[\mathbf{x}_1, \dots, \mathbf{x}_k]$ of polynomial functions in $k$ variables with coefficients in $R$. For $f$ in $R[\mathbf{x}]$, define the norm $\parallel f \parallel$ to be the supremum of the absolute values of the coefficients of $f$:

$$\parallel f \parallel = \sup_I |a_I| \tag{2.3}$$

where $f = \sum_{I=(i_1,\dots,i_k)} a_I \mathbf{x}^I$. By definition, the **Tate algebra** $R\langle \mathbf{x} \rangle$ is the completion of $R[\mathbf{x}]$ with respect to this norm. It coincides with the set of formal power series $f = \sum_I a_I \mathbf{x}^I$ converging (absolutely) on the closed unit polydisk $R^k$. Moreover, the absolute convergence is equivalent to $|a_I| \to 0$ as $\mathrm{length}(I) \to \infty$. Every element $g$ in $R\langle \mathbf{x} \rangle^k$ determines a **Tate analytic** map $g\colon \mathsf{U} \to \mathsf{U}$.

For $f$ and $g$ in $R\langle \mathbf{x} \rangle$ and $c$ in $\mathbf{R}_+$, the notation $f \in p^c R\langle \mathbf{x} \rangle$ means $\parallel f \parallel \leq |p|^c$ and the notation $f \equiv g \mod (p^c)$ means $\parallel f - g \parallel \leq |p|^c$; we then extend such notations component-wise to $(R\langle \mathbf{x} \rangle)^m$ for all $m \geq 1$.

For indeterminates $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_k)$ and $\mathbf{y} = (\mathbf{y}_1, \dots, \mathbf{y}_m)$, the composition $R\langle \mathbf{y} \rangle \times R\langle \mathbf{x} \rangle^m \to R\langle \mathbf{x} \rangle$ is well defined, and coordinatewise we obtain

$$R\langle \mathbf{y} \rangle^n \times R\langle \mathbf{x} \rangle^m \to R\langle \mathbf{x} \rangle^n. \tag{2.4}$$

When $m = n = k$, we get a semigroup $R\langle \mathbf{x} \rangle^k$. The group of (Tate) **analytic diffeomorphisms** of $\mathsf{U}$ is the group of invertible elements in this semigroup; we denote it by $\mathsf{Diff}^{an}(\mathsf{U})$. Elements of $\mathsf{Diff}^{an}(\mathsf{U})$ are bijective transformations $f\colon \mathsf{U} \to \mathsf{U}$ given by $f(\mathbf{x}) = (f_1, \dots, f_k)(\mathbf{x})$ where each $f_i$ is in $R\langle \mathbf{x} \rangle$ with an inverse $f^{-1}\colon \mathsf{U} \to \mathsf{U}$ that is also defined by power series in the Tate algebra.

The following result is due to Jason Bell and Bjorn Poonen (see [2, 12]).

**Theorem 2.1.** *Let $f$ be an element of $R\langle \mathbf{x} \rangle^k$ with $f \equiv \mathrm{id} \mod (p^c)$ for some real number $c > 1/(p-1)$. Then $f$ is a Tate diffeomorphism of $\mathsf{U} = R^k$ and there exists a unique Tate analytic map $\Phi\colon R \times \mathsf{U} \to \mathsf{U}$ such that*

(1) $\Phi(n, \mathbf{x}) = f^n(\mathbf{x})$ *for all $n \in \mathbf{Z}$;*
(2) $\Phi(s + t, \mathbf{x}) = \Phi(s, \Phi(t, \mathbf{x}))$ *for all $t$, $s$ in $R$.*

2.2.2. *Second proof of Theorem A.* Denote by $S$ the finite set of all the coefficients that appear in the polynomial formulas defining $f$. Let $R_S \subset \mathbf{k}$ be the ring generated by $S$ over $\mathbf{Z}$, and let $K_S$ be its fraction field:

$$\mathbf{Z} \subset R_S \subset K_S \subset \mathbf{k}. \tag{2.5}$$

Since $\mathrm{char}(\mathbf{k}) = 0$, there exists a prime $p > 2$ such that $R_S$ embeds into $\mathbf{Z}_p$ (see [10], §4 and 5, and [2], Lemma 3.1). We apply this embedding to the coefficients of $f$ and get an automorphism of $\mathbb{A}^k_{\mathbf{Q}_p}$ which is defined by polynomial formulas in $\mathbf{Z}_p[\mathbf{x}_1, \ldots, \mathbf{x}_k]$; for simplicilty, we keep the same notation $f$ for this automorphism (embedding $R_S$ in $\mathbf{Z}_p$ does not change the value of the degrees $\deg(f^n)$). Since $f$ is a polynomial automorphism with coefficients in $\mathbf{Z}_p$, it determines an element of $\mathsf{Diff}^{an}(\mathsf{U})$, the group of analytic diffeomorphisms of the polydisk $\mathsf{U} = \mathbf{Z}_p^k$.

There exists a positive integer $m$ such that $f^m$ fixes the origin $0 \in \mathsf{U}$ modulo $p^2$: $f^m(0) \equiv 0 \mod (p^2)$. Taking some further iterate, we may also assume that the differential $Df_0^m$ satisfies $Df_0^m \equiv \mathsf{Id} \mod (p)$. We fix such an integer $m$ and replace $f$ by $f^m$. The following lemma follows from the submultiplicativity of degrees (see Equation (1.2) in Section 1). It shows that replacing $f$ by $f^m$ is armless if one wants to bound the degrees of the iterates of $f$.

**Lemma 2.2.** *If the sequence* $\deg(f^{mn})$ *is bounded for some* $m > 0$*, then the sequence* $\deg(f^n)$ *is bounded too.*

Denote by $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_k)$ the coordinate system of $\mathbb{A}^k$, and by $m_p$ the multiplication by $p$: $m_p(\mathbf{x}) = p\mathbf{x}$. Change $f$ into $g := m_p^{-1} \circ f \circ m_p$; then $g \equiv \mathsf{Id}$ mod $(p)$ in the sense of Section 2.2.1. Since $p \geq 3$, Theorem 2.1 gives a Tate analytic flow $\Phi\colon \mathbf{Z}_p \times \mathbb{A}^k(\mathbf{Z}_p) \to \mathbb{A}^k(\mathbf{Z}_p)$ which extends the action of $g$: $\Phi(n, \mathbf{x}) = g^n(\mathbf{x})$ for every integer $n \in \mathbf{Z}$. Since $\Phi$ is analytic, one can write

$$\Phi(\mathbf{t}, \mathbf{x}) = \sum_J A_J(\mathbf{t}) \mathbf{x}^J \tag{2.6}$$

where $J$ runs over all multi-indices $(j_1, \ldots, j_k) \in (\mathbf{Z}_{\geq 0})^k$ and each $A_J$ defines a $p$-adic analytic curve $\mathbf{Z}_p \to \mathbb{A}^k(\mathbf{Q}_p)$. By submultiplicativity of the degrees, there is a constant $C > 0$ such that $\deg(g^{n_i}) \leq CB^m$. Thus, we obtain $A_J(n_i) = 0$ for all indices $i$ and all multi-indices $J$ of length $|J| > CB^m$. The $A_J$ being analytic functions of $t \in \mathbf{Z}_p$, the principle of isolated zeros implies that

$$A_J = 0 \text{ in } \mathbf{Z}_p\langle t \rangle, \ \forall J \text{ with } |J| > CB^m. \tag{2.7}$$

Thus, $\Phi(t,\mathbf{x})$ is a polynomial automorphism of degree $\leq CB^m$ for all $t \in \mathbf{Z}_p$, and $g^n(\mathbf{x}) = \Phi(n,\mathbf{x})$ has degree at most $CB^m$ for all $n$. By Lemma 2.2, this proves that $\deg(f^n)$ is a bounded sequence.

## 3. BIRATIONAL TRANSFORMATIONS

We now extend Theorem A to the case of birational transformations.

**Theorem B.–** *Let* $\mathbf{k}$ *be a field of characteristic* $0$. *Let* $X$ *be a projective variety and* $f\colon X \dashrightarrow X$ *be a birational transformation of* $X$*, both defined over* $\mathbf{k}$. *If the sequence* $(\deg(f^n))_{n\geq 0}$ *is not bounded, then it goes to* $+\infty$ *with* $n$*:*

$$\liminf_{n\to+\infty} \deg(f^n) = +\infty.$$

Urech's argument does not apply to this context, because the dimension of the space of rational transformations of $\mathbb{A}^k_{\mathbf{k}}$ of degree $\leq B$ is infinite. We shall therefore apply the $p$-adic method, adapting the proof given in Section 2.2.2.

Note that Theorem B can be combined with a theorem of Weil to obtain the following: *if* $f$ *is a birational transformation of the projective variety* $X$*, over an algebraically closed field of characteristic* $0$*, and if the degrees of its iterates are bounded along an infinite subsequence* $f^{n_i}$*, then there exist a birational map* $\psi\colon Y \dashrightarrow X$ *and an integer* $m > 0$ *such that* $f_Y := \psi^{-1}\circ f \circ \psi$ *is in* $\mathsf{Aut}(Y)$*, and* $f_Y^m$ *is in the connected component* $\mathsf{Aut}(Y)^0$ (see [3] and references therein).

In what follows, $f$ and $X$ are as in Theorem B; we also assume, without loss of generality, that $\mathbf{k} = \mathbf{C}$ and that $X$ is smooth. We suppose that there is an infinite sequence of integers $n_1 < n_2 < \ldots < n_j < \ldots$ and a positive number $B$ such that $\deg(f^{n_j}) \leq B$ for all $j$. We fix a finite set $S$ of complex numbers such that $X$ and $f$ are defined by equations and formulas with coefficients in $S$, and we embed the ring $R_S \subset \mathbf{C}$ generated by $S$ in some $\mathbf{Z}_p$, for some prime number $p > 2$. According to [5], Section 3, we may assume that $X$ and $f$ have good reduction modulo $p$.

### 3.1. The Hrushovski's theorem and $p$-adic polydisks.

According to a theorem of Hrushovski (see [8]), there is a periodic point $z_0$ of $f$ in $X(\mathbf{F})$ for some finite field extension $\mathbf{F}$ of the residue field $\mathbf{F}_p$, the orbit of which does not intersect the indeterminacy points of $f$ and $f^{-1}$. If $\ell$ is the period of $z_0$, then $f^\ell(z_0) = z_0$ and $Df^\ell_{z_0}$ is an element of the finite group $\mathsf{GL}\left((TX_{\mathbf{F}_q})_{z_0}\right) \simeq \mathsf{GL}\left(k,\mathbf{F}_q\right)$. Thus, there is an integer $m > 0$ such that $f^m(z_0) = z_0$ and $Df^m_{z_0} = \mathsf{Id}$.

Replace $f$ by its iterate $g = f^m$. Then, $g$ fixes $z_0$ in $X(\mathbf{F})$, $g$ is an isomorphism in a neighborhood of $z_0$, and $Dg_{z_0} = \mathsf{Id}$. According to [1] and [5] Section 3, this implies that there is

- a finite extension $K$ of $\mathbf{Q}_p$, with valuation ring $R \subset K$;
- a point $z$ in $X(K)$ and a polydisk $\mathsf{V}_z \simeq R^k \subset X(K)$ which is $g$-invariant and such that $g_{V_z} \equiv \mathsf{Id} \mod (p)$ (in the coordinate system $(\mathbf{x}_1, \ldots, \mathbf{x}_k)$ of the polydisk).

When the point $z_0$ is in $X(\mathbf{F}_p)$ and is the reduction of a point $z \in X(\mathbf{Z}_p)$, the polydisk $\mathsf{V}_z$ is the set of points $w \in X(\mathbf{Z}_p)$ with $|z - w| < 1$; one identifies this polydisk to $\mathsf{U} = (\mathbf{Z}_p)^k$ via some $p$-adic analytic diffeomorphism $\varphi \colon \mathsf{U} \to \mathsf{V}_z$; changing $\varphi$ into $\varphi \circ m_p$ if necessary, we obtain $g_{V_z} \equiv \mathsf{Id} \mod (p)$ (see Section 2.2.2 and [5], Section 3). In full generality, a finite extension $K$ of $\mathbf{Q}_p$ is needed because $z_0$ is a point in $X(\mathbf{F})$ for some extension of the residue field.

3.2. **Controling the degrees.** As in Section 2.2.1, denote by $\mathsf{U}$ the polydisk $R^k \simeq \mathsf{V}_z$; thus, $\mathsf{U}$ is viewed as the polydisk $R^k$ and also as a subset of $X(K)$. Applying Theorem 2.1 to $g$, we obtain a $p$-adic analytic flow

$$\Phi \colon R \times \mathsf{U} \to \mathsf{U}, \quad (t, \mathbf{x}) \mapsto \Phi(t, \mathbf{x}) \tag{3.1}$$

such that $\Phi(n, \mathbf{x}) = g^n(\mathbf{x})$ for every integer $n$. In other words, the action of $g$ on $\mathsf{U}$ extends to an analytic action of the additive compact group $(R, +)$.

Let $\pi_1 : X \times X \to X$ denote the projection onto the first factor. Denote by $\mathsf{Bir}_D(X)$ the set of birational transformations of $X$ of degree $D$; once birational transformations are identified to their graphs, this set becomes naturally a finite union of irreducible, locally closed algebraic subsets in the Hilbert scheme of $X \times X$ (see [3], Section 2.2, and references therein). Taking a subsequence, there is a positive integer $D$, an irreducible component $B_D$ of $\mathsf{Bir}_D(X)$, and a strictly increasing, infinite sequence of integers $(n_j)$ such that

$$g^{n_j} \in B_D \tag{3.2}$$

for all $j$. Denote by $\overline{B_D}$ the Zariski closure of $B_D$ in the Hilbert scheme of $X \times X$. To every element $h \in \overline{B_D}$ corresponds a unique algebraic subset $\mathcal{G}_h$ of $X \times X$ (the graph of $h$, when $h$ is in $B_D$). Our goal is to show that, for every $t \in R$, the graph of $\Phi(t, \cdot)$ is the intersection $\mathcal{G}_{h_t} \cap \mathsf{U}^2$ for some element $h_t \in \overline{B_D}$; this will conclude the proof because $g^n(\mathbf{x}) = \Phi(n, \mathbf{x})$ for all $n \geq 0$.

We start with a simple remark, which we encapsulate in the following lemma.

**Lemma 3.1.** *There is a finite subset $E \subset \mathsf{U} \subset X(K)$ with the following property. Given any subset $\tilde{E}$ of $\mathsf{U} \times \mathsf{U}$ with $\pi_1(\tilde{E}) = E$, there is at most one element $h \in \overline{B_D}$ such that $\tilde{E} \subset \mathcal{G}_h$.*

Fix such a set $E$, and order it to get a finite list $E = (x_1, \ldots, x_{\ell_0})$ of elements of $\mathsf{U}$. Let $E' = (x_1, \ldots, x_{\ell_0}, x_{\ell_0+1}, \ldots, x_\ell)$ be any list of elements of $\mathsf{U}$ which extends $E$.

For every element $h$ in $\overline{B_D}$, the variety $\mathcal{G}_h$ determines a correspondance $\mathcal{G}_h \subset X \times X$. The subset of elements $(h, (x_i, y_i)_{1 \leq i \leq \ell})$ in $\overline{B_D} \times (X \times X)^\ell$ defined by the incidence relation

$$(x_i, y_i) \in \mathcal{G}_h \tag{3.3}$$

for every $1 \leq i \leq \ell$ is an algebraic subset of $\overline{B_D} \times (X \times X)^\ell$. Add one constraint, namely that the first projection $(x_i)_{1 \leq i \leq \ell}$ coincides with $E'$, and project the resulting subset on $(X \times X)^\ell$: we get a subset $G(E')$ of $(X \times X)^\ell$.

Then, define a $p$-adic analytic curve $\Lambda \colon R \to (X \times X)^\ell$ by

$$\Lambda(t) = (x_i, \Phi(t, x_i))_{1 \leq i \leq \ell}. \tag{3.4}$$

If $t = n_j$, $g^{n_j}$ is an element of $B_D$ and $\Lambda(n_j)$ is contained in the graph of $g^{n_j}$; hence, $\Lambda(n_j)$ is an element of $G(E')$. By the principle of isolated zeros, the analytic curve $t \mapsto \Lambda(t) \subset (X \times X)^\ell$ is contained in $G(E')$ for all $t \in R$. Thus, for every $t$ there is an element $h_t \in \overline{B_D}$ such that $\Lambda(t)$ is contained in the subset $\mathcal{G}_{h_t}^\ell$ of $(X \times X)^\ell$. From the choice of $E$ and the inclusion $E \subset E'$, we know that $h_t$ does not depend on $E'$. Thus, the graph of $\Phi(t, \cdot)$ coincides with the intersection of $\mathcal{G}_{h_t}$ with $\mathsf{U} \times \mathsf{U}$. This implies that the graph of $g^n(\cdot) = \Phi(n, \cdot)$ coincides with $\mathcal{G}_{h_n}$, and that the degree of $g^n$ is at most $D$ for all values of $n$.

## 4. Lower bounds on degree growth

We now prove that the growth of $(\deg(f^n))$ can not be arbitrarily slow unless $(\deg(f^n))$ is bounded. For simplicity, we focus on birational transformations of the projective space; there is no restriction on the characteristic of $\mathbf{k}$.

4.1. **A family of integer sequences.** Fix two positive integers $k$ and $d$: later on, $k$ will be the dimension of the projective space $\mathbb{P}^k_{\mathbf{k}}$, and $d$ will be the degree of $f \colon \mathbb{P}^k \dashrightarrow \mathbb{P}^k$. Set

$$m = (d-1)(k+1). \tag{4.1}$$

Then, consider an auxiliary integer $D \geq 1$, which will play the role of the degree of an effective divisor in the next paragraphs, and define

$$q = (dD)^m (D+1). \tag{4.2}$$

Thus, $q$ depends on $k$, $d$ and $D$ because $m$ depends on $k$ and $d$. Then, set

$$a_0 = \binom{k+D}{k} - 1, \quad b_0 = 1, \quad c_0 = D+1. \tag{4.3}$$

Starting from the triple $(a_0, b_0, c_0)$, we define a sequence $((a_j, b_j, c_j))_{j\geq 0}$ inductively by

$$(a_{j+1}, b_{j+1}, c_{j+1}) = (a_j, b_j - 1, qc_j^2) \tag{4.4}$$

if $b_j \geq 2$, and by

$$(a_{j+1}, b_{j+1}, c_{j+1}) = (a_j - 1, qc_j^2, qc_j^2) = (a_j - 1, c_{j+1}, c_{j+1}) \tag{4.5}$$

if $b_j = 1$. By construction, $(a_1, b_1, c_1) = (a_0 - 1, qc_0^2, qc_0^2)$.

Define $\Phi\colon \mathbf{Z}^+ \to \mathbf{Z}^+$ by

$$\Phi(c) = qc^2. \tag{4.6}$$

**Lemma 4.1.** *Define the sequence of integers $(F_i)_{i\geq 1}$ recursively by $F_1 = q(D+1)^2$ and $F_{i+1} = \Phi^{F_i}(F_i)$ for $i \geq 1$ (where $\Phi^{F_i}$ is the $F_i$-iterate of $\Phi$). Then*

$$(a_{1+F_1+\cdots+F_i},\, b_{1+F_1+\cdots+F_i},\, c_{1+F_1+\cdots+F_i}) = (a_0 - i - 1,\, F_{i+1},\, F_{i+1}).$$

The proof is straightforward. Now, define the function $S\colon \mathbf{Z}^+ \to \mathbf{Z}^+$ as the sum of the $F_i$:

$$S(j) = 1 + F_1 + F_2 + \cdots + F_j \tag{4.7}$$

for all $j \geq 1$. The function $S$ is increasing and goes to $+\infty$ extremely fast with $j$. Then, set

$$\chi_{d,k}(n) = \max\left\{ D \geq 0 \mid S(\binom{k+D}{k} - 2) < n \right\}. \tag{4.8}$$

**Lemma 4.2.** *The function $\chi_{d,k}\colon \mathbf{Z}^+ \to \mathbf{Z}^+$ is non-decreasing and goes to $+\infty$ with $n$.*

**Remark 4.3.** *The function $S$ is primitive recursive* (see [6], Chapters 3 and 13). In other words, $S$ is obtained from the basic functions (the zero function, the successor $s(x) = x+1$, and the projections $(x_i)_{1\leq i\leq m} \to x_i$) by a finite sequence of compositions and recursions. Equivalently, one can write a program that computes $S$, all of whose instructions are limited to (1) the zero initialization $V \leftarrow 0$, (2) the increment $V \leftarrow V+1$, (3) the assignement $V \leftarrow V'$, and (4) loops of definite length. Writing such a program is an easy exercise. Now, consider the diagonal Ackermann function $A(n)$ (see [6], Section 13.3). It grows asymptotically faster than any primitive recursive function; hence, the inverse of the Ackermann diagonal function

$$\alpha(n) = \max\{D \geq 0 \mid \mathrm{Ack}(D) \leq n\}. \tag{4.9}$$

is, asymptotically, a lower bound for $\chi_{d,k}(n)$. A better lower bound is obtained by showing that $\chi_{d,k}$ is in the $\mathcal{L}_6$ hierarchy of [6], Chapter 13; this gives an asymptotic lower bound by the inverse of the function $f_7$ of [6], independent on the values of $d$ and $k$, but this is a very week bound too.

4.2. **Statement of the lower bound.** We can now state the result that will be proved in the next paragraphs.

**Theorem C.–** *Let $f$ be a birational transformation of the complex projective space $\mathbb{P}^k_{\mathbf{C}}$. If the sequence $(\max_{0\leq j\leq n}(\deg(f^j)))_{n\geq 0}$ is unbounded, then it is bounded from below by the sequence of integers $(\chi_{d,k}(n))_{n\geq 0}$.*

**Remark 4.4.** There are infinitely, but only countably many sequences of degrees $(\deg(f^n))_{n\geq 0}$ (see [14]). Consider the countably many sequences

$$\left(\max_{0\leq j\leq n}(\deg(f^j))\right)_{n\geq 0} \tag{4.10}$$

restricted to the family of birational maps for which $(\deg(f^n))$ is unbounded. We get a countable family of non-decreasing, unbounded sequences of integers. Now, let $(u_i)_{i\in\mathbf{Z}_{\geq 0}}$ be any countable family of non-decreasing and unbounded sequences of integers $(u_i(n))$. Define a sequence $w(n)$ as follows. First, set $v_j = \min\{u_0, u_1, \ldots, u_j\}$; this defines a new family of sequences, with the same limit $+\infty$, but now $v_j(n) \geq v_{j+1}(n)$ for every pair of non-negative integers. Then, set $m_0 = 0$, and define $m_{n+1}$ recursively to be the first positive integer such that $v_{n+1}(m_{n+1}) \geq v_n(m_n) + 1$. We have $m_{n+1} \geq m_n + 1$ for all $n \in \mathbf{Z}_{\geq 0}$. Set $w(n) := v_{r_n}(m_{r_n})$ where $r_n$ is the unique non-negative integer satisfying $m_{r_n} \leq n \leq m_{r_n+1} - 1$. By construction, $w(n)$ goes to $+\infty$ with $n$ and $u_i(n)$ is *asymptotically* bounded from below by $w(n)$.

In Theorem C, the result is more explicit. Firstly, the lower bound is explicitely given by the sequence $(\chi_{d,k}(n))_{n\geq 0}$. Secondly, the lower bound is not asymptotic: it works for every value of $n$. In particular, if $\deg(f^j) < \chi_{d,k}(n)$ for $0 \leq j \leq n$ and $\deg(f) = d$, then the sequence $(\deg(f^n))$ is bounded.

4.3. **Divisors and strict transforms.** To prove Theorem C, we consider the action of $f$ by strict transform on effective divisors. As above, $d = \deg(f)$ and $m = (d-1)(k+1)$ (see Section 4.1).

4.3.1. *Exceptional locus.* Let $X$ be a smooth projective variety and $\pi_1$ and $\pi_2 \colon X \to \mathbb{P}^k$ be two birational morphisms such that $f = \pi_2 \circ \pi_1^{-1}$; then, consider the exceptional locus $\mathrm{Exc}(\pi_2) \subset X$, project it by $\pi_1$ into $\mathbb{P}^k$, and list its irreducible components of codimension 1: we obtain a finite number

$$E_1, \ldots, E_{m(f)} \tag{4.11}$$

of irreducible hypersurfaces, contained in the zero locus of the jacobian determinant of $f$. Since this critical locus has degree $m$, we obtain:

$$m(f) \leq m, \quad \text{and } \deg(E_i) \leq m \quad (\forall i \geq 1). \tag{4.12}$$

4.3.2. *Effective divisors.* Denote by $M$ the semigroup of effective divisors of $\mathbb{P}^k_{\mathbf{k}}$; every element of $M$ is a finite sum of irreducible hypersurfaces with non-negative integer coefficients. There is a partial ordering $\leq$ on $M$, which is defined by $E \leq E'$ if and only if the divisor $E' - E$ is effective.

We denote by $\deg \colon M \to \mathbf{Z}_{\geq 0}$ the degree function. For every degree $D \geq 1$, we denote by $M_D$ the set $\mathbb{P}(H^0(\mathbb{P}^k_{\mathbf{k}}, O_{\mathbb{P}^k_{\mathbf{k}}}(D)))$ of effective divisors of degree $D$; thus, $M$ is the disjoint union of all the $M_D$, and each of these components will be endowed with the Zariski topology of $\mathbb{P}(H^0(\mathbb{P}^k_{\mathbf{k}}, O_{\mathbb{P}^k_{\mathbf{k}}}(D)))$. The dimension of $M_D$ is equal to the integer $a_0 = a_0(D,k)$ from Section 4.1:

$$\dim(M_D) = \binom{k+D}{k} - 1. \tag{4.13}$$

Let $G \subset M$ be the semigroup generated by the $E_i$:

$$G = \bigoplus_{i=1}^{m(f)} \mathbf{Z}_{\geq 0} E_i. \tag{4.14}$$

The elements of $G$ are the effective divisors which are supported by the exceptional locus of $f$. For every $E \in G$, there is a translation operator $T_E \colon M \to M$ which is defined by $T_E \colon E' \mapsto E + E'$; it is a linear projective embedding of the projective space $M_D$ into the projective space $M_{D+\deg(E)}$. We define

$$M_D^\circ = M_D \setminus \bigcup_{E \in G \setminus \{0\}, \deg(E) \leq D} T_E(M_{D-\deg(E)}). \tag{4.15}$$

Thus, $M_D^\circ$ is an open subset of $M_D$; it is the complement of finitely many proper linear projective subspaces. Also, $M_0^\circ = M_0$ and $M_1^\circ$ is obtained from

$M_1$ by removing finitely many points, corresponding to the $E_i$ of degree 1 (the hyperplanes contracted by $f$). Set $M^\circ = \bigcup_{D\geq 0} M_D^\circ$. This is the set of effective divisors without any component in the exceptional locus of $f$. The inclusion of $M^\circ$ in $M$ will be denoted by $\iota\colon M^\circ \to M$.

There is a natural projection $\pi_G\colon M \to G$; namely, $\pi_G(E)$ is the maximal element such that $E - \pi_G(E)$ is effective. We denote by $\pi_\circ\colon M \to M^\circ$ the projection $\pi_\circ = \mathsf{Id} - \pi_G$; this homomorphism removes the part of an effective divisor $E$ which is supported on the exceptional locus of $f$.

**Remark 4.5.** The restriction of the map $\pi_\circ$ to the projective space $M_D$ is piecewise linear, in the following sense. Consider the subsets $U_{E,D}$ of $M_D$ which are defined for every $E \in G$ with $\deg(E) \leq D$ by

$$U_{E,D} = T_E(M_{D-\deg(E)}) \setminus \bigcup_{E'>E, E'\in G, \deg(E')\leq D} T_{E'}(M_{D-\deg(E')}).$$

They define a stratification of $M_D$ by (open subsets of) linear subspaces, and $\pi_\circ$ coincides with the of the linear map inverse of $T_E$ on each $U_{E,D}$.

4.3.3. *Strict transform.* First, we consider the total transform $f^*\colon M \to M$, which is defined by $f^*(E) = (\pi_1)_*\pi_2^*(E)$ for every divisor $E \in M$. This is an injective homomorphism of semigroups. Let $[x_0,\ldots,x_k]$ be homogeneous coordinates on $\mathbb{P}^k$. If $f = [f_0 : \cdots : f_k]$ is defined by homogeneous polynomial functions $f_i \in \mathbf{k}[x_0,\ldots,x_k]$ of degree $d$, and if $E$ is defined by the homogeneous equation $P(x_0,\ldots,x_k) = 0$, then $f^*(E)$ is defined by $P \circ f = P(f_0,\ldots,f_k) = 0$. Thus, $f^*$ induces a linear projective embedding of $M_D$ into $M_{dD}$ for every $D$.

Then, we denote by $f^\circ\colon M^\circ \to M^\circ$ the strict transform. It is defined by

$$f^\circ(E) = (\pi_\circ \circ f^* \circ \iota)(E). \tag{4.16}$$

This is a homomorphism of semigroups. Removing the exceptional locus $(\pi_1)_*(E(\pi_2))$ from $\mathbb{P}^k_{\mathbf{k}}$, one gets a variety $Y$, and an induced birational transformation $f_Y\colon Y \dashrightarrow Y$. Then, every divisor $E \in M^\circ$ intersects $Y$ on a divisor $E_Y$ of the same degree: this provides a bijection between effective divisors of $Y$ and elements of $M^\circ$ that conjugates $(f_Y)^*$ to $f^\circ$. In particular, $(f^\circ)^n = (f^n)^\circ$.

4.4. **Proof of Theorem C.** Let $\eta$ be the generic point of $M_1^\circ$ ($\eta$ corresponds to a generic hyperplane of $\mathbb{P}^k_{\mathbf{k}}$). The degree of $f^*(\eta)$ is equal to the degree of $f$, and since $\eta$ is generic, $f^*(\eta)$ coincides with $f^\circ(\eta)$. Thus, $\deg(f) = \deg(f^\circ(\eta))$ and more generally

$$\deg(f^n) = \deg((f^\circ)^n\eta) \quad (\forall n \geq 1). \tag{4.17}$$

Fix an integer $D \geq 0$. Write $M^\circ_{\leq D}$ for the union of the $M^\circ_{D'}$ with $D' \leq D$, and define recursively $Z_D(0) = M^\circ_{\leq D}$ and

$$Z_D(i+1) = \{E \in Z_D(i) \mid f^\circ(E) \in Z_D(i)\} \tag{4.18}$$

for $i \geq 0$. A divisor $E \in M^\circ_{\leq D}$ is in $Z_D(i)$ if its strict transform $f^\circ(E)$ is of degree $\leq D$, and $f^\circ(f^\circ(E))$ is also of degree $\leq D$, up to $(f^\circ)^i(E)$ which is also of degree at most $D$. The subsets $Z_D(i)$ form a decreasing sequence of Zariski closed subsets (in the disjoint union $M^\circ_{\leq D}$ of the $M^\circ_{D'}$, $D' \leq D$). The strict transform $f^\circ$ maps $Z_D(i+1)$ in $Z_D(i)$. There exists a minimal integer $\ell(D) \geq 0$ such that

$$Z_D(\ell(D)) = \bigcap_{i\geq 0} Z_D(i); \tag{4.19}$$

we denote this subset by $Z_D(\infty) = Z_D(\ell(D))$. By construction, $Z_D(\infty)$ is stable under the operator $f^\circ$; more precisely, $f^\circ(Z_D(\infty)) = Z_D(\infty) = (f^\circ)^{-1}(Z_D(\infty))$.

Let $\tau\colon \mathbf{Z}_{\geq 0} \to \mathbf{Z}_{\geq 0}$ be a lower bound for the inverse function of $\ell$:

$$\ell(\tau(n)) \leq n \quad (\forall n \geq 0). \tag{4.20}$$

Assume that $\max\{\deg(f^m) \mid 0 \leq m \leq n_0\} \leq \tau(n_0)$ for some $n_0 \geq 1$. Then $\deg((f^\circ)^i(\eta)) \leq \tau(n_0)$ for every integer $i$ between 0 and $n_0$; this implies that $\eta$ is in the set $Z_{\tau(n_0)}(\ell(\tau(n_0))) = Z_{\tau(n_0)}(\infty)$, so that the degree of $(f^\circ)^m(\eta)$ is bounded from above by $\tau(n_0)$ for all $m \geq 0$. From Equation (4.17) we deduce that the sequence $(\deg(f^m))_{m\geq 0}$ is bounded. This proves the following lemma.

**Lemma 4.6.** *Let $\tau$ be a lower bound for the inverse function of $\ell$. If*

$$\max\{\deg(f^m) \mid 0 \leq m \leq n_0\} \leq \tau(n_0)$$

*for some $n_0 \geq 1$, then the sequence of degrees $(\deg(f^n))_{n\geq 0}$ is bounded.*

So, to conclude, we need to compare $\ell\colon \mathbf{Z}_{\geq 0} \to \mathbf{Z}^+$ to the function $S\colon \mathbf{Z}_{\geq 0} \to \mathbf{Z}^+$ of paragraph 4.1 (recall that $S$ depends on the parameters $k = \dim(\mathbb{P}^k_{\mathbf{k}})$ and $d = \deg(f)$ and that $\ell$ depends on $f$).

Let us describe $Z_D(i+1)$ more precisely. For each $i$, and each $E \in G$ of degree $\deg(E) \leq dD$ consider the subset $T_E(\overline{\iota(Z_D(i))}) \cap M_{dD}$; this is a subset of $M_{dD}$ which is made of divisors $W$ such that $\pi_\circ(W)$ is contained in $Z_D(i)$, and the union of all these subsets when $E$ varies is exactly the set of points $W$ in $M_{dD}$ with a projection $\pi_\circ(W)$ in $Z_D(i)$. Thus, we define

$$(f^*)^{-1}(T_E(\overline{\iota(Z_D(i))})) = \{V \in M^\circ_{\leq D} \mid f^*(\iota(V)) \in T_E(\overline{\iota(Z_D)})\}. \tag{4.21}$$

These sets are closed subsets of $M^{\circ}_{\leq D}$, and

$$Z_D(i+1) = Z_D(i) \bigcap \bigcup_{E \in G, \deg(E) \leq dD} \pi_{\circ}\left((f^*)^{-1}(T_E(\overline{\iota(Z_D(i))})\right). \tag{4.22}$$

Now, write $Z'_D(i) = Z_D(i) \setminus Z_D(\infty)$, and note that it is a decreasing sequence of open subsets with $Z'_D(j) = \emptyset$ for all $j \geq \ell(D)$.

We shall say that a closed subset $L$ of $M^{\circ}_{\leq D} \setminus Z_D(\infty)$ for the Zariski topology is **piecewise linear** if all its irreducible components are equal to the intersection of $M^{\circ}_{\leq D} \setminus Z_D(\infty)$ with a linear projective subspace of some $M_{D'}$, $D' \leq D$. Let $\mathrm{Lin}(a,b,c)$ be the family of closed piecewise linear subsets of $M^{\circ}_{\leq D} \setminus Z_D(\infty)$ of dimension $a$, with at most $c$ irreducible components, and at most $b$ irreducible components of maximal dimension $a$. Then:

(1) $Z'_D(i+1) = \{E \in Z'_D(i) \mid f^{\circ}(E) \in Z'_D(i)\} = \pi_{\circ}(f^* Z'_D(i)) \bigcap \cup_E T_E(Z'_D(i))$, where $E$ runs over the elements of $G$ of degree $\deg(E) \leq dD$.
(2) in this union, every irreducible component of $T_E(Z'_D(i))$ is piecewise linear.

Recall that $q = (dD)^m(D+1)$ was introduced in Section 4.1. If $Z$ is any closed piecewise linear subset of $M^{\circ}_{\leq D} \setminus Z_D(\infty)$ that contains exactly $c$ irreducible components, the set

$$\pi_{\circ}(f^* Z) \bigcap \bigcup_{E \in G,\ \deg(E) \leq dD} T_E(E) \tag{4.23}$$

has at most $qc^2 = (dD)^m(D+1)c^2$ irreducible components (this is just a crude estimate : the factor $(D+1)$ comes from the number of irreducible components of $M_{\leq D}$, and the factor $(dD)^m$ from the fact that $G$ contains at most $(dD)^m$ elements of degree $\leq dD$). Let us now use that the sequence $Z'_D(i)$ decreases strictly as $i$ varies from 0 to $\ell(D)$, with $Z'_D(\ell(D)) = \emptyset$. If $0 \leq i \leq \ell(D) - 1$, and if $Z'_D(i)$ is contained in $\mathrm{Lin}(a,b,c)$, we obtain

(1) if $b \geq 2$, then $Z'_D(i+1)$ is contained in $\mathrm{Lin}(a, b-1, qc^2)$;
(2) if $b = 1$, then $Z'_D(i+1)$ is contained in $\mathrm{Lin}(a-1, qc^2, qc^2)$.

This shows that

$$\ell(D) \leq S(\begin{pmatrix} k+D \\ k \end{pmatrix} - 2) + 1 \tag{4.24}$$

where $S$ is the function introduced in the Equation (4.7) of Section 4.1. Since $\chi_{d,k}$ satisfies $\ell(\chi_{d,k}(n)) \leq n$ for every $n \geq 1$, the conclusion follows.

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