The Cremona group in two variables

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Abstract. We survey a few results concerning the Cremona group in two variables.

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1. The Cremona group, and some of its subgroups

1.1. Cremona groups. Let $\mathbf{k}$ be a field and n be a positive integer. The Cremona group $\mathrm{Cr}_n(\mathbf{k})$ is the group of automorphisms of $\mathbf{k}(X_1, \ldots, X_n)$, the $\mathbf{k}$-algebra of rational functions in n independent variables. Given n rational functions $F_i \in \mathbf{k}(X_1, \ldots, X_n)$, $1 \leq i \leq n$, there is a unique endomorphism of this algebra that maps X_i onto F_i ; this endomorphism is an automorphism if and only if the rational transformation f defined by $f(X_1, \ldots, X_n) = (F_1, \ldots, F_n)$ is a birational transformation of the affine space $\mathbb{A}^n_{\mathbf{k}}$. After compactification of $\mathbb{A}^n_{\mathbf{k}}$ into the projective space $\mathbb{P}^n_{\mathbf{k}}$, one gets

$$\mathrm{Cr}_n(\mathbf{k}) = \mathrm{Bir}(\mathbb{A}^n_{\mathbf{k}}) = \mathrm{Bir}(\mathbb{P}^n_{\mathbf{k}}). \tag{1}$$

In homogeneous coordinates $[x_1 : \ldots : x_{n+1}]$, with $X_i = x_i/x_{n+1}$, every birational transformation f of $\mathbb{P}^n_{\mathbf{k}}$ can be written as

$$f[x_1 : \ldots : x_{n+1}] = [f_1 : \ldots : f_{n+1}] \tag{2}$$

where the f_i are homogeneous polynomials in the variables x_i, of the same degree d, and without common factor of positive degree. This degree d is the **degree** of f.

1.2. Examples, indeterminacy points, and dynamics. The group of automorphisms of $\mathbb{P}^n_{\mathbf{k}}$ is the group $\mathsf{PGL}_{n+1}(\mathbf{k})$ of linear projective transformations. As a subgroup of $\mathrm{Cr}_n(\mathbf{k})$, it coincides with the set of birational transformations of degree 1. In dimension 1, $\mathrm{Cr}_1(\mathbf{k})$ is equal to $\mathsf{PGL}_2(\mathbf{k})$, because a rational fraction $f(X_1) \in \mathbf{k}(X_1)$ is invertible if and only if its degree is equal to 1.

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1.2.1. Monomial transformations. The multiplicative group $\mathbb{G}_m^n$ of dimension n, which we identify to $(\mathbb{A}^1_{\mathbf{k}} \setminus \{0\})^n$, sits as a Zariski open subset in $\mathbb{P}^n_{\mathbf{k}}$. Consequently, $\mathrm{Cr}_n(\mathbf{k})$ contains the group of its algebraic automorphisms *i.e.* the group of monomial transformations $\mathsf{GL_n}(\mathbf{Z})$. For example, $(X_1, X_2) \mapsto (1/X_1, 1/X_2)$ and $(X_1, X_2) \mapsto (X_1^2 X_2, X_1 X_2)$ are two monomial transformations of the plane. The first is denoted by σ in what follows: it can be written as

$$\sigma[x_1 : x_2 : x_3] = [x_2 x_3 : x_3 x_1 : x_1 x_2] \tag{3}$$

in homogeneous coordinates, and is therefore an involution of degree 2. By definition, σ is the **standard quadratic involution**. If $\mathbf{k}$ is the field of complex numbers $\mathbf{C}$, the second transformation preserves the torus $\{(X_1, X_2) \in \mathbf{C}^*;\ |X_1| = |X_2| = 1\}$ and determines a diffeomorphism of Anosov type on this torus [15].

1.2.2. Indeterminacy points. Birational transformations may have indeterminacy points. For instance, σ is not defined at the three points $[1 : 0 : 0]$, $[0 : 1 : 0]$, and $[0 : 0 : 1]$. The set of indeterminacy points of $f \in \mathrm{Cr}_n(\mathbf{k})$ is an algebraic subset of $\mathrm{Cr}_n(\mathbf{k})$ of co-dimension at least 2, and is therefore finite when $n = 2$.

1.2.3. Hénon mappings. The group $\mathrm{Aut}(\mathbb{A}^n_{\mathbf{k}})$ of polynomial automorphisms of the affine space $\mathbb{A}^n_{\mathbf{k}}$ is contained in the Cremona group $\mathrm{Cr}_n(\mathbf{k})$. In particular, all transformations $(X_1, \ldots, X_n) \mapsto (X_1 + P(X_2, \ldots, X_n), X_2, \ldots, X_n)$, with P in $\mathbf{k}[X_2, \ldots, X_n]$, are contained in $\mathrm{Cr}_n(\mathbf{k})$. This shows that $\mathrm{Cr}_n(\mathbf{k})$ is "infinite dimensional" when $n \geq 2$.

A striking example of automorphism is furnished by the Hénon mapping

$$h_{a,c}(X_1, X_2) = (X_2 + X_1^2 + c, aX_1), \tag{4}$$

for $a \in \mathbf{k}^*$ and $c \in \mathbf{k}$. When $a = 0$, $h_{a,c}$ is not invertible: the plane is mapped into the line $\{X_2 = 0\}$ and, on this line, $h_{0,c}$ maps X_1 to $X_1^2 + c$. The dynamics of $h_{0,c}$ on this line coincides with the dynamics of the upmost studied transformation $z \mapsto z^2 + c$, which, for $\mathbf{k} = \mathbf{C}$, provides interesting examples of Julia sets (see [65]). For $a \in \mathbf{C}^*$, the main features of this dynamical system survive in the dynamical properties of the automorphism $h_{a,c} \colon \mathbb{A}^2_{\mathbf{C}} \to \mathbb{A}^2_{\mathbf{C}}$, such as positive topological entropy and the existence of infinitely many periodic points [6].

1.3. Subgroups of Cremona groups. Birational transformations are simple objects, since they are determined by a finite set of data, the coefficients of the homogeneous polynomials defining them. On the other hand, they may exhibit very rich dynamical behaviors, as shown by the previous examples. Another illustration of the beauty of $\mathrm{Cr}_n(\mathbf{k})$ comes from the study of its subgroups.

1.3.1. Mapping class groups. Let Γ be a group which is generated by a finite number of elements γ_i, $1 \leq i \leq k$. Consider the space R_Γ of all morphisms of Γ into $\mathsf{SL}_2(\mathbf{k})$: it is an algebraic variety over $\mathbf{k}$ of dimension at most $3k$. The group $\mathsf{SL}_2(\mathbf{k})$ acts on R_Γ by conjugacy; the quotient space $R_\Gamma /\!\!/ \mathsf{SL}_2(\mathbf{k})$, in the sense of geometric invariant theory, is an algebraic variety.

The group of all automorphisms of Γ acts on R_Γ by pre-composition. This determines an action of the outer automorphism group $\mathrm{Out}(\Gamma)$ by regular tranformations on $R_\Gamma /\!\!/ \mathsf{SL}_2(\mathbf{k})$, where $\mathrm{Out}(\Gamma)$ is the quotient of $\mathrm{Aut}(\Gamma)$ by the subgroup of all inner automorphisms. There are examples for which this construction provides an embedding of $\mathrm{Out}(\Gamma)$ in the group of automorphisms of $R_\Gamma /\!\!/ \mathsf{SL}_2(\mathbf{k})$. Fundamental groups of closed orientable surfaces of genus $g \geq 3$ or free groups $\mathbb{F}_g$ with $g \geq 2$ provide such examples. Thus, the mapping class groups $\mathrm{Mod}(g)$ and the outer automorphism groups $\mathrm{Out}(\mathbb{F}_g)$ embed into groups of birational transformations [59, 2].

1.3.2. Analytic diffeomorphisms of the plane. Consider the group $\mathrm{Bir}^\infty(\mathbb{P}^2_\mathbf{R})$ of all elements f of $\mathrm{Bir}(\mathbb{P}^2_\mathbf{R})$ with no real indeterminacy point: over $\mathbf{C}$, indeterminacy points of f come in complex conjugate pairs. Based on the work of Lukackiĭ, Kollár and Mangolte observed that $\mathrm{Bir}^\infty(\mathbb{P}^2_\mathbf{R})$ determines a dense subgroup in the group of diffeomorphisms of $\mathbb{P}^2(\mathbf{R})$ of class $\mathcal{C}^\infty$ (see [56]).

1.4. Aim and scope. These notes focus on the algebraic structure of (subgroups of) the Cremona group **in two variables**. Dynamical properties of birational transformations are not discussed; this would require a much longer report [22, 53]. Most results concerning $\mathrm{Bir}(\mathbb{P}^2_\mathbf{k})$ extend to $\mathrm{Bir}(X)$ for all projective surfaces X; when this is the case, I state the corresponding theorems in their greater generality.

2. Algebraic subgroups of $\mathrm{Cr}_2(\mathbf{k})$

2.1. Algebraic subgroups. The Cremona group $\mathrm{Cr}_2(\mathbf{k})$ contains two important algebraic subgroups. The first one is the group $\mathsf{PGL}_3(\mathbf{k})$ of automorphisms of $\mathbb{P}^2_\mathbf{k}$. The second is obtained as follows. Start with the surface $\mathbb{P}^1_\mathbf{k} \times \mathbb{P}^1_\mathbf{k}$, considered as a smooth quadric in $\mathbb{P}^3_\mathbf{k}$; its automorphism group contains $\mathsf{PGL}_2(\mathbf{k}) \times \mathsf{PGL}_2(\mathbf{k})$. By stereographic projection, the quadric is birationally equivalent to the plane, so that $\mathrm{Bir}(\mathbb{P}^2_\mathbf{k})$ contains a copy of $\mathsf{PGL}_2(\mathbf{k}) \times \mathsf{PGL}_2(\mathbf{k})$.

To introduce the notion of algebraic subgroups in $\mathrm{Cr}_n(\mathbf{k})$, note that the set of birational transformations of degree at most d is an algebraic variety, which we denote by $\mathrm{Cr}_n(\mathbf{k}; d)$. Let G be an algebraic group over $\mathbf{k}$. One says that G can be realized as an algebraic subgroup of $\mathrm{Cr}_n(\mathbf{k})$ if there is a positive integer d, and a rational map $\varphi \colon G \dashrightarrow \mathrm{Cr}_n(\mathbf{k}; d)$ such that φ is an injective homomorphism on the open subset on which it is well defined (see [34, 71] for precise definitions). Both $\mathsf{PGL}_3(\mathbf{k})$ and $\mathsf{PGL}_2(\mathbf{k}) \times \mathsf{PGL}_2(\mathbf{k})$ are algebraic subgroups of $\mathrm{Cr}_2(\mathbf{k})$. Similarly, all finite subgroups of $\mathrm{Cr}_2(\mathbf{k})$ are algebraic subgroups.

Example 2.1. An important subgroup of $\mathrm{Cr}_2(\mathbf{k})$ which is not algebraic is the **de Jonquières group** $\mathrm{Jonq}_2(\mathbf{k})$, of all transformations of $\mathbb{P}^1_\mathbf{k} \times \mathbb{P}^1_\mathbf{k}$ that permute the fibers of the projection onto the first factor. It is isomorphic to the semi-direct product $\mathsf{PGL}_2(\mathbf{k}) \ltimes \mathsf{PGL}_2(\mathbf{k}(\mathsf{x}))$; for example, it contains all transformations $(X_1, X_2) \mapsto (aX_1, Q(X_1)X_2))$ with a in $\mathbf{k}^*$ and Q in $\mathbf{k}(X_1) \setminus \{0\}$, so that its "dimension" is infinite.

2.2. Generating sets and relations. The first main result on $\mathrm{Cr}_2(\mathbf{k})$ is due to Noether and Castelnuovo [66, 28]. It exhibits two sets of generators for $\mathrm{Cr}_2(\mathbf{k})$.

Theorem 2.2 (Noether, Castelnuovo). *Let $\mathbf{k}$ be an algebraically closed field. The group $\mathrm{Cr}_2(\mathbf{k})$ is generated by $\mathsf{PGL}_3(\mathbf{k})$ and the standard quadratic involution σ. It is also generated by $\mathrm{Jonq}_2(\mathbf{k})$ and the involution $\eta(X_1, X_2) = (X_2, X_1)$.*

The group $\mathrm{Jonq}_2(\mathbf{k})$ can be identified with the group of birational transformations of $\mathbb{P}^2_{\mathbf{k}}$ that preserve the pencil of lines through the point $[1:0:0]$, and η to the involution $[x_1 : x_2 : x_3] \mapsto [x_2 : x_1 : x_3]$. With such a choice, η is in $\mathsf{PGL}_3(\mathbf{k})$ and σ in $\mathrm{Jonq}_2(\mathbf{k})$. Then, $\mathrm{Cr}_2(\mathbf{k})$ *is the amalgamated product of* $\mathrm{Jonq}_2(\mathbf{k})$ *and* $\mathsf{PGL}_3(\mathbf{k})$ *along their intersection, divided by one relation, namely* $\sigma \circ \eta = \eta \circ \sigma$ (see [12] and [50, 51] for former presentations of $\mathrm{Cr}_2(\mathbf{k})$).

Remark 2.3. (a).– Similarly, Jung's theorem asserts that the group of polynomial automorphisms of the affine plane is the free product of two of its subgroups, amalgamated along their intersection (see [57] for example).

(b).– For every smooth irreducible curve C, there is a birational transformation g of $\mathbb{P}^3_{\mathbf{k}}$ and a surface $X \subset \mathbb{P}^3_{\mathbf{k}}$ such that (i) X is birationally equivalent to $C \times \mathbb{P}^1_{\mathbf{k}}$ and (ii) g contracts X onto a subset of codimension ≥ 2. Consequently, one needs as many families as families of smooth curves to generate $\mathrm{Cr}_3(\mathbf{k})$ (see [68, 20]).

2.3. Algebraic tori and Weyl group. Let $\mathbf{k}$ be a field. Let G be a connected semi-simple algebraic group defined over $\mathbf{k}$. The group G acts on its Lie algebra $\mathfrak{g}$ by the adjoint representation; the $\mathbf{k}$-rank of G is the maximal possible dimension $\dim_{\mathbf{k}}(A)$ over all connected algebraic subgroups A of G which are diagonalizable over $\mathbf{k}$ in $\mathsf{GL}(\mathfrak{g})$. Such a maximal diagonalizable subgroup is called a **maximal torus**. For example, the $\mathbf{R}$-rank of $\mathsf{SL_n}(\mathbf{R})$ is $n-1$, and diagonal matrices form a maximal torus. If $\mathbf{k} = \mathbf{C}$ and the rank of G is equal to r, the centralizer of a generic element $g \in G$ has dimension r. Thus, the rank reflects well the commutation properties inside G.

Theorem 2.4 (Enriques, Demazure, [43, 34]). *Let $\mathbf{k}$ be an algebraically closed field, and $\mathbb{G}_m$ be the multiplicative group over $\mathbf{k}$. Let r be an integer. If $\mathbb{G}_m^r$ embeds as an algebraic subgroup in $\mathrm{Cr}_n(\mathbf{k})$, then $r \leq n$ and, if $r = n$, the embedding is conjugate to an embedding into the group of diagonal matrices in $\mathsf{PGL_{n+1}}(\mathbf{k})$.*

In other words, viewed from its algebraic subgroups, $\mathrm{Cr}_n(\mathbf{k})$ has rank n, and the group of diagonal matrices plays the role of a maximal torus in $\mathrm{Cr}_n(\mathbf{k})$. Its normalizer is the semi-direct product of itself with the group of monomial transformations $\mathsf{GL_n}(\mathbf{Z})$; hence, $\mathrm{Cr}_n(\mathbf{k})$ looks like a group of rank n with Weyl group isomorphic to $\mathsf{GL_n}(\mathbf{Z})$. Nevertheless, for $n = 2$, we shall explain in Section 4 that $\mathrm{Cr}_2(\mathbf{k})$ is better understood as a group of rank 1.

2.4. Finite subgroups. One of the rich and well understood chapters on $\mathrm{Cr}_2(\mathbf{k})$ concerns the study of its finite subgroups. While there is still a lot to due regarding arbitrary fields and conjugacy classes of finite groups, there is now a list of all possible finite groups and maximal algebraic subgroups that can be realized in $\mathrm{Cr}_2(\mathbf{C})$. We refer to [71, 41, 11, 9] for details and references, and to [70] for simple finite subgroups of $\mathrm{Cr}_3(\mathbf{C})$. For instance, a finitary version of Theorem 2.4 has been observed by Beauville in [3] for $n = 2$ (see [69] for $n = 3$). *Let $p \neq char(\mathbf{k})$ be a prime integer. Assume that the abelian group $(\mathbf{Z}/p\mathbf{Z})^r$ embeds into $\mathrm{Cr}_2(\mathbf{k})$; if $p \geq 5$, then $r \leq 2$ and, if $r = 2$, the image of $(\mathbf{Z}/p\mathbf{Z})^r$ is conjugate to a subgroup of the group of diagonal matrices of $\mathsf{PGL}_3(\mathbf{k})$.*

3. An infinite dimensional hyperbolic space

Most recent results are better understood if one explain how $\mathrm{Cr}_2(\mathbf{k})$ acts by isometries on an infinite dimensional hyperbolic space $\mathbb{H}_\infty(\mathbb{P}^2_{\mathbf{k}})$. This construction is due to Manin and Zariski.

Example 3.1. The standard quadratic involution σ maps a line to a conic. Thus, it acts by multiplication by 2 on the Picard group of the plane $\mathbb{P}^2_{\mathbf{k}}$ (or on the homology group $H_2(\mathbb{P}^2(\mathbf{C}), \mathbf{Z})$ if $\mathbf{k} = \mathbf{C}$). Since σ is an involution, the action of σ^2 on that group is the identity, not multiplication by 4. This shows that $\mathrm{Cr}_2(\mathbf{k})$ does not "act" on the Picard group. The forthcoming construction bypasses this difficulty by blowing up all possible indeterminacy points.

3.1. The Picard-Manin space.

3.1.1. General construction. Let X be a smooth, irreducible, projective surface. The Picard group $\mathrm{Pic}(X)$ is the quotient of the abelian group of divisors by the subgroup of principal divisors [54]. The intersection between curves of X determines a quadratic form, the so-called **intersection form**,

$$(C, D) \mapsto C \cdot D \tag{5}$$

on $\mathrm{Pic}(X)$; the quotient of $\mathrm{Pic}(X)$ by the subgroup of divisors E such that $E \cdot D = 0$ for all divisor classes D is denoted by $\mathrm{NS}(X)$. The group $\mathrm{NS}(X)$ is a free abelian group and its rank, the Picard number $\rho(X)$, is finite; when $\mathbf{k} = \mathbf{C}$, $\mathrm{NS}(X)$ can be identified to $H^{1,1}(X; \mathbf{R}) \cap H^2(X; \mathbf{Z})$. The Hodge index Theorem asserts that the signature of the intersection form is equal to $(1, \rho(X) - 1)$ on $\mathrm{NS}(X)$.

If $\pi \colon X' \to X$ is a birational morphism, the pull-back map π^* is an injective morphism from $\mathrm{NS}(X)$ to $\mathrm{NS}(X')$ that preserves the intersection form; hence $\mathrm{NS}(X')$ decomposes as the orthogonal sum of $\pi^*\mathrm{NS}(X)$ and a subspace generated by classes of curves contracted by π, on which the intersection form is negative definite. If $\pi_1 \colon X_1 \to X$ and $\pi_2 \colon X_2 \to X$ are two birational morphisms, there is a third birational morphism $\pi_3 \colon X_3 \to X$ that "covers" π_1 and π_2, meaning that $\pi_3 \circ \pi_1^{-1}$ and $\pi_3 \circ \pi_2^{-1}$ are morphisms (X_3 is obtained from X by blowing-up all points that are blown-up either by π_1 or by π_2).

One can therefore define the inductive limit of the groups $\mathrm{NS}(X')$, where $\pi \colon X' \to X$ describes all birational morphisms onto X. This limit

$$\mathcal{Z}(X) := \lim_{\pi \colon X' \to X} \mathrm{NS}(X') \tag{6}$$

is the **Picard-Manin space** of X. It is an infinite dimensional free abelian group. The intersection forms on $\mathrm{NS}(X')$ determine a quadratic form on $\mathcal{Z}(X)$, the signature of which is equal to $(1, \infty)$. By construction, $\mathrm{NS}(X)$ embeds naturally as a proper subspace of $\mathcal{Z}(X)$, and the intersection form is negative on $\mathrm{NS}(X)^{\perp}$.

Example 3.2. The group $\mathrm{Pic}(\mathbb{P}^2_{\mathbf{k}})$ is generated by the class $\mathbf{e}_0$ of a line. Blow-up one point q_1 of the plane, to get a morphism $\pi_1 \colon X_1 \to \mathbb{P}^2_{\mathbf{k}}$. Then, $\mathrm{Pic}(X_1)$ is a free abelian group of rank 2, generated by the class $\mathbf{e}_1$ of the exceptional divisor E_{q_1}, and by the pull-back of $\mathbf{e}_0$ under π_1 (still denoted $\mathbf{e}_0$ in what follows). More generally, after n blow-ups $X_i \to X_{i-1}$ of points $q_i \in X_{i-1}$ one obtains

$$\mathrm{Pic}(X_n) = \mathbf{Z}\mathbf{e}_0 \oplus \mathbf{Z}\mathbf{e}_1 \oplus \ldots \oplus \mathbf{Z}\mathbf{e}_n \tag{7}$$

where $\mathbf{e}_0$ (resp. $\mathbf{e}_i$) is the class of the total transform of a line (resp. of the exceptional divisor E_{q_i}) by the composite morphism $X_n \to \mathbb{P}^2_{\mathbf{k}}$ (resp. $X_n \to X_i$). The direct sum decomposition (7) is orthogonal with respect to the intersection form. More precisely,

$$\mathbf{e}_0 \cdot \mathbf{e}_0 = 1, \quad \mathbf{e}_i \cdot \mathbf{e}_i = -1 \;\forall\, 1 \leq i \leq n, \quad \text{and} \quad \mathbf{e}_i \cdot \mathbf{e}_j = 0 \;\forall\, 0 \leq i \neq j \leq n. \tag{8}$$

In particular, $\mathrm{Pic}(X) = \mathrm{NS}(X)$ for rational surfaces. Taking limits, one sees that the Picard-Manin space $\mathcal{Z}(\mathbb{P}^2_{\mathbf{k}})$ is a direct sum $\mathcal{Z}(\mathbb{P}^2_{\mathbf{k}}) = \mathbf{Z}\mathbf{e}_0 \oplus \bigoplus_q \mathbf{Z}\mathbf{e}_q$ where q runs over all possible points that can be blown-up (including infinitely near points).

3.1.2. Hyperbolic space. Fix an ample class $\mathbf{e}_0$ in $\mathrm{NS}(X) \subset \mathcal{Z}(X)$. Denote by $\mathcal{Z}(X, \mathbf{R})$ and $\mathrm{NS}(X, \mathbf{R})$ the tensor products $\mathcal{Z}(X) \otimes_{\mathbf{Z}} \mathbf{R}$ and $\mathrm{NS}(X) \otimes_{\mathbf{Z}} \mathbf{R}$. Elements of $\mathcal{Z}(X, \mathbf{R})$ are finite sums $u_X + \sum_i a_i \mathbf{e}_i$ where u_X is an element of $\mathrm{NS}(X, \mathbf{R})$, each $\mathbf{e}_i$ is the class of an exceptional divisor, and the coefficients a_i are real numbers. Allowing infinite sums $\sum_i a_i \mathbf{e}_i$ with $\sum_i a_i^2 < +\infty$, one gets a new space $\mathsf{Z}(X)$, on which the intersection form extends continuously [21].

The set of vectors u in $\mathsf{Z}(X)$ such that $u \cdot u = 1$ is a hyperboloïd. The subset

$$\mathbb{H}_\infty(X) = \{u \in \mathsf{Z}(X) \mid \quad u \cdot u = 1 \quad \text{and} \quad u \cdot \mathbf{e}_0 > 0\} \tag{9}$$

is the sheet of that hyperboloid containing ample classes of $\mathrm{NS}(X, \mathbf{R})$. With the distance $\mathrm{dist}(\cdot, \cdot)$ defined by

$$\cosh \mathrm{dist}(u, u') = u \cdot u', \tag{10}$$

$\mathbb{H}_\infty(X)$ becomes a complete, simply connected, infinite dimensional riemannian manifold with constant curvature -1 (see [52, 7, 29]).

The projection of $\mathbb{H}_\infty(X)$ in the projective space $\mathbb{P}(\mathsf{Z}(X))$ is injective. The boundary $\partial\mathbb{H}_\infty(X)$ of its image is the projection of the isotropic cone of the intersection form, and can be identified with the boundary of $\mathbb{H}_\infty(X)$ as a Gromov hyperbolic space [13]. The closure $\mathbb{H}_\infty(X) \cup \partial\mathbb{H}_\infty(X)$ is denoted by $\overline{\mathbb{H}_\infty(X)}$ (this space is not locally compact).

We denote by $\mathrm{Isom}(\mathsf{Z}(X))$ the group of isometries of $\mathsf{Z}(X)$ with respect to the intersection form, and by $\mathrm{Isom}(\mathbb{H}_\infty(X))$ the subgroup that preserves $\mathbb{H}_\infty(X)$.

3.1.3. Action of $\mathrm{Bir}(X)$. Given $f \in \mathrm{Bir}(X)$, there is a birational morphism $\pi\colon X' \to X$, obtained by blowing up indeterminacy points of f, such that f lifts to a morphism $f'\colon X' \to X$ (see [54]). By pull back, the transformation f' determines an isometry $(f')^*$ from $\mathcal{Z}(X)$ to $\mathcal{Z}(X')$: identifying $\mathcal{Z}(X)$ to $\mathcal{Z}(X')$ by π^*, we obtain an isometry f^* of $\mathcal{Z}(X)$. Since all points of X have been blown-up to define $\mathcal{Z}(X)$, birational transformations behave as regular automorphisms on $\mathcal{Z}(X)$, and one can show that the map $f \mapsto f_* = (f^{-1})^*$ is a morphism from $\mathrm{Bir}(X)$ to the group $\mathrm{Isom}(\mathcal{Z}(X))$; hence, after completion, $\mathrm{Bir}(X)$ acts on $\mathbb{H}_\infty(X)$ by isometries.

Theorem 3.3 (Manin, [60]). *Let X be a projective surface defined over an algebraically closed field $\mathbf{k}$. The morphism $f \mapsto f_*$ is an injective morphism from $\mathrm{Bir}(X)$ to the group of isometries of $\mathsf{Z}(X)$ (hence of $\mathbb{H}_\infty(X)$).*

3.2. Types and degree growth. Isometries of $\mathbb{H}_\infty(X)$ are classified into three types [16]. **Elliptic** isometries have a fixed point in $\mathbb{H}_\infty(X)$, and act as rotations around it. **Parabolic** isometries have a unique fixed point in $\overline{\mathbb{H}_\infty(X)}$, located on $\partial\mathbb{H}_\infty(X)$, and all orbits accumulate towards it. **Loxodromic** isometries have two fixed points in $\overline{\mathbb{H}_\infty(X)}$, both of them on $\partial\mathbb{H}_\infty(X)$, one repulsive and one attracting. Moreover, $s \in \mathrm{Isom}(\mathbb{H}_\infty(X))$ is loxodromic if and only if its translation length

$$L(s) = \inf\{\mathrm{dist}(x, s(x)) \mid x \in \mathbb{H}_\infty(X)\} \tag{11}$$

is positive. In that case, $\lambda(s) = \exp(L(s))$ is the largest eigenvalue of s as a linear transformation of $\mathsf{Z}(X)$ and, for all vectors u in $\mathbb{H}_\infty(X)$, the sequence $\lambda(s)^{-n}s^n(u)$ converges in $\mathsf{Z}(X)$ towards a non-zero isotropic vector; the isotropic line determined by this vector corresponds to the attracting fixed point of s on $\partial\mathbb{H}_\infty(X)$.

Since $\mathrm{Bir}(X)$ acts faithfully on $\mathbb{H}_\infty(X)$, there are three types of birational transformations: elliptic, parabolic, and loxodromic, according to the type of the associated isometry of $\mathbb{H}_\infty(X)$. We now describe how each type can be characterized in algebraic terms.

Let $\mathbf{h} \in \mathrm{NS}(X, \mathbf{R})$ be an ample class with self-intersection 1. Define the degree of f with respect to the polarization $\mathbf{h}$ by

$$\deg_{\mathbf{h}}(f) = f_*(\mathbf{h}) \cdot \mathbf{h} = \cosh(\mathrm{dist}(\mathbf{h}, f_*\mathbf{h})). \tag{12}$$

For instance, if f is an element of $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$, and $\mathbf{h} = \mathbf{e}_0$ is the class of a line, then $\deg_{\mathbf{h}}(f)$ is the degree of f, as defined in §1.1.

The sequence $\deg_{\mathbf{h}}(f^n)^{1/n}$ converges towards a real number $\lambda(f) \geq 1$, called the **dynamical degree** of f; its logarithm $\log(\lambda(f))$ is the translation length of the isometry f_*, because $\deg_{\mathbf{h}}(f) = \cosh(\mathrm{dist}(\mathbf{h}, f_*\mathbf{h}))$. Consequently, $\lambda(f)$ does not depend on the polarization and is invariant under conjugacy. In particular, f is loxodromic if and only if $\lambda(f) > 1$. Elliptic and parabolic transformations are also classified in terms of degree growth. Say that a sequence of real numbers $(d_n)_{n\geq 0}$ grows linearly (resp. quadratically) if $n/c \leq d_n \leq cn$ (resp. $n^2/c \leq d_n \leq cn^2$) for some $c > 0$.

Theorem 3.4 (Gizatullin, Cantat, Diller and Favre, see [49, 17, 18, 39]). *Let X be a projective surface defined over an algebraically closed field $\mathbf{k}$, f be a birational transformation of X, and $\mathbf{h}$ be a polarization of X.*

- *f is elliptic if and only if the sequence* $\deg_{\mathbf{h}}(f^n)$ *is bounded. In this case, there exists a birational map* $\phi\colon Y \dashrightarrow X$ *and an integer* $k \geq 1$ *such that* $\phi^{-1} \circ f \circ \phi$ *is an automorphism of* Y *and* $\phi^{-1} \circ f^k \circ \phi$ *is in the connected component of the identity of the group* $\mathrm{Aut}(Y)$.

- *f is parabolic if and only if the sequence* $\deg_{\mathbf{h}}(f^n)$ *grows linearly or quadratically with* n. *If* f *is parabolic, there exists a birational map* $\psi\colon Y \dashrightarrow X$ *and a fibration* $\pi\colon Y \to B$ *onto a curve* B *such that* $\psi^{-1} \circ f \circ \psi$ *permutes the fibers of* π. *The fibration is rational if the growth is linear, and elliptic (or quasi-elliptic if char*$(\mathbf{k}) \in \{2,3\}$*) if the growth is quadratic.*

- *f is loxodromic if and only if* $\deg_{\mathbf{h}}(f^n)$ *grows exponentially fast with* n*: there is a constant* $b(f) > 0$ *such that* $\deg_{\mathbf{h}}(f^n) = b(f)\lambda(f)^n + O(1)$.

Remark 3.5. If f is parabolic, the push forward of the fibration $\pi\colon Y \to B$ by ψ is the unique f-invariant (singular) algebraic foliation [25].

Example 3.6. All transformations $(X,Y) \mapsto (X, Q(X)Y)$ with $Q \in \mathbf{k}(X)$ of degree $\deg(Q) \geq 1$ are parabolic transformations of $\mathbb{P}^2_{\mathbf{k}}$ with linear degree growth.

Assume $\mathbf{k} = \mathbf{C}$. Let ι be a square or cubic root of -1 and E be the elliptic curve $\mathbf{C}/\mathbf{Z}[\iota]$. The linear transformation $(x,y) \mapsto (x+y, y)$ of $\mathbf{C}^2$ preserves $\mathbf{Z}[\iota] \times \mathbf{Z}[\iota]$: it determines an automorphism f of the abelian surface $X = E \times E$, that commutes to the automorphism $m(x,y) = (\iota x, \iota y)$. The sequence $\deg_{\mathbf{h}}(f^n)$ grows quadratically. The quotient X/m is rational, and f induces an automorphism of X/m, hence a birational transformation of $\mathbb{P}^2_{\mathbf{C}}$ with quadratic degree growth.

3.3. Comparison with mapping class groups. Let $g \geq 2$ be an integer, and $\mathrm{Mod}(g)$ be the mapping class group of the compact orientable surface of genus g. Theorem 3.4 parallels Nielsen-Thurston classification of isotopy classes of homeomorphisms $\varphi \in \mathrm{Mod}(g)$ (see [45, 22]).

The two types of parabolic transformations $f \in \mathrm{Bir}(X)$, those with linear or quadratic degree growth, are respectively called **de Jonquières twists** and **Halphen twists**. This is justified by the analogy with Dehn (multi-)twists $\varphi \in \mathrm{Mod}(g)$ and by the following two facts (for $X = \mathbb{P}^2_{\mathbf{k}}$). If the growth is linear, the invariant foliation can be transformed into a pencil of lines by an element of $\mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$; hence $f \in \mathrm{Jonq}_2(\mathbf{k})$ up to conjugacy. If the growth is quadratic, it can be transformed in a Halphen pencil [55, 40].

Loxodromic elements $f \in \mathrm{Bir}(X)$ should be compared to pseudo-Anosov classes $\varphi \in \mathrm{Mod}(g)$. The dynamical degree $\lambda(f)$ is a substitute for the stretching factor of φ. The action of f_* on $\mathbb{H}_\infty(X)$ is somehow analogous to the action of φ on the Teichmüller space. When $\mathbf{k} = \mathbf{C}$, fixed points on the boundary $\partial\mathbb{H}_\infty(X)$ are "represented" by f-invariant closed positive currents on X with a laminar structure, while fixed points of φ on the boundary of Thurston's compactification of the Teichmüller space correspond to invariant measured foliations. We refer to [45, 22, 21] for this dictionary, and to [4, 5, 21, 38, 42, 46] for dynamical properties of loxodromic birational transformations.

3.4. Dynamical degrees and automorphisms. If g is an automorphism of X, $\lambda(g)$ is equal to the spectral radius of the linear transformation $g^*\colon \mathrm{NS}(X) \to \mathrm{NS}(X)$. This shows that $\lambda(g)$ is an algebraic number because g^* preserves the integral structure of $\mathrm{NS}(X)$. A similar phenomenon occurs for $f \in \mathrm{Bir}(X)$; after a finite number of blow-ups, the action of f on $\mathrm{NS}(X)$ is multiplicative, *i.e.* $(f_*)^n = (f^n)_*$ for all $n \geq 1$ (here f_* denotes temporarily the action on $\mathrm{NS}(X)$), and $\lambda(f)$ is equal to the leading eigenvalue of f_* (see [39]). For example, if $f = \sigma$ is the standard quadratic involution, the three indeterminacy points need to be blown-up.

A Pisot number is a real algebraic integer $\alpha > 1$, all of whose conjugates $\alpha' \neq \alpha$ have modulus < 1. A Salem number is a real algebraic integer $\beta > 1$ such that $1/\beta$ is a conjugate of β, all other conjugates have modulus 1, and there is at least one conjugate β' on the unit circle. The set of Pisot numbers is countable, closed, and contains accumulation points (the smallest one being the golden mean); the smallest Pisot number is the root $\lambda_P \simeq 1.3247$ of $t^3 = t + 1$. Salem numbers are not well understood yet; its smallest known element is the Lehmer number $\lambda_L \simeq 1.1762$, a root of $t^{10} + t^9 - t^7 - t^6 - t^5 - t^4 - t^3 + t + 1 = 0$.

Theorem 3.7 (Diller and Favre, McMullen, Blanc and Cantat [39, 62, 63, 64, 23]). *Let X be a projective surface, defined over an algebraically closed field $\mathbf{k}$. Let f be a birational transformation of X with dynamical degree $\lambda(f) > 1$. Then $\lambda(f)$ is either a Pisot number or a Salem number and*

(a) *if $\lambda(f)$ is a Salem number, then there exists a birational map $\psi\colon Y \dashrightarrow X$ which conjugates f to an automorphism of Y;*

(b) *if f is conjugate to an automorphism, as in* (a), *$\lambda(f)$ is either a quadratic integer or a Salem number.*

Moreover, $\lambda(f) \geq \lambda_L$, where λ_L is the Lehmer number and there are examples of birational transformations of the complex projective plane (resp. of some complex K3 surfaces) such that $\lambda(f) = \lambda_L$.

4. Subgroups of finite type and normal subgroups

According to the previous section, the Cremona group acts by isometries on an infinite dimensional hyperbolic space, and there is a powerful dictionary between the classification of isometries and the classification of birational maps in terms of degree growth and invariant fibrations. In this section, we explain how this dictionary can be used to describe the structure of the group $\mathrm{Cr}_2(\mathbf{k})$.

4.1. Tits Alternative. A group G **satisfies Tits alternative** if the following property holds for all subgroups Γ of finite type in G: either Γ contains a finite index solvable subgroup or Γ contains a free non-abelian subgroup (i.e. a copy of the free group $\mathbb{F}_r$, with $r \geq 2$). Tits alternative holds for linear groups $\mathsf{GL}_\mathsf{n}(\mathbf{k})$ (see [72]), but not for the group of $\mathcal{C}^\infty$-diffeomorphisms of the circle $\mathbb{S}^1$ (see [14],

[48]). If G satisfies Tits alternative, it does not contain groups with intermediate growth; its finite type subgroups are tame, from a geometric point of view.

The main technique to prove that a group contains a non-abelian free group is the ping-pong lemma. Let g_1 and g_2 be two bijections of a set S. Assume that S contains two non-empty disjoint subsets S_1 and S_2 such that $g_1^m(S_2) \subset S_1$ and $g_2^m(S_1) \subset S_2$ for all $m \in \mathbf{Z}^*$. Then, according to the ping-pong lemma, the subgroup of $\mathrm{Bij}(S)$ generated by g_1 and g_2 is a free group on two generators [31]. Now, consider a group Γ that acts on a hyperbolic space $\mathbb{H}_\infty$ and contains two loxodromic isometries h_1 and h_2 with four distinct fixed points on $\partial\mathbb{H}_\infty$. Take two disjoint neighborhoods S_1 and S_2 of the sets of fixed points of h_1 and h_2 in $\overline{\mathbb{H}_\infty}$. Then, the ping-pong lemma applies to sufficiently high powers $g_1 = h_1^n$ and $g_2 = h_2^n$, and produce a free subgroup of Γ.

This strategy can be used for $\mathrm{Bir}(X)$, acting on $\mathbb{H}_\infty(X)$ by isometries. The difficulty resides in the study of subgroups that do not contain any ping-pong pair of loxodromic isometries; Theorem 3.4 comes in help to deal with this situation, and leads to the following result.

Theorem 4.1 ([21]). *If X is a projective surface over a field $\mathbf{k}$, the group $\mathrm{Bir}(X)$ satisfies Tits alternative.*

If M is a projective variety (resp. a compact kähler manifold), its group of automorphisms satisfies also the Tits alternative [21].

Question 4.2. Does $\mathrm{Cr}_n(\mathbf{k})$ satisfy Tits alternative for all $n \geq 3$?

Would the answer be yes, one would obtain a proof of Tits alternative for all subgroups of Cremona groups: this includes linear groups, mapping class groups of surfaces, and $\mathrm{Out}(\mathbf{F}_g)$ for all $g \geq 1$ (see §1.3.1; see [8] for Tits alternative in this context). In the same spirit – comparing subgroups of Cremona groups to subgroups of linear groups – the most basic question that has not found any answer yet is the following, which parodies Malcev's and Selberg's theorems.

Question 4.3. Are finitely generated subgroups of $\mathrm{Cr}_n(\mathbf{k})$ residually finite ? Does every finitely generated subgroup of $\mathrm{Cr}_n(\mathbf{k})$ contain a torsion free subgroup of finite index ? (see [2] for automorphisms of $\mathbb{A}^n_{\mathbf{k}}$)

4.2. Rank one phenomena. As explained in §2.3, the Cremona group $\mathrm{Cr}_2(\mathbf{k})$ behaves like an algebraic group of rank 2, with a maximal torus given by the group of diagonal matrices in $\mathsf{PGL}_3(\mathbf{k})$. On the other hand, generic elements of degree $d \geq 2$ in $\mathrm{Cr}_2(\mathbf{C})$ are loxodromic (not elliptic) and, as such, cannot be conjugate to elements of this maximal torus. This suggests that $\mathrm{Cr}_2(\mathbf{k})$ has rank 1 from the point of view of its generic elements. The following statement provides a strong version of this principle.

Theorem 4.4 ([21, 23]). *Let $\mathbf{k}$ be a field. Let X be a projective surface over $\mathbf{k}$ and f be a loxodromic element of $\mathrm{Bir}(X)$. Then, the infinite cyclic subgroup of $\mathrm{Bir}(X)$ generated by f has finite index in the centralizer $\{g \in \mathrm{Bir}(X) \mid g \circ f = f \circ g\}$.*

Another rank one phenomena comes from the rigidity of rank 2 subgroups of $\mathrm{Cr}_2(\mathbf{k})$. Let G be a real, almost simple, linear algebraic group and Γ be a lattice in G, *i.e.* a discrete subgroup such that G/Γ has finite Haar volume. When the $\mathbf{R}$-rank of G is at least 2, Γ inherits its main algebraic properties from G (see [61]). For instance, Γ has Kazhdan property (T), according to which all representations of Γ by unitary motions on a Hilbert space have a global fixed point.

Theorem 4.5 (Deserti, Cantat, [35, 21]). *Let $\mathbf{k}$ be an algebraically closed field and X be a projective surface over $\mathbf{k}$. Let Γ be a countable group with Kazhdan property (T). If $\rho\colon \Gamma \to \mathrm{Bir}(X)$ is a morphism with infinite image, then ρ is conjugate to a morphism into $\mathsf{PGL}_3(\mathbf{k})$ by a birational map $\psi\colon X \dashrightarrow \mathbb{P}^2_{\mathbf{k}}$.*

In [35, 36, 37], Déserti draws several algebraic consequences of this result; for instance, she can list all abstract automorphisms of $\mathrm{Cr}_2(\mathbf{C})$.

Let G be a simple real Lie group of rank r. As a byproduct of Theorem 4.5, $\mathrm{Cr}_2(\mathbf{C})$ does not contain any lattice of G if $r \geq 2$, except when G is isomorphic to $\mathsf{PSL}_3(\mathbf{R})$ or $\mathsf{PSL}_3(\mathbf{C})$. This supports Zimmer's conjecture, which predicts that such a lattice cannot act faithfully by diffeomorphisms on a compact manifold of dimension $< r$. We refer to [47] for a survey on Zimmer's program, to [19, 27] for the case of holomorphic diffeomorphisms of compact kähler manifolds, and to [33, 24] for the existence of rank 1 lattices in $\mathrm{Cr}_2(\mathbf{C})$.

4.3. Normal subgroups. Let us pursue the comparison between groups of birational transformations and groups of diffeomorphisms. If M is a connected compact manifold and $\mathrm{Diff}^\infty_0(M)$ denotes the group of infinitely differentiable diffeomorphisms of M which are isotopic to the identity, then $\mathrm{Diff}^\infty_0(M)$ is a simple group: it does not contain any normal subgroup except $\{\mathrm{Id}_M\}$ and the group $\mathrm{Diff}^\infty_0(M)$ itself (see [1]). From Noether-Castelnuovo Theorem, one can show that $\mathrm{Cr}_2(\mathbf{C})$ is "connected"; hence, there is no need to rule out connected components, as for diffeomorphisms. Enriques conjectured in 1894 that $\mathrm{Cr}_2(\mathbf{C})$ is a simple group, and this is indeed true from the point of view of its algebraic subgroups [44, 10]. On the other hand, as an abstract group, $\mathrm{Cr}_2(\mathbf{C})$ is far from being simple.

Theorem 4.6 (Cantat and Lamy, [26]). *Let $\mathbf{k}$ be an algebraically closed field. The group $\mathrm{Cr}_2(\mathbf{k})$ is not a simple group. If $\mathbf{k} = \mathbf{C}$ is the field of complex numbers, $\mathrm{Cr}_2(\mathbf{C})$ contains an uncountable family of distinct normal subgroups.*

To prove this theorem, one makes use of the action of $\mathrm{Cr}_2(\mathbf{k})$ on $\mathbb{H}_\infty(\mathbb{P}^2_{\mathbf{k}})$, and of ideas coming from small cancellation theory and the geometry of hyperbolic groups in the sense of Gromov, as in [32]. One obtains the existence of a constant $N > 1$ with the following property: *there is a loxodromic element g in $\mathrm{Cr}_2(\mathbf{k})$ such that all elements $h \neq Id$ of the smallest normal subgroup containing g^N are loxodromic elements with $\lambda(h) > \lambda(g)$.* When $\mathbf{k} = \mathbf{C}$, one can choose a generic element of degree 2 for g.

The same type of strategy is used in various contexts, as in the recent proof, by Dahmani, Guirardel and Osin, that high powers of pseudo-Anosov elements generate strict, non-trivial, normal subgroups in mapping class groups. Applied to the Cremona group, their techniques lead to the following.

Theorem 4.7 (Dahmani, Guirardel, and Osin, [26, 30]). *Let* $\mathbf{k}$ *be an algebraically closed field. The Cremona group* $\mathrm{Cr}_2(\mathbf{k})$ *is sub-quotient universal: every countable group can be embedded in a quotient group of* $\mathrm{Cr}_2(\mathbf{k})$*.*

Being sub-quotient universal, while surprising at first sight, is a common feature of hyperbolic groups [32, 67]. For instance, $\mathsf{SL}_2(\mathbf{Z})$ is sub-quotient universal [58].

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