RESOLUTION OF INDETERMINACY OF PAIRS

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In memory of Paolo Francia

Introduction

Suppose that G is a finite subgroup of the group of birational transformations $\mathrm{Bir}(Y)$ of a projective variety Y. Then a resolution of indeterminacy of the pair (Y, G) consists of a smooth variety X, birationally equivalent to Y, and a birational map $\phi : X \dashrightarrow Y$ such that for every $\tau \in G$ the composite map $\phi^{-1}\tau\phi$ is an automorphism of X.

One motivation for finding resolutions is the study of the group $\mathrm{Bir}(Y)$ itself. The general philosophy is that, by resolving the indeterminacy, we reduce to study isomorphisms of smooth varieties. See for instance [1] and [5], where this process of resolution is applied to classify cyclic subgroups of $\mathrm{Bir}(\mathbb{P}^2)$ up to conjugation.

The first non-trivial example of resolution of indeterminacy of pairs occurs when resolving the fundamental locus of a birational involution τ of a smooth surface by a minimal sequence of monoidal transformations $f : X \to Y$. Indeed one can show that in this case $f^{-1}\tau f \in \mathrm{Aut}(X)$. On the other hand, if for instance we consider birational transformations τ of any order greater that two, then in general this is not true, even in dimension two. Clearly the picture becomes more complicated if we consider non-cyclic groups or increase the dimension.

One can see that resolutions of indeterminacy of pairs (Y, G) always exist. This is probably well known to the specialists. We give an elementary proof of this basic result in Section 1.

In this paper, we study the two dimensional case in detail. In Section 2 we formalize the terminology concerning infinitely near points in the language of algebraic valuations. We use a theorem of Zariski on algebraic valuations to define a topology on the set of algebraic valuations of a given smooth projective surface, and establish a correspondence between birational morphisms of smooth surfaces and finite closed subsets of algebraic valuations. In Section 3, we apply these results to show that a minimal resolution of a pair (Y, G) exists, when Y is a smooth projective surface.

In the last section we introduce a birational invariant of each subgroup of prime order of $\mathrm{Bir}(Y)$ when Y is a projective surface. We use

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this invariant to distinguish whether two subgroups of the same prime order of $\mathrm{Bir}(\mathbb{P}^2)$ are conjugate.

We would like to mention Cheltsov's paper [4], where an explicit construction of resolution of indeterminacy of pairs (called there *regularization*) is given in dimension three using the Minimal Model Program.

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Throughout this paper, all varieties are assumed to be defined over an algebraically closed field k of characteristic zero.

1. The resolution of pairs in its general form

Let Y be a projective variety defined over an algebraically closed field k, and let G be a group.

Definition 1.1. We say that G *acts birationally* on Y if a homomorphism $\eta : G \to \mathrm{Bir}(Y)$ is assigned; G is said to *act faithfully* if η is injective. The action of G on Y is said *biregular* if $\eta(G) \subset \mathrm{Aut}(Y)$.

We fix a group G, and consider pairs (Y, G), where Y is a projective variety and G acts birationally on Y. We will always assume that the action of G is faithful, and look at G as a subgroup of $\mathrm{Bir}(Y)$.

Definition 1.2. Let (X, G) and (Y, G) two pairs. A dominant rational map $\phi : X \dashrightarrow Y$ is said to be a *G-equivariant rational map*, or *rational map of pairs*, if the birational actions of G on X and Y commute with ϕ. We also write $\phi : (X, G) \dashrightarrow (Y, G)$. Two pairs (X, G) and (Y, G) are said to be *birationally equivalent* if there is a birational map of pairs $\phi : (X, G) \dashrightarrow (Y, G)$. In this case, we write $(X, G) \sim_\phi (Y, G)$.

Definition 1.3. Given a pair (Y, G), a second pair (X, G) is called *resolution of indeterminacy of* (Y, G) if the following three conditions are satisfied:

(i) X is a smooth projective variety,
(ii) there is a birational map $\phi : X \dashrightarrow Y$ such that $(X, G) \sim_\phi (Y, G)$,
(iii) $G \subset \mathrm{Aut}(X)$ inside $\mathrm{Bir}(X)$.

We also say that ϕ *resolves* the indeterminacy of (Y, G).

Theorem 1.4. (Existence of resolution – General form.) *Let Y be a variety and G be a finite group acting birationally on Y. Then there exists a smooth projective variety X and a birational map $\phi : X \dashrightarrow Y$ which resolves the indeterminacy of (Y, G).*

Proof. Choose W to be a smooth projective variety such that $K(W) = K(Y)^G$. Let Z be the normalization of W in the function field $K(Y)$. Now observe that G acts biregularly on Z. Then a G-equivariant resolution of the singularities of Z (cf. [8]) gives a resolution of indeterminacy (X, G) of (Y, G). □

2. Algebraic valuations and birational morphisms

Throughout this section Y will denote a smooth projective surface over an algebraically closed field k.

Since M. Noether, mathematicians realized the importance of understanding the geometry of a surface Y at the level of infinitely near points in order to study rational maps upon it. Classically, an *infinitely near point* of Y is a (reduced closed) point lying on some exceptional divisor over Y. For our purpose, it is more convenient to describe an infinitely near point as the discrete valuation along the exceptional divisor obtained by blowing up such point. So, we recall the following definition:

Definition 2.1. An *algebraic valuation ring* R of Y is a discrete valuation ring in $K(Y)$ which is determined, through a birational map $\phi : X \dashrightarrow Y$, by the local ring of an irreducible divisor E of a smooth projective surface X. An *algebraic valuation* v of Y is the discrete valuation determined by an algebraic valuation ring R of Y. The center on Y of v is the image of the closed point of $\operatorname{Spec} R$ in Y. The set of algebraic valuations of Y is denoted by $\mathrm{Val}(Y)$. We denote by $\mathrm{Val}_0(Y)$ the set of valuations along irreducible divisors of Y, and set $\mathrm{Val}_+(Y) := \mathrm{Val}(Y) - \mathrm{Val}_0(Y)$. [1]

In this section we will introduce a natural topology on the set $\mathrm{Val}(Y)$ of algebraic valuations of Y. This is probably well know to the experts, but we would like to set up some standard notation which will be used in Section 3.

An important consequence in dimension two of Zariski's Theorem [7, Lemma 2.45] is the following property:

Lemma 2.2. *For any algebraic valuation $v \in \mathrm{Val}_+(Y)$ of a smooth surface Y there is a unique minimal sequence of blowups of centers of v such that v is the valuation along the exceptional divisor E of the last blow up of the sequence.*

Definition 2.3. We will refer to the minimal sequence of blowups mentioned in Lemma 2.2 as the *minimal extraction* of v.

Lemma 2.2 allows us to define a partial ordering in $\mathrm{Val}_+(Y)$.

Definition 2.4. Let $v, w \in \mathrm{Val}_+(Y)$. We put $v \geq w$, and say that v *dominates* w, if w is the valuation along one of the prime exceptional divisors occurring in the minimal extraction of v. The *level* (over Y) of v is defined as the number i of valuations $w \in \mathrm{Val}_+(Y)$ dominated by v (in other words, i is the number of prime exceptional divisors occurring in the minimal extraction of v).

[1] In [7, Pag. 50], algebraic valuations of a variety Y are defined as the valuations along irreducible exceptional divisors over Y. These correspond to the elements in $\mathrm{Val}_+(Y)$ in our notation.

Remark 2.5. The level 1 elements of $\mathrm{Val}_+(Y)$ are precisely the minimal elements with respect to the partial order defined above.

If we denote by $\mathrm{Val}_i(Y)$ the set of valuations of Y of level i, then

$$\mathrm{Val}(Y) = \mathrm{Val}_0(Y) \sqcup \mathrm{Val}_+(Y) = \sqcup_{i\geq 0} \mathrm{Val}_i(Y).$$

Definition 2.6. Let $V \subset \mathrm{Val}_+(Y)$ be a subset. The *support* of V in Y is the union of the centers on Y of the elements of V. An element v of V is said to be *maximal* in V if it is not dominated by any other element of V.

We can define a topology on $\mathrm{Val}_+(Y)$ by choosing as closed sets the subsets $V \subset \mathrm{Val}_+(Y)$ which satisfy the following condition: if $v \in V$ and $w < v$, then $w \in V$. It is immediate to verify indeed that

Lemma 2.7. *The closed subsets of* $\mathrm{Val}_+(Y)$*, as defined above, satisfy the axioms for a topology on* $\mathrm{Val}_+(Y)$*.*

Note that, in this topology, if V is a closed set in $\mathrm{Val}_+(Y)$ containing all maximal elements of another set $W \subset \mathrm{Val}_+(Y)$, then $V \supset W$. Note also that the support of a closed set $V \subset \mathrm{Val}_+(Y)$ is the union of the centers on Y of the elements of level 1 of V.

Remark 2.8. The topology of $\mathrm{Val}_+(Y)$ extends to a topology of $\mathrm{Val}(Y)$ by choosing as closed subsets the sets of the form $\mathrm{Val}_0(Y) \sqcup V$, where V is closed in $\mathrm{Val}_+(Y)$, and the empty set. See also [9] for other natural topology on the set of valuations.

Let $\mathcal{V}$ be the category whose objects are finite closed subsets $V \subset \mathrm{Val}_+(Y)$ and whose morphisms are inclusions of sets. Let $\mathcal{B}$ be the category of smooth projective Y-surfaces birational to Y. An object in $\mathcal{B}$ is a birational morphism $f : X \to Y$, where X is a smooth projective surface, and, for $f, f' \in \mathrm{Obj}(\mathcal{B})$, a morphism $\alpha : f \to f'$ is a commutative diagram

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\ & Y. & \end{tikzcd}$$

Then we define

$$F : \mathcal{V} \to \mathcal{B}$$

by associating to any $V \in \mathrm{Obj}(\mathcal{V})$ the morphism

$$f_V = f_h \cdots f_0 : X := Y_{h+1} \to Y_0 := Y,$$

where $f_i : Y_i \to Y_{i-1}$ is recursively defined as the blow up of the centers on Y_{i-1} of the elements of V of level i. Conversely, consider

$$G : \mathcal{B} \to \mathcal{V}$$

which associates to any $f \in \mathrm{Obj}(\mathcal{B})$ the set $V(f)$ of valuations of the prime exceptional divisors of f.

Theorem 2.9. *F and G establish an equivalence of categories $\mathcal{V} \cong \mathcal{B}$.*

Proof. It is immediate to verify that F and G are well defined contravariant functors, and that each one is the inverse of the other. $\square$

We observe that, if $\phi : X \dashrightarrow Y$ is a birational map of smooth projective surfaces, then ϕ induces a bijection $\phi_* : \mathrm{Val}(X) \to \mathrm{Val}(Y)$. Moreover, if T is another smooth projective surface and $\psi : Y \dashrightarrow T$ is a birational map, then $\psi_*\phi_* = (\psi\phi)_*$ (as a bijection $\mathrm{Val}(X) \to \mathrm{Val}(T)$). In particular, we see that $(\phi_*)^{-1} = (\phi^{-1})_* =: \phi_*^{-1}$ (as a bijection $\mathrm{Val}(Y) \to \mathrm{Val}(X)$).

Now we consider a birational morphism of smooth surfaces

$$f : X \to Y.$$

Let $V(f) \subset \mathrm{Val}_+(Y)$ be the associated closed set. We introduce the following notation:

$$V(Y, f) := \mathrm{Val}_0(Y) \sqcup V(f).$$

We observe that

$$f_*^{-1} V(Y, f) = \mathrm{Val}_0(X).$$

In other words, $V(Y, f)$ is the set of algebraic valuations of Y which correspond, through f_*^{-1}, to valuations of X along effective divisors (on X). Note that this set depends on the particular morphism f from X to Y. We have the following characterization of resolution of birational transformations:

Proposition 2.10. *Let f, X and Y as above, and let $\tau \in \mathrm{Bir}(Y)$. Then $f^{-1}\tau f \in \mathrm{Aut}(X)$ if and only if*

$$\tau_*^{-1} V(Y, f) = V(Y, f).$$

Before proving this proposition, we investigate some properties, concerning resolutions of birational maps of surfaces, which follow from Theorem 2.9. Let $\phi : X \dashrightarrow Y$ be a birational map of smooth projective surfaces, and let

$$(*) \qquad \begin{tikzcd} & Z \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \arrow[rr, dashed, "\phi"] & & Y \end{tikzcd}$$

be a resolution of indeterminacy of ϕ.

Lemma 2.11. (1) *With the above notation,*

$$V(X, p) = \phi_*^{-1} V(Y, q) \supset \phi_*^{-1} \mathrm{Val}_0(Y)$$

(2) *If $E \subset Z$ is an irreducible divisor and $v_E \in \mathrm{Val}(X)$ and $w_E \in \mathrm{Val}(Y)$ are the respective valuations determined by E, then $\phi_*(v_E) = w_E$. Moreover, if E is both p-exceptional and q-exceptional, then v_E is maximal in $V(p)$ if and only if w_E is maximal in $V(q)$.*

Proof. The first part follows from
$$p_*^{-1}V(X,p) = \mathrm{Val}_0(Z) = q_*^{-1}V(Y,q).$$
Regarding the second part, we see that $w_E(h) = v_E(\phi^*(h))$ for every function $h \in K(Y)$. The last assertion follows by observing that the valuation along E is maximal in either direction if and only if E is a (-1)-curve on Z. □

Now we review the definition of minimal resolution of indeterminacy:

Definition 2.12. Diagram $(*)$ is a *minimal resolution* (of indeterminacy) of ϕ if there are no maximal elements v of $V(p)$ such that $\phi_*(v)$ is a maximal element of $V(q)$.

This definition clearly coincides with the usual notion of minimal resolution.

Lemma 2.13. *Consider the resolution given by the commutative diagram $(*)$.*

(1) *The diagram is a minimal resolution of ϕ if and only if it is a minimal resolution of ϕ^{-1}.*
(2) *The resolution is minimal if and only if*
$$V(X,p) = \overline{\phi_*^{-1}\,\mathrm{Val}_0(Y)}.$$
In particular, the minimal resolution of ϕ exists and is unique.
(3) *If the resolution is minimal, then every maximal element v of $V(p)$ is in $\phi_*^{-1}\,\mathrm{Val}_0(Y)$.*

Proof. The definition of minimal resolution is symmetric by Lemma 2.11, thus (1) follows. To prove (2), assume that $V(X,p) \neq \overline{\phi_*^{-1}\,\mathrm{Val}_0(Y)}$. Note that
$$\mathrm{Val}_0(X) \subset \overline{\phi_*^{-1}\,\mathrm{Val}_0(Y)} \subset V(X,p).$$
Thus we can find a maximal element v of $V(p)$ such that $\phi_*(v) \in V(q)$. Then, by Lemma 2.11 (2), $\phi_*(v)$ is maximal in $V(q)$, thus the resolution is not minimal. Note that the morphism p which gives the minimal resolution of ϕ is uniquely determined by the condition in (2). Now, (3) follows directly from (2) and the way the topology is defined. □

The following lemma characterizes morphisms and isomorphisms among birational maps of smooth projective surfaces.

Lemma 2.14. $\mathrm{Val}_0(X) \supset \phi_*^{-1}\,\mathrm{Val}_0(Y)$ *if and only if ϕ is a morphism, and equality holds if and only if ϕ is an isomorphism.*

Proof. One direction is clear, so assume that $\mathrm{Val}_0(X) \supset \phi_*^{-1}\,\mathrm{Val}_0(Y)$. Then
$$\mathrm{Val}_0(X) = \overline{\phi_*^{-1}\,\mathrm{Val}_0(Y)}.$$
We conclude that ϕ is a morphism by Lemma 2.13 (2). The last statement follows by the same argument applied to ϕ^{-1}. □

Proof of Proposition 2.10. Since $f_*^{-1}V(Y, f) = \mathrm{Val}_0(X)$, the assertion follows from Lemma 2.14 applied to $\phi := f^{-1}\tau f$. □

3. Explicit resolution of pairs in dimension two

Let G be a finite group acting birationally on a smooth projective surface Y. For every $\tau \in G$ we consider the minimal resolution

$$\begin{tikzcd} & Z_\tau \arrow[dl, "p_\tau"'] \arrow[dr, "q_\tau"] & \\ Y \arrow[rr, dashed, "\tau"] & & Y \end{tikzcd}$$

of the indeterminacy of τ. Let $V(p_\tau)$ be the closed subset of $\mathrm{Val}_+(Y)$ associated to p_τ by Theorem 2.9.

Definition 3.1. The subset of $\mathrm{Val}_+(Y)$ given by

$$V_G := \bigcup_{\tau \in G} V(p_\tau).$$

is called *set of indeterminacy of the pair* (Y, G). We also set

$$V(Y, G) := \mathrm{Val}_0(Y) \sqcup V_G.$$

Theorem 3.2. (Construction of resolution.) *Let Y, G and V_G be as above. Let $f : X \to Y$ be the birational morphism associated to V_G by Theorem 2.9. Then f resolves the indeterminacy of (Y, G). In other words, G acts biregularly on X via f.*

Proof. Note that $V(Y, f) = V(Y, G)$. By Proposition 2.10, we need to prove that, for every $\sigma \in G$,

$$\sigma_*^{-1}V(Y, G) = V(Y, G).$$

Since $(\sigma_*^{-1})^{-1} = (\sigma^{-1})_*^{-1}$, it is enough to show only one inclusion. In fact, since

$$V(Y, G) = \bigcup_{\tau \in G} V(Y, p_\tau),$$

we reduce to show that, for every $\sigma, \tau \in G$,

$$\sigma_*^{-1}V(Y, p_\tau) \subset V(Y, G).$$

We consider the birational morphism $g : Z \to Y$ associated to the closed set

$$V(p_\sigma) \cup V(p_{\tau\sigma}).$$

We observe that g resolves the indeterminacy of both σ and $\tau\sigma$. We have the following commutative diagram

$$\begin{tikzcd}
 & & Z \arrow[ddll, bend right, "g"'] \arrow[dl, "g_\sigma"] \arrow[ddrr, bend left, "h"] & & \\
 & Z_\sigma \arrow[dl, "p_\sigma"'] \arrow[dr, "q_\sigma"] & & Z_\tau \arrow[dl, "p_\tau"'] \arrow[dr, "q_\tau"] & \\
Y \arrow[rr, dashed, "\sigma"'] & & Y \arrow[rr, dashed, "\tau"'] & & Y.
\end{tikzcd}$$

Then, by the universal property applied to the minimal resolution of τ, there is a morphism $h_\tau : Z \to Z_\tau$ such that $h = q_\tau h_\tau$. Now, Lemma 2.11 (1) applied to the diagram

$$\begin{tikzcd}
 & Z \arrow[dl, "g"'] \arrow[dr, "p_\tau h_\tau"] & \\
Y \arrow[rr, dashed, "\sigma"] & & Y
\end{tikzcd}$$

gives

$$\sigma_*^{-1} V(Y, p_\tau h_\tau) \subset V(Y, g).$$

Then, observing that $V(Y, p_\tau) \subset V(Y, p_\tau h_\tau)$ and recalling that $V(Y, g) = V(Y, p_\sigma) \cup V(Y, p_{\tau\sigma})$, we conclude that

$$\sigma_*^{-1} V(Y, p_\tau) \subset \sigma_*^{-1} V(Y, p_\tau h_\tau) \subset V(Y, G).$$

Therefore the theorem is proved. $\square$

The following theorem implies that the resolution constructed in Theorem 3.2 is minimal among all resolutions of indeterminacy of pairs determined by birational morphisms.

Theorem 3.3. (Universal property of resolution of pairs.) *In the notation of Theorem 3.2, assume that there is another smooth projective surface X' and a birational morphism $f' : X' \to Y$ which resolves the indeterminacy of (Y, G). Then f' factors through f.*

Proof. All $\tau \in G$ lift to automorphisms of X'. Then, for every $\tau \in G$, $V(p_\tau)$ is contained in $V(f')$, hence f' factors through f. $\square$

The following theorem follows directly from the existence of equivariant resolution of singularities. On the other hand, it is interesting to observe that, for finite groups acting on surfaces, equivariant resolution of singularities follows from Theorem 3.2 (and, of course, usual resolution of singularities).

Theorem 3.4. (Strong equivariant factorization.) *Let G be a finite group acting biregularly on two smooth projective surfaces X and Y, and $\phi : X \dashrightarrow Y$ be a G-equivariant birational map. Then there exists a smooth projective surface Z, a biregular action of G on Z,* *and G-equivariant birational morphisms p and q, giving the following commutative diagram of pairs:*

$$\begin{tikzcd} & (Z,G) \arrow[dl, "p"'] \arrow[dr, "q"] & \\ (X,G) \arrow[rr, dashed, "\phi"] & & (Y,G). \end{tikzcd}$$

Proof. Let Z_0 be the closure of the graph of ϕ inside $X \times Y$. Since ϕ is G-equivariant, the componentwise action of G on $X \times Y$ induces a biregular action on Z_0. Then the theorem follows by taking a G-equivariant resolution of singularities of Z_0. □

Remark 3.5. Assume that, in the set up introduced at the beginning of the section, G is a cyclic group, and let τ be a generator of G. Then, in order to construct the minimal resolution of (Y,G), we can follow an explicit procedure that involves recursive resolutions and lifts of the generator τ. We start with $Y_0 := Y$ and $\tau_0 := \tau$, and then define recursively:

(1) $f_i : Y_i \to Y_{i-1}$ to be the morphism giving the minimal resolution of indeterminacy of τ_{i-1}, and
(2) $\tau_i \in \mathrm{Bir}(Y_i)$ to be the lift of $\tau_{i-1} \in \mathrm{Bir}(Y_{i-1})$ to Y_i via f_i.

One can see that the composition $g_i := f_1 \cdots f_i : Y_i \to Y_0$ determines resolutions of indeterminacy for $\tau, \tau^2 \ldots, \tau^i$. On the other hand, if Z is a smooth surface and $h : Z \to Y$ is a birational morphism which determines resolutions of indeterminacy for both τ and τ^{-1}, then it follows, by the universal property of the blow up for smooth surfaces, that τ lifts to an automorphism on Z via h. Therefore, if n denotes the order of G, the above sequence of recursive resolutions stops after a number $l \leq n-1$ of steps, producing a surface Y_l and an automorphism $\tau_l \in \mathrm{Aut}(Y_l)$, so that (Y_l, τ_l) is a resolution of indeterminacy of (Y, τ). In fact, it turns out that this is the minimal resolution, i.e., $Y_l = X$ in the notation of Theorem 3.2.

Note that, if τ and τ^{-1} have the same set of indeterminacy, this process stops at the first step, giving $Y_1 = X$. However, in general the resolution of the indeterminacy of the generator of the group does not produce a resolution of indeterminacy of the pair. For example, consider the pair $(\mathbb{P}^2, \tau)$, where τ is the birational transformation of order 5 given, in coordinates (x,y,z) of $\mathbb{P}^2$, by

$$\tau : (x,y,z) \to (x(z-y), z(x-y), xz). \tag{†}$$

In this case, one can check that $V(p_\tau) \not\supset V(p_{\tau^2})$, thus the resolution of indeterminacy of τ would not resolve the birational action of the group generated by τ.

4. A birational invariant of pairs

Throughout this section (X, σ) and (Y, τ) will denote two pairs consisting respectively of smooth projective surfaces X and Y, defined over an algebraically closed field k of characteristic 0, and automorphisms of finite order σ and τ of X and Y, respectively.

Definition 4.1. A birational morphism of pairs $f : (X, \sigma) \to (Y, \tau)$ is said to be a *minimal equivariant blow up* if any times it is written as a composition $f = f_1 f_2$ of two equivariant birational morphisms f_i, then either f_1 or f_2 (but not both) is an isomorphism.

Lemma 4.2. *Let $f : (X, \sigma) \to (Y, \tau)$ be a minimal equivariant blow up. Assume that the order of σ (and thus of τ) is finite. Then f is the blow up of Y along the τ-orbit of a point q. Moreover, if $g : (X, \sigma) \to (Y, \tau)$ is a birational morphism of pairs, then g factors as a composition of minimal equivariant blowups (and an automorphism).*

Proof. Let $q \in Y$ be a point where f^{-1} is not defined. Since f is equivariant, f^{-1} is not defined at $\tau^k q$ for all $k = 1, \dots, n-1$, where n is the order of τ. Then, by the universal property of the blow up for smooth surfaces, f factors through the blow up $h : \mathrm{Bl}_\Sigma Y \to Y$, where Σ is the τ-orbit of q. □

We can attach to a pair (X, σ) a number which is birational invariant modulo the order of σ. Let $r_i(X, \sigma)$ be the number of connected components of dimension i (for $i = 0, 1$) of the locus of points of X which are fixed by σ.

Proposition 4.3. *Assume that (X, σ) and (Y, τ) are birational equivalent pairs. Assume moreover that the order of σ and τ is a prime number p. Then*

$$(\ddagger) \qquad \rho(X) - r_0(X, \sigma) - 2r_1(X, \sigma) \equiv \rho(Y) - r_0(Y, \tau) - 2r_1(Y, \tau)$$

modulo p, where $\rho(X)$ and $\rho(Y)$ are the ranks of the respective Neron-Severi groups.

Proof. By Theorem 3.4 and Lemma 4.2, we can reduce to the case when (X, σ) is a minimal equivariant blow up of (Y, τ). Let $f : (X, \sigma) \to (Y, \tau)$ denote the blowing up.

If f is the blow up of Y along an orbit consisting of p distinct points, then $\rho(X) = \rho(Y) + p$ and $r_i(X, \sigma) = r_i(Y, \tau)$ for $i = 1, 2$. Thus $(\ddagger)$ is satisfied.

Otherwise f is the blow up of Y at a fixed point q. In this case $\rho(Y) = \rho(X) + 1$. Note that the exceptional curve $E = f^{-1}(q)$ is σ-invariant. If $\sigma|_E \neq 1_E$, then there are two fixed points contained in E, which correspond to the two distinct eigenspaces of the action of τ on the tangent space of Y at q. Then, independently of the fact that q is an isolated fixed point or is contained in a fixed curve, we have

$$r_0(X, \sigma) = r_0(Y, \tau) + 1 \quad \text{and} \quad r_1(X, \sigma) = r_1(Y, \tau).$$

Suppose now that $\sigma|_E = 1_E$. In the complete local ring $\mathcal{O}_{Y,q}$, the action of τ can be linearized [3, Lemma 2]. Hence, after been linearized, the action is given by $\lambda \cdot \mathrm{Id}$, for some $\lambda \neq 1$. Therefore q is an isolated fixed point, and

$$r_0(X, \sigma) = r_0(Y, \tau) - 1 \quad \text{and} \quad r_i(X, \sigma) = r_i(Y, \tau) + 1.$$

We see then that in both cases (‡) is satisfied. □

We can apply this invariant to distinguish conjugacy classes of given birational transformations. For instance, we can consider the birational transformations occurring in the classification, up to conjugation, of the elements of prime order in $\mathrm{Bir}(\mathbb{P}^2)$ (cf. [2], [6], [1], [5]). The previous method, used to distinguish the conjugacy classes of elements of $\mathrm{Bir}(\mathbb{P}^2)$ is based on the consideration of the geometric genus of the fixed curve. Consider the following examples of birational transformations of $\mathbb{P}^2$:

- E1. Let $\tau \in \mathrm{Bir}(\mathbb{P}^2)$ be the transformation induced by the order 3 Galois automorphism σ of the cyclic covering given by a cubic $X \subset \mathbb{P}^3$ over $\mathbb{P}^2$.
- E2. Let $\tau \in \mathrm{Bir}(\mathbb{P}^2)$ be the transformation induced by the order 3 Galois automorphism σ of the cyclic covering given by a sextic $X \subset \mathbb{P}(1,1,2,3)$ of equation of the form $z^3 = F(x, y, w)$ over $\mathbb{P}(1,1,3)$. In other words, σ is given by the diagonal action on the coordinates sending $(x, y, z, w) \to (x, y, e^{2\pi i/3}z, w)$.
- E3. Let $\tau \in \mathrm{Bir}(\mathbb{P}^2)$ be the transformation induced by the order 5 Galois automorphism σ of the cyclic covering given by a sextic $X \subset \mathbb{P}(1,1,2,3)$ of equation of the form $xy^5 = F(x, z, w)$ over $\mathbb{P}(1,2,3)$. In other words, σ is given by the diagonal action on the coordinates sending $(x, y, z, w) \to (x, e^{2\pi i/5}y, z, w)$.

We refer to [5] for more details concerning these transformations. In E2 and E3, (x, y, z, w) denote the weighted coordinates of $\mathbb{P}(1,1,2,3)$, and all cyclic coverings mentioned in these examples are the ones induced by the obvious linear projections of the ambient spaces. The pairs (X, σ) coincide with cases A1–A3 of [5, Theorem A].

Proposition 4.4. *All three examples E1–E3 are not birational equivalent to linear automorphisms of* $\mathbb{P}^2$*. Moreover, E*1 *is not birational equivalent to E*2*.*

Proof. By considering the corresponding eigenspaces, we see that, if α is any linear automorphism of finite order of $\mathbb{P}^2$, then

$$\rho(\mathbb{P}^2) - r_0(\mathbb{P}^2, \alpha) - 2r_1(\mathbb{P}^2, \alpha) = -2.$$

On the other hand, if τ is as in one of examples E1–E3, then the pair $(\mathbb{P}^2, \tau)$ is birational equivalent to the corresponding pair (X, σ). Then we see that

$$\begin{aligned} \rho(X) - r_0(X,\sigma) - 2r_1(X,\sigma) &= 5 \not\equiv -2 \mod 3 \quad \text{for case E1} \\ \rho(X) - r_0(X,\sigma) - 2r_1(X,\sigma) &= 6 \not\equiv -2 \mod 3 \quad \text{for case E2} \\ \rho(X) - r_0(X,\sigma) - 2r_1(X,\sigma) &= 6 \not\equiv -2 \mod 5 \quad \text{for case E3.} \end{aligned}$$

Therefore these transformations can not be birationally equivalent to any linear automorphism α. Moreover, $5 \not\equiv 6$ modulo 3 shows that the two transformations in examples E1 and E2 are not birationally equivalent. □

Remark 4.5. Unfortunately, both the above method and the method of considering the genus of the fixed curve are not sufficient to determine whether the birational transformation defined by (†) in Remark 3.5 is conjugate to a linear automorphism of $\mathbb{P}^2$.

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