COMPLEX DYNAMICS IN HIGHER DIMENSION. II

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Introduction.

In this paper we continue the investigations started in [FS1] in order to construct a Fatou-Julia theory for holomorphic (respectively meromorphic) self maps in $\mathbb{P}^2$. We start by considering maps arising in two dimensions from Newton's method. This leads to the study of iteration of meromorphic maps in $\mathbb{P}^2$. More precisely if we consider the problem in $\mathbb{C}^2$ to find the zeroes of a complex polynomial map, we are led to study iteration of maps on $\mathbb{P}^2$. The results can afterwards be interpreted back down in $\mathbb{C}^2$.

It turns out that generically, when one applies Newton's method, the map one has to study in $\mathbb{P}^2$ is not holomorphic but just meromorphic. This is why we are also interested in the dynamics of meromorphic maps.

One of the main tools in iteration theory in one complex variable is the Montel Theorem, i.e. the fact that $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ is Kobayashi hyperbolic. This approach was explored in [FS1].

It is however possible to prove many results of the Fatou-Julia theory in one variable using potential theory. This was started by Brolin [Br] and continued in [Si], [T]. In paragraph 6 we show how to work out this approach in the context of rational maps in $\mathbb{P}^1$ in order to obtain recent results due to Lyubich [Ly] and Lopez-Freire-Mane [FLM], see also Hubbard-Papadopol [HP]. In the above mentioned articles the use of the Koebe distortion theorem is crucial in order to construct a measure of maximal entropy for a rational map in $\mathbb{P}^1$ and to prove convergence results. Such a distortion theorem is not valid in several variables.

After generalities on meromorphic maps in $\mathbb{P}^2$, we consider the Green function associated to a "generic meromorphic" map in $\mathbb{P}^2$. Such a function was studied in the context of Hénon maps by Hubbard, from the topological point of view, and it was extended to the case of holomorphic maps by Hubbard and Papadopol [HP].

To the Green function we associate an invariant closed positive $(1,1)$ current T in $\mathbb{P}^2$. Such currents and their wedge products where considered in the context of Hénon maps by Sibony. Bedford and Sibony established their first properties. Some of the results they obtained appeared in § 3 of Bedford-Smillie [BS1], see also the introduction of [BLS]. The structure of these Green-currents was studied extensively by Bedford-Lyubich-Smillie [BS1], [BS2], [BLS] and Fornaess-Sibony [FS]. The notion was adapted in the context of holomorphic maps in $\mathbb{P}^k$ by Hubbard-Papadopol [HP].

Let f be a holomorphic self map in $\mathbb{P}^2$ of degree $d \geq 2$. Let J_0 denote the Julia set of f, i.e. $p \in J_0$ if the family (f^n) is not normal in a neighborhood of p. We show that there exists a closed positive $(1,1)$ current T satisfying the functional equation $f^*T = dT$ and whose support is exactly J_0, (this is done in the context of "generic meromorphic normal" maps). Moreover T is extremal among the currents satisfying the previous functional equation. We then show that the Julia set J_0 is always connected and the Fatou set i.e. $\mathbb{P}^2 \backslash J_0$ is a domain of holomorphy. Hence the critical set of f always intersects the Julia set J_0.

In § 6 we study the probability measure $\mu := T \wedge T$ and we show that it is an invariant ergodic measure of maximal entropy.

The analysis of a simple example such as $f[z : w : t] = [z^d : w^d : t^d]$ shows that the Julia set is not indecomposable as in one variable. We introduce a first order Julia set J_1, see Definition 5.8, and we show that $J_1' := \operatorname{supp} \mu \subset J_1$. If J_1' has nonempty interior then $J_1' = \mathbb{P}^2$. If K is a compact invariant set, hyperbolic and of unstable dimension 2, then necessarily $K \subset J_1$. We also show that no f can be hyperbolic on the whole Julia set J_0. Finally if the nonwandering set of a holomorphic map of degree $d \geq 2$ is hyperbolic, then the Fatou components are preperiodic to finitely many periodic basins.

It will be natural to consider similar questions in $\mathbb{P}^k$, $k \geq 2$. For simplicity we have written this article, restricting ourselves to the case of $\mathbb{P}^2$. We will continue our study in forthcoming articles, see [FS4].

1. Newton's method and meromorphic maps in $\mathbb{P}^2$.

Given two complex polynomials P, Q in two variables z, w, Newton's method provides a way to approximate the roots (z_0, w_0) of the equations $P(z,w) = Q(z,w) = (0,0)$ by starting with an initial guess (z_1, w_1) and inductively define $(z_{n+1}, w_{n+1}) := (z_n, w_n) - F'^{-1}(z_n, w_n)F(z_n, w_n)$ where $F : \mathbb{C}^2 \to \mathbb{C}^2, F := (P, Q)$ and the $2 * 2$ matrix F' denotes the derivative of F.

Hence Newton's method consists of iteration of the map

$$R(z,w) := (z,w) - F'^{-1}F.$$

The natural way to consider this map is as a map on $\mathbb{P}^2$ rather than $\mathbb{C}^2$ since it depends on quotients of polynomials. To write this as a map on $\mathbb{P}^2$, introduce first the determinant D of F', and the two determinants $D_w := Q_w \cdot P - P_w \cdot Q$ and $D_z := P_z \cdot Q - Q_z \cdot P$ where we have used subscripts to denote partial derivatives. Hence we write R as a map on $\mathbb{P}^2$ in homogeneous coordinates as $R : [z : w : 1] \to [zD - D_w : wD - D_z : D]$.

We will assume that the map F has maximal rank 2, otherwise $F = 0$ would in general have no root. Also we can assume that the map F is at least of degree two. Otherwise Newton's method immediately gives the root after one step. This amounts to the same thing as requiring that R is non constant.

Observe that at points where $D_z = D_w = 0$, which means that $\begin{pmatrix} P_z \\ Q_z \end{pmatrix}, \begin{pmatrix} P \\ Q \end{pmatrix}$ are linearly dependent and $\begin{pmatrix} P_w \\ Q_w \end{pmatrix}, \begin{pmatrix} P \\ Q \end{pmatrix}$ are linearly dependent we have generically that $D = 0$, i.e. $\begin{pmatrix} P_z \\ Q_z \end{pmatrix}, \begin{pmatrix} P_w \\ Q_w \end{pmatrix}$ are linearly dependent. So in general R has poles.

If the map R has maximal rank 1, then it is easy to show that the image X of R is a $\mathbb{P}^1$ and that R restricted to X is a rational map. We consider, for the purpose of this paper, this as a known case. So we will assume from now on that R has maximal rank 2. For the one variable theory we refer to [B], [Ca] or [Mi].

Notice that if $R = [A : B : C]$ are homogeneous polynomials of degree d, we may assume they have at most finitely many common zeroes (lines of common zeroes in $\mathbb{C}^3$). If not, they have a common factor which can be divided out. The remaining points p if any in $\mathbb{P}^2$ are called points of indeterminacy.

In the case where R is linear, the dynamics is rather simple. Hence we will in the rest of the paper restrict ourselves to the case when R has at least degree 2.

We will confine ourselves to giving one situation in which a nontrivial polynomial equation gives rise to a linear Newton's map - in complete analogy with an important case in one variable.

PROPOSITION 1.1. - *If $(P,Q)(z,w)$ are homogeneous of the same degree $n > 1$, and $F = (P,Q)$ has maximal rank 2, then $R(z,w) = ((1-1/n)z, (1-1/n)w)$. If (P,Q) are two polynomials of degree $n > 1$ such that the highest degree terms (P_n, Q_n) have maximal rank 2, then R is the identity map at infinity $(t = 0)$.*

Proof. Immediate. □

The situation can be quite different for homogeneous polynomials of different degree. Let $F = (P,Q) = (z^3 + w^3, z^2 + w^2)$. Then Newton's map $R = [4z^3w - 3z^2w^2 + w^4 : -z^4 + 3z^2w^2 - 4zw^3 : 6zwt(z-w)]$. Then the point $(0,0)$ is a root of $F = 0$. However Newton's method applied to points $[\epsilon : \epsilon : 1]$ arbitrarily close to the root, $\epsilon \neq 0$, are all mapped to the fixed point $[1 : -1 : 0]$ at infinity, so Newton's method diverges arbitrarily close to the root ! The problem is that at such points $D = 0$ but D_w, $D_z \neq 0$. The root $[0 : 0 : 1]$ is a point of indeterminacy for R while $(z = w)$ is mapped to a different fixed point.

Schröder ([1871]) was the first to study Newton's method in one complex variable. He observed that there are infinitely many variations of Newton's method $R = z - (f'(z))^{-1}f(z)$. For example, one can replace $f'(z)$ by the derivative at some fixed point close to the root. Doing this in two dimensions increases the class of maps obtained from Newton's method significantly. Any polynomial map $R = [A : B : t^d]$ can be written in the form, $(t = 1)$,
$(A,B) = (z,w) - M^{-1}(P,Q)$ where M is an invertible constant matrix and (P,Q) is polynomial. There is also the relaxed Newton's method, $R = z - \epsilon F'^{-1}$. F for some constant $0 < \epsilon < 1$ (which approximates the Newton Flow $X = -F'^{-1} \cdot F$). Based on these remarks, we will from now on study the class of all meromorphic maps R on $\mathbb{P}^2$ with maximal rank 2 and degree at least two. (The Newton Flow would lead rather to the study of complex foliations of $\mathbb{P}^2$).

With the degree d of any homogeneous map $R = [A : B : C]$ we mean the degrees of A, B or C, which are equal, after cancellation of all common irreducible factors. So we assume that $d \geq 2$. Let $I = I(R) = \{q_k\}$ denote the (finite) indeterminacy set consisting of the points q_k in $\mathbb{P}^2$ where $A = B = C = 0$. Also let $V = \cup V_j$ denote the finite union of irreducible compact complex curves V_j on each of which R has a constant value (at least outside I). Say $R(V_j) = p_j$. We call such curves V_j, *R-constant.*

PROPOSITION 1.2. - *If V is nonempty, then also I is nonempty. In fact each irreducible branch of V contains at least one point of indeterminacy. It can happen that V is empty while I is nonempty.*

Proof. Suppose that V is nonempty. Then we may after a rotation assume that some irreducible component $V_j = \{h = 0\}$ for some irreducible homogeneous polynomial h and that $R(V_j) = [0 : 0 : 1]$. Hence h divides both A and B. But then the set $C = 0$ and $h = 0$ must intersect and such an intersection point is a point of indeterminacy.

For the converse, consider the example $R = [zw : z^2 + wt : t^2]$. For this example, there is one point of indeterminacy, [0:1:0], while the map is not constant on any curve. □

Let R be a meromorphic map $\mathbb{P}^2 \to \mathbb{P}^2$. Let I be the indeterminacy set. Given $a \in \mathbb{P}^2$ we want to discuss the number of preimages of a. Recall that Bezout's Theorem asserts that if $(P_1, \cdots, P_k)$ are k homogeneous polynomials in $\mathbb{P}^k$ with discrete set of zeroes then the number of zeroes counted with multiplicities is equal to the product of the degrees.

PROPOSITION 1.3. - *Let $R : \mathbb{P}^2 \to \mathbb{P}^2$ be a meromorphic map of degree d. Assume $I \neq \emptyset$.*
Assume R is of rank 2. Then for any a which is not one of the finitely many points which is the image of an $R-$ constant curve, $R^{-1}(a) = d' < d^2$. Here we count the number of points with multiplicity.

Proof. Consider the map $\mathbb{C}^4 \to \mathbb{C}^3$, $(z_0, z_1, z_2, t) = (z, t) \to R(z) - at^d$. We have 3 polynomials in $\mathbb{P}^3$. Assume there is no 1 dimensional variety in $\mathbb{P}^2$ such that $R(V) = a$. Then the number of zeroes of $R(z) - at^d$ is finite in $\mathbb{P}^3$. So it is d^3 counting multiplicity. For $p \in I$, $[p : 0]$ is a zero of multiplicity at least d. Hence number of zeroes in $t = 1$ is $< d^3$. Since rotation of t by a d^{th} root of unity produces an equivalent solution in $\mathbb{P}^2$ we get that $d' < d^2$. □

Remark 1.4. - *If a point $a \notin R(\mathbb{P}^2 \backslash I)$ then $\sum_{p \in I}$ multiplicity of $[p : 0]$ for $R(z) - at^d = 0$ is d^3.*

Consider the forward orbit of the points p_j. The variety V_j is called degree lowering if for some (smallest) $n = n_j \geq 0$, $R^n(p_j) \in I$.

We will next discuss the growth of the degrees of the iterates of maps R.

When there is a degree lowering variety, all the components of the iterates of R^{n_j+1} vanish on $V_j = \{h = 0\}$. Hence one need to factor out a power of h in order to describe the map properly. Hence the degree of the iterate will drop below d^{n_j+1}.

We will not study the class of maps with degree lowering varieties in this paper. If a map R has an $R-$ constant variety V with $R(V) = p$ which is not degree lowering, most likely the complement of the preimages of this variety is Kobayashi hyperbolic. So on these varieties, the map eventually lands on the R constant variety after which the orbit reduces to the orbit of a point, and in the complement of these varieties, the iterates is a normal family, even on any subvariety disjoint from $\bigcup_{n\geq 0} R^{-n}(V)$. If the variety is degree lowering, the iterations can be much more complicated and worthy of further study. However, the method we will be pursuing in this paper, pluripotential theory, is more difficult to carry out for such maps. So we will pursue these in a separate paper with other methods.

2. Green Function.

In this section we will study generic meromorphic maps on $\mathbb{P}^2$, i.e. meromorphic maps of maximal rank 2, which have degree at least 2 and which have no degree lowering curves.

We have first to define Fatou sets and Julia sets of $R : \mathbb{P}^2 \to \mathbb{P}^2$.

Since we would like to have a notion of Fatou and Julia set which is invariant under R, we need to be precise about what one means with the preimage of a point. We say that for a given p, a point q is a preimage of p if R is defined at q and $R(q) = p$. (If $q \in I(R)$ and $p \in W$, the blow up of q, so (q,p) is in the closure of the graph of R, then q is with this convention *not* in the preimage of p). Here I denote as usual the indeterminacy set of the meromorphic map R.

DEFINITION 2.1. - *An orbit $\{p_n\}_{n=-k}^{0}$ is called complete if*

(i) $R(p_n) = p_{n+1}$,
(ii) $p_0 \in I$,
(iii) $p_n \notin I$, $n < 0$,
(iv) *If k is finite, $p_{-k} \notin R(\mathbb{P}^2 \backslash I)$.*

We call $k+1$ the length of the orbit.

LEMMA 2.2. - *A point $p \in \mathbb{P}^2$ is a point of indeterminacy for R^n if and only if $\{p, R(p), \cdots, R^{k-1}(p)\}$ is a right tail of some complete orbit for some $1 \leq k \leq n$.*

Proof. Immediate. □

COROLLARY 2.3. - *If I_n denotes the indeterminacy set of R^n, then $I_n \subset I_m$ $\forall m > n$.*
Proof. Immediate. □

The set $I(R)$ should belong naturally to the "Julia set". So does $\cup I_n$. Hence the closure $E := \overline{\cup I_n}$, *the extended indeterminacy set*, belongs naturally to the Julia set as well.

PROPOSITION 2.4. - *If* $p \in \mathbb{P}^2 \backslash E$ *and* $R^n(p) \in E$, $n \geq 1$, *then* p *is on an* R^n-*constant curve.*

Proof. If p is not on an R^n- constant curve there are arbitrarily small neighborhoods $U(R^n(p))$, $V(p)$ so that $R^n \mid V : V \to U$ is a finite, proper, surjective holomorphic map. Moreover we may assume that $V \cap E = \emptyset$. Since every such open set U contains a point from some I_k, it follows that V contains a point from some I_{n+k}. Hence $p \in E$, a contradiction. □

DEFINITION 2.5. - *Let* $R : \mathbb{P}^2 \to \mathbb{P}^2$ *be a generic meromorphic map. A point* $p \in \mathbb{P}^2$ *is in the Fatou set if and only if there exists for every* $\epsilon > 0$ *some neighborhood* $U(p)$ *such that* $\operatorname{diam} R^n(U \backslash I_n) < \epsilon$ *for all* n.

Note that this implies that p cannot belong to the extended indeterminacy set. We say that the Julia set is the complement of the Fatou set. By a normal family argument it follows that the Fatou set is an open set and that the Julia set is closed. Also we conclude that the extended indeterminacy set belongs to the Julia set. We denote the Julia set of R by $J(R)$ or J_0, since we introduce also higher order Julia sets.

We have complete invariance of the Julia set :

PROPOSITION 2.6. - *Suppose that* $p \in \mathbb{P}^2 \backslash I$ *and that* $R(p) \in J(R)$, *the Julia set of* R. *Then* p *belongs to the Julia set also. On the other hand, if* $p \in J(R)$, *and* $p \notin I$, *then* $R(p) \in J(R)$.

LEMMA 2.7. - *Suppose that* $p \in \mathbb{P}^2 \backslash I$ *and that* $R(p) \in F(R)$, *the Fatou set of* R. *Then* $p \in F(R)$ *also.*

Proof. This is obvious from the definition since R is continuous at p.

LEMMA 2.8. - *Suppose that* $p \in F(R)$. *So in particular,* $p \in \mathbb{P}^2 \backslash I$. *Then* $R(p)$ *is also in* $F(R)$.

Proof. Suppose first that p does not belong to any $R-$ constant curve. Then R is locally finite to one and proper near p. Hence arbitrarily small neighborhoods of p are mapped properly onto arbitrarily small neighborhoods of $R(p)$. The conclusion follows.

Suppose next that p belongs to an $R-$ constant curve X. We can select a small complex disc Δ centered at p such that $\Delta \backslash \{p\}$ does not intersect X nor the critical set of R. Considering a small neighborhood of $\Delta \backslash \{p\}$, and taking the image of it, we obtain a piece of a complex curve Y through $R(p)$ and a neighborhood V of $Y \backslash R(p)$ on which the iterates of R is an equicontinuous family. Note that $R(p)$ cannot belong to $\cup I_n$ since this would imply that p belongs to this set also, contradicting that p belongs to the Fatou set. Pushing discs, we can conclude that any iterate R^n is holomorphic in a fixed neighborhood of $R(p)$ so $R(p) \notin E$. But then it follows that equicontinuity extends to a neighborhood of $R(p)$, so $R(p)$ is in the Fatou set.

So the proposition follows from the two lemmas. □

DEFINITION 2.9. - *A point* $p \in \mathbb{P}^2$ *has a nice orbit if there is an open neighborhood* $U(p)$ *and an open neighborhood* $V(I)$ *so that* $R^n(U) \cap V = \emptyset$ *for all* $n \geq 0$.

So if p has a nice orbit, R^n is well defined for all n on some fixed neighborhood of p. The set of nice points is an open subset of $\mathbb{P}^2 \backslash E$.

DEFINITION 2.10. - *A generic meromorphic map is said to be normal if* N*, the set of nice points equals* $\mathbb{P}^2 \backslash \overline{\cup I_n}$ *.*

Let R be a generic meromorphic map in $\mathbb{P}^2$. With an abuse of notation we will also denote by R a lifting of R to $\mathbb{C}^3$. If $\| \ \|$ is a norm on $\mathbb{C}^3$ we define the n^{th} Green function G_n on $\mathbb{C}^3$ by the formula $G_n := \frac{1}{d^n} \mathrm{Log} \| R^n \|$. Here d is the common degree of the components of R. Observe that if R is meromorphic, G_n has other poles in $\mathbb{C}^3$ than just the origin. Let π denote the canonical map $\mathbb{C}^3 \mid \{0\} \to \mathbb{P}^2$.

PROPOSITION 2.11. - *The functions* G_n *converge u. c. c. to a function* G *on the set* $\pi^{-1}(N)$ *of points with nice orbits.*

Proof. If $p \in N$ there exists $U(p)$ and $c > 0$ such that on $\pi^{-1}(U(p))$

$$\| R^{n+1}(z) \| \geq c \| R^n(z) \|^d .$$

On the other hand the reverse inequality

$$\| R^{n+1}(z) \| \leq C \| R^n(z) \|^d$$

holds always. Hence the sequence $\frac{\log \| R^n \|}{d^n}$ converges u. c. c. on $\pi^{-1}(U(p))$. □

Because of the second inequality, the limit G always exists and is a plurisubharmonic function on $\mathbb{C}^3$, possibly $\equiv -\infty$, although we don't believe this can happen. (You just need one periodic orbit to show that the limit is not identically $-\infty$.)

Obviously $N \subset \bigcup_n \mathrm{int}\{G > -n\}$.

The other inclusion does not hold in general, as the following example shows.

Example 2.1. - *Let* $R = [z^d : w^d : t^{d-1}w]$, $d \geq 3$. *This map has one point of indeterminacy,* $[0:0:1]$, *which has no preimage. Hence* $E = [0:0:1]$. *Also, the only* $R-$ *constant curve is* $(w = 0)$ *whose image is the fixed point* $[1:0:0]$. *Hence the map is generic. We get that* $R^n = [z^{d^n} : w^{d^n} : t^{(d-1)^n} w^{d^n - (d-1)^n}]$. *Computing the Green's function* G,

$$G_n = 1/d^n \log \| (z^{d^n}, w^{d^n}, t^{(d-1)^n} w^{d^n - (d-1)^n}) \|$$

and we obtain $G = \max\{\log \mid z \mid, \log \mid w \mid\}$. *Letting* $\Omega := \{\mid z \mid < \mid w \mid < \mid t \mid\}$, G *is pluriharmonic, on* $\pi^{-1}(\Omega)$, *but* $\Omega \not\subset N$ *since* $R^n \to [0:0:1]$. *Furthermore this is pluriharmonic when* $\mid z \mid < \mid w \mid$, $\mid w \mid$ *is close to 1 and* $t = 1$. *But then* $R^n = [(z/w)^{d^n} : 1 : (1:w)^{(d-1)^n}]$. *Notice that when* $\mid w \mid < 1$ *the limit becomes* $[0:0:1]$ *while if* $\mid w \mid > 1$ *the limit becomes* $[0:1:0]$ *so in particular the points where* $\mid z \mid < \mid w \mid = \mid t \mid$ *are in the Julia set even though* G *is pluriharmonic there. In the region* $t = 1$, $\mid z \mid > \mid w \mid$, $R^n = [1 : (w/z)^{d^n} : \left(\frac{w}{zw^{(d-1)^n/d^n}}\right)^{d^n}]$ *so* $R^n \to [1:0:0]$, *so the Julia set also contains* $\{\mid z \mid = \mid w \mid\}$. *There are three Fatou components :* $\Omega_1 = \{\mid z \mid < \mid w \mid < \mid t \mid\}$ *on which* $R^n \to [0:0:1]$, $\Omega_2 = \{\mid w \mid < \mid z \mid\}$, *on which* $R^n \to [1:0:0]$, $\Omega_3 = \{\mid z \mid < \mid w \mid, \mid t \mid < \mid w \mid\}$ *on which* $R^n \to [0:1:0]$. *The blow up of* $[0:0:1]$ *is the* z*-axis (which happens to coincide with the* $R-$ *constant curve).*

For the behaviour on the Julia set : If $\mid z \mid = \mid w \mid < 1 = t$, *then* $R^n \to [0:0:1]$. *If* $\mid z \mid = \mid w \mid > 1 = t$, *then* $R^n \to [1:1:0]$. *If* $\mid z \mid < \mid w \mid = 1 = t$, *then* R^n *converges to the invariant circle* $z = 0$, $\mid w \mid = \mid t \mid$. *Notice that these points are not in a nice component. Finally the set* $\mid z \mid = \mid w \mid = \mid t \mid$ *is an invariant torus.*

We just recall that plurisubharmonic (p.s.h. for short) functions on a complex manifold are upper semicontinuous functions that are subharmonic on one dimensional analytic discs, see [Le] or [Kl].

THEOREM 2.12. - *Let* R *be a generic meromorphic map on* $\mathbb{P}^2$. *Then* $G(z) = \lim 1/d^n * \log \| R^n \| = \lim \searrow H_n$ *where* $H_n := G_n + \sum_{k=n+1}^{\infty} \frac{\log M}{d^k}$, M *is some constant.*

(i) *The function* G *is plurisubharmonic in* $\mathbb{C}^3$ *(or* $\equiv -\infty$*).*
(ii) G *is pluriharmonic on* $\pi^{-1}(\Omega)$ *if* Ω *is a Fatou component.*
(iii) *If* N *is the set of nice points of* R *then* G *is continuous on* $\pi^{-1}(N)$ *and if* G *is pluriharmonic on* $\pi^{-1}(\omega)$ *where* ω *is an open subset of* N, *then* ω *is contained in a Fatou component.*

Proof. Let $M := \sup\{\| R(z) \| ; \| z \| = 1\}$. Then

$$\| R^{n+1}(z) \| \leq M \| R^n(z) \|^d$$

by homogeneity. Hence

$$G_{n+1}(z) \leq \frac{\log M}{d^{n+1}} + G_n(z).$$

Replacing G_n by $H_n := G_n + \sum_{k=n+1}^{\infty} \frac{\log M}{d^k}$ we get

$$H_{n+1} = G_{n+1} + \sum_{k=n+2}^{\infty} \frac{\log M}{d^k} \leq G_n + \sum_{k=n+1}^{\infty} \frac{\log M}{d^k} = H_n.$$

The function G is a decreasing limit of p.s.h. functions, hence it is p.s.h. or $\equiv -\infty$.

Next we prove ii).

Suppose that $p \in U$ is a point in the Fatou set, and U is a small neighborhood inside the Fatou set. Choose a subsequence R^{n_k} which converges uniformly to a holomorphic map R^∞ on U. Shrinking U if necessary, taking a thinner subsequence and renaming the coordinates, we may assume that $R^{n_k}(U) \subset \{| z |, | w |< 2, t = 1\}$. We can then write $R^{n_k} = R_3^{n_k}(A_k, B_k, 1)$ for uniformly bounded holomorphic functions A_k, B_k over U in $\mathbb{C}^3$. Hence $G_{n_k} = 1/d^{n_k} * \log | R_3^{n_k} | + 1/d^{n_k} * \log \| (A_k, B_k, 1) \|$. Since the last term converges uniformly to 0 and the first term is always pluriharmonic, the result follows.

We prove iii). On a compact subset of N, we have

$$| G_{n+k} - G_n | \leq C/d^n$$

since $| R^{n+1}(z) | \geq c_1 | R^n(z) |^d$, c_1 independent of n, k. So $| G - G_n | \leq C/d^n$ and if $G = \log | h |$, h is a nonvanishing holomorphic function, we get $| 1/d^n \log \frac{\| R^n \|}{| h^{d^n} |} | \leq C/d^n$ i.e.

$$e^{-C} \leq \frac{\| R^n \|}{| h^{d^n} |} \leq e^C. \ \square$$

Remark 2.13. - The same theorem holds for holomorphic maps on $\mathbb{P}^k \geq 1$.
COROLLARY 2.14. - *Let $p \in E$. The Hausdorff dimension of J near p is at least 2. If R is normal, then the Hausdorff dimension near any point of J is at least 2.*

Proof. Let $q \in \mathbb{C}^3 \backslash \{0\}$ such that $\pi(q) = p$. Assume the 4 dimensional Hausdorff measure $\Lambda^4(\pi^{-1}(J) \cap U(q)) = 0$ for some neighborhood $U(q)$ of q. Then using a theorem of Harvey-Polking [Ha.P] G would extend as a pluriharmonic function in $U(q)$. But $G = -\infty$ on $\pi^{-1}(\cup I_n) \cap U(q)$, a contradiction. In the normal case we apply the same extension result and Theorem 2.12.

PROPOSITION 2.15. - *In the generic meromorphic case, the Green function G satisfies the functional equation*

$$G(R(z)) = dG(z).$$

Moreover for $\lambda \in \mathbb{C}$, $G(\lambda z) = \log | \lambda | + G(z)$.

Proof. Direct computation gives

$$G(R(z)) = \lim \frac{\log \| R^n(R(z)) \|}{d^n} = d \lim \frac{\log \| R^{n+1}(z) \|}{d^{n+1}} = dG(z).$$

The proof of the second assertion is clear. $\square$

We give an example showing that the pole set of G is not just $\pi^{-1}(\cup I_n)$. The example has also interesting dynamics.

Example 2.2. - $R = [w^4 : w^2(w - 2z)^2 : t^4]$.

The indeterminacy set I consists of one point, $[1:0:0]$. The only $R-$ constant variety is $(w=0)$ which is mapped to the fixed point $[0:0:1]$, so the map is generic. The map R is a polynomial map, sending the hyperplane at infinity, $(t=0)$, to itself. At infinity, the map is $w \to \left(\frac{w-2}{w}\right)^2$. This map is a critically finite map on $\mathbb{P}^1$, and all critical points are preperiodic. Hence the Julia set of this map on $\mathbb{P}^1$ is all of $\mathbb{P}^1$, [Ca]. In particular, $\cup I_n$ is dense in $(t=0)$, so $E=(t=0)$. Since for example $[1:1:0]$ is a fixed point for R, $G \mid (t=0) \not\equiv -\infty$. However, $G \equiv -\infty$ on $\bigcup_n I_n$. Since $(G=-\infty)$ is a G_δ dense set in $(t=0)$, $(G=-\infty)$ is uncountable. Hence $(G=-\infty) \neq \bigcup_n I_n$.

The point $[0:0:1]$ is a superattractive fixed point and the punctured line $(w=0)\backslash[1:0:0] =: L$ is mapped to it. Hence L is contained in the attractive basin Ω of $[0:0:1]$. But R maps lines through $[0:0:1]$ to lines through $[0:0:1]$. Since the map $R \mid (t=0)$ is chaotic, it follows that Ω contains a dense set of lines through $[0:0:1]$, punctured at $(t=0)$.

On the fixed line $(w=2iz)$ the map is $z \to 16z^4$. Hence the Julia set contains the disc $\{[z:2iz]; \mid z \mid \geq 2^{4/3}\}$ centered at $(t=0)$. It follows that for a dense set of points p in $(t=0)$, the straight line through $[0:0:1]$ and p contains a disc centered at p which is contained in $J(R)$.

3. - Special Generic Meromorphic Maps.

If one wants to study various well known phenomena from complex dynamics in the context of generic meromorphic maps one sees that many such phenomena are often no longer always true, but are true for large subclasses. The subclass depends often on which phenomena one wishes to study. We will illustrate this here by discussing some such phenomena, showing by examples that they don't generally hold, and give conditions which make them hold.

There are two particular cases of generic meromorphic maps which have been previously studied. First there are the complex Hénon maps. These are invertible polynomial maps which have one point of indeterminacy at infinity. Also the hyperplane at infinity is the unique $R-$ constant variety and its image is a fixed point at infinity, different from the point of indeterminacy. This point of view is developped in [FS2]. Also there is the class of holomorphic maps on $\mathbb{P}^2$, maps without points of indeterminacy.

We will define various subclasses of the meromorphic maps. These classes will be usually large enough to contain all holomorphic maps and all Hénon maps, i.e. maps of the form $R[z:w:1]=[p(z)+aw:z:1]$ where p is a one variable polynomial of degree $d \geq 2$.

DEFINITION 3.1. - *A generic meromorphic map is said to belong to the class of indeterminacy repellors - IR - if there exist arbitrarily small neighborhoods $U \subset\subset V$ of the* *indeterminacy set for which* $R(\mathbb{P}^2\backslash U) \subset \mathbb{P}^2\backslash V$.

Both Hénon maps and holomorphic maps belong to IR. Nevertheless the definition is rather strong. It implies that every point in $\mathbb{P}^2\backslash I$ has a nice orbit, which also implies that the points in I have no preimages (recall that points in I are not considered as preimages).

DEFINITION 3.2. - *We say that a generic meromorphic map belongs to the class with no* $R-$ *constant blow ups, NRB, if there is no point of indeterminacy* q *for which the blow up is* R^n- *constant for some* $n \geq 1$. *The complement of this class is the set RB.*

Hénon maps are in RB. The map $R = [zw : z^2 + wt : t^2]$ is in NRB since it has no R-constant variety.

DEFINITION 3.3. - *We say that a generic meromorphic map* R *is a meromorphic Hénon map, MH, if there exists a generic meromorphic map* S *such that* $R \circ S = Id = S \circ R$ *in the complement of some hypersurface. We say that* S *is the inverse of* R.

Example 3.1. - *Example of a meromorphic Hénon map :* $R[z : w : t] = [t^2 : zt : z^2 + ct^2 + awt]$, $a \neq 0$. $I = \cup I_n = [0 : 1 : 0]$ $(t = 0) \to [0 : 0 : 1] \to [1 : 0 : c] \to [c^2 : c : 1 + c^3] \to \cdots$ *Note that* $[0 : 1 : 0]$ *has no preimage so* $(t = 0)$ *cannot be degree lowering. It follows that the map is generic. If* $c = 0$ *we have a cycle of period 2. Let* $S = \left[wz : \frac{tz - w^2 - cz^2}{a} : z^2\right]$. *Then* S *is also generic and* $RS = Id$ *out of* $z = 0$ *and* $SR = Id$ *out of* $t = 0$. *The map* R *belongs also to the class* IR if $|a| < 1$.

Note that a shear S, $S[z : w : t] = [zt : wt + z^2 : t^2]$ has an inverse which is also a shear, but they are not generic, so are not in the class MH.

PROPOSITION 3.4. - *Suppose that* R *and* S *are meromorphic maps on* $\mathbb{P}^2$ *such that* $R \circ S = Id$ *outside a hypersurface. Suppose that* R *is a generic meromorphic map. Then they are both meromorphic Hénon maps.*

Proof. It suffices to show that S is a generic meromorphic map. Clearly S has maximal rank 2. Since R is not linear, S cannot be linear either. Suppose that S is not generic. Then there must exist an irreducible compact curve V which is $S-$ constant and which is degree lowering. Let $q = q_0$ denote the image $S(V)$. Let $q_{n+1} := S(q_n)$ be the inductively defined orbit for q_0 up to q_m which is a point of indeterminacy for S. Note that this means that there exists a curve W such that $R(W) = q_m$. Hence W is $R-$ constant. Moreover $R^{m+1}(W) = q_0$ and q_0 is a point of indeterminacy for R. (Unless some q_j already is a point of indeterminay for R, which is also fine.) Hence W is degree lowering, a contradiction. □

PROPOSITION 3.5. - *If a generic meromorphic map* R *belongs to NRB or is normal, then* $F(R) = F(R^n)$ *for all* $n \geq 1$.

Proof. Clearly $F(R) \subset F(R^n)$. Suppose next that $p \in F(R^n)\backslash F(R)$. Then there exists a subsequence $\{R_m^{j+nk_m}\}$, $0<j<n$, such that R^{nk_m} converges uniformly on some neighborhood $U(p)$ while R^{j+nk_m} diverges on all neighborhoods of p. This implies that $R^{nk_m}(p)\to I_j$. The diameter of the images under R^{j+nk_m} must remain large. Since R belongs to NRB, this must remain true for R^{n+nk_m}. But this contradicts that R^n is a normal family. If R is normal, this is a consequence of Theorem 2. 12.

4. The invariant current T.

Let R be a generic meromorphic map in $\mathbb{P}^2$. Let G be the Green function in $\mathbb{C}^3$ associated to R. We study, in this section, the properties of the current T defined by the relation $\pi^*T=dd^cG$.

We refer to de Rham [de Rh] for general properties of currents. For results concerning positive currents on complex analytic varieties, see [Le] or [Kl]. We recall here a few facts. For simplicity we restrict to domains Ω in $\mathbb{C}^n$, but since the definitions are invariant under holomorphic change of coordinates, they make sense on any complex manifold.

Let $\Omega\subset\mathbb{C}^n$ be an open set, let $\mathcal{D}^{p,q}$ denote the smooth compactly supported (p,q) forms $\varphi=\sum\varphi_{IJ}dz^I\wedge d\bar z^J$, $\mid I\mid=p$, $\mid J\mid=q$, with its usual topology [de Rh]. The dual $\mathcal{D}_{p,q}$ of $\mathcal{D}^{p,q}$ is the space of currents of bidimension (p,q). So a current is just a differential form with distributions as coefficients.

Let $f:M\to N$ be a proper holomorphic map from the complex manifold M to the manifold N. If S is a current on M, the direct image of S under f, which we denote f_*S is defined by

$$<f_*S,\varphi>:=<S,f^*\varphi>\ . \tag{1}$$

Observe that the definition make sense if for every compact $K\subset N$, $f^{-1}(K)\cap\text{Support}(S)$ is compact. If S has locally integrable coefficients, the form f_*S is obtained by integrating S on the fibres of f.

When f is a submersion and S is a current on N we define f^*S as the current acting on test forms as follows, see [Sc],

$$<f^*S,\varphi>:=<S,f_*\varphi>\ . \tag{2}$$

The operation f^*, f_* have the same functorial properties as when they are applied to smooth forms.

A current S of bidegree $(n-p,n-p)$ on a complex analytic manifold M is positive if for any given forms $\varphi_1,\cdots,\varphi_p$ of bidegree $(0,1)$, smooth with compact support the distribution

$$S\wedge i\varphi_1\wedge\bar\varphi_1\wedge\cdots\wedge i\varphi_p\wedge\bar\varphi_p$$

is positive. When S is of bidegree $(1,1)$ and is written in coordinates as

$$S=i\sum_{j,k}S_{jk}dz_j\wedge d\bar z_k,$$

the condition of positivity is equivalent to

$$\sum_{j,k} S_{jk}\lambda_j\bar{\lambda}_k \geq 0 \text{ for all } \lambda_j \in \mathbb{C}.$$

The 0-currents S_{jk} are then measures.

Recall also that $d = \partial + \bar{\partial}$ and $d^c = i(\bar{\partial} - \partial)/(2\pi)$. An upper semicontinuous function V with values in $[-\infty, \infty($ is p.s.h. iff $dd^cV \geq 0$. Recall that $E \subset \Omega$ is pluripolar if $E \subset \{u = -\infty\}$ where u is p.s.h. and $u \not\equiv -\infty$.

If S is a positive current and ω is a smooth $(1,1)$ positive form, then the current $S \wedge \omega$ is positive. If S is positive and f^*S, f_*S are well defined, then they are positive.

The mass norm of a current S is given by

$$M(S) = \sup_{|\varphi| \leq 1} |< S, \varphi >| \, .$$

The norm on the test form φ, is just the supremum over all coefficients after fixing an atlas, [de Rh]. When S is a positive distribution then $M(S)$ is comparable to the total variation of the positive measure $\sum_{i=1}^n S_{ii}$. The mass of a positive current on a compact set K is given by

$$M_K(S) = \sup_{|\varphi| \leq 1} |< \chi_K S, \varphi >| = M(\chi_K S)$$

here χ_K denotes the characteristic function of K. The multiplication of S by χ_K makes sense since S has measure coefficients.

We now describe the current on $\mathbb{P}^2$ associated to a p.s.h. function on $\mathbb{C}^3$, with the right homogeneity properties.

Note that G has the following homogeneity of a plurisubharmonic function H

$$H(\alpha z, \alpha w, \alpha t) = \log \mid \alpha \mid + H(z, w, t) (*).$$

We denote by $[H]$ the class of functions equal to H up to a constant. Let P denote the class of plurisubharmonic function classes $[H]$ on $\mathbb{C}^3$ with $H(\alpha z) = c \log \mid \alpha \mid + H(z)$, $c \geq 0$, and let Q denote the class of closed, positive $(1,1)$ currents T on $\mathbb{P}^2$.

Let $\pi : \mathbb{C}^3 \backslash 0 \to \mathbb{P}^2$ be the natural projection. Consider any local holomorphic inverse $s : U \to \mathbb{C}^3 \backslash 0$ such that $\pi \circ s = Id$. Then we can define $T = T_s$ on U by $T_s = dd^c(H \circ s)$, $H \in P$. The important fact is that T_s is independent of s : If s' is another section of U, then $s' = \varphi s$ for some invertible holomorphic function φ on U. Hence

$$T_{s'} = dd^c(H \circ s') = dd^c(H(\varphi s)) =$$

$$dd^c c \log \mid \varphi \mid + dd^c H(s) = T_s.$$

So using this local definition we can define $T = L_1(H)$, $L_1 : P \to Q$ globally on $\mathbb{P}^2$ and write with abuse of notation $T = dd^cH$. Since H is plurisubharmonic, it follows that T

is positive [Le], so T is a positive, closed $(1,1)$ current. For the reader's convenience we prove the following result.

THEOREM 4.1. - *The map L_1 is a bijection between P and Q.*

Proof. We may suppose that we are in the coordinate system where $t \neq 0$. Use the section $s(z,w) = (z,w,1)$. Using that π^* and dd^c commutes ([deRh]), we get that

$$\pi^* T = \pi^*(dd^c(H \circ s)) = dd^c \pi^*(H \circ s) =$$

$$dd^c(H \circ s \circ \pi) =$$

$$dd^c(H(z/t, w/t, 1)) =$$

$$dd^c(H \circ (1/t)(z,w,t)) =$$

$$dd^c(H(z,w,t) - c \log \mid t \mid) =$$

$$dd^c H.$$

Suppose next that $L_1(H) = L_1(H')$. By the first part of the proof, $dd^c H = dd^c H'$. Hence it follows that $H - H'$ is a pluriharmonic function on $\mathbb{C}^3 \backslash 0$. But then $H - H'$ extends through the origin as a pluriharmonic function. Since $H - H'$ also grows at most like $\log \parallel z \parallel$ at infinity, it follows that $H - H'$ is constant and hence that $[H] = [H']$ and hence L_1 is $1 \to 1$.

Next we study the inverse. So let T be a positive, closed $(1,1)$ current on $\mathbb{P}^2$. Define $\nu = \pi^* T$ on $\mathbb{C}^3 \backslash (0)$. Since π is a submersion on $\mathbb{C}^3 \backslash 0$, ν is a closed, positive $(1,1)$ current [deRh]. Since the Hausdorff dimension of (0) is zero, it follows by a theorem of [HaP] that the trivial extension of ν is positive and closed. Hence, by Lelong's Theorem [Le], there exists a plurisubharmonic function H on $\mathbb{C}^3$ such that $\nu = dd^c H$. Then H is unique modulo pluriharmonic additions.

LEMMA 4.2. - There is a unique plurisubharmonic $[H]$, $H = O(\log \mid z \mid)$ at infinity, such that $\nu = dd^c H$.

Proof. Pick any H with $dd^c H = \nu$. For $\theta \in \mathbb{R}$, define $H_\theta(z) = H(e^{i\theta} z)$. We show at first that $dd^c H_\theta = \nu$ as well. Let $T_\theta(z) := e^{i\theta} z$ on $\mathbb{C}^3$. Then

$$dd^c H_\theta = dd^c(H \circ T_\theta) = T_\theta^* dd^c H =$$

$$T_\theta^* \nu = T_\theta^*(\pi^* T) = (\pi \circ T_\theta)^* T =$$

$$\pi^* T = \nu.$$

Hence it follows that if we define

$$\tilde{H}(z) = \frac{1}{2\pi} \int_0^{2\pi} H_\theta(z) d\theta,$$

then $dd^c\tilde{H} = \nu$ also. Now $\tilde{H} = \tilde{H}_\theta \ \ \forall\theta$. Since $\tilde{H}$ therefore is a radial subharmonic function on any complex line through zero, either $\tilde{H} \equiv -\infty$ (which can happen at most on a set of lines of measure zero) on the line or $\tilde{H} = -\infty$ at most at the center. Pluriharmonic functions like this are constant.

This implies that except for constant additions, $\tilde{H}$ is the only function with $\tilde{H} = \tilde{H}_\theta$, $\forall\theta$ such that $dd^c\tilde{H} = \nu$. However, for any $\alpha \in \mathbb{C}^*$, $\tilde{H}_\alpha(z) := \tilde{H}(\alpha z)$ is also such a solution. Hence

$$\tilde{H}(\alpha z) \equiv C(\alpha) + \tilde{H}(z)(**)$$

for some constant $C(\alpha)$. So

$$\tilde{H}(\alpha^n z) = nC(\alpha) + \tilde{H}(z).$$

Hence $\tilde{H}$ is of sublogarithmic growth at ∞. The Lemma follows, since pluriharmonic functions of sublogarithmic growth at ∞ are constant. □

LEMMA 4.3. - *There is a constant $c \geq 0$ so that $\tilde{H}(\alpha z) = c\log \mid \alpha \mid + \tilde{H}(z)$, $\tilde{H}$ as in the previous lemma, so we write $[\tilde{H}] = L_2(T)$, $L_2 : Q \to P$.*

Proof. Let $\alpha = 2$ and define $H' = \tilde{H} - \frac{C(2)}{\log 2}\log \parallel z \parallel$. Then H' is radial and subharmonic on each punctured complex line through 0. Moreover

$$H'(2z) = \tilde{H}(2z) - \frac{C(2)}{\log 2}\log \parallel 2z \parallel =$$

$$C(2) + \tilde{H}(z) - \frac{C(2)}{\log 2}\log 2 - \frac{C(2)}{\log 2}\log \parallel z \parallel = H'(z).$$

Hence H' is bounded from above, hence constant on each complex line. Therefore on the complex line through z, we have

$$\tilde{H}(\tau z) = H'(\tau z) + \frac{C(2)}{\log 2}\log \parallel \tau z \mid = H'(z) + \frac{C(2)}{\log 2}\log \mid \tau \mid + \frac{C(2)}{\log 2}\log \parallel z \parallel =$$

$$\tilde{H}(z) + \frac{C(2)}{\log 2}\log \mid \tau \mid . \ \square$$

LEMMA 4.4. - *The maps L_1, L_2 are inverses of each other.*

Proof. Suppose $[H] \in P$. We define $T = L_1(H)$. Then $\pi^*T = dd^cH$. Hence, $H = L_2(T)$, so $L_2 \circ L_1 = Id$.

Next, let $T \in Q$ and define $\tilde{H} = L_2(T)$ as above. Then $\tilde{H} \in P$ and $dd^c\tilde{H} = \pi^*T$. Next, consider $T' = L_1(\tilde{H})$. Then T' is a current so that $\pi^*T = \pi^*T'$. Composing with a section of π we see that $T' = T$. Hence $L_1 \circ L_2 = Id$ as well. With these lemmas the theorem follows. □

PROPOSITION 4.5. - *Let f be a generic meromorphic normal map in $\mathbb{P}^2$ with $G \not\equiv -\infty$. Then support $T = J$. If (f_λ) is a holomorphic family of holomorphic self maps of $\mathbb{P}^2$ of degree d, then the function $(\lambda, z) \to G_\lambda(z)$ is p.s.h. and the currents (T_λ) vary continuously.*

Proof. That Support $T = J$ is a consequence of Theorem 2.12. Assume λ varies in a complex manifold Δ. For $\lambda \in \Delta$ let F_λ be a lifting of the mapping f_λ, we can assume that $\lambda \to F_\lambda$ is also a holomorphic family. Fix $\lambda_0 \in \Delta$, and define

$$M_\delta = \sup\{\| F_\lambda(z) \| \; ; \; \| z \| = 1, \quad \lambda \in \Delta_\delta \text{ a } \delta \text{ neighborhood of } \lambda_0\}.$$

If δ is small enough M_δ is finite. Then $G_\lambda(z)$ is a limit of an almost decreasing sequence of p.s.h. functions and is p.s.h. with respect to (λ, z), the proof is just as in Theorem 2.12 with parameter.

When (f_λ) is a holomorphic family of holomorphic maps of degree d then, given $\epsilon > 0$ if λ is in a small enough neighborhood of λ_0, Δ_δ, we have

$$\frac{1}{C} \| z \|^d \leq \| F_\lambda(z) \| \leq C \| z \|^d \quad \text{uniformly.}$$

Hence for $\lambda \in \Delta_\delta$ and $n \in \mathbb{N}$

$$\left| \frac{1}{d^{n+1}} \log \| F_\lambda^{n+1}(z) \| - \frac{1}{d^n} \log \| F_\lambda^n(z) \| \right| \leq \frac{\log C}{d^{n+1}}.$$

Hence $G_{n,\lambda}$ converges uniformly to G_λ. Since each $G_{n,\lambda}$ is continuous in (λ, z) it follows that $G_\lambda(z)$ is continuous in (λ, z). As a consequence G_λ varies continuously with λ and hence $\lambda \to T_\lambda$ is a continuous map with values in Q. Here Q carries the weak topology of currents. □

Remark 4.6. - This remark is a continuation of example 2.1. Let $R = [z^d : w^d : t^{d-1}w]$. Then $G(z, w, t) = \sup(\log | z |, \log | w |)$ and $J = (| z | = | w |) \cup (| z | \leq | w | = | t |)$. As we observed G is pluriharmonic near in $(| z | < | w | = | t |)$ which is in J. Hence the support of T does not coincide with the Julia set J. In this example there is no positive closed $(1, 1)$ current whose support is equal to J.

Suppose S is such a current. Let $\sigma : (| w | = | z |) \to S^1$, $\sigma(z, w) = w/z$ and let $[c^{-1}(t)]$ denote the current of integration on $\{[z : w : t]\}; w/z = t, | w | > | t |$. It follows from a Theorem by Demailly [De] that there exists a measure ν whose support is S^1 such that $S = \int [\sigma^{-1}(t)] d\nu(t)$ on a neighborhood of $\{| w | = | z | > | t |\}$. Let $\tilde{S} = \int [\tilde{\sigma}^{-1}(t)] d\nu(t)$ where $[\tilde{\sigma}^{-1}(t)]$ stands for the current of integration on the whole line, not only for $| w | = | z | > | t |$. Then $\tilde{S}$ is closed and the support of $S - \tilde{S}$ is $K := \{| z | \leq | w | = | t |\} \cup \{| w | = | z | \leq | t |\}$. This is a compact set in $\mathbb{C}^2$. No compact set in $\mathbb{C}^2$ is the support of a nonzero positive closed $(1, 1)$ current. If S_1 is such a current, $< S_1, dd^c \| z \|^2 >= 0$ since S_1 have compact support and on the other hand positivity of S_1 implies that $< S_1, dd^c \| z \|^2 >$ represents the mass of S_1. Here $z = (z_1, z_2)$.

We study now sets of zero mass for the currents T.

PROPOSITION 4.7. - *Let f be a generic meromorphic map in $\mathbb{P}^2$, with Green function $G \not\equiv -\infty$. Let X, Y be closed sets in $\mathbb{P}^2$ with $Y \subset X$ and $\Lambda^2(Y) = 0$. Assume that G is locally bounded on $\pi^{-1}(X\backslash Y)$ and that X is locally $\mathbb{R}^4$-polar. Then T has no mass on X.*

Proof. Recall that a positive current is of order zero and hence has measure coefficients in a local chart. To say that T has no mass on a set E, means that all such measures, in the expression of T in a chart, have zero mass on E. For the reader's convenience we prove a well known lemma. Here Δ denotes the Laplacian in $\mathbb{R}^k$. Recall also that a polar set is a set contained in $(v = -\infty)$ where v is a subharmonic function in $\mathbb{R}^k$, not identically $-\infty$. In particular, analytic varieties in $\mathbb{P}^2$ are locally $\mathbb{R}^4$ polar.

LEMMA 4.8. - *Let v be a bounded subharmonic function on an open set $U \subset \mathbb{R}^k$. Then Δv (which is a locally finite measure), has no mass on $\mathbb{R}^k$ polar sets.*

Proof. Since a polar set is contained in a G_δ polar set and since Δv is a regular measure, it is enough to show that if K is a compact polar set then $(\Delta v)(K) = 0$. Suppose not. Then choose u such that $\Delta u = \chi_K \Delta v$, u is subharmonic, u is harmonic out of K. If u is locally bounded below near K it would have a harmonic extension, which is impossible. Also, let u' be a subharmonic function with $\Delta u' = \Delta v - \chi_K \Delta v$. Then $u' + u$ must differ from v by a harmonic function, a contradiction since v is bounded, it would follow that u is bounded below. □

We now continue the proof of the proposition. Since G is locally bounded near $\pi^{-1}(X\backslash Y)$ it follows that $\pi^*(T)$ has no mass on $\pi^{-1}(X\backslash Y)$. Since $\pi^*(T)$ is a $(1,1)$ positive current in $\mathbb{C}^3$, it has no mass on $\pi^{-1}(Y)$ which is locally of 4-Hausdorff measure 0. It follows that π^*T and hence T has no mass on X. □

We want to show next that, under quite general conditions, the current T satisfies the functional equation $f^*T = dT$ and that it is an extremal current among the currents satisfying this equation. Given a $(1,1)$ closed positive current S on $\mathbb{P}^2$ and f a generic meromorphic map of degree d, we want to define the current f^*S. Observe however that f is not a submersion.

Let F be a lifting for f. Using Theorem 4.1 we can find $u \in L$ such that $\pi^*S = dd^c u$. We define f^*S as the closed positive $(1,1)$ current such that $\pi^*(f^*S) = dd^c(u \circ F)$. If f is a submersion on $\Omega \subset \mathbb{P}^2$, then F is a submersion on $\pi^{-1}(\Omega)$ and hence $dd^c(u \circ F) = F^* dd^c u = F^*\pi^*S$. So if C denotes the critical set of f, we have, on $\Omega = \mathbb{P}^2\backslash C$, $\pi^* f^* S = F^*\pi^*S$. We also have that on $\mathbb{P}^2\backslash C$, f^*S defined above coincides with f^*S defined by relation (2) when f is a submersion.

Let Ω be a complex manifold and A a closed set in Ω. Let S be a closed positive (p,p) current on $\Omega\backslash A$, with locally bounded mass near A. We call the trivial extension of S to Ω, the extension of S to Ω giving zero mass to A. A Theorem of Skoda [Sk] asserts that such an extension is closed if A is an analytic variety.

DEFINITION 4.9. - *Let $\mathcal{N}$ denote the set of $(1,1)$ positive currents S on $\mathbb{P}^2$ which do not charge any compact complex curve V. In other words, S agrees with the trivial extension* *of* $S_{|\mathbb{P}^2\setminus V}$.

THEOREM 4.10. *Let R be a generic meromorphic map in $\mathbb{P}^2$ of degree d, with Green function $G \not\equiv -\infty$.*

(i) The currents T and R^*T belong to $\mathcal{N}$ and satisfy the functional equation $R^*T = dT$.
(ii) If R is normal and E, the extended indeterminacy set, has Lebesgue measure zero, then the current T is on an extremal ray of the cone of positive closed currents satisfying $R^*S = dS$.

Proof. We first show that $T \in \mathcal{N}$. Let V be an irreducible analytic variety in $\mathbb{P}^2$. If T has mass on V, then a Theorem of Siu [Siu] implies that the nonzero current $\chi_V T$ is closed and there exists a constant $C > 0$ such that $\chi_V T = c[V]$, here χ_V denotes the characteristic function of V and $[V]$ denotes the current of integration on V.

Let h be a polynomial of degree ℓ such that $h^{-1}(0) = V$ and $\pi^*[V] = dd^c\log \mid h \mid$. Hence $G = c\log \mid h \mid + U$, where U is p.s.h. But we have

$$G(R(z)) = dG(z) = cd\log \mid h(z) \mid + dU(z)$$

$$= c\log \mid h(R(z)) \mid + U(R(z)).$$

Hence

$$G(z) = \frac{c}{d}\log \mid h(R(z)) \mid + \frac{1}{d}U(R(z)).$$

So, the current T has also mass $c\ell$ on $(h \circ R = 0)$. Since the mass of T on $\mathbb{P}^2$ is bounded the varieties $(h \circ R^s = 0)$ cannot be all distinct as s varies. Without loss of generality assume $(h \circ R = 0) = (h = 0)$. So $R : V \to V$.

If V is R-constant then $R(V) = p \notin I$, since R is generic. Hence p is fixed and if $\pi(q) = p$ we cannot have $G(q) = -\infty$. So we can assume that R is a non constant self map on V. If the normalisation $\hat{V}$ of V is Kobayashi hyperbolic or a $\mathbb{P}^1$ then R has periodic points and we can conclude as above. If $\hat{V}$ is a torus we also have periodic points except if R is an irrational translation on $\hat{V}$. In this case the argument is more delicate since we don't have periodic points. Then V has no cusp singularities. But we know that Green's function is identically $-\infty$ on $\pi^{-1}(V)$. Hence there are points of indeterminacy of R on V, otherwise all orbits stay away from I and G is not $-\infty$. For simplicity we write the argument assuming that there is just one point of indeterminacy $p \in V$. We are going to show that the area on V of the set where $\frac{1}{d^n}\log \parallel R^n \parallel < \log\epsilon$ is small if ϵ is small. Given a point $z \in \pi^{-1}(V)$ we can write $\parallel R(z) \parallel = \delta([z]) \parallel z \parallel^d$ where $\delta([z])$ satisfies $\frac{1}{c}d([z],[p])^l \leq \delta([z]) \leq cd([z],[p])$ for some finite $c, l \geq 1$ and d is the distance in $\mathbb{P}^2$. If (z_n) is the orbit of z_0, $\parallel z_0 \parallel = 1$, $z_n = R^n(z_0)$, $\delta_k = \delta([z_k])$ we have

$$\parallel z_1 \parallel = \delta_0 \parallel z_0 \parallel^d \quad , \quad \parallel z_{n+1} \parallel = \delta_n \delta_{n-1}^d \cdots \delta_0^{d^n} .$$

If all $\delta_k \geq \epsilon^{d^k/(k+2)^2}$ then $\mid R(z_n) \mid > \epsilon^{d^n}$. So we measure the area of

$$N_k =: \{z; \delta_k(z) < \epsilon^{d^k/(k+2)^2}\}.$$

Since in this case R is essentially area preserving on V, N_k is contained in a disc of radius $(c'\epsilon^{d^k/(k+2)^2})^{1/r'}$ for some $c'>0$, $r'\geq 1$ measured in a smooth metric on V, the sum of the area of $\bigcup_k N_k$ is very small. On $\pi^{-1}(V\setminus \cup N_k)$, G is not $-\infty$, hence no such V exists and $T\in\mathcal{N}$. The potential for the current $\frac{1}{d}R^*T$ is $\frac{G(R(z))}{d}$, so the same analysis shows that $R^*T\in\mathcal{N}$.

We have

$$\pi^*R^*T = dd^cG(R) = d(dd^cG) = d\pi^*T.$$

Hence

$$R^*T = dT.$$

We prove now, that the ray λT, $\lambda>0$, is extremal in the cone of positive closed currents S, satisfying the functional equation $R^*S=dS$.

Assume $T=T_1+T_2$. By positivity $T_j\in\mathcal{N}$. Let G_1,G_2 be plurisubharmonic functions in L such that $\pi^*T_j=b_jdd^cG_j$, $j=1,2$, $b_j>0$. Since $G-(b_1G_1+b_2G_2)$ is pluriharmonic of logarithmic growth, it is constant. We can assume, without loss of generality, that $G=b_1G_1+b_2G_2$. Let $\Omega:=\mathbb{P}^2\setminus E$. Since R is normal, given $z_0\in\Omega$ there is a neighborhood $B(z_0,r)$ of z_0 such that on $\pi^{-1}\left(\bigcup_{n=0}^{\infty}R^n(B(z_0,r))\right)$ we have : $\log \parallel z\parallel -a\leq G(z)\leq \log\parallel z\parallel +a$. The functions G_j are u.s.c. so, there are constants a_j, such that $\log\parallel z\parallel -a_j\leq G_j(z)\leq \log\parallel z\parallel +a_j$ on $\pi^{-1}\left(\bigcup_{n=0}^{\infty}R^n(B(z_0,r)\right)$. If we compose by R^n divide by d^n and let $n\to\infty$, we find that $\frac{G_j(R^n(z))}{d^n}\to G(z)$. But since $R^*T_j=dT_j$ we have that $\frac{G_j\circ R}{d}-G_j=c_j$ where c_j is a constant and $b_1c_1+b_2c_2=0$. Adding a suitable constant to G_j we can assume $c_j=0$, so $G_j=G$ on $\pi^{-1}(\Omega)$. If E is of Lebesgue measure zero on $\mathbb{P}^2$, $\pi^{-1}(E)$ is of Lebesgue measure zero in $\mathbb{C}^3$, hence $G_j=G$ and therefore T is extremal. □

We want next to show that, under quite general conditions, given a positive closed $(1,1)$ current S on $\mathbb{P}^2$, $\frac{(f^n)^*S}{d^n}\to cT$ where c is a positive constant.

We need some preliminary results.

Let K be a compact set in $\mathbb{C}^2$, and let $f:U\to\mathbb{C}^2$ be a holomorphic map on a bounded neighborhood U of K. Assume that for every $w\in f(U)$, the fiber $S_w=f^{-1}(w)$ is discrete.

We prove a Lojasiewicz type inequality.

PROPOSITION 4.11. - *Let n be the maximum multiplicity of points in S_w, $w\in f(K)$. There is a constant $c>0$ such that if $w\in f(U)$, $z\in K$, then*

$$\parallel f(z)-w\parallel\geq c\,\mathrm{dist}(z,S_w)^n.$$

Proof. It suffices to prove the proposition locally. Assume that $f(0)=0$ and that f has multiplicity n at 0. We will suppose z,w are close enough to 0.

The graph of f is a branched covering with multiplicity n of the w-plane. The branches are locally given by $z=\{g_1(w),\cdots,g_n(w)\}$, $g_j(w)=(g_j^1(w),g_j^2(w))$. Hence we can form the symmetric products

$$\prod_{j=1}^{n}(z_1-g_j^1(w)) = z_1^n + a_{n-1}(w)z_1^{n-1}+\cdots+a_0(w) = P_1(z_1,w)$$

$$\prod_{j=1}^{n}(z_2 - g_j^2(w)) = z_2^n + b_{n-1}(w)z_2^{n-1} + \cdots + b_0(w) = P_2(z_2, w)$$

to obtain two Weierstrass polynomials.

Fix z^0, w^0 close to 0. Then $(z^0, f(z^0))$ belongs to the graph of f. Hence $P_1(z_1^0, f(z^0)) = P_2(z_2^0, f(z^0)) = 0$. We will try to find a point $(\tilde{z}, w^0)$ on the graph with $\tilde{z}$ close to z^0. In that case $\tilde{z} \in S_{w^0}$ and we obtain the proposition by proving a good estimate on $z^0 - \tilde{z}$.

Let C be a constant to be determined below. This constant only depends on the size of the first derivatives of the coefficients $a_j(w)$, $b_i(w)$. Say $w^0 \neq f(z^0)$. If $w^0 = f(z_0)$ we are done.

There exists an integer $2 \leq k \leq 4n$ so that $P_r(t, f(z^0))$ has no root t with

$$(k-1)C\left(\sqrt[n]{\| w^0 - f(z^0) \|}\right) \leq | t - z_1^0 | \leq (k+1)C\left(\sqrt[n]{\| w^0 - f(z^0) \|}\right), \quad r = 1 \text{ or } 2.$$

For any $\theta \in \mathbb{R}$, let $\zeta_r(\theta) = \zeta_r = kC\left(\sqrt[n]{\| w^0 - f(z^0) \|}\right)e^{i\theta} + z_r^0$ and consider the symmetric product

$$\prod_{j=1}^{n}(g_j^r(w) - \zeta_r) = G_\theta^r(w).$$

Then $G_\theta^r(w)$ is uniformly Lipschitz. Moreover, $| G_\theta^r(f(z^0)) | = | P_r(\zeta_r, f(z^0)) | \geq C^n \| w^0 - f(z^0) \|$, using the choice of k.

Hence G_θ^r has no zeroes in the ball $\{w; \| w - f(z^0) \| \leq 2 \| w^0 - f(z^0) \|\}$, if C is chosen large enough. Hence $(\zeta_1(\theta), \zeta_2(\psi), w)$ is not on the graph for any w in this ball. By continuity this means that when $w = w^0$ there must exist a point (z, w^0) on the graph with $(| z_r - z_r^0 | \leq kC(\| w^0 - f(z^0) \|)^{1/n}$, $r = 1, 2$.

The proposition now follows immediately. □

Let $\mathcal{H}_d$ denote the space of non degenerate holomorphic self maps of degree d in $\mathbb{P}^2$.

COROLLARY 4.12. *Let $f \in \mathcal{H}_d$ be holomorphic on $\mathbb{P}^2$, $d \geq 2$. Then there exists a $c > 0$ so that if $z, w \in \mathbb{P}^2$, then*

$$\operatorname{dist}(f(z), w) \geq c \operatorname{dist}(z, f^{-1}(w))^{(d^2)}.$$

Proof. The maximum multiplicity possible is d^2. □

Let us return to the notation of the proposition 4.11.

COROLLARY 4.13. - *There are constants $a > 0$, $r_0 > 0$, so that if $z \in K$, $0 < r < r_0$, then*

$$f(\mathbb{B}(z, r)) \supset \mathbb{B}(f(z), ar^n).$$

Proof. The conclusion of proposition 4.11 holds in a neighborhood of K. Pick r_0 so small that $\overline{\mathbb{B}(z,r_0)}$ is in this neighborhood for all $z \in K$, and such that no point has more than n preimages in any such ball.

Let $z_0 \in K$, $0 < r < r_0$. Then there exists k, $2n \leq k \leq 4n$, so that there is no preimage of $f(z_0)$ in $\{r \cdot \frac{k-1}{4n+1} \leq \| z - z_0 \| \leq r \cdot \frac{k+1}{4n+1}\}$. Hence if $z \in \partial\mathbb{B}(z_0, r \cdot \frac{k}{4n})$, then $\operatorname{dist}(z, S_{f(z_0)}) \geq \frac{r}{4n+1}$. Hence $\| f(z) - f(z_0) \| \geq ar^n$ for some $a > 0$. But then

$$f(\mathbb{B}(z_0,r)) \supset f(\mathbb{B}(z_0, \frac{rk}{4n})) \supset \mathbb{B}(f(z_0), ar^n). \square$$

Applying this to $f \in \mathcal{H}_d$, we get

COROLLARY 4.14. - *Let $f \in \mathcal{H}_d$. Then there exist constants $c > 0$, $r_0 > 0$ so that for $z \in \mathbb{P}^2$ and $0 < r < r_0$; then $f(\mathbb{B}(z,r)) \supset \mathbb{B}(f(z), cr^{d^2})$.*

Next we discuss the size of the image of a ball $B(z,r)$ under iteration of f.

THEOREM 4.15. - *Assume $f \in \mathcal{H}_d$ is holomorphic on $\mathbb{P}^2$, $d \geq 3$. Suppose that the local multiplicity of f is at most $(d-1)$ except on a finite set S. We assume that S contains no periodic points. There exists a constant $c > 0$, so that if $\{z_j\}_{j=0}^{\infty}$ is any orbit of f and $0 < r < 1$, then there exist radii $\{r_j\}_{j=0}^{\infty}$ with $f(\mathbb{B}(z_j,r_j)) \supset \mathbb{B}(z_{j+1}, r_{j+1})$ for every j. Moreover $r_0 = r$, $r_{j+1} = cr_j^{d_j}$ where $1 \leq d_j \leq d^2$ is an integer and $d_0 d_1 \cdots d_n \leq \frac{1}{c}(d - \frac{1}{2})^n$ for every n.*

Proof. The hypothesis on f implies that if N is sufficiently large the local multiplicity of f^N is at most

$$(d^2)^{\ell}(d-1)^{N-\ell} \leq (d - \frac{1}{2})^N$$

where ℓ denotes the number of points in S. But then the result follows easily from Proposition 4.11 and Corollary 4.13.

THEOREM 4.16. - *Let $R \in \mathcal{H}_d$ be a holomorphic map on $\mathbb{P}^2$. Assume that the local multiplicity of R is at most $(d-1)$, except possibly on a finite set without periodic points. Let $u \in P$, $u(\lambda z) = \log \mid \lambda \mid + u(z)$. Then $u(R^n)/d^n \to G$ in L^1_{loc}. Hence if $\pi^* S = dd^c u$, then $(R^n)^* S/d^n \to T$ in the sense of currents.*

Proof. The sequence $u_n := u(R^n)/d^n$ is uniformly bounded above on $\{\| z \| \leq 1\}$. We show first that no subsequence $u_{n_i} \to -\infty$ uniformly on compact sets. If so, then

$$\begin{aligned} \frac{1}{d^{n_i}} u\left(\frac{R^{n_i}}{\| R^{n_i} \|}\right) &= \frac{1}{d^{n_i}} u(R^{n_i}) - \frac{1}{d^{n_i}} \log \| R^{n_i} \| \\ &= \frac{1}{d^{n_i}} u(R^{n_i}) - G_{n_i} \to -\infty. \end{aligned}$$

Hence the map $\frac{R^{n_i}}{\| R^{n_i} \|}$ on $\mid z \mid = 1$ cannot be surjective, a contradiction.

Assume that $u_{n_i} \to G_1$ in L^1_{loc}. We want to show that $G_1 = G$. Clearly $G_1 \leq G$. Since G_1 is upper semi continuous and G is continuous, $\{G_1 < G\}$ is open. Let $\omega \in \mathbb{P}^2$ be an open set such that $G_1 < G - 2\delta$ on $\pi^{-1}(\omega)$, $\delta > 0$. By Hartogs Lemma it follows that for n_i large enough,

$$\frac{1}{d^{n_i}} u(tR^{n_i}/ \parallel R^{n_i} \parallel) < -\delta$$

on $\pi^{-1}(\omega)$, $t \in [\frac{1}{2}, 1]$ arbitrary. Hence, $\pi^{-1}(R^{n_i}(\omega)) \cap \{\frac{1}{2} \leq \parallel z \parallel \leq 1\}$ is contained in $X := \{u < -\delta d^{n_i}\}$. Let L be any line on which u is not identically $-\infty$. Since u has sublogarithmic growth at infinity, the logarithmic capacity of $X \cap L$ is at most $e^{-\delta d^{n_i}}$. But a classical estimate, see [Ts], shows that any disc contained in $X \cap L$ has a radius of order of magnitude at most $e^{-\delta d^{n_i}}$. However, by the previous theorem $X \cap \{\frac{1}{2} \leq \parallel z \parallel \leq 1\}$ contains balls of radius of order of magnitude $\epsilon^{(d-1/2)^{n_i}}$ for some $\epsilon > 0$ in the image of $\pi^{-1}(\omega)$. This contradiction completes the proof. □

PROPOSITION 4.17. - *The hypothesis of Theorem 4.16 is satisfied in the complement of a countable union of closed, proper, subvarieties of $\mathcal{H}_d$, $d \geq 3$.*

Proof. For $f \in \mathcal{H}_d$, let $(J_f = 0)$ be the equation for the critical set. Then $\sum := \{(f, z) \in \mathcal{H}_d \times \mathbb{P}^2 \; ; \; \text{grad}\, J_f = 0, \; J_f = 0\}$ is an analytic variety. It follows from the example in the proof of Lemma 5.9 in [FS1] that the projection $\pi(\sum)$ in $\mathcal{H}_d$ of $\sum$ is not all of $\mathcal{H}_d$. Hence $\pi(\sum)$ is a proper subvariety of $\mathcal{H}_d$, and for $f \notin \pi(\sum)$, the local multiplicity is at most 2 except for finitely many points in $(J_f = 0)$.

For ϵ small, consider the map

$$R_\epsilon = \left[z^d + \epsilon z(w^{d-1} + 2t^{d-1}) : w^d + \epsilon w(z^{d-1} + 2t^{d-1}) : t^d + \epsilon t(z^{d-1} + 2w^{d-1})\right].$$

One sees easily that the periodic orbits are the points where two axes cross, or are in the axes close to the unit circles or close to the torus |z|=|w|=|t|=1.

However one easily checks that none of these points are on the critical set for $\epsilon \neq 0$, ϵ small enough.

For every n, consider the analytic set $\sum_n \subset \mathcal{H}_d \times \mathbb{P}^2$ given by

$$\sum_n = \{(f, z) \; ; \; f^n(z) = z \;\; \text{and} \;\; J_f(z) = 0\}.$$

Then each $\pi(\sum_n)$ is a proper subvariety of $\mathcal{H}_d$.

The hypothesis of the theorem is then satisfied for any $f \in \mathcal{H}_d \backslash (\pi(\sum) \cup \bigcup_n \pi(\sum_n))$. □

Remark 4.18. - *If f is a normal, generic, meromorphic map on $\mathbb{P}^2$, and if the local multiplicity of f is $\leq d - 1$ in the set of normality, except on a finite set without points belonging to periodic orbits, then the conclusion of the theorem holds provided E has zero volume.*

Remark 4.19. - *A natural question is whether positive closed currents S satisfying $f^*S = dS$ are unique. The answer is no in general. Let $F = (F_1(z,w), F_2(z,w), t^d)$, then $(t=0)$ is exceptional, i.e. $f^{-1}(t=0) = (t=0)$. Define $G_1(z,w,t) = G(z,w,0)$. Then $G_1 \circ F = G(F_1, F_2, 0) = G(F(z,w,0)) = dG(z,w,0) = dG_1(z,w,t)$. It is easy to check in this case that $G(z,w,t) = \sup(G(z,w,0), \log |\, t \,|)$. If $F_1(z,w) = (z-2w)^2$, $F_2(z,w) = z^2$ and if $T_1 : \pi^* T_1 = dd^c G_1$ we find that $\operatorname{supp} T_1 = \mathbb{P}^2$ but that the Julia set is not $\mathbb{P}^2$. Hence a hypothesis on the map R is necessary in order to prove Theorem 4.16.*

Remark 4.20. - *Let $f(z,w) = (z^2 + c + aw, z)$ be the standard Hénon map. One defines $G^+(z,w) = \lim_n \frac{1}{2^{n+1}} \log^+[|\, f_1^n \,|^2 + |\, f_2^n \,|^2]$. Let $R[z:w:t] = [z^2 + ct^2 + awt : zt : t^2]$* be *the corresponding map on $\mathbb{P}^2$ and G the associated Green function on $\mathbb{C}^3$. We have $G^+(z,w) = G(z,w,1)$ and $G(z,w,0) = \log |\, z \,|$. If $\mu^+ = dd^c G^+$ considered as a current in $\mathbb{C}^2$, it was observed in [FS2] that μ^+ has a closed positive extension to $\mathbb{P}^2$ and it is easy to check that this extension is just T. Convergence results for the current μ^+ where obtained in [BS1], [BS2], [FS1], more recently the structure of μ^+ has been studied in [BLS].*

5. - Connectedness of Julia sets.

One of the main developments in the theory of several complex variables is the solution of the Levi problem. Here we show a dynamical consequence of this fundamental result.

DEFINITION 5.1. - *We say that a compact subset X of $\mathbb{C}^2$ satisfies the local maximum principle if for every $p \in X$, all small enough $r > 0$ and all complex polynomials h, $|\, h(p) \,| \leq \mathrm{Max} \,|\, h \,|_{\partial \mathbb{B}(p,r) \cap X}$.*

THEOREM 5.2. - *Suppose f is a normal generic meromorphic map on $\mathbb{P}^2$. Then the Fatou components are domains of holomorphy. The Julia set J is connected and satisfies the local maximum principle. If J is a $\mathcal{C}^1$ manifold in a neighborhood of a point on J, then J is laminated on that neighborhood by Riemann surfaces.*

Proof. We first show that $\mathbb{C}^3 \backslash (\Pi^{-1}(J) \cup (0))$ is Stein. Observe that by Theorem 2.12 the function G is plurisubharmonic in $\mathbb{C}^3$ (or $\equiv -\infty$) and pluriharmonic precisely on $\mathbb{C}^3 \backslash (\Pi^{-1}(J) \cup (0)$.

The following result is due to Cegrell [Ce], for the reader's convenience we give a proof.

LEMMA 5.3. - *Let M be a complex manifold and u a plurisubharmonic function on M. If Ω is the maximal open set where u is pluriharmonic, then Ω is pseudoconvex.*

Proof of lemma. Fix $0 < r < 1$. It is enough to show that if $H := \{(z,w), z \in \mathbb{C},\ w = (w_1, \cdots, w_{n-1}) \in \mathbb{C}^{n-1};\ |\, z \,| < 1,\ \|\, w \,\| < r$ or $r < \|\, z \,\| < 1,\ \|\, w \,\| < 1\}$ and if u is p.s.h. in a neighborhood of the closed unit polydisc $\bar{D}^n$ and pluriharmonic on H, then u is pluriharmonic on D^n. Since u is pluriharmonic on H, $u = Re\, h$, where h is holomorphic on each vertical disc, h being unique after normalization. Then h is holomorphic on H. Let $\tilde{h}$ be the holomorphic extension of h to D^n. We clearly have $u \leq Re\,\tilde{h}$ on D^n and $u - Re\,\tilde{h} = 0$ on H. Hence $u = Re\,\tilde{h}$ on D^n. □

Continuation of the proof of the theorem. Since $\mathbb{C}^3 \backslash (\Pi^{-1}(J) \cup (0))$ is pseudoconvex it follows that any Fatou component is locally pseudoconvex. Hence by the solution of the Levi Problem in $\mathbb{P}^2$, the Fatou components are domains of holomorphy (or all of $\mathbb{P}^2$, which cannot happen).

That the Julia set is connected follows from the Hartogs extension phenomenon : If $K \subset \Omega$ is a compact subset of a domain of holomorphy, then $\Omega \backslash K$ has exactly one unbounded connected component Ω' and all holomorphic functions on Ω' extend across K, hence $\Omega \backslash K$ cannot be a domain of holomorphy. So if J is not connected, we can write $J = K_1 \cup K_2$ for disjoint nonempty compact sets, and let $\Omega = \mathbb{P}^2 \backslash K_1$ and $K = K_2$ to obtain a contradiction to the solvability of the Levi problem.

If J is $\mathcal{C}^1$ near a point, since the complement is pseudoconvex, then J cannot be a real curve. If J is two dimensional, J must be a Riemann surface. If J is three dimensional, then J is laminated by Riemann surfaces, (Scherbina [Sch]), and if J is four dimensional we can also laminate a neighborhood locally by Riemann surfaces. That J satisfies the local maximum principle is a theorem by Wermer [We]. All these result use just the fact that the complement of J is a domain of holomorphy. □

PROPOSITION 5.4. - *For a normal generic meromorphic map the Julia set J does not have a Stein neighborhood.* Hence J intersects the support of any positive closed $(1,1)$ current, and in particular any compact complex curve.

Proof. If $G \equiv -\infty$, then $E = \mathbb{P}^2$, so $J = \mathbb{P}^2$ and we are done since $\mathbb{P}^2$ is not Stein. So assume $G \not\equiv -\infty$. In that case T is well defined. Suppose $U \supset J$ is a Stein neighborhood. Let ρ be a strictly p.s.h. function in U and let θ be a test function supported in U, with value 1 in a neighborhood of J. Since T is closed we have

$$< T, dd^c\rho >=< T, dd^c(\theta\rho) >= 0.$$

But $< T, dd^c\rho >$ bounds the mass of T. So U does not even have p.s.h. functions, strictly plurisubharmonic near a point of J. The complement of the support of a nonzero, positive closed $(1,1)$ current is Stein, as can be deduced from Theorem 4. 1 and Lemma 5. 3. Recall that a compact complex hypersurface is the support of a positive, closed $(1,1)$ current [Le]. □

The following result shows that an open set is in the Fatou set if a subsequence of (R^n) is equicontinuous.

PROPOSITION 5.5. - *Let R be a generic meromorphic map on $\mathbb{P}^2$. Let N be the open set of nice points. Assume that on an open set $\Omega \subset N$ there is a subsequence R^{n_i} uniformly convergent on compact sets. Then Ω is contained in the Fatou set of R.*

Proof. Let $G = \lim \frac{1}{d^n} \log \| R^n \|$ where $R : \mathbb{C}^3 \to \mathbb{C}^3$ denotes also a lifting of R. We know that locally on Ω there exist λ_{n_i} nonvanishing holomorphic functions such that $\frac{R^{n_i}}{\lambda_{n_i}} \to h$. Write on $\pi^{-1}(\Omega)$,

$$G = \lim \left[\frac{1}{d^{n_i}} \log \| R^{n_i}/\lambda_{n_i} \| + \frac{1}{d^{n_i}} \log | \lambda_{n_i} | \right].$$

Since the first term converges to 0 on $\pi^{-1}(\Omega)$ we have $dd^c G = \lim dd^c \left[\frac{1}{d^{n_i}} \log | \lambda_{n_i} |\right] = 0$, hence G is pluriharmonic on $\pi^{-1}(\Omega)$. Then Theorem 2.12 shows that Ω is contained in a Fatou component. □

It is natural to introduce the meromorphic Fatou set $F'(R)$.

DEFINITION 5.6. - *Let R be a generic meromorphic map on $\mathbb{P}^2$. A point $p \in \mathbb{P}^2$ is in $F'(R)$ if there exists a neighborhood $U(p)$ on which any subsequence of iterates has a convergent subsequence to a meromorphic map. Here we say that a sequence g_k of meromorphic maps converges to a meromorphic map g if there exists an isolated set of points S in U such that for every compact set $K \subset U \backslash S$ there exists a k_0 so that the sequence $\{g_k\}_{k>k_0}$ is equicontinuous on K and converges to g on K.*

THEOREM 5.7. - *Let R be a generic normal meromorphic map on $\mathbb{P}^2$, with Green function G. The function G is pluriharmonic on $\pi^{-1}(F'(R))$. Hence $F'(R)$ contains no point of indeterminacy and $F'(R) = F(R)$.*

Proof. Fix a convergent subsequence on $U \subset F'(R)$. As in Proposition 5.5, and using the notations of Definition 5.6, the function G is pluriharmonic on $\pi^{-1}(U \backslash S)$. But $\pi^{-1}(S)$ is just a union of complex lines in $\mathbb{C}^3$, hence G is pluriharmonic on $\pi^{-1}(U)$. Consequently $F'(R)$ does not intersect E. If R is normal or if U is contained in the set of nice points of R then $U \subset F(R)$ as follows from Theorem 2.12. □

Let R be a generic map on $\mathbb{P}^2$. If $G \not\equiv -\infty$ and if R is normal, the Julia set $J(R)$, which we also denote J_0, can be described as the support of the current T. From the example $R = [z^2 : w^2 : t^2]$ we see that there is a natural stratification of the Julia set

$$J_0 = \{[z : w : t]; | z |=| w |\geq| t |; | z |=| t |\geq| w |; | z |=| t |\geq| w |\}.$$

On the set $| z |=| t |= 1$, $| w |< 1$ the sequence (R^n) is not normal, but this set is foliated by complex discs and on each of them (R^n) is normal. We introduce the following definition.

DEFINITION 5.8. - *Let R be a generic meromorphic map on $\mathbb{P}^2$. A point $p \in \mathbb{P}^2$ is in the one dimensional Fatou set F_1 if there is a neighborhood Ω of p and for every point $q \in \Omega$, there exists a (germ of a) complex curve X_q through q such that the family of iterates R^n is equicontinuous on X_q The one dimensional Julia set J_1 is the complement of F_1. The point $p \in \mathbb{P}^2$ is in the one dimensional Fatou set F_1' if there is a neighborhood Ω of p and for every point $q \in \Omega$, a (germ of a) complex curve X_q through q such that the Green function G is harmonic on X_q (after normalization).*

Notice that we don't require that the X_q is the same in the definition of F_1, F_1'. Also note that when restricted to complex curves the finitely many points of indeterminacy of any iterate R^n are always removable.

We denote by F_0 the Fatou set of R and by F_0' the maximal open set in $\mathbb{P}^2$ such that G is pluriharmonic on $\pi^{-1}(F_0')$. In other words F_0' is the complement of the support of T. Recall that N denotes the set of nice points. Using this notation, we can reformulate Theorem 2.12 and the definitions as follows.

PROPOSITION 5.9. - *If R is a generic meromorphic map, then $F_0 \subset F_0'$. Moreover $F_0' \cap N \subset F_0$. Equivalently $J_0' \subset J_0$ and $J_0 \cap N \subset J_0'$. Also $F_0 \subset F_1$, $F_0' \subset F_1'$, $J_1 \subset J_0$.*

PROPOSITION 5.10. - *If R is a generic meromorphic map on $\mathbb{P}^2$, then $F_1 \cap N = F_1' \cap N$.*

Proof. On compact subsets of N,

$$|1/d^n \log \parallel F^n \parallel - G| \leq c/d^n.$$

If G is harmonic on the normalization of a curve X_q, then $G_{|X_q} = \log \mid h \mid$ for a holomorphic function $h \neq 0$ there. So $\{F^n/h^{d^n}\}$ is a normal family. Hence $F_1' \cap N \subset F_1 \cap N$. Suppose $\{F^n/h^{d^n}\}$ is a normal family on $X_q \subset N$. Then it follows that G is harmonic on the normalization. So $F_1 \cap N \subset F_1' \cap N$. □

Example. - The following map is studied in [FS3]

$$g([z : w : t]) = [(z - 2w)^2 : (z - 2t)^2 : z^2].$$

The point $p = [1 : 1 : 1]$ is fixed and repelling, hence $p \in J_1$. On the other hand it is shown in [FS3] that $\bigcup_{n \geq 0} g^{-n}(p)$ is dense in $\mathbb{P}^2$. Therefore for this map g, we have $J_1 = \mathbb{P}^2$.

6. $\mathbf{T} \wedge \mathbf{T} =: \mu$.

In the theory of dynamical systems, invariant measures are very useful. In this section we will discuss invariant measures for holomorphic maps in $\mathbb{P}^k$, $k = 1, 2$.

Let f be a holomorphic map on $\mathbb{P}^k$ of degree $d \geq 2$. For a continuous function φ on $\mathbb{P}^k$ define

$$f_* \varphi(x) = \sum_{f(y)=x} \varphi(y).$$

If x is a critical value we take into account the multiplicity, this coincides with the direct image of φ considered as a $(0,0)$ current as defined in paragraph 4, see [deRh].

If ν is a measure on $\mathbb{P}^k$, define the measure $f^*\nu$ by the relation

$$< f^*\nu, \varphi > = < \nu, f_*\varphi > .$$

This coincides with the pullback of ν considered as a (k,k) current provided ν has no mass on $f(C)$ where C is the critical set of f. We also define

$$< f_*\nu, \varphi >=< \nu, f^*\varphi > .$$

In $\mathbb{P}^1$ the current T is identified with the probability measure μ such that $\pi^*\mu = dd^cG$. The fact that μ has mass 1 can be checked as follows. Let ω be the standard Kähler form on $\mathbb{P}^1$ such that $\int_{\mathbb{P}^1} \omega = 1$. By the change of variable formula we have

$$\int_{\mathbb{P}^1} d\mu = \lim_n \int \frac{(f^n)^*\omega}{d^n} = 1.$$

Since μ does not give mass to locally polar sets in $\mathbb{P}^1$ (G is continuous and we apply lemma 4.8) $f^*\mu$ defined above coincides with f^*T as defined in paragraph 4.

If ν is a probability measure on $\mathbb{P}^1$ we define in $\mathbb{C}^2$

$$u(z,w) = \int_{\mathbb{P}^1} \log \frac{\mid z\zeta_2 - w\zeta_1 \mid}{\parallel \zeta \parallel} d\nu(\zeta).$$

Clearly $u \in P$ and it is easy to check that $\pi^*\nu = dd^cu$. This makes explicit the correspondance L_2 in Paragraph 4 between probability measures on $\mathbb{P}^1$ and plurisubharmonic functions u in $\mathbb{C}^2$ such that

$$u(z,w) \leq \log^+ \mid (z,w) \mid + 0(1)$$

$$u(\lambda(z,w)) = \log \mid \lambda \mid + u(z,w).$$

Let f be a holomorphic map of degree d on $\mathbb{P}^1$ and let $F = (P,Q)$ be a lifting of f to $\mathbb{C}^2$. If $a = [a_1, a_2] \in \mathbb{P}^1$ we can assume $\parallel a \parallel = (\mid a_1 \mid^2 + \mid a_2 \mid^2)^{1/2} = 1$. Then the potential u_a associated to the Dirac mass ϵ_a at a is $u_a(z,w) = \log \mid za_2 - wa_1 \mid$. It is easy to check that if a is not a critical value the potential associated to $\frac{f^*(\epsilon_a)}{d}$ is $\frac{u_a \circ F}{d}$, by continuity this holds also if a is a critical value. Similarly we prove that if ν is a probability measure with potential u then the potential associated to $\frac{f^*\nu}{d}$ is $\frac{u \circ F}{d}$.

The results we describe in the following theorem are well known, from a different point of view, see [Ly] [LFM], compare also with [HP]. For background on ergodic theory, see [Wa].

THEOREM 6.1. - *Let $f \in \mathcal{H}_d$ on $\mathbb{P}^1$. Then*

(i) $f^*\mu = d\mu$ *and hence* $f_*\mu = \mu$, $\operatorname{supp} \mu = J(f)$.
(ii) *If $u \in P$ and $\nu = L_1(u)$, then $f^{*n}\nu/d^n \to \mu$ except if ν has positive mass on an exceptional point. Similarly $\frac{u(F^n)}{d^n} \to G$ in L^1_{loc}* with the same exception.
(iii) *μ is ergodic and of maximal entropy.*
(iv) *Let $\{z_i\}$ denote the periodic points of order n (or a factor of n), and let $\mu_n := \sum 1/d^n \epsilon_{z_i}$. Then $\lim \mu_n = \mu$.*

Proof. (i) The fact that $f^*\mu = d\mu$ is a special case of Theorem 4.10 and is just a consequence of the functional equation $G(f(z)) = dG(z)$. Since μ does not have mass on points, for every continuous function φ on $\mathbb{P}^1$,

$$d < f_*\mu, \varphi >= d < \mu, \varphi \circ f >=< f^*\mu, \varphi \circ f >=$$

$$< \mu, f_ast(\varphi \circ f) >=< \mu, d\varphi >= d < \mu, \varphi > .$$

Hence $f_ast\mu = \mu$. The fact that the support of μ coincides with the Julia set is clear from the remark after Theorem 2.12.

(ii) Let F be a lifting for f. The sequence of plurisubharmonic functions in $\mathbb{C}^2$, $u_n = \frac{u(F^n)}{d^n}$ is relatively compact in L^1_{loc} :

If u_{n_i} converges to $-\infty$ the sequence v_{n_i} also converges to $-\infty$, where

$$v_n = u(F^n)/d^n - 1/d^n \log \parallel F^n \parallel = u(F^n/ \parallel F^n \parallel)/d^n.$$

Given $M > 0$ then for $\parallel z \parallel = 1$ we have $u(F^{n_i}/ \parallel F^{n_i} \parallel) < -Md^{n_i}$ for n_i large enough. This contradicts the surjectivity of $F^n/ \parallel F^n \parallel$ from $\parallel z \parallel = 1$ to $\parallel z \parallel = 1$ (u cannot be arbitrarily small on all of $\parallel z \parallel = 1$).

So assume $u(F^{n_i})/d^{n_i} \to \varphi$ in L^1_{loc}. Since $u \leq \log \parallel z \parallel + O(1)$ then $\varphi \leq G$. If $\varphi = G$ everywhere, we are done. Otherwise since G is continuous $\varphi < G$ is open. Let $\omega \subset\subset \pi\{\varphi < G\} =: \Omega$. By Hartogs lemma, for i large enough

$$1/d^{n_i} u(F^{n_i}) < G - 2\delta \quad \text{on} \quad \pi^{-1}(\omega)$$

i.e.

$$1/d^{n_i} u(F^{n_i}/ \parallel F^{n_i} \parallel) < -\delta \quad \text{on} \quad \pi^{-1}(\omega)$$

$$u(F^{n_i}/ \parallel F^{n_i} \parallel) < -\delta d^{n_i}.$$

This implies that the image under f^{n_i} of ω avoids a fixed set of positive measure in $\mathbb{P}^1$, i. e. f^{n_i} is a normal family and hence by Proposition 5.5, ω is contained in a Fatou component. Hence G is pluriharmonic on $\pi^{-1}(\Omega)$. So $\psi := \varphi - G$ is a strictly negative subharmonic function of Ω and is zero on $\partial\Omega$. Hence, by the maximum principle, Ω is a Fatou component.

Let σ be the measure on $\mathbb{P}^1$ such that $\pi^*\sigma = dd^c\varphi (= dd^c\psi$ on $\Omega)$. We have identified $(1, 1)$ positive currents on $\mathbb{P}^1$ and positive measures on $\mathbb{P}^1$. We want to show that $\sigma = 0$ on Ω.

Let $\theta \geq 0$ be a smooth function with compact support in (any connected component of) Ω

$$< \sigma, \theta >= \lim_{n_i} 1/d^{n_i} \int \sum_{j; f^{n_i}(z_j)=z} \theta(z_j) d\nu(z) \leq C_\theta \nu(f^n(\Omega).$$

If $(f^{n_i}(\Omega))$ are pairwise disjoint then $< \sigma, \theta >= 0$. So we can assume Ω is preperiodic. If $f^{-1}(\Omega)$ has other components than Ω we also have $< \sigma, \theta >= 0$ because the number of points in $\Omega \supset \operatorname{supp}\theta$ grows at most like $(d-1)^n$. So it follows that $f^{-1}(\Omega) = \Omega$, i.e. f is a map of degree d on Ω. Then $f^{-n}(z_0) \to \partial\Omega$ see [Be] or [Mi], except for a z_0 which is exceptional. Since θ is of compact support in Ω we find $<\sigma, \theta>= 0$ except when ν has mass on an exceptional point, i.e. a point such that $f^{-1}(z_0) = \{z_0\}$. So σ is supported on $\pi(\{\varphi = G\})$. Hence φ is continuous on support of $\Delta\varphi$. Therefore φ is continuous [Ts]. So $\varphi = G$ by the maximum principle.

(iii) Let $E \subset \mathbb{P}^1$, a totally invariant set. Assume $\mu(E) = c > 0$. Define $\nu = \chi_E \mu/c$. Then ν is a probability measure and $\frac{f^*\nu}{d} = \nu$. But since $\frac{(f^n)^*\nu}{d^n} \to \mu$ this implies that $\mu = \nu$, hence $c = 1$, therefore μ is ergodic. It is known [Gr], [Ly] that the entropy of f is $\log d$. Since $f^*\mu = d\mu$ the conditional measures are just Dirac masses of mass $\frac{1}{d}$ in almost every fiber of f. It follows by classical arguments [Ro] that μ is of maximal entropy, see [Ly] or [FLM].

(iv) One has to show that $\mu_n := \frac{1}{d^n} \sum \epsilon_{z_i^n} \to \mu$, where z_i^n are the periodic points of order n. Define

$$v_n(z, w) = \frac{1}{d^n} \log \mid P^n(z, w)w - Q^n(z, w)z \mid$$

where $F^n = (P^n, Q^n)$.

We have $\pi^*\mu_n = dd^c v_n$. We want to show that $v_n \to G$. Since

$$\mid P^n(z, w)w - Q^n(z, w)z \mid \leq \| (z, w) \| \| F^n \|,$$

any limit φ of any subsequence v_{n_i} satisfies $\varphi \leq G$. Assume $\varphi < G$ somewhere. Since G is continuous, this set is open. Let Ω be a component of $\{\varphi < G\}$. For every compact $K \subset \Omega$, there is, by Hartogs Theorem, a $\delta > 0$ so that $v_{n_i} < G - \delta$ on K for all sufficiently large i. Hence

$$\mid P^{n_i} w - Q^{n_i} z \mid \leq e^{-\delta d^{n_i}} (\mid P^{n_i} \mid + \mid Q^{n_i} \mid),$$

i.e. $f^{n_i} \to Id$ u. c. c. on $\pi(\Omega)$. So, there is at most one periodic point in $\pi(\Omega)$ and consequently $dd^c\varphi = 0$ on Ω. Hence φ and G are both pluriharmonic on Ω and agree on $\partial\Omega$ where G is continuous. It follows that $\varphi = G$ on Ω, a contradiction. Hence $\frac{1}{d^n} \sum \epsilon_{z_i^n} \to \mu$. $\square$

Observe that we already proved an analogue of Theorem 6.1 for $\mathbb{P}^2$ i.e. Theorem 4.16, for the current T instead of μ.

We want now to consider the case of a holomorphic map $f \in \mathcal{H}_d$ on $\mathbb{P}^2$ and construct a measure of maximal entropy. Let ω denote the standard Kähler form on $\mathbb{P}^2$ such that $\int_{\mathbb{P}^2} \omega^2 = 1$.

PROPOSITION 6.2. - *Let $f : \mathbb{P}^2 \to \mathbb{P}^2$ be a holomorphic map in $\mathcal{H}_d$, $d \geq 2$. Define μ by the identity $\pi^*\mu = (dd^c G)^2$. Then μ is a probability measure which satisfies the equations $f^*\mu = d^2\mu$ and $f_*\mu = \mu$. Moreover μ does not charge locally pluripolar sets.*

Proof. Let $\nu = (dd^cG)^2$ in $\mathbb{C}^3\backslash(0)$, where $(dd^cG)^2 := dd^c(G\, dd^cG)$. The definition makes sense since G is continuous. Moreover, the $(2,2)$ current ν extends to $\mathbb{C}^3$ as a positive, closed current [HaP]. As for $(1,1)$ currents, we can define μ by the functional equation $\pi^*\mu = \nu$. Say, in the chart $(t \neq 0)$, $\mu = (dd^cG(z,\omega,1))^2$. Since G is bounded, it follows that μ has no mass on locally pluripolar sets ([BT1], [CLN] or [Kl]).

We show that $\mu = \lim\limits_{n\to\infty} \frac{(f^n)^*\omega^2}{d^{2n}}$. We have

$$\begin{aligned}
\frac{\pi^*[(f^n)^*\omega^2]}{d^{2n}} &= \frac{(F^n)^*\pi^*\omega^2}{d^{2n}} \\
&= (F^n)^* \frac{(dd^c\log \parallel z \parallel)^2}{d^{2n}} \\
&= \left(\frac{dd^c\log \parallel F^n \parallel}{d^n}\right)^2 \\
&= (dd^cG_n)^2.
\end{aligned}$$

So in the chart $t \neq 0$, $\frac{(f^n)^*\omega^2}{d^{2n}} = (dd^cG_n(z,w,1))^2$.
By the change of variable formula, $\{(f^n)^*\omega^2/d^{2n}\}$ have uniformly bounded mass. Let ν be any weak limit. Since G_n converge uniformly on compact sets in $\mathbb{C}^3\backslash 0$ to G, by ($[CLN]$), $(dd^cG_n)^2 \to (dd^cG)^2$ in $\mathbb{C}^3\backslash 0$. Hence $\nu = (dd^cG)^2 = \mu$, for example in the chart $t \neq 0$, so $\nu = \mu$.

It follows from the change of variable formula, since f is a d^2 to 1 map, that μ is a probability measure.

We prove that $f^*\mu = d^2\mu$. On $\mathbb{P}^2\backslash f^{-1}(f(C))$, f is a submersion. We then have

$$\begin{aligned}
\pi^*(f^*\mu) &= F^*(\pi^*\mu) \\
&= F^*(dd^cG)^2 \\
&= (dd^c(G\circ F))^2 \\
&= d^2(dd^cG)^2 \\
&= d^2\pi^*\mu.
\end{aligned}$$

So $f^*\mu = d^2\mu$ on $\mathbb{P}^2\backslash f^{-1}(f(C))$. Since μ does not charge complex curves, then $f^*\mu = d^2\mu$ everywhere.

To prove that $f_*\mu = \mu$, observe that $f_*(\varphi\circ f)(x) = d^2\varphi(x)$ if $x \notin f(C)$ which is a set of μ measure 0. Hence,

$$\begin{aligned}
< f_*\mu, \varphi > &= < \mu, \varphi\circ f > \\
&= \frac{1}{d^2} < f^*\mu, \varphi\circ f > \\
&= \frac{1}{d^2} < \mu, f_*(\varphi\circ f) > \\
&= < \mu, \varphi >,
\end{aligned}$$

so $f_*\mu = \mu$. $\square$

We recall first some results in pluripotential theory, see [AT], [BT1], [Kl].

Let B be the unit ball in $\mathbb{C}^k$ and K a compact set in $B(0,r)$, $r<1$. Following [BT1] define

$$C(K,B) = \sup\left\{\int_B (dd^c u)^k,\quad 0 \le u \le 1,\quad u \text{ p.s.h.}\right\}.$$

It is shown in [BT1] that C extends to a Choquet capacity whose zero sets are the pluripolar sets.

Define also the Siciak function u_K of the compact K as follows

$$u_K(z) = \sup\{v(z),\ v \le 0 \text{ on } K,\ v(z) \le \log \mid z \mid + 0(1) \text{ at infinity}\}.$$

The uppersemicontinuous regularization of u_K is a p.s.h. function of logarithmic growth iff K is not pluripolar.

Alexander and Taylor [AT] proved the following estimate. If $K \subset B(0,r)$, $r<1$ then, there exists a constant $A(r)$ such that

$$m_k := \sup_B u_K \le \frac{A(r)}{C(K,B)}. \tag{1}$$

Using maximum principle one shows easily that

$$\log^+ \parallel z \parallel \le u_K(z) \le m_K + \log^+ \parallel z \parallel .$$

Now we prove the following result.

THEOREM 6.3. *Let $f \in \mathcal{H}_d$ on $\mathbb{P}^2$. The measure μ is mixing and of maximal entropy.*

Proof. It is enough, [Wa], to show that given two non negative smooth test functions φ, ψ we have

$$\lim_{n\to\infty} \int \psi(f^n)\varphi\, d\mu = (\int \varphi\, d\mu)(\int \psi\, d\mu).$$

Define

$$\lambda_n(a,\varphi) := \frac{(f^n)_*\varphi(a)}{d^{2n}} = \frac{1}{d^{2n}} \sum_i \varphi(f_i^{-1}(a)).$$

We show first a lemma.

LEMMA 6.4. *There exists a constant M such that*

$$\mu(\mid \lambda_n(a,\varphi) - c \mid \ge s) \le \frac{M \mid \varphi \mid_2}{s\, d^n}.$$

Here $c = \int \varphi \, d\mu$ *and* $|\varphi|_2$ *denotes the* $\mathcal{C}^2$ *norm of* φ.

Proof. It is enough to prove the above estimate locally in $\mathbb{P}^2$. So fix coordinates, say $t = 1$ and define

$$K_s = \{a = (z, w) \in B(0, \frac{1}{2}), \quad \lambda_n(a, \varphi) - c \geq s\}.$$

Let u_s be the Siciak function for K_s in $\mathbb{C}^2$ and define v_s in $\mathbb{C}^3$ by

$$v_s(z, w, t) = u_s\left(\frac{z}{t}, \frac{w}{t}\right) + \log |t| .$$

Let S be the closed $(1,1)$ current in $\mathbb{P}^2$ such that $\pi^* S = dd^c v_s$. Recall that $\nu_s := S \wedge S$ is a probability measure, Theorem 4.4 in [FS4]. Since ν_s is supported where $\lambda_n(a, \varphi) - c \geq s$, we have

$$\begin{aligned} s \leq \int (\lambda_n(a, \varphi) - c) d\nu_s &= \int \lambda_n(a, \varphi) d\nu_s - c \\ &= \int \lambda_n(a, \varphi) d\nu_s - \int \lambda_n(a, \varphi) d\mu. \end{aligned}$$

We have used that $\frac{(f^n)^*\mu}{d^{2n}} = \mu$, hence $c = < \mu, \varphi > = < \mu, \frac{f^n_* \varphi}{d^{2n}} >$. So

$$\begin{aligned} s \leq \int \lambda_n(a, \varphi)[S \wedge S - T \wedge T] &= \int \lambda_n(a, \varphi)[S - T] \wedge [S + T] \\ &= \int \frac{\varphi}{d^n}(f^n)^*[S - T] \wedge \frac{(f^n)^*(S + T)}{d^n}. \end{aligned}$$

Now we have

$$\pi^*(f^{n*})(S - T) = (F^n)^* \pi^*(S - T) = dd^c[(v_s - G) \circ F^n].$$

The function $v_s - G$ is well defined in $\mathbb{P}^2$ so

$$(f^n)^*(S - T) = dd^c[(v_s - G) \circ f^n].$$

We then have

$$\begin{aligned} s &\leq \int \frac{dd^c\varphi}{d^n}(v_s - G)(f^n) \wedge \frac{(f^n)^*(S + T)}{d^n} \\ &\leq \frac{|\varphi|_2}{d^n} \sup |v_s - G| \int \omega \wedge \frac{(f^n)^*(S + T)}{d^n}, \end{aligned}$$

since the last integral equals 1 we get

$$s \leq \frac{|\varphi|_2}{d^n} \sup |v_s - G| . \tag{2}$$

We now estimate $\sup |v_s - G|$. Let $m(s) = \sup_B u_s$. Since

$$\log^+ \| z, w \| \leq u_s(z, w) \leq m(s) + \log^+ \| z, w \|,$$

we get

$$u_s(\frac{z}{t},\frac{w}{t}) - G(\frac{z}{t},\frac{w}{t},1) \le m(s) - G\left(\frac{z/t, w/t, 1}{\| (z/t, w/t) \| \vee 1}\right)$$
$$\le m(s) + M$$

where M is a constant independent of s. Similarly

$$G\left(\frac{z}{t},\frac{w}{t},1\right) - u_s\left(\frac{z}{t},\frac{w}{t}\right) \le G\left(\frac{z/t, w/t, 1}{\| z/t, w/t \| \vee 1}\right) \le M.$$

So relation (2) gives

$$s \le \frac{|\varphi|_2}{d^n}\sup | v_s - G | \le \frac{|\varphi|_2}{d^n}(m(s) + M). \tag{3}$$

Using the Alexander-Taylor inequality (1) we get

$$m(s) + M \le \frac{A(1/2)}{C(K_s, B)} + M \le \frac{A(1/2) + MC(K_s, B)}{C(K_s, B)} \le \frac{M'}{C(K_s, B)}$$

since $C(K_s, B) \le C(B, B)$.

So using (3)

$$C(K_s, B) \le \frac{1}{s} \mid \varphi \mid_2 \frac{M'}{d^n}.$$

In the chart $t \ne 0$, $\mu = (dd^c G(z, w, 1))^2$. The function $\lambda = \frac{1}{2}\left(\frac{G(z,w,1)}{\sup_B |G(z,w,1)|} + 1\right)$ is p.s.h. on B and $0 \le \lambda \le 1$. So by the very definition of $C(K_s, B)$ we have the existence of a constant α such that

$$C(K_s, B) \ge \int_B (dd^c\lambda)^2 \ge \mu(K_s)\alpha^{-1}.$$

Finally

$$\mu(K_s) \le \alpha\, C(K_s, B) \le \frac{\alpha}{s}\frac{M'}{d^n} \mid \varphi \mid_2 .$$

A similar computation with the set $H_s = \{a/c - \lambda_n(a, \varphi) \ge s\}$ finishes the proof of the lemma.

End of proof of Theorem 6.3. Define

$$I_n := < \mu, \psi(f^n)\varphi > - < \mu, \psi >< \mu, \varphi > .$$

Observe that

$$< \mu, \psi(f^n)\varphi > = < \frac{(f^n)^*}{d^{2n}}\mu, \psi(f^n)\varphi >$$
$$= < \mu, \frac{(f^n)_*}{d^{2n}}\psi(f^n)\varphi >$$
$$= < \mu, \psi \cdot \frac{f^n_*\varphi}{d^{2n}} > .$$

So

$$I_n =< \mu, \psi[\lambda_n(a,\varphi) - c] > .$$

Let $q > 1$ and p such that $\frac{1}{p} + \frac{1}{q} = 1$. Using Hölder's inequality we have, if $L = 2\sup \mid \varphi \mid$,

$$\begin{aligned}
\mid I_n \mid &\leq \left(\int \psi^q d\mu\right)^{1/q} \left(\int \mid \lambda_n(a,\varphi) - c \mid^p d\mu\right)^{1/p} \\
&\leq \parallel \psi \parallel_q \left(\int_0^L p\, s^{p-1} \mu(\mid \lambda_n(a,\varphi) - c \mid \geq s) ds\right)^{1/p} \\
&\leq \parallel \psi \parallel_q \left(\int_0^L p\, s^{p-2} M \frac{\mid \varphi \mid_2}{d^n} ds\right)^{1/p} \\
&\leq \parallel \psi \parallel_q \left(\frac{p}{p-1}\right)^{1/p} (2 \parallel \varphi \parallel_\infty)^{\frac{p-1}{p}} \mid \varphi \mid_2^{1/p} d^{-n/p} M^{1/p} \\
&\leq C_p \parallel \psi \parallel_q \parallel \varphi \parallel_\infty^{(1-1/p)} \mid \varphi \mid_2^{1/p} d^{-n/p}.
\end{aligned}$$

So $\lim I_n = 0$ and μ is mixing.

Remark 6.5. Observe that we have given in the proof of Theorem 6.3 an estimate of the decay of the coefficient of correlation. Indeed we have shown that if φ is $\mathcal{C}^2$ and ψ is bounded then for $\epsilon > 0$ there exists C_ϵ such that

$$\left|\int \psi(f^n)\varphi\, d\mu - (\int \psi\, d\mu)(\int \varphi\, d\mu)\right| \leq C_\epsilon \mid \varphi \mid_2 \mid \psi \mid_\infty \cdot d^{-n(1-\epsilon)}.$$

Given $f \in \mathcal{H}_d$ on $\mathbb{P}^2$ we define J_1' as the support of μ. We don't know whether J_1' is equal to the Julia set of order one as defined in Definition 5.8. It has however some properties showing that it is the right analogue of the Julia set in one variable.

THEOREM 6.6. - *Let $f \in \mathcal{H}_d$, $f : \mathbb{P}^2 \to \mathbb{P}^2$. Let U be an open set intersecting J_1'. Define $E := \mathbb{P}^2 \setminus \bigcup_{n=0}^{\infty} f^n(U)$. Then E is a closed locally pluripolar set in $\mathbb{P}^2$.*

Proof. Let $W := \bigcup_{n=0}^{\infty} f^n(U)$ and let χ be the characteristic function of W. Since $f(W) \subset W$, we have that $\chi \leq \chi \circ f$. The ergodic theorem [Wa] applied to μ implies that

$$\frac{1}{N} \sum_0^{N-1} \chi \circ f^n(x) \to \int \chi\, d\mu, \qquad \mu \ a.\ e.$$

So μ a.e., $\chi \leq \int \chi\, d\mu$ hence $\int \chi\, d\mu = 1$, so W is an open set of full measure for μ.

Assume there is a small ball B such that the closed set $K := E \cap \bar{B}$ is not pluripolar. We can consider that K is contained in the chart $t \neq 0$, identified with $\mathbb{C}^2$, let

$$U_K(z,w) = (\sup\{v(z,w); v \leq 0 \text{ on } K,\ v \leq \log \parallel (z,w) \parallel +0(1) \text{ at } \infty\})^*,$$

where * denotes the uppersemicontinuous regularization. The fact that K is not pluripolar is equivalent to the fact that U_K is not $+\infty$ and hence is a locally bounded p.s.h. function, such that

$$U_K(z,w) \leq \log \parallel (z,w) \parallel +O(1)$$

at infinity. Let $\nu := (dd^c U_K)^2$. It is known that ν is supported on K see [Kl]. Define $v(z,w,t) = U_K(\frac{z}{t}, \frac{w}{t}) + \log \mid t \mid$. The function v belongs to P.

We have : $\frac{v(F^n)}{d^n} - \frac{1}{d^n}\log \parallel F^n \parallel = \frac{1}{d^n} v\left(\frac{F^n}{\parallel F^n \parallel}\right) = O\left(\frac{1}{d^n}\right)$. Hence $v_n := \frac{v(F^n)}{d^n}$ converges uniformly on compacts subsets of $\mathbb{C}^3 \backslash 0$ to G. So by [CLN] or [BT1] $(dd^c v_n)^2 \to (dd^c G)^2$ in the sense of currents. Let ν_n be the probability measures such that $\pi^* \nu_n = (dd^c v_n)^2$. We get that $\nu_n \to \mu$. So $\nu_n(\chi) \to 1$, hence support of $\nu_n \subset f^{-n}(K)$ intersects W, contradicting that $K \subset E$. Therefore E is locally pluripolar.

COROLLARY 6.7. - *Let $f \in \mathcal{H}_d$, $f : \mathbb{P}^2 \to \mathbb{P}^2$. If J_1' contains a nonempty open set then $J_1' = \mathbb{P}^2$.*

Proof. If U is an open set in J_1', then $\bigcup_{n=0}^{\infty} f^n(U)$ is dense in $\mathbb{P}^2$ and contained in J_1', so $J_1' = \mathbb{P}^2$.

Remark 6.8. - *Let $f : \mathbb{P}^2 \to \mathbb{P}^2$ be a meromorphic, non holomorphic, generic map. When the product $\mu = T \wedge T$ is defined, it turns out that $\mu = 0$. This is clear from the functional equation $f^* \mu = d^2 \mu$ if we apply the change of variable formula since f is generically a d' to 1 map with $d' < d^2$, as shown in Proposition 1.3. We will consider the problem of constructing interesting invariant measures for meromorphic maps in a forthcoming paper.*

As we have said, we don't know whether J_1' is equal to the Julia set of order one. But we have the following result, that we will use when we discuss hyperbolicity.

PROPOSITION 6.9. - *Let $f \in \mathcal{H}_d$, f $\mathbb{P}^2 \to \mathbb{P}^2$. Then $J_1' \subset J_1$. In particular J_1 is non empty.*

We first prove a lemma.

LEMMA 6.10. - *Let u be a continuous function on a closed ball $\bar{B}$ in $\mathbb{C}^k$. Assume that u is p.s.h. in B and that through any point p in B, there is a holomorphic disc Δ_p such that $u_{|\Delta_p}$ is harmonic. Then $(dd^c u)^k = 0$ in B.*

Proof. A holomorphic disc is by definition the image of the unit disc under a non constant holomorphic map φ. That $u \mid_{\Delta_p}$ is harmonic means that $u \circ \varphi$ is harmonic on the unit disc. Let v be a continuous function on $\bar{B}$, p.s.h. in B. Assume $v \leq u$ on ∂B. We show first that $v \leq u$ on B. Suppose not. Let $K = \{z/(v-u)(z) = M\}$ where M is the positive maximum of $v-u$ on $\bar{B}$. Let p be a peak point for a function $h \in \mathcal{C}(K)$ which is a uniform limit on K of holomorphic polynomials, i.e., $h(p) = 1$ and $| h |< 1$ on $K \backslash \{p\}$. Since on Δ_p, $v \leq u + M$ and $v - u$ reaches its maximum at p : we get that $v - u = M$ on Δ_p, hence $\Delta_p \subset K$, contradicting that p is a peak point. It follows that u is the solution of the Bremerman Dirichlet problem with boundary data $u |_{\partial B}$. Hence by a result due to Bedford-Taylor [BT2] $(dd^c u)^k = 0$.

Proof of Proposition 6.9. We have to show that μ vanishes on F_1. Let B be a ball $B \subset\subset F_1$. Given any point in B, there is an analytic variety of dimension one through p, X_p, such that $f^n_{|X_p}$ is normal. This means that we can find holomorphic functions λ_{n_i} on $\pi^{-1}(X_p)$ such that $\frac{F^{n_i}}{\lambda_{n_i}}$ is normal on $\pi^{-1}(X_p)$. So if $G_n = \frac{1}{d^n}\log \| F^n \|$ we have

$$G_{n_i} = \frac{1}{d^{n_i}}\log \| \frac{F^{n_i}}{\lambda_{n_i}} \| + \frac{1}{d^{n_i}}\log | \lambda_{n_i} | .$$

The first term in the sum converges uniformly to zero so $\frac{1}{d^{n_i}}\log | \lambda_{n_i} |$ converges uniformly to G on $\pi^{-1}(X_p)$, hence G is pluriharmonic on $\pi^{-1}(X_p)$. If σ is a holomorphic section of π and $G_\sigma = G \circ \sigma$ we get that G_σ is harmonic on X_p, consequently by Lemma 6.10 we have $(dd^c G_\sigma)^2 = 0$ on B. So $\mu(B) = 0$.

7. - Hyperbolicity.

In this section we consider $f : \mathbb{P}^2 \to \mathbb{P}^2$, $f \in \mathcal{H}_d$.

We define first hyperbolicity (See Ruelle [Ru]). Let $K \subset \mathbb{P}^2$ be a compact set. We assume that K is *surjectively forward invariant*, that is $f(K) = K$. The space $\hat{K} = K^N$ of orbits $\{x_n\}_{n=-\infty}^0$, $f(x_n) = x_{n+1}$ is compact in the product topology. By the tangent bundle T_K of $\hat{K}$ we mean the space of (x, ξ) where $x = \{x_n\} \in \hat{K}$ and $\xi \in T_{\mathbb{P}^2}(x_0)$ is a tangent vector. We give this tangent bundle the obvious topology. Then f lifts to a homeomorphism $\hat{f} : \hat{K} \to \hat{K}$ and f' lifts to a map $\hat{f}'$ on T_K in the obvious way.

DEFINITION 7.1. - *Let $K \subset \mathbb{P}^2$ be a compact surjectively forward invariant set. Then f is hyperbolic on K if there exists a continuous splitting $E^u \oplus E^s$ of the tangent bundle of $\hat{K}$ such that $\hat{f}'$ preserves the splitting and for some constants $C, c > 0, \lambda > 1, \mu < 1$ depending on the choice of a Hermitian metric on $\mathbb{P}^2$,*

$$| D\hat{f}^n(\xi) | \geq c\lambda^n | \xi |, \quad \xi \in E^u$$

$$| D\hat{f}^n(\xi) | \leq C\mu^n | \xi |, \quad \xi \in E^s, \quad n = 1, 2, \cdots$$

THEOREM 7.2. - *Let $f : \mathbb{P}^2 \to \mathbb{P}^2$ be a holomorphic map of degree $d \geq 2$. Then f cannot be hyperbolic on $\mathbb{P}^2$ nor on J_0.*

Proof. Assume f is hyperbolic on $\mathbb{P}^2$. Since the critical set is nonempty the fibre dimension of E^u is ≤ 1. If $\dim E^s = 2$ then all periodic orbits are attractive. Pick one, p, with immediate basin of attraction Ω. Since f is surjective, $\partial\Omega$ is a non empty, compact, forward invariant subset of $\mathbb{P}^2$. Hyperbolicity implies that orbits of points $q \in \Omega$ close to $\partial\Omega$ converge to $\partial\Omega$ contradicting that they are in the attractive basin of p. Hence $\dim E^s = 1$. Then we have a lamination of $\mathbb{P}^2$ by stable curves, and on each curve the family (f^n) is equicontinuous, so $\mathbb{P}^2 \subset F_1$. We get $J_1 = \emptyset$. This contradicts Proposition 6.9.

Assume J_0 is hyperbolic. Necessarily J_0 intersects C, the critical set (Proposition 5.4). Hence the fibre dimension of $E^u = 1$, so $\dim E^s = 1$. This implies that through every point p in $\mathbb{P}^2$ there exists an analytic disc Δ_p on which $f^n \mid \Delta_p$ is equicontinuous (clearly this is true for points not in J_0, and for points p in J_0, we consider the stable manifold through p). So $F_1 = \mathbb{P}^2$ and this is again impossible since $J_1 \neq \emptyset$. □

Next we consider the question whether there exist maps which are hyperbolic on the nonwandering set.

DEFINITION 7.3. *The nonwandering set of a map $f : \mathbb{P}^2 \to \mathbb{P}^2$ is the set of points p such that for every open neighborhood $U(p)$, there exists a positive integer n such that $f^n(U) \cap U \neq \emptyset$.*

It is clear that the nonwandering set is compact, surjectively forward invariant.

THEOREM 7.4. - *Let S_l be a compact hyperbolic, surjectively forward invariant set of unstable dimension l. For $l = 0$, $S_0 \subset F_0$ and S_0 is a finite union of attractive periodic orbits. If $l = 1$, $S_1 \subset J_0$ and if $l = 2$, $S_2 \subset J_1$.*

Proof. There is an arbitrarily small finitely connected neighborhood V of S_0 such that $f^n(V) \subset\subset V$ for all n large enough and f^n is strictly distance decreasing on each component of V. Hence $f^n \mid V$ converges to attractive periodic orbits.

Case l = 1. We need to show that $S_1 \subset J_0$. Let $x \in S_1 \cap F_0$. Then f^n is equicontinuous on some neighborhood of x. Let $\xi \neq 0$ be an unstable tangent vector at x. The iterates $(f^n)'(x)(\xi)$ have to blow up, a contradiction.

Case l = 2. We need to show that $S_2 \subset J_1$. Suppose $x \in S_2 \cap F_1$. Then there is a complex curve X through x so that $\{f^n \mid X\}$ is equicontinuous. We can assume that X is irreducible at x. If x is a regular point, let ξ be a nonzero tangent vector to X at x. Then this is an unstable tangent vector, a contradiction. So it remains to consider the case when x is a singular point of X.

We parametrize X, $\varphi : \Delta \to X$, $t \to (t^p, t^q + \cdots) = (z, w)$ in local coordinates, $q > p$. We assume $x = O$. The sequence $\{f^n \circ \varphi\}$ is equicontinuous and $f^n(\varphi(t)) - f^n(\varphi(0)) = O(t^p)$ independently of n. This contradicts that 0 is an unstable point in all directions. □

Assume from now on that the nonwandering set Ω is hyperbolic. We divide

$\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$ where Ω_j has unstable dimension j.

THEOREM 7.5. - *If the nonwandering set Ω of a holomorphic map f of degree at least 2 is hyperbolic, then all Fatou components are preperiodic to finitely many attractive periodic basins.*

Proof. Pick a Fatou component U. Assume at first that $f^n \mid U$ does not converge u. c. c. to the Julia set. So for some compact subset $K \subset U$ and some subsequence f^{n_k}, the iterates converge uniformly to a holomorphic map with values outside some neighborhood of J. These values must then be in a periodic Fatou component, which we may assume is U. Replacing f by an iterate, we may assume that U is fixed. Hence we may assume that for some other subsequence $f^{n_{k+1}-n_k}$ the iterates converge to a holomorphic map with a fixed point in U. This point is then necessarily nonwandering so by the above theorem is an attractive periodic point for f.

On the other hand assume that the iterates $f^n \mid U$ converge u. c. c. to J. Pick $q \in U$. Let p be any cluster point of the iterates $f^n(q)$. Then p must be nonwandering. Any such cluster point belongs to $S_1 \cup S_2$. Note however because of the repelling nature of S_2, it is impossible to only cluster at S_2 without also clustering at other points arbitrarily close to S_2. Since S_1 and S_2 are disjoint compact sets, the cluster set must be contained in S_1. However in a small neighborhood of S_1 we can use the hyperbolicity on sectors in the tangent space to conclude that the iterates of $f \mid U$ must be diverging. Since unstable sectors are mapped to corresponding unstable sectors, the derivatives blow up since we always stay in a neighborhood of S_1, contradicting that we are in a Fatou component. So this case is impossible. We have shown then that all Fatou components are preperiodic to a finite number of attractive basins. □

Question 7.6. - *Let S_1 be a compact hyperbolic, surjectively forward invariant set of unstable dimension 1. Is $S_1 \subset F_1 \backslash F_0$?*

Example 7.7. - *Consider the map $f = [z^2 : w^2 : t^2]$. Then this map has three superattractive fixed points, $[0:0:1]$, $[0:1:0]$, $[1:0:0]$. The complement of the three basins of attraction is the Julia set. More precisely, $J_0 = \{[z : w : t]$; such that two coordinates have modulus one and the third has modulus at most one$\}$. In addition to the set of superattractive fixed points S_0 the nonwandering set contains the set S_1 consisting of the three circles where one homogeneous coordinate is 0 and the other two have modulus one and also the set S_2 consisting of the totally real torus where all three coordinates have modulus one. This example is hyperbolic on the nonwandering set.*

Question 7.8. - *Assume that X is a closed totally invariant set for f and that X is disjoint from the closure of the forward orbit of the critical set. Is f hyperbolic on X ?*

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