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Dynamics of P^2 (Examples)

by

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Contents

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1 Introduction

The Fatou-Julia theory of the dynamics of rational maps in $\mathbf{P}^1$ has received a lot of attention in the past 20 years. The notes by J. Milnor ([Mi]) or the monograph by Carleson-Gamelin ([CG]) give an introduction to the classical theory. It is quite natural to extend the theory to holomorphic endomorphisms of $\mathbf{P}^k$, $k \geq 2$. Some progress has been made in this direction and we refer to the survey articles ([F]) and ([S1]) for a description of the main results obtained in this direction.

The purpose of the present article is to study some families of endomorphisms of $\mathbf{P}^2$ that exhibit interesting dynamical properties. Before discussing more precisely the content, we recall some basic properties, see ([F]), ([S1]) for the proofs.

Let f be a holomorphic endomorphism of $\mathbf{P}^k$. In homogeneous coordinates, $f = [f_0 : \cdots : f_k]$ where all the f_j are homogeneous polynomials of degree d. We will say that d is the algebraic degree of f. The topological degree is d^k. We will assume that $d \geq 2$.

The Fatou set for f is the maximal open set where the family of iterates (f^n) is locally equicontinuous. The complement of the Fatou set is the Julia set $\mathcal{J} = \mathcal{J}_f$. One has the following characterization for the Julia set (FS2]). Let ω denote the standard Kahler form in $\mathbf{P}^k$. The sequence of forms $\left(\frac{(f^n)^*\omega}{d^n}\right)$ converges in the sense of currents to a positive closed current T. The support of T is exactly the Julia set of f. The current T satisfies the functional equation $f^*T = dT$.

The saddle periodic points with their stable manifolds are contained in the Julia set. The periodic points which are repelling are also contained in the Julia set. So in contrast with the one dimensional theory, the Julia set is not contained in the non wandering set of f. The Julia set can be decomposed using the currents of bidegree (ℓ, ℓ), $T^\ell := T \wedge \cdots \wedge T$, ℓ times. We will restrict our attention here to the measure $\mu := T \wedge \cdots \wedge T$, k times. The measure μ is an invariant measure of maximal entropy $\log d^k$. The map f is mixing for μ. Moreover as was recently proved by Briend-Duval ([BT]) the Lyapunov exponents for μ are strictly positive. Hence the measure μ is a limit of $\mu_n := \frac{1}{d^{nk}} \sum_j \epsilon_{a_j}$ where the sum is over the repelling periodic points of order n.

A point p is in the support of μ if and only if for every $r > 0$ the set $\mathbf{P}^k \setminus \cup_{n>0} f^n(B(p,r))$ is pluripolar, in some sense it means that the set is "small", for example it is of Hausdorff dimension at most $2(k-1)$ ([FS2]). Since the map f is mixing for μ, the support S_μ of μ is contained in the non wandering set of f. It is in some sense the analogue of the Julia in $\mathbf{P}^1$, for example periodic repelling points are dense in S_μ.

Besides the Fatou components and S_μ, there are other pieces of $\mathbf{P}^k$ where interesting dynamics occurs, for example attractors, first investigated in this setting in ([FW]) . Recall that a closed set $A \subset \mathbf{P}^k$ is an <u>*attracting*</u> set if there is an open set $U \supset A$ such that $f(U) \subset\subset U$ and $A = \cap f^n(U)$. The open set U is called a trapping region. The set A is an <u>*attractor*</u>, in Ruelle's sense ([Ru]), if A is a minimal equivalence class for the relation $x \succ y$ defined by $x \succ x$ for all x and $x \succ y$ for $x \neq y$ if there is for every $\epsilon > 0$ an ϵ chain from x to y, i.e. there are $\{x = x_0, \ldots, x_n = y\}$ such that for all $1 \leq j \leq n$, dist $(f(x_{j-1}), x_j)) < \epsilon$. For example let $f_0 = [P(z,w) : Q(z,w)]$ be a rational map on $\mathbf{P}^1$ with Julia set equal to $\mathbf{P}^1$. Then the map

$$\overline{f}([z:w:t]) := [P(z,w) : Q(z,w) : t^d]$$

has the hyperplane $(t = 0)$ as an attractor. We study in the second section perturbations of the map $\overline{f}$, we show that for some special perturbations f_ϵ there is an attractor A_ϵ where periodic points are dense and where the corresponding unstable manifolds are dense. The attractor A_ϵ is not algebraic in general, a first example of that phenomenon was given by Jonsson ([J1]), see also ([JW]).

In paragraph 3 we give examples in $\mathbf{P}^2$ where the Julia set is equal to $\mathbf{P}^2$. We also generalize to polynomial maps of $\mathbb{C}^k$ some standard facts for polynomials in $\mathbb{C}$. More precisely let f be a polynomial map of $\mathbb{C}^k$, which extends holomorphically to $\mathbf{P}^k$. Let

$$K := \{z; (f^n(z)) \text{ is bounded}\}.$$

If K does not intersect the critical set, then f is strictly expanding on K and $K = S_\mu$, so repelling periodic points are dense in K.

In paragraph 4 we study perturbations of generalized Hénon maps in $\mathbb{C}^2$. Consider the automorphisms of $\mathbb{C}^2$ defined by

$$f_0(z, w) = (p(z) + aw, z)$$

where p is a polynomial of degree d. We consider the perturbation

$$f_\epsilon([z : w : t]) = [t^d p(z/t) + awt^{d-1} : zt^{d-1} + \epsilon w^d : t^d].$$

For $\epsilon > 0$, f_ϵ is an endomorphism of $\mathbf{P}^2$. We show that the nonwandering set Ω_ϵ of a composition of such maps might contain a countable discrete set of repelling periodic orbits. When $f_0 = (z^2 + aw, z)$ and $|a| >> 1$ we give a more complete description of the dynamics of

$$f_\epsilon([z : w : t]) = [z^2 + awt : zt + \epsilon w^2 : t^2]$$

in P^2.

In paragraph 5 we give examples where $\mathrm{Supp}(\mu)$ equals the Julia set.

2 Attractors

The general form of maps on $\mathbf{P}^2$ preserving complex lines through $[0 : 0 : 1]$ is

$$f = [P(z, w) : Q(z, w) : R(z, w, t)].$$

We will consider the special case

$$f = f_\epsilon = [P(z, w) : Q(z, w) : t^d + \epsilon Q_1(z, w)].$$

We will assume in what follows that the map

$$f_0 = [P(z, w) : Q(z, w)]$$

has Julia set equal $\mathbf{P}^1$. In the first three subsections we will also normalize so that $[1 : 0]$ is a fixed point.

LEMMA 2.1 *For all small enough complex valued ϵ, the only Fatou component is the superattractive basin of* $[0 : 0 : 1]$.

Proof: If p is not in the basin of zero, let D be a disk through p that projects (via lines through the origin) onto an open subset of the line $\mathbf{P}^1$ at infinity. Since the map preserves lines through the origin and its restriction to the line at infinity has Julia set equal to $\mathbf{P}^1$, we see that forward images of D will eventually project onto the entire line at infinity. Hence iterates of f cannot be normal on D, unless $f^n(D)$ tends uniformly to zero, which our assumptions disallow. We conclude that p is in the Julia set.

∎

LEMMA 2.2 *Let U be any neighborhood of $(t = 0)$. Then f_ϵ has an attracting set A contained in U for any small enough ϵ. The attracting set intersects all lines through* $[0 : 0 : 1]$.

Proof: When $\epsilon = 0$ the map f fixes $(t = 0)$ and this set is attracting. Hence there exists a small neighborhood $V \subset U$ of $(t = 0)$ so that for all small enough ϵ, $f_\epsilon(V) \subset\subset V$. Set $A = \cap f_\epsilon^n(V)$, then since $f_\epsilon^n(t = 0)$ intersects all lines, the same must be true for A.

∎

We note that A is connected since we may take V to be connected.

The following is obvious.

THEOREM 2.3 *The periodic points in A are the following: Consider any periodic orbit in the space of lines. Next there is a unique periodic orbit in those lines belonging to A and this is attracting inside those lines. In particular, A does not contain any repelling periodic orbit.*

Note that on the fixed line and all its preimages the lines are divided in two by the preimages of the quasicircle in the fixed line. On the fixed line the boundary of the basin of 0 and the basin of A agree.

Remark 2.4 *If an attracting set A contains a repelling periodic point p of period ℓ, then A contains a nonempty open set. Indeed if r is small enough, $B(p,r) \subset A$ and so*
$\lim_{n\to\infty} f^{n\ell}(B(p,r)) \subset A$, *but $f^{n\ell}(B(p,r))$ increases, (at least after a linear coordinate change,) to an open set $V \subset A$. If f^ℓ is linearizable in a*

neighborhood of p, then V is a non degenerate image of $\mathbb{C}^2$, *using a map* $\lim_{n\to\infty} f^{n\ell} \circ \left[(f^\ell)'(0)\right]^{-n}$.

The following theorem is proved in the same way as Proposition 6.1 of ([FS2]).

THEOREM 2.5 *Take any forward invariant closed positive* $(1,1)$ *current* σ, *i.e.* $f_*\sigma = d\sigma$, *d being the algebraic degree of* f, *on* A, *for example a Cesaro mean of the forward orbits of the current* $(t = 0)$. *Then* $\nu := \sigma \wedge T$ *is an invariant measure.*

Let B_0 be the basin of attraction of 0 and B_A the basin of attraction of A, i.e. $B_A = \cup_n f^{-n}(U)$ where U is any small neighborhood of A. Both are totally invariant connected open sets.

THEOREM 2.6 *We have that Supp*$(\mu) \subset \partial B_0 \subset \partial B_A$. *Moreover Supp*$(\mu)$ *projects onto* $\mathbb{P}^1$.

Proof: The basin B_A contains the line at infinity $(t = 0)$. So $\mathbb{P}^2 \setminus \overline{B}_A$ is a bounded invariant open set in $\mathbb{C}^2$. As a consequence $\mathbb{P}^2 \setminus \overline{B}_A$ is a union of Fatou components. Since there is only one Fatou component, namely B_0, we have $\mathbb{P}^2 = \overline{B}_A \cup B_0$. It follows that $\partial B_0 \subset \partial B_A$.

Let $p \in$ Supp μ, and W any neighborhood of p. Then $\cup f^n(W)$ ([FS1]) is dense in $\mathbb{P}^2$. Hence W must contain points in B_0 and points in B_A, so Supp $\mu \subset \partial B_0 \cap \partial B_A = \partial B_0$.

Let p be a repelling periodic point in Supp(μ). By the Briend-Duval Theorem ([BD]) such p exists. It is a repelling fixed point for some iterate $g_n := f^n_{|L}$ where L is a line through p. The preimages of p under inverse iterates of g_n are dense in the Julia set of g_n. The preimages of p under f are in Supp(μ). Hence Supp(μ) projects onto $\mathbb{P}^1$.

■

We consider the maps f_ϵ on the space of lines, i.e. the map f_0 on $\mathbb{P}^1 = (t = 0)$. Let $\mathcal{F}$ be the compact set of histories $(x_{-n})_{n\leq 0}, (f_0(x_{-n-1}) = x_{-n})$ in the product topology of $\mathbb{Z}^{\mathbb{P}^1}$. We define a map $\Phi : \mathcal{F} \to A$. But we first need a more precise trapping region.

2.1 Trapping region

Let $\eta > 0$. We define the neighborhood W_η about $(t = 0)$ as follows:

$$\begin{aligned} W_\eta^1 &= \{[z:w:t]; |z| \leq |w|, \ |t| < \eta|w|\} \\ W_\eta^2 &= \{[z:w:t]; |w| \leq |z|, \ |t| < \eta|z|\} \\ W_\eta &= W_\eta^1 \cup W_\eta^2. \end{aligned}$$

So W_μ is the exterior of the bidisc of radius $1/\eta$ in the (z, w) coordinates. Clearly, if we fix any small $\eta_0 > 0$ then for all small enough ϵ and all $\eta < \eta_0$ we have $f(W_\eta) \subset W_1$. Then, for example, if $[z : 1 : t] \in W_\eta^1$ and $f([z : w : t]) \in W_1^1$ we can write

$$f([z:1:t]) = [\frac{P(z,1)}{Q(z,1)} : 1 : \frac{t^d + \epsilon Q_1(z,1)}{Q(z,1)}].$$

Here $|z| \leq 1$ and $|P(z,1)| \leq |Q(z,1)|$ so necessarily $Q(z,1)$ is bounded uniformly away from zero. So we see that we get:

LEMMA 2.7 *There exists a constant $C > 0$ independent of ϵ so that: If we restrict to any line and any $0 < \delta < \epsilon < \eta$, the image of the disc $\Delta(0, \delta)$ around $t = 0$ is contained in $\Delta(0, C(\delta^d + \epsilon))$ in the image-line and the derivative in the t direction is bounded by $C\delta^{d-1}$.*

Hence we get a trapping region of the form $W_{2C\epsilon}$ and on these the derivative in the t direction is bounded by $2^{d-1}C^d\epsilon^{d-1}$. It follows that this trapping region contains a nontrivial attracting set, intersecting all lines. In fact we see that the attracting set is an attractor by using the fact that in the space of lines there are arbitrarily long pseudo-orbits connecting two points.

2.2 The map Φ.

Let (x_{-n}) be any element of $\mathcal{F}$. For any n consider the image $f^n(\Delta(0, 2C\epsilon))$ where we take the disc in the line x_{-n}. It follows that the intersection of these images contains exactly one point $p \in A$. We set $\Phi((x_{-n})) = p$. So this map sends $\mathcal{F}$ to A. On the other hand, if $p \in A$, then there is a sequence (p_{-n}) of preimages in A (perhaps several), $p_0 = p$. Each of these p_{-n} are in some line x_{-n} and necessarily $(x_{-n}) \in \mathcal{F}$. Then $\Phi((x_{-n})) = p$. Hence Φ is onto.

The following Lemma is then obvious.

LEMMA 2.8 *$\Phi : \mathcal{F} \to A$ is continuous and onto. Also Φ is a semiconjugacy. Φ maps periodic points to periodic points and hence periodic points are dense in the attractor.*

Since $[0 : 0 : 1] \notin A$, we can define a projection $\pi : A \to \mathbf{P}^1, \pi([z : w : t]) = [z : w]$. Then π is also a semiconjugacy, $\pi \circ f = f_0 \circ \pi$.

Notice that this construction works even if the map on $(t = 0)$ is non-chaotic. (Of course periodic points are then not necessarily dense. But we always get a semiconjugacy $\Phi : \mathcal{F} \to A$.)

LEMMA 2.9 *Let (x_j) be an arbitrary repelling periodic orbit for the dynamics in the space of lines and let $(p_j) = \Phi(x_j)$ be the corresponding periodic orbit in A. Then (p_j) is a saddle orbit. The unstable curve of any of the p_j and with respect to this periodic history is contained in A and is dense in A. Also the stable curve of p_j intersects A in a dense set.*

Proof: We prove that the unstable curve W^u for a periodic orbit is dense in A. It is clear that W^u intersects all lines. Let $q \in A$. Let $q_{-\ell} \in A$ be such that $f^\ell(q_{-\ell}) = q$. Now W^u intersects the line through $q_{-\ell}$ at Q. Then $f^\ell(Q)$ belongs to W^u and must be close to $f^\ell(q_{-\ell}) = q$ if $\ell >> 1$ since f^ℓ is contracting along the lines through 0 in B_A.

Also the stable manifold must contain all the intersections of the trapping region with the lines in the preimages of the periodic lines. Suppose $q \in A$, and let $y \in W^u$ be in the same line as q, close to q. Let Σ be a small irreducible neighborhood in W^u of y (in some branch of W^u). Then this Σ must intersect all nearby periodic lines. So W^s is dense in A.

■

LEMMA 2.10 *The set A is an attractor. Moreover $f|A$ is topologically transitive.*

Proof: The first statement was observed above. Recall that $f_{|A}$ is topologically transitive if given two relatively open sets U and V in A, there is an integer $n \geq 1$ so that $f^n(U) \cap V \neq \emptyset$. Since by Lemma 2.8 the periodic points are dense in A, we can assume that U contains a periodic saddle point p. The unstable curve associated to p is dense in A (Lemma 2.9) hence $\cup_n f^n(U)$ is dense in A.

■

2.3 Non-Algebraicity of A

In this section we will restrict to the case of degree $d = 2$ to simplify calculations.

Let z_f be the fixed point on A contained in the fixed line $(w = 0)$. Also let L be the unique other line in the preimage of $(w = 0)$ in the space of lines. We will make a condition on the coefficients of the polynomials P, Q, Q_1 that will imply that the image of the trapping region in L is disjoint from z_f. This will imply that the intersection of the attractor with the fixed line is infinite and hence non-algebraic.

We fix notation: Write:

$$\begin{aligned} P(z,w) &= z^2 + azw + bw^2 \\ Q(z,w) &= czw + dw^2 \\ Q_1(z,w) &= ez^2 + fzw + gw^2 \\ z_f &= [\alpha : 0 : 1] \end{aligned}$$

We assume that $e, \epsilon, c \neq 0$. We fix P, Q, Q_1 and let ϵ be small enough. The following is immediate from the form $z \longrightarrow \frac{z^2}{1+\epsilon e z^2}$ of f on the fixed line.

LEMMA 2.11

$$\alpha = \frac{1+\sqrt{1-4\epsilon e}}{2\epsilon e} \sim \frac{1}{\epsilon e}$$

Next we observe that the line $cz + dw = 0$ is the other preimage in the space of lines of the fixed line:

LEMMA 2.12 *The image of the point at infinity on the line $cz + dw = 0$ is the point $[\frac{B}{\epsilon} : 0 : 1]$ where*

$$B = \frac{d^2 - acd + bc^2}{(ed^2 - fcd + gc^2)}$$

The image under f of the trapping region on the line $cz+dw = 0$ is contained in a disc around $\frac{B}{\epsilon}$ of radius about C for some fixed constant C independent of ϵ. In particular if $B \neq \frac{\pm 1}{e}$ then for all small enough ϵ the image under f of the attractor in $cz+dw = 0$ does not contain z_f nor it's preimage $-z_f$ on the fixed line. The second image is already closer to z_f than C (we can assume that C is the same constant as above). Since f is $1-1$ there, it follows that the intersection of the attractor with the fixed line is infinite and hence that the attractor is non-algebraic.

COROLLARY 2.13 *The fixed point z_f has only one preimage in the attractor (itself). The same is then true for a whole neighborhood in A. The set of points with only one preimage is open. In particular, these attractors are not completely invariant.*

We don't need in the above arguments that the Julia set of $f_0 = [P : Q]$ is $\mathbf{P}^1$. It suffices to assume that f_0 has no attracting periodic points. Hence we have:

PROPOSITION 2.14 *Let $f_0 = [P(z,w) : Q(z,w)]$ be a rational map on $\mathbf{P}^1$ without attracting periodic points. Then for $\epsilon \neq 0$ small enough the map*

$$f_\epsilon = [P(z,w) : Q(z,w) : t^d + \epsilon Q_1(z,w)]$$

has a nontrivial attractor A. If the Julia set of f_0 is different from $\mathbf{P}^1$, then $f_{\epsilon|A}$ is not topologically transitive and periodic points are not dense in A. When f_0 is of degree 2 and $[1 : 0]$ is a repelling fixed point for f_0 then A is non algebraic with the above conditions on the coefficients.

Remark 2.15 *Assume f_0 has a Siegel disc D centered at the fixed point $[z_0 : 1]$. Then the sequence of iterates $f^n_{\epsilon\,|D}$ converges uniformly to a limit disc which is invariant and with the same rotation number as the original Siegel disc. This Siegel disc is contained in A.*

PROPOSITION 2.16 *Let $f \in \mathcal{H}_d$. If A is an attractor, $A \neq \mathrm{P}^k$, then A is disjoint from S_μ, the support of μ.*

Proof: Let U be a neighborhood of A such that $f(U) \subset\subset U$. Assume that $S_\mu \cap U$ is nonempty. Since by ([FS1]), $\cup_n f^n(S_\mu \cap U)$ covers P^k except for a pluripolar set, we get a contradiction by choosing U, $\overline{U} \neq \mathrm{P}^k$.

∎

PROPOSITION 2.17 *Let $\delta > 0$. Then for all small enough ϵ each slice of the attractor for f_ϵ in a line through $[0:0:1]$ has Hausdorff dimension less than δ.*

Proof: We know from Lemma 2.7 that there exists a trapping region $W_{c\epsilon}$ where the derivative in the t direction is bounded by $C_1|\epsilon|^{d-1}$. It follows that the image of a disc of radius r in the t direction is contained in a disc of radius $C_1|\epsilon|^{d-1}r$, here $r << 1$.

For a given line ℓ_0 through 0, let $A_0 := A \cap \ell_0$. The closed set $A \cap f_\epsilon^{-n}(A_0)$ is contained in d^{2n} discs of radius $\epsilon_0 < 1$. Hence A_0 is contained in d^{2n} discs of radius $C_1^n|\epsilon|^{n(d-1)}\epsilon_0$. To estimate the Hausdorff measure of dimension δ, we calculate $d^{2n}(C_1^n|\epsilon|^{n(d-1)}\epsilon_0)^\delta$. This is finite if $d^2C_1|\epsilon|^{\delta(d-1)} < 1$. So it suffices to choose $\epsilon < \left(\frac{1}{d^2C_1}\right)^{\frac{1}{\delta(d-1)}}$.

∎

THEOREM 2.18 *([Gu]) Every neighborhood of the support of a positive closed current of bidegree $(1,1)$ contains a compact complex curve. Hence every neighborhood of a nontrivial attractor contains a compact complex curve.*

2.4 Subhyperbolicity of attractors.

In this subsection we want to extend the subhyperbolicity of the critically finite maps $f_0 := [(z-2w)^2 : z^2]$ on P^1 to attractors for $f_\epsilon := [(z-2w)^2 : z^2 : t^2 + \epsilon Q_1(z,w)]$ and show that unstable curves are dense. This generalizes to other post-critically finite maps, but we use f_0 for computational simplicity.

The critical points of f_0 in $(w = 1)$ are $\{0, 2\}$ with orbits $2 \to 0 \to \infty \to 1 \hookleftarrow$. Then there is a metric $d\sigma$ on $\mathbf{P}^1$ ([CG]) which is smooth except at the points $0, \infty, 1,$ for which the map is strictly expanding. The singularites are $d\sigma \sim |dz|/\sqrt{|z|}$ in local coordinates near 0 and $d\sigma \sim |dz|/|z|^{3/4}$ in local coordinates around ∞ and 1.

Next, we define a pseudometric $d\sigma^*$ on $\mathbf{P}^2$ near $(t = 0)$. Pick a smooth Hermitian metric ds, and pick a point (p, ξ) in the tangent bundle of $\mathbf{P}^2$, p near $(t = 0)$. We can decompose $\xi = \xi_1 + \xi_2$ where ξ_2 is along the line through 0 and p in $(t = 1)$ and ξ_1 is perpendicular to ξ_2 in the $ds-$ metric. There is a natural projection to $(t = 0)$ and this projects ξ_1 to a vector tangent to $(t = 0)$ which we also denote by ξ_1. We set

$$d\sigma^*(\xi) := \sqrt{(d\sigma)^2(\xi_1) + (ds)^2(\xi_2)}.$$

With this singular metric on the attractor, the map f is contracting in one direction and expanding along the line at infinity. This generalizes the concept of subhyperbolicity in one dimension.

Pick any p in the attractor A for the maps above. Let $\{p_{-n}\}$ be any history in A. For each n let Δ_n denote small discs of about the same radius around $[p_{-n}]$, the projection of p_{-n} to $(t = 0)$. Let $\Sigma_n := f^n(\Delta_n)$.

THEOREM 2.19 *The curves Σ_n converge to a nonconstant image of $\mathbb{C}$ contained and dense in A, and passing through p.*

Proof: First we give a brief outline of the proof. In the first part of the proof we show that the inverse maps in the space of lines are strongly contracting. Next we show that the inverse images of a small disc therefore will always stay far away from the critical orbit. This implies that the forward images in $\mathbf{P}^2$ of these small discs are all graphs, and hence have good convergence properties. This defines the local unstable manifolds and then the forward images of the discs Δ_n converge to the global unstable manifolds.

To proceed with the details of the proof, we will first assume that $[p]$ is not one of the points on the critical orbit. Fix a small $\delta-$ neighborhood, V of $[1]$.

Note that none of the points in the history are in the critical orbit. In local coordinates near $[1]$, the map is $z \to -4z$. We can assume that $[p]$ is further than δ from the critical orbit. To prove the theorem we will first construct the local unstable manifold through p with the given history. Our first step is to estimate the expansion of the map along the history and close to it.

If $[p_{-n}] \to [1]$, the growth of $|(f^n)'| \sim 4^n$. We assume next that $[p_{-n}]$ does not converge to $[1]$. Let $I = [p_{-k}, \ldots, p_{-k+\ell}]$ be a maximal interval in V. Then the derivative of $f^{-\ell}$ at $p_{-k+\ell}$ is about $4^{-\ell}$ and the distance from $[p_{-k}]$ to $[1]$ is about $\delta/4^\ell$. This implies that p_{-k-1} has distance about $\delta/4^\ell$ from ∞ and that the derivative of $f^{-\ell-1}$ is about $4^{-\ell}$. Therefore p_{-k-2} is at distance about $\sqrt{\frac{\delta}{4^\ell}}$ from 0 and the derivative of $f^{-\ell-2}$ is about

$$\frac{1}{2\sqrt{\frac{\delta}{4^\ell}}}4^{-\ell} \sim \frac{1}{2\sqrt{\delta}}\frac{1}{\sqrt{4^\ell}}.$$

Next $[p_{-k-3}]$ is at distance about $\left(\frac{\delta}{4^\ell}\right)^{\frac{1}{4}}$ from $[2]$ and the derivative of $f^{-\ell-3}$ is about

$$\frac{1}{2\left(\frac{\delta}{4^\ell}\right)^{\frac{1}{4}}}\frac{1}{2\sqrt{\delta}}\frac{1}{\sqrt{4^\ell}} \sim \frac{1}{4\delta^{\frac{3}{4}}}\frac{1}{4^{\frac{\ell}{4}}}.$$

Next $[p_{-k-4}]$ is away from the critical orbit so the derivative follows the general expansion of the mapping measured in a given subhyperbolic metric (which is smooth there). Let next $\Delta'_n \subset \Delta_n$ denote the local preimages in $(t = 0)$ of Δ_0 containing $[p_{-n}]$. Since these estimates hold uniformly, we get that the diameters of the discs Δ'_n shrink exponentially and since they are always contained in sectors of small angles when they pass near critical points, we also get that $f^{-n} : \Delta_0 \to \Delta_n$ have well defined branches and are biholomorphic.

Next we consider the forward images in P^2 of Δ'_n. Set $f^k(\Delta'_n) := D_{n,k},\ 0 \leq k \leq n$.

Suppose Γ is a graph over Δ'_n (or a subset), Γ contained in the trapping region. Since f, as a map on the space of lines is $1-1$ on Δ'_n and maps Δ'_n to Δ'_{n-1}, $f(\Gamma)$ is a graph over Δ'_{n-1}. In particular, the sets $D_{n,k}$ are graphs over Δ_{n-k}.

In particular, we then get the graphs $f^n(\Delta'_n)$, Γ_n over Δ_0. Moreover, these Γ_n converge uniformly over Δ_0 as graphs to a limit graph Γ_0 because of the uniform contraction in the t direction. Moreover, Γ_0 is contained in the attractor and is the local unstable manifold of p with the given history. Since $\{f^{-n}\}$ are strongly contracting in the space of lines, we can replace Δ'_n by minimal larger round discs $\Delta"_n \supset\supset (1+\delta)f^{-1}(\Delta''_{n-1})$ for a fixed $\delta > 0$. (Here the factor $(1+\delta)$ refers to an expansion of the radius.) For small enough $\delta > 0$ the discs $\Delta"_n$ are still shrinking exponentially. The inductive limit of $\{\Delta"_n\}$ is a $\mathbb{C}$. In particular, as map on the space of lines, for any n, $f^k(\Delta''_{n+k})$ cover P^1 in the space of lines for all $k \geq k(n)$. The forward images $f^n(\Delta"_n)$ converges to the global unstable set, which is dense in the attractor.

To complete the proof, we consider the case when the point $[p]$ is on the critical orbit. But in this case the points $[p_{-4}]$ are not in the critical orbit and we can forward iterate the unstable variety of p_{-4}.

■

3 When the compact set of points with bounded orbit is disjoint from the critical set

3.1 $\mathcal{J} = \mathrm{P}^2$

We investigate examples where $\mathcal{J} = \mathrm{P}^2$, the line $(t = 0)$ is preserved, and $\mathrm{Supp}(\mu)$ is a Cantor set.

Fix in this section $[P(z,w) : Q(z,w)]$ a map with Julia set P^1. Let $L_1, \dots, L_r$ be those complex lines on which $\det(P,Q)' = 0$ and let $\tilde{L}_j := (P,Q)(L_j)$. Pick any point $(a,b) \notin \cup\tilde{L}_j$.

THEOREM 3.1 *If the complex number c is sufficiently large and*

$$f_c = [P(z,w) - cat^d : Q(z,w) - cbt^d : t^d]$$

then $\mathcal{J}_{f_c} = \mathrm{P}^2$.

Proof: We can assume that $a \neq 0$. There exist strictly positive constants A, B so that

$$A\|(z,w)\|^d \leq \|(P(z,w),Q(z,w))\| \leq B\|(z,w)\|^d.$$

Set $R = 2\left(\frac{\|(ac,bc)\|}{A}\right)^{\frac{1}{d}}$ and $U = \{\max(|z|,|w|) < R\}$.

LEMMA 3.2 *If* $(z,w) \notin U$ *and* $|c|$ *is sufficiently large, then* $f_c^n(z,w) \to \infty$.

Proof of the Lemma: Suppose that $(z,w) \notin U$. Then we get:

$$\begin{aligned}
\|f_c(z,w)\| &= \|(P(z,w) - ca, Q(z,w) - cb)\| \\
&\geq \|(P(z,w), Q(z,w))\| - |c|\|(a,b)\| \\
&\geq A\|(z,w)\|^d - \frac{\|(z,w)\|}{R}|c|\|(a,b)\| \\
&\geq \|(z,w)\|\left[A\|(z,w)\|^{d-1} - \frac{|c|\|(a,b)\|}{R}\right] \\
&\geq \|(z,w)\|\left[AR^{d-1} - \frac{|c|\|(a,b)\|}{R}\right] \\
&\geq \|(z,w)\|\left[A2^{d-1}\left(\frac{\|(ac,bc)\|}{A}\right)^{\frac{d-1}{d}} - \frac{|c|\|(a,b)\|}{2\left(\frac{\|(ac,bc)\|}{A}\right)^{\frac{1}{d}}}\right] \\
&\geq \|(z,w)\|A^{\frac{1}{d}}\|(ac,bc)\|^{\frac{d-1}{d}}\left[2^{d-1} - \frac{1}{2}\right] \\
\text{If } |c| \text{ is large enough:} &\geq 2\|(z,w)\|
\end{aligned}$$

So $|c| \geq c_0 := \frac{2^{\frac{d}{d-1}}}{A^{\frac{1}{d-1}}} \frac{1}{\left[2^{d-1}-\frac{1}{2}\right]^{\frac{d}{d-1}}} \frac{1}{\|(a,b)\|}$ will suffice.

■

Continuation of the Proof of the Theorem: It follows from the previous Lemma that the set K of points whose orbits are bounded is contained in

$$U_1 := \{(z,w) \in U; |P - ca| < R \wedge |Q - cb| < R\}.$$

LEMMA 3.3 *If* $|c|$ *is sufficiently large, the map* f_c *is uniformly expanding on* U_1. *More precisely, for some constant* $\delta_0 > 0$, $|f_c'(z,w)(\xi)| \geq \delta_0 |c|^{1-\frac{1}{d}}|\xi|$ *on* U_1.

Proof of the Lemma: If $(z,w) \in U_1$, then $|P(z,w)| \geq |ca| - R$. Hence

$$\begin{aligned} B\|(z,w)\|^d &\geq |P(z,w)| \\ &\geq |ca| - R, \\ \|(z,w)\|^d &\geq \left|\frac{ca}{B}\right| - \frac{R}{B} \\ \|(z,w)\|^d &\geq \left|\frac{ca}{B}\right| - \frac{2\left(\frac{\|(ac,bc)\|}{A}\right)^{\frac{1}{d}}}{B} \\ \|(z,w)\| &\geq t|c|^{\frac{1}{d}} \text{ (for some fixed } t > 0,\ |c| \text{ large enough)} \end{aligned}$$

Hence, if we set $s = 4\left(\frac{\|(a,b)\|}{A}\right)^{\frac{1}{d}}$,

$$U_1 \subset \{t|c|^{\frac{1}{d}} \leq \|(z,w)\| \leq 2R \leq s|c|^{\frac{1}{d}}\}.$$

By homogeneity, there exists a continuous, nonnegative function $\lambda(\rho)$ on the space of lines, $\lambda(\rho) = 0 \Leftrightarrow L \in \{L_j\}$, and increasing near these lines, so that

$$|(P,Q)'(z,w)(\xi)| \geq \lambda([z:w])\|(z,w)\|^{d-1}|\xi|$$

for any tangent vector ξ.

Suppose that $(z,w) \in U_1$.

Then

$$\begin{aligned} |P(z,w) - ca| &\leq s|c|^{\frac{1}{d}} \\ |Q(z,w) - cb| &\leq s|c|^{\frac{1}{d}} \end{aligned}$$

Hence for large c the spherical distance

$$d([P(z,w):Q(z,w)],[a:b]) \lesssim |c|^{\frac{1}{d}-1}.$$

It follows that the spherical distances

$$d([P(z,w):Q(z,w)],\{\tilde{L}_j\}) \geq \frac{1}{2}d([a:b],\{\tilde{L}_j\})$$

for large enough $|c|$. Hence, for some $\tau > 0$ independent of c, $d([z:w], \{L_j\}) \geq \tau$ and

$$\begin{aligned} |(P,Q)'(z,w)(\xi)| &\geq \lambda([z:w])\|(z,w)\|^{d-1}|\xi| \\ &\geq \lambda([z:w])(t|c|^{\frac{1}{d}})^{d-1}|\xi| \\ &\geq 2|\xi| \text{ for large enough } |c| \end{aligned}$$

■

Remark 3.4 *We have in particular shown that there are no critical points in U_1 for large $|c|$.*

End of the Proof of the Theorem: The set K of points with bounded orbits is contained in U_1 and f_c is expanding there, hence these points are in the Julia set. However, points outside K are in the basin of the line $(t = 0)$. Since the map is chaotic on $(t = 0)$ it follows that $\mathcal{J}_{f_c} = \mathrm{P}^2$.

■

PROPOSITION 3.5 *If $|c|$ is large enough, K is a Cantor set, Supp(μ)=K and repelling periodic points are dense in K.*

Proof: The map $f_c : U_1 \to U$ is proper and has no critical value by the above remark. So we have d^2 well defined inverse branches on U, $g_1, \ldots, g_{d^2}$. Each g_j is uniformly contracting and $g_j(U) \cap g_i(U) = \emptyset$ for $i \neq j$. Consequently every connected component of K is a point, and by symbolic dynamics, K has no isolated points, i.e. K is a Cantor set.

Using symbolic dynamics again, it is clear that periodic orbits are dense in K, and they are all repelling since f is expanding. Suppose $p \in \mathrm{P}^2 \setminus K$. Then there exists an $r > 0$ so that $\overline{\cup f^n(B(p,r))} \cap K = \emptyset$. By ([FS1]), see below, Supp $\mu \subset K$. To show the other inclusion, let $p \in K$ and $r > 0$. It is again clear from symbolic dynamics that $\cup f^n(B(p,r))$ contains K. As Supp μ is completely invariant and closed, this implies that p, hence $K \subset$ Supp μ.

Here we are using the following result from ([FS1]).

■

PROPOSITION 3.6 *Let $f \in \mathcal{H}_d$. A point q is in S_μ if and only if $\cup_{n\geq 0} f^n(B(q,r)) = \mathbb{P}^k \setminus E$ where E is pluripolar.*

Proof: If $q \in S_\mu$ then $\cup_{n\geq 0} f^n(B(q,r))$ is pluripolar, see ([FS1]). If $q \notin S_\mu$ then since S_μ is totally invariant and nonpluripolar ([FS1]), then $\cup_{n\geq 0} f^n(B(q,r))$ omits S_μ for r small enough.

■

PROPOSITION 3.7 *The Hausdorff dimension $\alpha(c)$ of K satisfies*

$$\alpha(c) \leq \frac{2\log d}{\log \delta_0 + \left(1 - \frac{1}{d}\right) \log |c|}$$

for all large enough $|c|$.

Proof: Recall from Lemma 3.3 that the map f_c is expanding on U_1 by $\delta_0 |c|^{1-\frac{1}{d}}$ where δ_0 is a fixed constant.

The d^2 components of $f^{-1}(U)$ have diameter at most $\mathrm{diam}(U)\frac{|c|^{\frac{1}{d}-1}}{\delta_0}$ so at each step the diameter of a component is multiplied by at most the (small) constant $\frac{|c|^{\frac{1}{d}-1}}{\delta_0}$. At the n^{th} step K is covered by d^{2n} open sets each of diameter at most $\mathrm{diam}(U)\frac{|c|^{n(\frac{1}{d}-1)}}{\delta_0^n}$. It follows that $\alpha \leq \frac{2\log d}{\log \delta_0 + \left(1-\frac{1}{d}\right)\log|c|}$.

■

3.2 Support of μ.

We now consider the following situation. Let f be a polynomial map of algebraic degree d, $f : \mathbb{C}^k \to \mathbb{C}^k$. We assume that f extends as a holomorphic map into $\mathbf{P}^k$. So the hyperplane at infinity given by $(t = 0)$ is an attracting set for f. We define

$$K = \{z; (f^n)(z) \text{ is bounded}\}.$$

The assumption on f implies that K is a compact polynomially convex set in $\mathbb{C}^k$. Indeed, there is R such that for $\|z\| \geq R$, $(f^n(z))$ converge to infinity. So we have

$$K = \{z; |f^n(z)| \leq R \text{ for } n \geq 0\}.$$

Clearly the set K is totally invariant.

THEOREM 3.8 *Let f be a polynomial map as above. Let C be the critical set for f. Assume $K \cap C = \emptyset$. Then*
i) The map f is strictly expanding on K.
ii) Repelling periodic points are dense in K.
iii) K = Support μ.

Proof: From the assumption, the critical set C is in the basin of attraction of the hyperplane at infinity. Let B denote the open ball of center 0 and radius R. Let $B_n := f^{-n}(B)$. We have $B_{n+1} \subset\subset B_n$ for every n. The map $f : B_{n+1} \to B_n$ is proper. We can choose n large enough so that B_n does not intersect the critical set C. Let K_ℓ denote the infinitesimal Kobayashi metric for B_ℓ.

Since f is a covering map from B_{n+1} to B_n we have

$$K_n(f(z), f'(z)\xi) = K_{n+1}(z, \xi)$$

for any vector ξ. Since $B_{n+1} \subset\subset B_n$, there is a constant $c > 0$ such that $K_{n+1}(z, \xi) \geq (1 + c)K_n(z, \xi)$, for z in B_{n+1}. We then have

$$K_n(f(z), f'(z)\xi) \geq (1 + c)K_n(z, \xi).$$

This proves that f is strictly expanding on K.

ii) Let $p \in K$. Assume that for a sequence n_j, $p_j := f^{n_j}(p)$ converges to q. We are going to show that q is a limit of periodic points (they are necessarily contained in repelling orbits). Fix $r > 0$ small. Then for j

large enough, there is an Ω_j, $p_j \in \Omega_j \subset\subset B(p_j, r/2) \subset B(p_{j+1}, r)$ for which $f^{n_{j+1}-n_j} : \Omega_j \to B(p_{j+1}, r)$ is a biholomorphism. Hence there is a fixed point q_j for $f^{n_{j+1}-n_j}$ in $B(p_j, r)$. So q is a limit of periodic orbits.

To prove that p is a limit of periodic points, it is enough to show that preimages of repelling periodic points are approximable by periodic orbits. We can assume that q_j as above has a preimage in $B(p, r)$. Indeed for n large enough $f^n(B(p, r))$ contains $B(p_j, r)$. Since q_j is on a periodic orbit we will then also have a repelling periodic orbit passing through $B(p, r)$. Assume $f^m(q) = q \in K$ and $(f^j(q))$ is repelling. Let $f^\ell(p_0) = q$. There is a curve γ from p_0 to q and a neighborhood U of γ, such that $U \cap [\cup_{n=0}^\infty f^n(C)] = \emptyset$. This is because C is in the domain of attraction of the hyperplane at infinity. It is then possible to define in U, inverse branches g_n of f^n such that $g_n(q) = q$. It follows from a Theorem by Ueda, ([U]) that the family g_n is equicontinuous. Hence (g_n) converges in U to the periodic orbit $(f^j(q))$. Let $B(p_0, r)$ be a small neighborhood of p_0 and consequently, since g_{nm} converges to q, there is a set $\Omega \subset\subset B(p_0, r)$ for which $f^{nm+\ell}(\Omega) = B(p, r)$ is biholomorphic. Therefore there is a periodic orbit passing through $B(p, r)$. This is the classical Julia's argument to construct homoclinic orbits, see Milnor ([Mi]) and Jonsson ([J2]).

iii) We show first that $S_\mu :=$ Support $\mu \subset K$. But if $p \notin K$, there exist small $r > 0$ for which $\overline{\cup f^n(B(p, r))} \cap K = \emptyset$. So by ([FS1]), $p \notin S_\mu$.

We want to show next that $K \subset S\mu$. We have proved that every point in K is non wandering. Since S_μ is totally invariant there is a neighborhood V of S_μ such that $f^{-1}(V) \subset\subset V$. So points in $V \setminus f^{-n}(V)$ are wandering. Hence $K \cap V = S_\mu$. Let $K_1 := K \setminus S_\mu$. We just proved that K_1 is closed and totally invariant. Since f is expanding on K_1 there is a neighborhood $V_1 \supset K$ such that $f^{-1}(V_1) \subset\subset V_1$. Given any point $q \in S_\mu$, if $B(q, r) \cap V_1 = \emptyset$ then $\cup f^n(B(q, r))$ is disjoint from V_1. This contradicts that $q \in S_\mu$. Hence K_1 is empty and $K = S_\mu$.

■

We now consider again in $\mathbb{C}^2$ the family of polynomial maps

$$f_c = (P(z, w) - ca, Q(z, w) - cb)$$

introduced in this paragraph. We assume that on $\mathbf{P}^1$ the map $[P : Q]$ is chaotic, i.e. it's Julia set is equal to $\mathbf{P}^1$.

The critical set for f_c is $(t=0) \cup_j L_j$. Define

$$\begin{aligned} K_c &= \{(z,w); (f_c^n(z,w)) \text{ is bounded}\} \\ \mathcal{H} &= \{(z,w,c); (f_c^n(z,w)) \text{ is bounded}\} \end{aligned}$$

We have proved in Lemma 3.2 that

$$\mathcal{H} \subset \{\|(z,w)\| < 4\frac{\|(a,b)c\|^{\frac{1}{d}}}{A}\}$$

if

$$|c| \geq c_0 := 2^{\frac{d}{d-1}} \frac{1}{\|(a,b)\|} \frac{1}{A^{\frac{1}{d-1}}} \frac{1}{\left[2^{d-1} - \frac{1}{2}\right]^{\frac{d}{d-1}}}.$$

So $\mathcal{H}$ is closed in $\mathbb{C}^3$ and each slice K_c is polynomially convex. By analogy with the Mandelbrot set for the quadratic family we define:

$$M := \{c; (\cup_j L_j) \cap K_c \neq \emptyset\}.$$

PROPOSITION 3.9 *With the previous notation the set M is closed and bounded. When c is in the unbounded component C_∞ of $\mathbb{C} \setminus M$, then K_c is a Cantor set.*

Proof: Lemma 3.3 implies that M is bounded. That M is closed is obvious. The result that K_c is a Cantor set was proved for large $|c|$, in Proposition 3.5. The previous result implies that for $c \notin M$, the map f_c is strictly expanding on K_c. It follows that in the connected component C_∞, we have a smooth family of strictly expanding map on K_c. A theorem in Ruelle ([Ru]) implies that for any $c_0 \in C_\infty$ there is a neighborhood $\Delta(c_0, \delta)$ and for every $c \in \Delta(c_0, \delta)$ a homeomorphism $\phi_c : K_{c_0} \to K_c$ such that $f_c \circ \phi_c = \phi_c \circ f_{c_0}$. It follows that K_c is a Cantor set for every $c \in C_\infty$.

∎

PROPOSITION 3.10 *Theorem 3.8 remains valid for holomorphic maps on* $\mathbb{P}^2$ *when the critical set is in the basin of an attractor.*

PROPOSITION 3.11 *Let* f *be a polynomial map on* $\mathbb{C}^2$ *which extends as a holomorphic map on* $\mathbb{P}^2$. *Assume* $K \cap C = \emptyset$ *and that the restriction to the hyperplane at infinity is hyperbolic on its Julia set. Then* f *is* $s-$*hyperbolic.*

Proof: The notion of $s-$hyperbolicity was introduced in ([FS2]); It follows from Theorem 3.8 that the set S_2 is totally invariant. It is also clear that S_1 is totally invariant which is more strict than the notion of $s-$hyperbolicity.

■

4 Isolated repelling points

4.1 Isolated repelling orbits

In this section we investigate small perturbations of polynomial automorphisms of $\mathbb{C}^2$, in order to construct infinitely many isolated repelling points in the nonwandering set of an endomorphism of $\mathbb{P}^2$.

Let $f_0(z,w) = (P(z,w), Q(z,w))$ be a biholomorphism of $\mathbb{C}^2$ of degree d. Assume that the indeterminacy set of f_0 as a rational map on $\mathbb{P}^2$ is $I_+ = [0:1:0]$ and the indeterminacy set of f_0^{-1} is $I_- = [1:0:0]$. Let $\tilde{P}, \tilde{Q}$ denote the homogeneous polynomials of degree d, such that $\tilde{P}(z,w,1) = P(z,w), \tilde{Q}(z,w,1) = Q(z,w)$. Notice that $f((t=0) \setminus I_+) = I_-$. Hence $\tilde{Q}(z,w,0) = 0$. Since I_+ consists of only $[0:1:0]$ it follows that $\tilde{P}(z,w,0) = az^d$. We may assume that $a = 1$.

Using a result of Jung, see ([FM]), we can assume that f_0 is a finite composition of Hénon maps. We consider the endomorphisms f_ϵ of $\mathbb{P}^2$ defined by

$$f_\epsilon[z:w:t] = [\tilde{P}(z,w,t) : \tilde{Q}(z,w,t) + \epsilon w^d : t^d].$$

The restriction of f_ϵ to $(t=0)$ is given by $[z^d : \epsilon w^d : 0]$, whose Julia set is the circle $|z| = \epsilon^{\frac{1}{d-1}}$ in $w = 1$. Observe that $(t=0)$ is an attracting set.

Moreover $f_\epsilon^{-1}(I_+) = I_+$ and $f_\epsilon^{-1}(I_-) = I_-$. So I_+ and I_- are superattractive fixed points for f_ϵ.

Let B_+, B_- denote the basins of attraction of I_+ and I_- respectively, they are clearly totally invariant and $\overline{B}_+ \cup \overline{B}_-$ contains a neighborhood in $\mathbf{P}^2$ of $(t = 0)$, the common boundary near $(t = 0)$ is made of the stable manifolds corresponding to the Julia set $S_1 := \{|z| = \epsilon^{\frac{1}{d-1}}\}$ in $(t = 0)$. (The unstable manifold is just $(t = 0), z, w \neq 0$ for every such point.) We will consider also B_A, the basin of attraction of $(t = 0)$, it consists of $B_+ \cup B_-$ and the stable set of S_1.

For $R > 0$, define

$$\begin{aligned} V_R &= \{|z| \leq R, |w| \leq R\} \\ V_R^+ &= \{|z| \geq R, |w| \leq |z|\} \\ V_R^- &= \{|w| \geq R, |w| \geq |z|\} \end{aligned}$$

This decomposition of $\mathbb{C}^2$ was introduced by Friedland-Milnor ([FM]). They showed that for the automorphism f_0, there is R such that $f_0(V_R \cup V_R^+) \subset V_R \cup V_R^+$.

It is easy to check that for f_0, I_- is a superattractive fixed point.

LEMMA 4.1 *There exist $\epsilon_0 > 0$ and $R > 0$ such that for all $0 < \epsilon \leq \epsilon_0$, $f_\epsilon(V_R \cup V_R^+) \subset V_R \cup V_R^+$. Moreover every point in V_R has at most one preimage in V_R.*

Proof: In $\mathbb{C}^2$ we have

$$f_\epsilon(z, w) = (P(z, w), Q(z, w) + \epsilon w^d) = (z_1, w_1).$$

We can choose R and $\epsilon_1 > 0$ so that in $V_{\frac{R}{2}}^+$

$$|P(z, w)| \geq \frac{|z|^d}{2}$$

and

$$|w_1| \leq \frac{|z_1|}{2} \text{ if } |\epsilon| \leq \epsilon_1.$$

If $(z,w) \in V_R$ it is clear that, after possibly increasing R and decreasing ϵ_1, for $|\epsilon| \leq \epsilon_1$ we have that $f_\epsilon(V_R) \subset V_R \cup V_R^+$.

Since for f_0 there is only at most one preimage of $(z,w) \in V_R$, the same holds for f_ϵ if ϵ is small enough.

∎

From now on we omit the subscript R and we assume that the Jacobian of f_0, is larger than one in modulus.

LEMMA 4.2 *Assume p is a repelling periodic point for f_ϵ, $f_\epsilon^m(p) = p$, and $p \in V$. Then p is isolated in the nonwandering set Ω_ϵ of f_ϵ.*

Proof: Assume first that $f_\epsilon(p) = p$. Let U_0 be a neighborhood of p such that $U_0 \cap f^{-1}(U_0) \subset\subset U_0$, and $U_0 \subset V$.

Define by induction $U_j := f^{-1}(U_{j-1}) \cap U_0$, $j \geq 1$. We want to show that every point in $U_1 \setminus \{p\}$ is wandering. Suppose $q \in \Delta_j := U_{j-1} \setminus U_j$. By modifying U_0 slightly we can assume that q is in the interior of Δ_j. If q is nonwandering, there is a first n such that $f^n(\Delta_j) \cap U_{j-1} \neq \emptyset$.

So there is $y \in \Delta_j$ with $f^n(y) = x \in U_{j-1}$, but $f^{n-1}(y) \notin U_{j-1}$. On the other hand, since $F(V^+) \subset V^+$, $f^{n-1}(y) \in f^{-1}(U_{j-1}) \cap U_0 = U_j \subset U_{j-1}$, a contradiction. So q is isolated in Ω_ϵ.

If q is periodic, $f_\epsilon^m(q) = q$, the whole orbit of q is in V. We can just apply the above argument to f_ϵ^m.

∎

THEOREM 4.3 *There is an endomorphism f_ϵ of P^2 such that the nonwandering set Ω_ϵ contains a countable discrete set of repelling periodic orbits.*

Proof: We can start with a one parameter family g_t of volume decreasing complex Hénon maps with real coefficients so that g_0 has a generic homoclinic tangency for some saddle point in $V \cap \mathbb{R}^2$. It is known, see ([Ga]) that for generic perturbations the perturbed maps have infinitely many sinks. Define $f_t = g_t^{-1}$. For ϵ small enough the family $((f_t)_{\epsilon|V})^{-1})$ has a generic homoclinic tangency and hence for some t, $((f_t)_{\epsilon|V})^{-1})$ has infinitely many sinks. Consequently $(f_t)_\epsilon$ has infinitely many repelling orbits in V. It follows from Lemma 4.2 that they are isolated in Ω_ϵ.

∎

Remark 4.4 *Support of μ_ϵ is disjoint from V. Indeed for q in V, $\cup_{n\geq 0} f^n(V)$ omits V^- so by Proposition 3.6, the point q cannot belong to Support(μ_ϵ). The existence of a repelling periodic point not in the support of μ_ϵ is already stated in ([HP], p. 345). We thank M. Jonsson for mentioning that.*

Remark 4.5 *It is possible to construct hyperbolic sets of dimension 1 and unstable dimension 3, for endomorphisms of $\mathbb{P}^3$, that are disjoint from Support of μ. It is enough to consider the map $g_\epsilon([z : w : t : u]) = [P(z,w,t) : Q(z,w,t) + \epsilon w^d : t^d : u^d]$. So the first three components are the components of f_ϵ. A repelling periodic orbit in V for f_ϵ gives rise to a circle of unstable dimension 3.*

We now restrict attention to the family

$$f_\epsilon(z,w) = (z^2 + aw, z + \epsilon w^2), \ |a| >> 1.$$

The critical set for f_ϵ is defined by the equation $4\epsilon zw - a = 0$.

PROPOSITION 4.6 *Suppose $|a| > 1$. Then for $0 < |\epsilon| < \epsilon_0(a)$, the only Fatou components for f_ϵ are B_+ and B_-. The compact set $K := \mathbb{C}^2 \setminus (B_+ \cup B_- \cup W^s(S_1)) = \mathbb{C}^2 \setminus B_A$ is the set of points in $\mathbb{C}^2$ with bounded orbit.*

Proof: For a given a, define V, V^+ and V^- as:

$$\begin{aligned} R &= 10|a|+3 \\ V &= \{(z,w); |z| \leq R,\ |w| \leq 8R\} \\ V^+ &= \{(z,w,); |z| > R,\ |w| \leq 8|z|\} \\ V^- &= \{(z,w); |w| > 8R,\ |z| < |w|/8\} \end{aligned}$$

If $(z,w) \in V^+$, $(z_1,w_1) = f_\epsilon(z,w)$, then

$$\begin{aligned} |z_1| &= |z^2+aw| \\ &\geq |z|^2 - 8|a||z| \\ &\geq |z|(|z|-8|a|) \\ &\geq 3|z| \\ |w_1| &= |z+\epsilon w^2| \\ &\leq |z| + 64|\epsilon||z|^2. \end{aligned}$$

We get that $|w_1| \leq 8|z_1|$ provided $|z|(1+64|\epsilon||z|) \leq 8|z|(|z|-8|a|)$ which follows if $|\epsilon| < \frac{15}{640}$. Next, f_ϵ maps V^+ to itself and multiplies the z variable by at least 3. If $(z,w) \in V$, then $w_1 = z + \epsilon w^2$, so $|w_1| \leq R + 64|\epsilon|R^2$. Then $|w_1| \leq 8R$ provided that $|\epsilon| \leq \frac{7}{64(10|a|+3)}$. In that case $f_\epsilon(V) \subset V \cup V^+$.

It follows that V^+ is in the basin of the line at infinity. Next we observe that points in V^- with $|w| \geq 2/\epsilon$ are also in the basin of the line at infinity. Also note that points in V^- with $|w| \leq 3/(4\epsilon)$ are iterated forward until they reach $V \cup V^+$.

We get on V that the Jacobian determinant $|4\epsilon zw - a| \geq |a| - 32|\epsilon|R^2 \geq \frac{|a|+1}{2}$ provided that $|\epsilon| \leq \frac{|a|-1}{64(10|a|+3)^2}$. Hence the only Fatou component intersecting V will be the one containing V^+, i.e. the basin of attraction of $[1:0:0]$.

Hence if there is a Fatou component B other than the basins of attraction of $[1:0:0], [0:1:0]$, then it must be contained in $U := \{(z,w,\in V^-; 3/(4|\epsilon|) < |w| < 2/|\epsilon|\}$. Since the Jacobian determinant is $4\epsilon zw - a$, B

must intersect the set of points in U for which $|4\epsilon zw - a| \leq 1$. At such a point

$$\begin{aligned} |z| &= \frac{4}{3}|z|\frac{3}{4} \\ &\leq \frac{4}{3}|z\epsilon w| \\ &\leq \frac{1}{3}\left[|4z\epsilon w - a| + |a|\right] \\ &\leq |a| + 1. \end{aligned}$$

However f_ϵ maps such points to V^+.

■

4.2 The compact set K of points with bounded orbit.

It is clear that the compact set $K := \mathbb{C}^2 \setminus B_A$ is the set of points defined by

$$K = \{(z, w); |f^n(z, w)| \leq \frac{4}{\epsilon} \text{ for every } n.\}$$

We define $K_1 := K \cap V$ and

$$K_2 := \{(z, w); f^n(z, w) \in K \cap \{(z, w); |z| < \frac{|w|}{8}, \frac{3}{4\epsilon} < |w| < \frac{2}{\epsilon}\} \ \forall\, n\}.$$

Observe that K_2 is totally invariant.

LEMMA 4.7 *f_ϵ is expanding on K_2 and Supp$(\mu_\epsilon) = K_2$.*

Proof: Let (u, v) be an arbitrary tangent vector at $(z, w) \in K_2$. Then

$$(u', v') := f'_\epsilon(u, v) = (2zu + av, u + 2\epsilon wv).$$

$$
\begin{aligned}
\text{if } |u| &\leq |v|/4 : \\
\text{then } |v'| &\geq \frac{3}{2}|v| - \frac{|v|}{4} \\
&\geq \frac{5}{4}|v| \\
&\Rightarrow \\
\max\{|u'|, |v'|\} &\geq \frac{5}{4}\max\{|u|, |v|\} \\
\text{if } |u| &\geq |v|/4 : \\
\text{then } |u'| &\geq 2|z||u| - 4|a||u| \\
(i)\ |z| &\geq 2|a| + 3 \\
|u'| &\geq 6|u| \\
&\geq \frac{3}{2}\max\{|u|, |v|\}. \\
(ii)\ |z| &\leq 2|a| + 3 \\
&\Rightarrow \\
f_\epsilon(z, w) &\in V^+ \\
&\Rightarrow \\
(z, w) &\notin K_2.
\end{aligned}
$$

Since $\mathrm{Supp}(\mu_\epsilon)$ is totally invariant and contained in the non-wandering set, it is necessarily contained in K_2. (Because of Proposition 3.6, the support of the measure cannot intersect the line at infinity.)
That $\mathrm{Supp}(\mu_\epsilon)$ is equal to K_2 follows from the argument in Theorem 3.8 of ([FS2]).

■

Let Ω denote the non wandering set for f_ϵ in $\mathbb{C}^2$. Define

$$
\begin{aligned}
\Omega_1 &:= \Omega \cap V, \\
U &:= V^- \cap \{\frac{3}{4|\epsilon|} < |w| < \frac{2}{|\epsilon|}\}.
\end{aligned}
$$

PROPOSITION 4.8 *The non wandering set Ω is the union of the two disjoint closed sets Ω_1, Supp μ_ϵ. The set Ω does not intersect the critical set C. The map f_ϵ is bijective on Ω_1.*

Proof: It is clear that Ω_1 is closed. Since f_ϵ is expanding on $\Omega \cap U$, and since Supp μ_ϵ is totally invariant, there is a neighborhood $U_1 \supset$ Supp μ_ϵ such that $f_\epsilon^{-\ell}(U_1) \subset\subset U_1$ for large ℓ. If a point in U is non wandering it has to be in K_2 hence in Supp(μ_ϵ).

The critical set C is disjoint from V so $C \cap \Omega = \emptyset$. We also know that any point in V has at most one preimage in V.

■

PROPOSITION 4.9 *The set K has no isolated point. The local Hausdorff dimension of any point in $K \setminus$ Supp μ_ϵ is at least 2.*

Proof: Given any $q \in$ Supp μ_ϵ, we know that Supp(μ_ϵ) is not pluripolar near q ([FS2]), so q is not isolated in K. We next show that the local Hausdorff dimension of any point in $K \setminus$Supp μ_ϵ is at least 2. Let $q \in K \setminus$Supp μ_ϵ. Fix $r > 0$ such that the ball $B(q,r) \cap$Supp $\mu_\epsilon = \emptyset$. So on $B(q,r)$ we have that $(dd^cG^+)^2 = 0$ and $K \cap B(q,r) = \{(z,w); (z,w) \in B(q,r), G^+(z,w) = 0\}$. The function G^+ is the solution in $B(q,r)$ of the Dirichlet problem for the Monge Ampère equation $(dd^cu)^2 = 0$ with $u_{|\partial B(q,r)} = G^+$. The solution u is equal to the maximum of plurisubharmonic functions on $B(q,r)$ which are smaller than G^+ on $\partial B(q,r)$ ([BT]). Hence q is in the polynomially convex hull of $K \cap \partial B(q,r)$. Hence, for every $r > 0$ the Hausdorff dimension of $K \cap \partial B(q,r)$ is at least one. Consequently the Hausdorff dimension of K at q is at least 2, see ([S2]).

■

We want to show next that for $|a| >> 1$ the maps f_ϵ satisfy strong hyperbolicity conditions.

Let $f = (z^2 + aw, az)$ be a hyperbolic Hénon map. Define

$$\begin{aligned} V &= \{|z| \leq 2|a|^2 + 2, |w| \leq 4|a|^3 + 4|a|\}, \\ V^+ &= \{|z| > 2|a|^2 + 2, |w| \leq 2|a||z|\}, \\ V^- &= \{|w| > 4|a|^3 + 4|a|, 2|a||z| < |w|\}, \\ K^+ &= \{(z,w)\} \text{ with bounded orbits}, \\ K^- &= \{(z,w)\} \text{ with bounded inverse orbits}, \\ K &= K^+ \cap K^-. \end{aligned}$$

Then $K \subset \text{int}(V)$. Let next $S \subset K$ consist of the nonwandering points of f. We assume that f is hyperbolic on S. So $S = S_0 \cup S_1 \cup S_2$, disjoint compact sets, S_j has stable dimension j. (Of course S_0 or S_2 must be empty and they are anyhow finite sets.) By ([BS1]) periodic points are dense in S, i.e. S satisfies Axiom A.

THEOREM 4.10 *For all $\epsilon \neq 0$, $|\epsilon|$ small enough, the maps $f_\epsilon := [z^2+awt : azt + \epsilon w^2 : t^2]$ satisfy Axiom A on the non wandering set.*

Proof: Let $\Omega = \Omega_\epsilon$ denote the nonwandering set. Then $\Omega_0^\epsilon := \Omega \cap (t=0)$ contains two attracting fixed points, $[1:0:0]$ and $[0:1:0]$ in S_0 and a saddle set $\{|z| = |\epsilon w|\} \cap (t=0) \subset S_1$. Moreover, since $(t=0)$ is an attracting set, Ω_0^ϵ is isolated in the non wandering set. Also, f_ϵ satisfies Axiom A there.

The set V^+ is contained in the basin of attraction of $[1:0:0]$. The set V is mapped to $V \cup V^+$. If $(z,w) \in V^-, |w| \geq \frac{3}{2|\epsilon|}$, then $|az + \epsilon w^2| > \epsilon|w|^2 - \frac{|w|}{2} \geq |w|$ so (z,w) is in the basin of attraction of $(t=0)$. Also if $(z,w) \in V^-, |w| \leq \frac{1}{2|\epsilon|}$, then $|az + \epsilon w^2| < \frac{|w|}{2} + \frac{|w|}{2} = |w|$. So the orbit lands eventually in $V \cup V^+$. Hence the set $\Omega_\epsilon = \Omega_0^\epsilon \cup \Omega_1^\epsilon \cup \Omega_2^\epsilon$ where $\Omega_1^\epsilon \subset V$ and $\Omega_2^\epsilon \subset \tilde{V}^- := \{(z,w) \in V^-; \frac{1}{2|\epsilon|} < |w| < \frac{3}{2|\epsilon|}\}$.

Next consider the nonwandering points Ω_2^ϵ. Since $\Omega_2^\epsilon \subset K_2$, hence f_ϵ is expanding on Ω_2^ϵ. We prove as in Theorem 3.8 that periodic points are dense in Ω_2^ϵ. So f_ϵ satisfies Axiom A on Ω_2^ϵ.

Finally, it follows from Ruelle ([Ru]) that f_ϵ also is hyperbolic on Ω_1^ϵ and satisfies Axiom A there.

∎

5 Examples of endomorphisms such that Supp μ = Julia set.

In general the Julia set for an endomorphism of $\mathbf{P}^2$ doesn't coincide with the support of the equilibrium measure μ. The simplest example is

$$f([z:w:t]) = [z^d : w^d : t^d].$$

In this paragraph we give examples where the Julia set for f coincides with the support of μ. The Briend-Duval theorem ([BD]) will then guarantee that repelling periodic orbits are dense in the Julia set.

We first give a general criterion.

THEOREM 5.1 *Let $(f_\lambda)_{\lambda\in\Delta}$ be a holomorphic family of holomorphic endomorphisms of algebraic degree $d \geq 2$ in $\mathbf{P}^2$. Here Δ denotes the unit disc in $\mathbb{C}$. Assume*
(a) i) There is an algebraic curve V_0 in $\mathbf{P}^2$ such that $f_\lambda(V_0) = V_0$, $\lambda \in \Delta$ and the Julia set for the restriction $(f_\lambda)_{|V_0}$ is equal to V_0.
ii) There is an integer ℓ, such that $\mathbf{P}^2 \setminus \cup_{j=0}^{\ell} f_\lambda^{-j}(V_0)$ is Kobayashi hyperbolic and hyperbolically embedded for $\lambda \in \Delta$.
(b) There exists $p \in V_0 \cap Supp(\mu_0)$, a repelling fixed point for f_0.
Then there exists a $\delta > 0$ such that Supp $\mu_\lambda = \mathcal{J}_\lambda$ for $\lambda \in \Delta(0,\delta)$.

Remark 5.2 *If f_λ has in addition an attracting fixed point, we also have that Supp $\mu_\lambda = \mathcal{J}_\lambda \neq \mathbf{P}^2$. This is the case in examples below.*

LEMMA 5.3 *Let $(f_\lambda)_{\lambda\in\Delta}$ be a holomorphic family of endomorphisms in $\mathbf{P}^2$ of algebraic degree $d \geq 2$. Assume that $f_\lambda(p) = p$, $\lambda \in \Delta$ and that $p \in$ Supp μ_0 and is repelling for f_0. Then there is a $\delta > 0$ such that $p \in$ Supp μ_λ for $\lambda \in \Delta(0,\delta)$.*

Proof: Recall that the support of $\mu_\lambda =: S_\lambda$ is a totally invariant closed set. A point q belongs to S_λ if and only if for every $r > 0$, $\cup_n f_\lambda^n(B(q,r))$ omits at most a pluripolar set. To simplify notation, assume p is the origin of coordinates in $\mathbb{C}^2$. Since it is a fixed repelling point, there is an $\ell > 1$, and $\delta_0 > 0, \rho > 0, c > 0$ such that

$$|f_\lambda^\ell(z)| \geq (1+c)|z|,\ |z| \leq \rho \text{ and } \lambda \in \Delta(0,\delta_0).$$

Hence for all $\lambda \in \Delta(0, \delta_0)$, given $r_\lambda > 0$ we have

$$\cup_{n\geq 1} f_\lambda^n(B(0, r_\lambda)) \supset B(0, \rho/2).$$

The total invariance of S_λ implies that if $0 \notin S_\lambda$ then there is $r_\lambda > 0$ so that $\mu_\lambda(\cup_{n\geq 1} f_\lambda^n(B(0, r_\lambda)) = 0$. As a consequence $\mu_\lambda(B(0, \rho/2) = 0$. On the other hand we know ([FS1]) that $\lambda \to \mu_\lambda$ is weakly continuous. Since we have assumed that $0 \in S_0$, $\mu_\lambda(B(0, \rho/2)) > 0$. It follows that we cannot have $0 \notin S_{\lambda_k}$ for a sequence λ_k converging to 0.

■

LEMMA 5.4 *Let $f_0 \in \mathcal{H}_d$. Assume there is an algebraic curve V_0 such that $f_0(V_0) = V_0$ and $(f_0)_{|V_0}$ is chaotic. Assume that there is a repelling fixed point $p_0 \in V_0 \cap S_{\mu_0}$. If there is an integer ℓ such that $\mathbb{P}^2 \setminus \cup_{j=0}^{\ell} f_0^{-j}(V_0)$ is Kobayashi hyperbolic and hyperbolically embedded, then Supp $\mu_0 = \mathcal{J}_0$, the Julia set of f_0.*

Proof: The total invariance of $S_0 =$ Support μ_0 implies that $V_0 \subset S_0$. The assumption on Kobayashi hyperbolicity implies that $\mathbb{P}^2 \setminus S_0$ is Kobayashi hyperbolic and hyperbolically embedded. Since $\mathbb{P}^2 \setminus S_0$ is invariant under f_0, the family of maps $(f_0^n)_n$ is equicontinuous on $\mathbb{P}^2 \setminus S_0$, hence the Julia set $\mathcal{J}_0$ of f_0 satisfies $\mathcal{J}_0 \subset S_0$. The other inclusion is obvious.

■

Proof of the Theorem: Lemma 5.3 implies that the point p is repelling and in Supp μ_λ for $\lambda \in \Delta(0, \delta)$. One can then apply Lemma 5.4 to the maps f_λ for $\lambda \in \Delta(0, \delta)$.

■

We next give an example of the situation of the Theorem. Let $f = [P(z, w) : Q(z, w)]$ be a map of degree $d \geq 3$ on $\mathbb{P}^1$. Assume that f has no exceptional points. Also, let $S(z, w)$ be a homogeneous polynomial of degree $d - 1$. Let (p_α) be those points in $\mathbb{P}^1$ on which S vanishes.

We assume next that we have the following nondegeneracy condition: Let $p \in \mathrm{P}^1$. Then $f^{-1}(p)$ contains at least one point not in $\{p_\alpha\}$.

We extend f to P^2 as a two parameter family:

$$f_{\lambda,c} := [P + ct^d : Q + ct^d : t^d + \lambda t S].$$

The next result shows that points in the Julia set of f belongs to S_μ.

THEOREM 5.5 *(i) There exists a $\lambda_0 > 0$ and for all $\lambda, |\lambda| \geq \lambda_0$ there is a $c_0(\lambda)$ so that if $c \in \mathbb{C}, |c| < c_0(\lambda)$, then $[p : q : 0]$ belongs to Support $\mu_{\lambda,c}$ whenever $[p : q]$ belongs to the Julia set of f. Moreover the maps $f_{\lambda,c}$ have an attracting fixed point.*
(ii) There exists $\lambda_0 > 0$ so that if $|\lambda| \geq \lambda_0$ and $|c| > 1$, then $[p : q : 0]$ belongs to Supp$(\mu_{\lambda,c})$ whenever $[p : q]$ belongs to the Julia set of f.

Proof: We show that if U is any neighborhood of $[p : q : 0]$, then $\cup_n f^n_{\lambda,c}(U)$ contains the complement of a pluripolar set (actually an attracting fixed point) provided we impose the indicated conditions on λ, c. It will follow then from ([FS1]) that $[p : q : 0]$ belongs to Supp $\mu_{\lambda,c}$.

Note that after finitely many iterations, $f^n_{\lambda,c}(U)$ contains a neighborhood of the line at infinity.

Let $R = R_{\lambda,c,U} \geq 0$ denote the smallest number so that $\|(z,w)\| > R \Rightarrow [z : w : 1] \in V := \cup_n f^n_{\lambda,c}(U) \setminus (t = 0)$. We want to show that $R = 0$.

We fix (small) discs $D_\alpha \subset (t = 0)$ around the p_α. If the discs are small enough, the nondegeneracy condition on f implies that $f(\mathrm{P}^1 \setminus \cup_\alpha \overline{D}_\alpha) = \mathrm{P}^1$.

Notice that if $\lambda, c = 0$ then the line at infinity $(t = 0)$ is an attracting set, which is why we need some assumptions on the constants. We will first consider the case when λ is large, but $c = 0$, and then consider the case $c \neq 0$ as a small perturbation.

There exist constants $C_1, C_2 > 0$ so that if $[z : w] \notin \cup D_\alpha$ then $|S(z,w)| \geq C_1\|(z,w)\|^{d-1}$, while $|S(z,w)| \leq C_2\|(z,w)\|^{d-1}$ everywhere. Choose constants $A, B > 0$ so that

$$A\|(z,w)\|^d \leq \|(P(z,w), Q(z,w))\| \leq B\|(z,w)\|^d.$$

Suppose that $\lambda_0 \geq \frac{4B}{C_1} + 2$ and that $\|(z,w)\|^{d-1}C_1 \geq 1$. If $[z:w] \notin \cup D_\alpha$ and $|\lambda| \geq \lambda_0$, then

$$\begin{aligned}
\|f_{\lambda,0}(z,w)\| &\leq \frac{\|(P(z,w),Q(z,w)\|}{|1+\lambda S(z,w)|} \\
|1+\lambda S(z,w)| &\geq |\lambda||S(z,w)|-1 \\
&\geq \left(\frac{4B}{C_1}+2\right)C_1\|(z,w)\|^{d-1}-1 \\
&\geq 1>0. \\
\|f_{\lambda,0}(z,w)\| &\leq \frac{B\|(z,w)\|^d}{|\lambda||S(z,w)|-1} \\
&\leq \frac{B\|(z,w)\|^d}{\frac{4B}{C_1}C_1\|(z,w)\|^{d-1}} \\
&\leq \frac{\|(z,w)\|}{4}
\end{aligned}$$

Since the maps $f_{\lambda,0}$ maps lines to lines, it follows that the forward images of U cover at least the complement of the ball of radius $\left(\frac{1}{C_1}\right)^{\frac{1}{d-1}}$.

Next, in case (i) assume that $\|(z,w)\| = \left(\frac{1}{C_1}\right)^{\frac{1}{d-1}}$.

$$\begin{aligned}
\|f_{\lambda,0}\|(z,w) &\leq \frac{\|(P,Q)\|}{\frac{|\lambda|}{2}C_1\|(z,w)\|^{d-1}} \\
&\leq \frac{B}{\frac{|\lambda|}{2}C_1}\|(z,w)\| \\
&\leq \frac{\tilde{K}}{|\lambda|}
\end{aligned}$$

for some constant $\tilde{K}$. Hence it follows that the forward images of U cover the complement of the ball of radius $\frac{\tilde{K}}{|\lambda|}$.

Next, we estimate the image of points in the ball of radius $\frac{\tilde{K}}{|\lambda|}$. We get, using the condition $d \geq 3$:

$$
\begin{aligned}
\|f_{\lambda,0}(z,w)\| &\leq \frac{B\|(z,w)\|^d}{1-|\lambda|C_2\left(\frac{\tilde{K}}{|\lambda|}\right)^{d-1}} \\
&\leq 2B\left(\frac{\tilde{K}}{|\lambda|}\right)^{d-1}\|(z,w)\| \\
&\leq \frac{\|(z,w)\|}{2}
\end{aligned}
$$

if λ_0 is chosen larger if necessary.

This shows that the forward images of U cover all of $\mathrm{P}^2\setminus(0)$ when $c=0$. If $c\neq 0$, but small, depending on λ, $f_{\lambda,c}$ has an attracting fixed point P near the origin and by a continuity argument, $f^n_{\lambda,c}(U)$ still covers the complement of the basin of attraction of P, hence $\mathrm{P}^2\setminus(P)$.

Next, in case (ii), we estimate the images when $c\neq 0$. This is done by a perturbation argument. First of all notice that there are small discs Δ_z around each point z in $(t=0)\setminus\cup\overline{D'_\alpha}$ where D'_α are slight extensions of D_α of approximately the same size so that the image of the boundary of Δ_z does not contain the image of z. Also notice that for $\|(z,w)\|$ sufficiently large, dependent on c, we still have $\|f_{\lambda,c}\|(z,w)\leq\|(z,w)\|/2$. More precisely, there exists then an $L>0$ such that if $\|(z,w)\|\geq L|c|^{\frac{1}{d}}$ in addition to the above conditions, we still get that the forward iterates of U cover the complement of the ball of radius $\max L|c|^{\frac{1}{d}},\left(\frac{1}{C}\right)^{\frac{1}{d-1}}$.

Since we may assume that $|c|\geq 1$, we can, by increasing L, assume that $L|c|^{\frac{1}{d}}\geq\left(\frac{1}{C}\right)^{\frac{1}{d-1}}$ and hence that the forward iterates of U cover the complement of the ball of radius $L|c|^{\frac{1}{d}}$.

Next, assume that $\|(z,w)\|=L|c|^{\frac{1}{d}}$.

$$
\begin{aligned}
\|f_{\lambda,c}\|(z,w) &\leq \frac{\|(P+c,Q+c)\|}{\frac{|\lambda|}{2}C\|(z,w)\|^{d-1}} \\
&\leq \frac{B}{\frac{|\lambda|}{2}C}\|(z,w)\|+\frac{|c|}{\frac{|\lambda|}{2}C\|(z,w)\|^{d-1}}
\end{aligned}
$$

$$\leq \frac{\tilde{K}}{|\lambda|}|c|^{\frac{1}{d}}$$

Hence it follows that the forward images of U cover the complement of the ball of radius $\frac{\tilde{K}}{|\lambda|}|c|^{\frac{1}{d}}$.

Next, we estimate the image of the ball of radius $\frac{\tilde{K}}{|\lambda|}|c|^{\frac{1}{d}}$. We get

$$\begin{aligned}
\|f_{\lambda,c}(z,w)\| &\geq \frac{|c| - \|(P,Q)\|}{|1+\lambda S|} \\
&\geq \frac{|c| - \|(P,Q)\|}{1 + |\lambda|\frac{C'}{|\lambda|^{d-1}}|c|^{\frac{d-1}{d}}} \text{ for some constant } C' \\
&\geq \frac{\tilde{K}}{|\lambda|}|c|^{\frac{1}{d}} \text{ if} \\
|\lambda| &\geq \text{some}\lambda_0 >> 1.
\end{aligned}$$

But this implies that the ball of radius $\frac{\tilde{K}}{|\lambda|}|c|^{\frac{1}{d}}$ is in the image of it's complement. This shows that the forward images of U cover all of $\mathbf{P}^2$.

■

We next make some further hypothesis.

(1) $[P:Q] = [\nu(z-2w)^d : z^d]$, $\left(\frac{\nu-2}{\nu}\right)^d = 1$, which is critically finite and chaotic with repelling fixed point $[\nu:1]$. Then S_μ contains $(t=0)$ and all its preimages.

(2) $S(z,w) = z^{d-1}$, $d \geq 3$.

Then the preimages of $(t=0)$ contains the lines in $(t=1): z = \left(-\frac{1}{\lambda}\right)^{\frac{1}{d-1}} =: c_{\lambda,j}, j = 1, \ldots, d-1$. The preimages of these lines are given by

$$\Sigma_j = \{\frac{\nu(z-2w)^d + c}{1+\lambda z^{d-1}} = c_{\lambda,j}\}.$$

Note that for each value of $z \neq c_{\lambda,j}$, there is at least one value for w for which $(z, w) \in \Sigma_j$, and as j varies these must be different. Also note that these are $(d-1)$ branched covers over the z axis. Hence Kobayashi hyperbolicity of the complement of S_μ follows. To show that $\mathrm{P}^2 \setminus \cup f^{-j}(V_0)$ is hyperbolically embedded, observe that for $c \neq 0$, each Σ_j intersects the line $\{z = c_{\lambda,i}\}$ at d distinct points.

If we apply Lemma 5.4 to each $f_{\lambda,c}$ and the repelling fixed point $[\nu : 1 : 0]$, we find that $\mathrm{Supp}(\mu_{\lambda,c}) = \mathcal{J}_{\lambda,c}$. We have thus obtained:

THEOREM 5.6 *There exists a $\lambda_0 > 0$ such that if $|\lambda| \geq \lambda_0$ and $|c| < c_0(\lambda)$, then Supp$(\mu_{\lambda,c}) = \mathcal{J}_{\lambda,c} \neq \mathrm{P}^2$ for the maps*

$$f_{\lambda,c} = [\nu(z-2w)^d + ct^d : z^d + ct^d : t^d + \lambda t z^{d-1}].$$

When $|c| > 1$ we only know that $\mathrm{Supp}(\mu_{\lambda,c}) = \mathcal{J}_{\lambda,c}$.

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