J. London Math. Soc. (2) 91 (2015) 180–202
doi:10.1112/jlms/jdu064

Polynomial composition rigidity and plane polynomial automorphisms

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Abstract

In the first part of this paper, we briefly present a conjecture dealing with polynomial composition and we prove it in some particular cases. In the second and longest part, we prove our main result, which consists of an application of the conjecture to plane polynomial automorphisms. More precisely, we describe the closure of the set of plane polynomial automorphisms having a prescribed multidegree of length 2.

1. *Introduction*

The theory of polynomials in one indeterminate and the theory of plane polynomial automorphisms are intimately connected. The most striking illustration of this phenomenon is probably the easy way to deduce the theorem of Jung from the theorem of Abhyankar, Moh and Suzuki (for this deduction, see, for example, [**6**, Proof of Corollary 5.3.6, p. 100]). Let us recall these famous statements (see [**1**, **19**, **35**]).

Theorem 1.1 (Abhyankar, Moh and Suzuki). *Let K be a field of characteristic 0. If $a, b \in K[X]$ are two polynomials which generate the whole algebra $K[X]$, that is, such that $K[a, b] = K[X]$, then $\deg a$ divides $\deg b$ or $\deg b$ divides $\deg a$.*

Theorem 1.2 (Jung). *Let K be a field of characteristic 0. Any polynomial automorphism of the affine plane $\mathbb{A}^2_K$ is a composition of affine and triangular automorphisms, where by definition an affine automorphism is of the form*

$$(x, y) \longmapsto (\alpha x + \beta y + \gamma, \delta x + \varepsilon y + \zeta) \quad \textit{with } \alpha, \ldots, \zeta \in K \textit{ and } \alpha\varepsilon - \beta\delta \neq 0$$

and a triangular automorphism is of the form

$$(x, y) \longmapsto (\alpha x + p(y), \beta y + \gamma) \quad \textit{with } \alpha, \beta, \gamma \in K,\ p \in K[y] \textit{ and } \alpha\beta \neq 0.$$

In the same vein, the main result of this paper consists in proving that the following rigidity conjecture $R(m, n)$ dealing with polynomials in one indeterminate has some applications to the theory of plane polynomial automorphisms (see Theorem B). Using Gröbner bases, we have checked $R(m, n)$ when $m \leqslant 8$ and $n \leqslant 5$.

Conjecture 1.3 ($R(m, n)$). Let $a = X(1 + a_1X + \cdots + a_mX^m)$, $b = X(1 + b_1X + \cdots + b_nX^n) \in \mathbb{C}[X]$, where the $a_i, b_j \in \mathbb{C}$. Let us write $a \circ b = X(1 + c_1X + \cdots + c_NX^N)$, where $N = (m + 1)(n + 1) - 1$ and the $c_k \in \mathbb{C}$. If $c_1 = \cdots = c_{m+n} = 0$, then $a = b = X$.

Received 12 October 2013; published online 1 December 2014.

2010 *Mathematics Subject Classification* 14R10 (primary), 13P99 (secondary).

The author gratefully acknowledges the French National Research Agency Grant 'BirPol', ANR-11-JS01-004-01.

REMARK 1.4. This conjecture being obvious for m or $n = 0$, we generally assume that m and n are $\geqslant 1$.

Let us begin to explain why this conjecture may be interpreted as a rigidity statement. Considering a_i, b_i as indeterminates of degree i allows us to see each c_i as a homogeneous polynomial of degree i in $\mathbb{C}[a_1, \dots, a_m, b_1, \dots, b_n]$. The conjecture $R(m, n)$ means that $c_1, \dots, c_{m+n}$ is a homogeneous system of parameters (hsop) of $\mathbb{C}[a_1, \dots, a_m, b_1, \dots, b_n]$. For the definition of an hsop and several characterizations, we refer to Subsection 3.2 and in particular to Lemma 3.5. This terminology (hsop) is, for example, used in [**34**, Chapter I, Definition 5.1, p. 33]. By Lemma 3.5, the conjecture $R(m, n)$ is still equivalent to asserting that the polynomial endomorphism of the complex affine space $\mathbb{A}^{m+n}$ sending $(a_1, \dots, a_m, b_1, \dots, b_n)$ to $(c_1, \dots, c_{m+n})$ is quasi-finite, that is, admits finite fibres. In some sense, this means that polynomial composition is rigid.

In this connection, let us recall the famous result of Ritt on polynomial composition (see [**29**]). A polynomial $a \in \mathbb{C}[X]$ of degree at least 2 is prime if the relation $a = b \circ c$ implies that b or c has degree 1. If $\deg f \geqslant 2$, then it is clear that f admits a decomposition $f = f_1 \circ \cdots \circ f_r$ into prime polynomials. Let us note that $a \circ b = c \circ d$ in the three following cases.

(i) We have $c = a \circ l$, $d = l^{-1} \circ b$, where a, b are any polynomials, l is a polynomial of degree 1 and l^{-1} is its inverse for the composition.

(ii) We have $a = d = t_m$ and $b = c = t_n$, where $t_m(X) = \cos(m \arccos X)$ is the mth Chebyshev polynomial.

(iii) We have $a = d = X^m$, $b = X^n p(X^m)$ and $c = X^n p(X)^m$, where p is any polynomial, and the converse situation, that is, $b = c = X^m$, $d = X^n p(X^m)$ and $a = X^n p(X)^m$.

If $f = f_1 \circ \cdots \circ f_r$ is a prime decomposition, then, by replacing an adjacent pair $(f_i, f_{i+1}) = (a, b)$ by (c, d) where a, b, c, d are as in (i)–(iii) above, we obtain a new prime decomposition. This process is called an elementary transformation.

THEOREM 1.5 (Ritt). *The prime decomposition is unique modulo elementary transformations, that is, if we have two prime decompositions of some polynomial, we can pass from one to the other by applying a finite number of elementary transformations.*

Many problems related to polynomial composition (and iteration) are intricate. For example, the famous Mandelbrot set is defined as the set of complex c-values for which the orbit of 0 under iteration of $p(X) = X^2 + c$ remains bounded. In fact, the conjecture $R(m, n)$ is related to the following one which perhaps looks more attractive.

CONJECTURE 1.6 ($R(m)$). Let $a = X(1 + a_1X + \cdots + a_mX^m) \in \mathbb{C}[X]$ and let $a^{-1} \in \mathbb{C}[\![X]\!]$ be its formal inverse for the composition. If m consecutive coefficients of a^{-1} vanish, then $a = X$.

If $a^{-1} = X(1 + \sum_{k \geqslant 1} \tilde{a}_k X^k)$, then the vanishing condition means that there exists an integer $n \geqslant 0$ such that $\tilde{a}_{n+k} = 0$ for $1 \leqslant k \leqslant m$. In words, the conjecture $R(m)$ means that the inverse for the composition of a nontrivial polynomial is badly approximated by polynomials. Let us note the analogy with Heisenberg's uncertainty principle in quantum mechanics asserting that one cannot reduce arbitrarily the uncertainty as to the position and the momentum of a free particle.

Here is a mathematical statement. Let f be a nonzero element of $L^2(\mathbb{R})$ describing a particle: the probability density that this particle is located at t is $(1/\| f \|^2)|f^2(t)|$. Let $\hat{f} \in L^2(\mathbb{R})$ be the Fourier transform of f. The probability density that the momentum of this particle is equal to ω is $(1/2\pi \parallel f \parallel^2)|\hat{f}^2(\omega)|$. If σ_t (respectively, σ_ω) denotes the variance of the location (respectively, momentum), then Heisenberg uncertainty is expressed by the following inequality:

$$\sigma_t \sigma_\omega \geqslant \tfrac{1}{2}.$$

For more details, we refer the reader to [**36**, p. 77], [**25**, Theorem 2.5] or [**20**, Theorem 1, p. 311].

Many mathematical results contain a close idea. For example, rationals are badly approximated by rationals: if α is any real number, then Hurwitz has proved that α is irrational if and only if there are infinitely many rationals p/q such that $|\alpha - p/q| < 1/\sqrt{5}q^2$ (see [**18**]). Algebraic numbers are also badly approximated by rationals. Roth has proved that for any algebraic number α and any real $\varepsilon > 0$, there exist only finitely many rationals p/q such that $|\alpha - p/q| < 1/q^{2+\varepsilon}$ (see [**30**]). This idea was first used by Liouville to construct transcendental numbers (see [**23**, **24**]). We show below (see Lemma 2.2) that the conjecture $R(m)$ holds if and only if the conjecture $R(m,n)$ holds for any n. Coming back to our initial conjecture $R(m,n)$, our first result is the following theorem.

Theorem A. *If m or $n \leqslant 2$, then the conjecture $R(m,n)$ holds.*

Before giving an application to plane polynomial automorphisms, we need some notation. A polynomial endomorphism of $\mathbb{A}^2_K$ will be identified with its sequence $f = (f_1, f_2)$ of coordinate functions $f_j \in K[X,Y]$. We define its degree by $\deg f = \max_j \deg f_j$.

The space $\mathcal{E} := \mathbb{C}[X,Y]^2$ of polynomial endomorphisms of the complex affine plane $\mathbb{A}^2$ is naturally an infinite-dimensional algebraic variety (see [**32**, **33**] for the definition). This roughly means that $\mathcal{E}_{\leqslant m} := \{f \in \mathcal{E},\ \deg f \leqslant m\}$ is a (finite-dimensional) algebraic variety for any $m \geqslant 1$, which comes from the fact that it is an affine space. If $Z \subseteq \mathcal{E}$, then we set $Z_{\leqslant m} := Z \cap \mathcal{E}_{\leqslant m}$. The space $\mathcal{E}$ is endowed with the topology of the inductive limit, in which Z is closed (respectively, open) if and only if $Z_{\leqslant m}$ is closed (respectively, open) in $\mathcal{E}_{\leqslant m}$ for any m.

Let $\mathcal{G}$ be the group of polynomial automorphisms of $\mathbb{A}^2$. Since $\mathcal{G}$ is locally closed in $\mathcal{E}$ (see [**12**] and also [**2**, **32**, **33**]), it is naturally an infinite-dimensional algebraic variety. Let

$$\mathcal{A} := \{(aX + bY + c, dX + eY + f),\ a,b,c,d,e,f \in \mathbb{C},\ ae - bd \neq 0\}$$

be the subgroup of affine automorphisms and

$$\mathcal{B} := \{(aX + p(Y), bY + c),\ a,b,c \in \mathbb{C}, p \in \mathbb{C}[Y],\ ab \neq 0\}$$

be the subgroup of (upper) triangular automorphisms ($\mathcal{B}$ may be viewed as a Borel subgroup of $\mathcal{G}$). By [**19**, **22**, **28**], any element f of $\mathcal{G}$ admits an expression

$$f = \alpha_1 \circ \beta_1 \circ \cdots \circ \alpha_k \circ \beta_k \circ \alpha_{k+1},$$

where the automorphisms α_j (respectively, β_j) belong to $\mathcal{A}$ (respectively, $\mathcal{B}$). By contracting such an expression, one might as well suppose that it is reduced, that is, for all j, $\beta_j \notin \mathcal{A}$ and for all j, $2 \leqslant j \leqslant k$, $\alpha_j \notin \mathcal{B}$. It follows from the amalgamated structure of $\mathcal{G}$ that if

$$f = \alpha'_1 \circ \beta'_1 \circ \cdots \circ \alpha'_l \circ \beta'_l \circ \alpha'_{l+1}$$

is another reduced expression of f, then $k = l$ and there exist $(\gamma_j)_{1 \leqslant j \leqslant k}$, $(\delta_j)_{1 \leqslant j \leqslant k}$ in $\mathcal{A} \cap \mathcal{B}$ such that $\alpha'_1 = \alpha_1 \circ \gamma_1^{-1}$, $\alpha'_j = \delta_{j-1} \circ \alpha_j \circ \gamma_j^{-1}$ (for $2 \leqslant j \leqslant k$), $\alpha'_{k+1} = \delta_k \circ \alpha_{k+1}$ and $\beta'_j = \gamma_j \circ \beta_j \circ \delta_j^{-1}$ (for $1 \leqslant j \leqslant k$). Following [**7**, **9**], we define the multidegree and the length of f by

$$\operatorname{mdeg}(f) = (\deg \beta_1, \ldots, \deg \beta_k) \quad \text{and} \quad l(f) = k.$$

These notions of multidegree and length could be defined in the same way for a polynomial automorphism of $\mathbb{A}^2_K$, where K is any field. We recall (see [**7**, **37**]) that degree and multidegree are related by

$$\deg f = \deg \beta_1 \times \cdots \times \deg \beta_k.$$

Let $\mathcal{D}$ be the set of multidegrees, that is, of finite sequences of integers at least 2 (including the empty sequence). If $d = (d_1, \ldots, d_k) \in \mathcal{D}$, then $\mathcal{G}_d$ will denote the set of automorphisms whose multidegree is equal to d. By [**7**, **11**], $\mathcal{G}_d$ is an irreducible smooth, locally closed subset of $\mathcal{G}$ of dimension $d_1 + \cdots + d_k + 6$. Let us note that $\mathcal{G}_d \subseteq \mathcal{G}_{\leqslant n}$ as soon as $n \geqslant d_1 \cdots d_k$ and that we have a partition of $\mathcal{G}$ as a disjoint union $\mathcal{G} = \coprod_{d \in \mathcal{D}} \mathcal{G}_d$. What can be said on the closure of $\mathcal{G}_d$?

By [**9**], the length of an automorphism is lower semicontinuous. Therefore, any element of $\bar{\mathcal{G}}_d$ has length at most k. The simplest nontrivial related example is induced by the Nagata automorphism (see [**28**]):

$$N := (X - 2YW - ZW^2, Y + ZW, Z) \quad \text{where } W = XZ + Y^2.$$

This automorphism of the complex affine space $\mathbb{A}^3$ can be seen as an automorphism of $\mathbb{A}^2_{\mathbb{C}[Z]}$ inducing as well a morphism from the complex affine line $\mathbb{A}^1$ to $\mathcal{G}$ sending z to

$$N_z = (X - 2Y(Xz + Y^2) - z(Xz + Y^2)^2, Y + z(Xz + Y^2)) \in \mathcal{G}.$$

If $z \neq 0$, then the factorization

$$N_z = (X - z^{-1}Y^2, Y) \circ (X, Y + z^2 X) \circ (X + z^{-1}Y^2, Y)$$

shows us that N_z has multidegree $(2, 2)$. If $z = 0$, then we have $N_0 = (X - 2Y^3, Y)$ so that N_0 has multidegree (3). This yields $\mathcal{G}_{(3)} \cap \bar{\mathcal{G}}_{(2,2)} \neq \emptyset$. Inspired by the analysis of similar examples, we hoped (erroneously) that the equality $\bar{\mathcal{G}}_d = \bigcup_{e \preceq d} \mathcal{G}_e$ might always be true, where $\preceq$ is defined in the following way.

Definition 1.7. The partial order $\preceq$ on $\mathcal{D}$ is induced by the following relations:

(i) $\emptyset \preceq d$ (for any d);
(ii) $(d_1, \ldots, d_k) \preceq (e_1, \ldots, e_k)$ when $d_j \leqslant e_j$ for any j;
(iii) $(d_1, \ldots, d_{j-1}, d_j + d_{j+1} - 1, d_{j+2}, \ldots, d_k) \preceq (d_1, \ldots, d_k)$ when $1 \leqslant j \leqslant k - 1$.

However, by [**5**], there cannot exist any partial order $\sqsubseteq$ such that $\bar{\mathcal{G}}_d = \bigcup_{e \sqsubseteq d} \mathcal{G}_e$ when $d = (11, 3, 3)$! Actually, it is proved there that $\mathcal{G}_{(19)} \cap \bar{\mathcal{G}}_{(11,3,3)} \neq \emptyset$ and on the grounds of dimension we cannot have $\mathcal{G}_{(19)} \subseteq \bar{\mathcal{G}}_{(11,3,3)}$. As a matter of fact reality is often complex. We now think that the equality $\bar{\mathcal{G}}_d = \bigcup_{e \preceq d} \mathcal{G}_e$ is actually true in length $0, 1$ and 2. In length 0, it is obvious. In length 1, it is proved in [**8**]. This paper settles the length 2 case assuming that the conjecture $R(m, n)$ holds. Indeed, here is the main result of our paper.

Theorem B. *If the conjecture $R(m, n)$ holds and if we set $d = (m + 1, n + 1)$, then we have*

$$\bar{\mathcal{G}}_d = \bigcup_{e \preceq d} \mathcal{G}_e.$$

Remark 1.8. (1) If m divides n or n divides m, without assuming that the conjecture $R(m, n)$ holds, and if we set $d = (m + 1, n + 1)$, then the following inclusion is proved in [**4**]:

$$\bigcup_{e \preceq d} \mathcal{G}_e \subseteq \bar{\mathcal{G}}_d.$$

(2) If the equality $\bar{\mathcal{G}}_d = \bigcup_{e \preceq d} \mathcal{G}_e$ holds when d is of length 2, then a straightforward induction on the length on d would establish that for any d the following inclusion holds:

$$\bigcup_{e \preceq d} \mathcal{G}_e \subseteq \bar{\mathcal{G}}_d.$$

Theorems A and B directly imply the following theorem.

Theorem C. *If $d = (d_1, d_2)$ is a multidegree with d_1 or $d_2 \leqslant 3$, then $\bar{\mathcal{G}}_d = \bigcup_{e \preceq d} \mathcal{G}_e$.*

Remark 1.9. We would like to stress that even if the conjecture $R(m, n)$ is obvious when m or $n = 1$ (cf. the proof of conjecture $R(1)$ in Subsection 2.4), its consequence for polynomial automorphisms (given via Theorem B) is not obvious at all. Indeed, it asserts that if $d = (d_1, d_2)$ is a multidegree such that d_1 or $d_2 \leqslant 2$, then $\bar{\mathcal{G}}_d = \bigcup_{e \preceq d} \mathcal{G}_e$. The simplest case where $d_1 = d_2 = 2$ means that

$$\bar{\mathcal{G}}_{(2,2)} = \mathcal{G}_\emptyset \cup \mathcal{G}_{(2)} \cup \mathcal{G}_{(3)} \cup \mathcal{G}_{(2,2)}.$$

In particular, this equality implies the two following points.

(i) We have $\mathcal{G}_{(3)} \cap \bar{\mathcal{G}}_{(2,2)} \neq \emptyset$. As explained above, this point can be established using the family of plane automorphisms induced by the Nagata automorphism.

(ii) We have $\bar{\mathcal{G}}_{(2,2)} \cap \mathcal{G}_{(4)} = \emptyset$. This equality follows from [**11**, Theorem A] asserting that for any multidegree $d = (d_1, \ldots, d_l)$, the set $\mathcal{G}_d$ is closed in the set of all automorphisms of degree $d_1 \cdots d_l$. The weaker fact that $\mathcal{G}_{(4)}$ is not included into $\bar{\mathcal{G}}_{(2,2)}$ was a consequence of [**8**, Theorem 2] and was also the object of [**8**, Proposition 12].

As a funny consequence of Theorem C, we will show the following theorem.

Theorem D. *Any closed subgroup of $\mathcal{G}$ containing the affine group and an automorphism of length 1 is equal to $\mathcal{G}$.*

Remark 1.10. (1) Let us note that any subgroup of $\mathcal{G}$ strictly containing the affine group is dense in $\mathcal{G}$ for the Krull topology (see [**10**, Theorem A]). This means that any element of $\mathcal{G}$ can be approximated at the origin and at any order by an element of this subgroup.

(2) We recall that any closed subgroup of $\mathcal{G}$ which is a finite-dimensional algebraic variety is conjugate to either a subgroup of $\mathcal{A}$ or $\mathcal{B}$ (see [**32**, Theorem 8], [**21**, Theorem 4.3] or [**13**, Theorem 7]).

We finish this introduction with the following natural question.

Question 1.11. Does there exist a nontrivial closed subgroup of $\mathcal{G}$ containing the affine group?

This paper is divided into two parts. The first part is the shortest and is composed of Section 2 only, where we study the rigidity conjecture $R(m, n)$ and prove Theorem A. The second part is composed of Sections 3–5. In Section 3, we give some preliminary results. In Sections 4 and 5, we prove Theorems B and D, respectively.

2. *The rigidity conjecture*

In Subsection 2.1, we prove that the conjectures $R(m,n)$ and $R(n,m)$ are equivalent. In Subsection 2.2, we relate the conjectures $R(m,n)$ and $R(m)$. In Subsection 2.3, we give some generalities on the conjecture $R(m)$. In Subsection 2.4, we prove the conjectures $R(1)$ and $R(2)$. In Subsection 2.5, we give some information on the conjecture $R(3)$. We explain why we were not able to solve it and hope that the given information might help or motivate some readers for further investigations.

2.1. *The conjectures $R(m,n)$ and $R(n,m)$ are equivalent*

LEMMA 2.1. *The conjectures $R(m,n)$ and $R(n,m)$ are equivalent.*

Proof. Let val: $\mathbb{C}((X)) \to \mathbb{Z} \cup \{+\infty\}$ be the valuation associated to the discrete valuation ring $\mathbb{C}[\![X]\!]$. The conjecture $R(m,n)$ means that if $a = X(1 + a_1X + \cdots + a_mX^m)$ and $b = X(1 + b_1X + \cdots + b_nX^n)$ are such that $\mathrm{val}(a \circ b - X) \geqslant m + n + 2$, then $a = b = X$. Therefore, it is enough to check that $\mathrm{val}(a \circ b - X) = \mathrm{val}(b \circ a - X)$. Indeed, if $k = \mathrm{val}(a \circ b - X)$, then we have $a \circ b(X) \equiv X \mod X^k$. Let $a^{-1}(X) \in \mathbb{C}[\![X]\!]$ be the formal inverse of a for the composition. We get $a^{-1} \circ (a \circ b) \circ a(X) \equiv a^{-1} \circ a(X) \mod X^k$, that is, $b \circ a(X) \equiv X \mod X^k$, so $\mathrm{val}(b \circ a - X) \geqslant \mathrm{val}(a \circ b - X)$. We would show the other inequality in the same way, so that finally $\mathrm{val}(a \circ b - X) = \mathrm{val}(b \circ a - X)$. □

2.2. *Relations between the conjectures $R(m,n)$ and $R(m)$*

LEMMA 2.2. *If $m \geqslant 1$, then the two following assertions are equivalent.*

(i) *The conjecture $R(m,n)$ holds for any $n \geqslant 0$.*
(ii) *The conjecture $R(m)$ holds.*

Proof. If $a = X(1 + a_1X + \cdots + a_mX^m) \in \mathbb{C}[X]$, let us set

$$a^{-1} = X\left(1 + \sum_{k\geqslant 1} \tilde{a}_k X^k\right) \in \mathbb{C}[\![X]\!].$$

By definition, the conjecture $R(m)$ holds if and only if for any integer $n \geqslant 0$ the following assertion $\tilde{R}(m,n)$ holds.

ASSERTION 2.3 ($\tilde{R}(m,n)$). $(\tilde{a}_{n+1} = \cdots = \tilde{a}_{n+m} = 0) \implies a(X) = X.$

However, if $b = X(1 + b_1X + \cdots + b_nX^n) \in \mathbb{C}[X]$, then we have

$$a \circ b(X) \equiv X \operatorname{mod} X^{m+n+2} \iff a^{-1}(X) \equiv b(X) \operatorname{mod} X^{m+n+2}.$$

Using this equivalence, let us show that the assertions $\tilde{R}(m,n)$ and $R(m,n)$ are equivalent. This will be enough to show the lemma.

Assume that the conjecture $R(m,n)$ holds and that we have $\tilde{a}_{n+1} = \cdots = \tilde{a}_{n+m} = 0$.

$$\text{Set} \quad b(X) = X\left(1 + \sum_{1\leqslant k\leqslant n} \tilde{a}_k X^k\right) = X\left(1 + \sum_{1\leqslant k\leqslant m+n} \tilde{a}_k X^k\right).$$

We get $a^{-1}(X) \equiv b(X) \operatorname{mod} X^{m+n+2}$, so $a \circ b(X) \equiv X \operatorname{mod} X^{m+n+2}$ and finally $a(X) = X$.

Assume that the assertion $\tilde{R}(m,n)$ holds and that we have $a \circ b(X) \equiv X \bmod X^{m+n+2}$. Then, we have $a^{-1}(X) \equiv b(X) \bmod X^{m+n+2}$, so $\tilde{a}_{n+1} = \cdots = \tilde{a}_{n+m} = 0$ and we get $a(X) = X$. The equality $b(X) = X$ follows at once. □

2.3. *Generalities on the conjecture $R(m)$*

Let $m \geqslant 1$ be a fixed integer. Set $a(X) = X(1-\lambda_1 X)\cdots(1-\lambda_m X) \in \mathbb{C}[X]$, where the λ_k are complex numbers, and express the formal inverse for the composition of a as

$$a^{-1}(X) = X\left(1 + \sum_{n\geqslant 1} \frac{u_n}{n+1} X^n\right) \in \mathbb{C}[\![X]\!],$$

where the u_n, $n \geqslant 1$ are complex numbers.

Lemma 2.4. *If $\oint$ denotes integration over a little circle around the origin, then we have*

$$u_n = \frac{1}{2\pi i}\oint \frac{dX}{a^{n+1}(X)} = \sum_{j_1+\cdots+j_m=n} \binom{n+j_1}{n} \cdots \binom{n+j_m}{n} \lambda_1^{j_1}\cdots\lambda_m^{j_m}.$$

Proof. By the Lagrange formula, we have

$$\begin{aligned} u_n &= \frac{n+1}{2\pi i}\oint \frac{a^{-1}(X)}{X^{n+2}}\,dX \\ &= \frac{n+1}{2\pi i}\oint \frac{Y}{a^{n+2}(Y)} a'(Y)\,dY \quad \text{by setting } X = a(Y) \\ &= \frac{1}{2\pi i}\oint \frac{dY}{a^{n+1}(Y)} \quad \text{by integrating by parts.} \end{aligned}$$

Therefore, u_n is the X^n-coefficient of the formal series $\prod_{1\leqslant k\leqslant m}(1/(1-\lambda_k X)^{n+1}) \in \mathbb{C}[\![X]\!]$. We conclude thanks to the Taylor expansion $1/(1-X)^{n+1} = \sum_{j\geqslant 0}\binom{n+j}{n} X^j$. □

Note that u_n is a homogeneous polynomial of degree n in the $\lambda_1,\ldots,\lambda_m$ (where each λ_k has weight 1). The conjecture $R(m)$ means that if m consecutive u_n vanish, then the λ_i vanish also. In other words, m consecutive polynomials u_n always constitute an hsop of $\mathbb{C}[\lambda_1,\ldots,\lambda_m]$.

Remark 2.5. In order to prove $R(m)$, it is sufficient to prove that if m consecutive u_n vanish, then $u_k = 0$ when k is large enough. Indeed, in this case a^{-1} is a polynomial and the relation $X = a \circ a^{-1}$ shows $a = X$ (by taking the degree).

The proof of the following lemma is due to Laurent Manivel.

Lemma 2.6. *The sequence $n \mapsto u_n$ satisfies a linear recurrence relation with polynomial coefficients of length m. More precisely, there exist polynomials $\mu_0(n),\ldots,\mu_m(n)$ in $\mathbb{C}[\lambda_1,\ldots,\lambda_m][n]$, not all zero, such that*

$$\forall n \geqslant 1, \qquad \sum_{0\leqslant k\leqslant m} \mu_k(n) u_{n+k} = 0.$$

Proof. By Lemma 2.4, it is enough to show an analogous linear recurrence relation for the sequence

$$n \longmapsto v_n := \oint \frac{dX}{a^n(X)}.$$

Set

$$v_{n,k} = \oint \frac{X^k}{a^n}\,dX \quad \text{and} \quad V_n = \begin{pmatrix} v_{n,0} \\ \vdots \\ v_{n,m-1} \end{pmatrix}.$$

Claim 2.7. There exists a square matrix M_n whose entries belong to the field $\mathbb{C}(\lambda_1, \ldots, \lambda_m)(n)$ such that

$$V_n = M_n V_{n+1}.$$

Set $\gamma(X) = X^{-1}a(X) = (1-\lambda_1 X)\cdots(1-\lambda_m X)$. We have $v_{n,k} = \oint (X^{k-n}/\gamma^n)\,dX$. If $k \neq n-1$, then we get $v_{n,k} = (n/(k-n+1)) \oint X^{k-n+1}\gamma'/\gamma^{n+1}\,dX$, that is,

$$\frac{k-n+1}{n} v_{n,k} = \oint \frac{X^{k-n+1}\gamma'}{\gamma^{n+1}}\,dX.$$

Note that this last equality still holds for $k = n-1$. By Euclidean division, there exist polynomials A_k, B_k such that

$$X^{k+1}\gamma' = A_k\gamma - B_k$$

and we have $\deg A_k = k$ and $\deg B_k < \deg \gamma = m$. We have

$$\frac{k-n+1}{n} v_{n,k} = \oint \frac{X^{-n}(A_k\gamma - B_k)}{\gamma^{n+1}}\,dX = \oint \frac{A_k}{a^n}\,dX - \oint \frac{XB_k}{a^{n+1}}\,dX.$$

This gives us a relation of the following form:

$$\frac{n-k-1}{n} v_{n,k} + \sum_{j=0}^{k} a_{k,j} v_{n,j} = \sum_{j=1}^{m} b_{k,j} v_{n+1,j}. \qquad (*)$$

But we have $\oint a'/a^{n+1}\,dX = 0$ and $a(X) = X\,\sum_{k=0}^{m} (-1)^k \sigma_k X^k$, where

$$\sigma_k = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}$$

denotes the kth elementary symmetric polynomial in the variables $\lambda_1, \ldots, \lambda_m$ (with the usual convention $\sigma_0 = 1$), so

$$\sum_{k=0}^{m} (-1)^k (k+1) \sigma_k v_{n+1,k} = 0. \qquad (**)$$

This relation allows us to express $v_{n+1,m}$ in terms of the elements $v_{n+1,j}$, $0 \leqslant j \leqslant m-1$. By making such a substitution in the right-hand side of $(*)$, we obtain a relation of the form

$$\frac{n-k-1}{n} v_{n,k} + \sum_{j=0}^{k} a_{k,j} v_{n,j} = \sum_{j=0}^{m-1} c_{k,j} v_{n+1,j}. \qquad (***)$$

However, the relations $(***)$ for $0 \leqslant k \leqslant m-1$ can be expressed with matrices as

$$K_n V_n = C V_{n+1},$$

where the matrix C does not depend on n, that is, has entries in $\mathbb{C}(\lambda_1, \ldots, \lambda_m)$, and the matrix K_n is lower triangular with diagonal entries the numbers

$$\frac{n-k-1}{n} + a_{k,k} = \frac{n-k-1}{n} + m, \quad 0 \leqslant k \leqslant m-1.$$

In particular, the matrix K_n is invertible. By setting $M_n := K_n^{-1}C$, we have proved the claim.

By using several times the claim, we get

$$\begin{aligned} V_{n+m} &= V_{n+m}, \\ V_{n+m-1} &= M_{n+m-1}V_{n+m}, \\ &\vdots \\ V_n &= (M_n M_{n+1} \cdots M_{n+m-1})V_{n+m}. \end{aligned}$$

Therefore, the $m+1$ elements $v_n, \ldots, v_{n+m}$ can be expressed as a linear combination, with coefficients in $\mathbb{C}(\lambda_1, \ldots, \lambda_m)(n)$, of the m elements $v_{n+m,0}, \ldots, v_{n+m,m-1}$. As a consequence, the elements $v_n, \ldots, v_{n+m}$ are linearly dependent over the field $\mathbb{C}(\lambda_1, \ldots, \lambda_m)(n)$ and hence also over the ring $\mathbb{C}[\lambda_1, \ldots, \lambda_m][n]$. □

2.4. *Proof of Theorem* A

According to Lemmas 2.1 and 2.2, it is sufficient to prove the conjectures $R(1)$ and $R(2)$.

2.4.1. *Proof of* $R(1)$. We use the notation of Subsection 2.3 with $m = 1$. Note that $u_n = \binom{2n}{n} \lambda_1^n$. If $u_n = 0$, then we want to prove $\lambda_1 = 0$. This is obvious.

2.4.2. *Proof of* $R(2)$. We use the notation of Subsection 2.3 with $m = 2$. Note that

$$u_n = \sum_{i+j=n} \binom{n+i}{n} \binom{n+j}{n} \lambda_1^i \lambda_2^j.$$

We rely on the following linear recurrence relation.

LEMMA 2.8. *For any* $n \geqslant 3$, *we have*

$$\begin{aligned} &n(n-1)(\lambda_1 - \lambda_2)^2 u_n + (n-1)(2n-1)(\lambda_1 + \lambda_2)(\lambda_1 - 2\lambda_2)(\lambda_2 - 2\lambda_1)u_{n-1} \\ &\quad - 3(3n-4)(3n-2)\lambda_1^2\lambda_2^2 u_{n-2} = 0. \end{aligned}$$

Proof. We follow the beginning of the proof of Lemma 2.6 in the case where $m = 2$ and we compute the relations $(*), (**)$ and $(***)$ given there. We get

$$A_0 = 2, \quad B_0 = 2 - \sigma_1 X, \quad A_1 = \frac{\sigma_1}{\sigma_2} + 2X \quad \text{and} \quad B_1 = \frac{\sigma_1}{\sigma_2} + \frac{2\sigma_2 - \sigma_1^2}{\sigma_2} X,$$

so that the relations $(*)$ for $k = 0, 1$ are the following:

$$\begin{cases} \dfrac{3n-1}{n} v_{n,0} = 2v_{n+1,1} - \sigma_1 v_{n+1,2}, \\ \sigma_1 v_{n,0} + \dfrac{3n-2}{n} \sigma_2 v_{n,1} = \sigma_1 v_{n+1,1} + \left(2\sigma_2 - \sigma_1^2\right) v_{n+1,2}. \end{cases} \tag{$*$}$$

We get at once

$$v_{n+1,0} - 2\sigma_1 v_{n+1,1} + 3\sigma_2 v_{n+1,2} = 0, \tag{$**$}$$

so that the relations $(***)$ for $k = 0, 1$ are the following:

$$\begin{cases} 3\dfrac{3n-1}{n}\sigma_2 v_{n,0} = \sigma_1 v_{n+1,0} + (6\sigma_2 - 2\sigma_1^2)v_{n+1,1}, \\ 3\sigma_1\sigma_2 v_{n,0} + 3\dfrac{3n-2}{n}\sigma_2^2 v_{n,1} = (\sigma_1^2 - 2\sigma_2)v_{n+1,0} + (7\sigma_1\sigma_2 - 2\sigma_1^3)v_{n+1,1}, \end{cases}$$

that is,

$$(6\sigma_2 - 2\sigma_1^2)v_{n+1,1} = 3\frac{3n-1}{n}\sigma_2 v_n - \sigma_1 v_{n+1} \tag{A}$$

and

$$3\sigma_1\sigma_2 v_n + 3\frac{3n-2}{n}\sigma_2^2 v_{n,1} = (\sigma_1^2 - 2\sigma_2)v_{n+1} + (7\sigma_1\sigma_2 - 2\sigma_1^3)v_{n+1,1}. \tag{B}$$

The relation (A) expresses $v_{n+1,1}$ in terms of v_n, v_{n+1}. Replacing n by $n-1$ allows $v_{n,1}$ to be expressed in terms of v_{n-1}, v_n:

$$(6\sigma_2 - 2\sigma_1^2)v_{n,1} = 3\frac{3n-4}{n-1}\sigma_2 v_{n-1} - \sigma_1 v_n. \tag{A$'$}$$

By substituting (A) and (A)$'$ in (B), we get the following relation between v_{n-1}, v_n, v_{n+1}:

$$\begin{aligned} &3\sigma_1\sigma_2(6\sigma_2 - 2\sigma_1^2)v_n + 3\frac{3n-2}{n}\sigma_2^2\left[3\frac{3n-4}{n-1}\sigma_2 v_{n-1} - \sigma_1 v_n\right] \\ &\quad = (\sigma_1^2 - 2\sigma_2)(6\sigma_2 - 2\sigma_1^2)v_{n+1} + (7\sigma_1\sigma_2 - 2\sigma_1^3)\left[3\frac{3n-1}{n}\sigma_2 v_n - \sigma_1 v_{n+1}\right], \end{aligned}$$

that is,

$$\mu_2 v_{n+1} + \mu_1 v_n + \mu_0 v_n = 0,$$

where

$$\begin{aligned} \mu_2 &= (\sigma_1^2 - 2\sigma_2)(2\sigma_1^2 - 6\sigma_2) + \sigma_1(7\sigma_1\sigma_2 - 2\sigma_1^3) \\ &= -3\sigma_2(\lambda_1 - \lambda_2)^2, \\ \mu_1 &= 3\sigma_1\sigma_2(6\sigma_2 - 2\sigma_1^2) - 3\frac{3n-2}{n}\sigma_1\sigma_2^2 - 3\frac{3n-1}{n}(7\sigma_1\sigma_2 - 2\sigma_1^3)\sigma_2 \\ &= 3\sigma_1\sigma_2(9\sigma_2 - 2\sigma_1^2)\frac{2n-1}{n} \\ &= 3\sigma_1\sigma_2(\lambda_1 - 2\lambda_2)(\lambda_2 - 2\lambda_1)\frac{2n-1}{n}, \\ \mu_0 &= 3^2\sigma_2^3\frac{3n-2}{n}\frac{3n-4}{n-1}. \end{aligned}$$

Since $v_{k+1} = 2\pi i u_k$ for any k, the result follows. □

REMARK 2.9. Another way to prove Lemma 2.8 is more elementary, but tedious. The left-hand side being homogeneous of degree $n+2$ in λ_1, λ_2, it is enough to check that for $0 \leqslant i \leqslant n+2$, the coefficient of $\lambda_1^i\lambda_2^{n+2-i}$ vanishes.

Let us prove $R(2)$. If $u_k = u_{k+1} = 0$, then we want to prove that $\lambda_1 = \lambda_2 = 0$. We begin by showing by contradiction that $\lambda_1 = \lambda_2$. Otherwise, Lemma 2.8 shows us that the following implication holds:

$$\forall n \geqslant 3, \quad (u_{n-2} = u_{n-1} = 0) \implies u_n = 0.$$

Therefore, $u_n = 0$ for $n \geqslant k$ and by Remark 2.5, this implies the equality $\lambda_1 = \lambda_2 = 0$: a contradiction. Therefore, $\lambda_1 = \lambda_2$. Set $\lambda := \lambda_1 = \lambda_2$. Lemma 2.8 gives us for any $n \geqslant 3$,

$$2(n-1)(2n-1)\lambda^3 u_{n-1} - 3(3n-4)(3n-2)\lambda^4 u_{n-2} = 0.$$

Let us show by contradiction that $\lambda = 0$. Otherwise, the following implication would hold:

$$\forall n \geqslant 3, \quad u_{n-2} = 0 \implies u_{n-1} = 0.$$

We still get $u_n = 0$ for $n \geqslant k$, so $\lambda = 0$: a contradiction. We have indeed $\lambda = 0$, so $\lambda_1 = \lambda_2 = 0$.

2.5. *The conjecture R*(3)

The first aim of this subsection is to explain why we were not able to solve conjecture $R(3)$ in the previous way. The second aim is to give some information (and motivation) for the reader who might be interested to undertake some investigations on this subject.

We use the notation of Subsection 2.3 with $m = 3$, so

$$u_n = \sum_{i+j+k=n} \binom{n+i}{n}\binom{n+j}{n}\binom{n+k}{n} \lambda_1^i \lambda_2^j \lambda_3^k \in \mathbb{Z}[\lambda_1,\ \lambda_2,\ \lambda_3].$$

Before giving the linear recurrence relation satisfied by the u_n, we need some notation. If $\mu = (\mu_1, \mu_2, \mu_3)$ where μ_i are integers satisfying $\mu_1 \geqslant \mu_2 \geqslant \mu_3 \geqslant 0$, then we define $m_\mu \in \mathbb{Z}[\lambda_1, \lambda_2, \lambda_3]$ by $m_\mu := \sum \lambda_1^{\nu_1}\lambda_2^{\nu_2}\lambda_3^{\nu_3}$ where (ν_1, ν_2, ν_3) describes all distinct permutations of the triple (μ_1, μ_2, μ_3). We identify (μ_1, μ_2) with $(\mu_1, \mu_2, 0)$ as well as (μ_1) with $(\mu_1, 0, 0)$. Hence $m_{(1)} = \lambda_1 + \lambda_2 + \lambda_3$, $m_{(1,1)} = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3$, $m_{(1,1,1)} = \lambda_1\lambda_2\lambda_3$ and $m_{(3,2,1)} = \lambda_1^3\lambda_2^2\lambda_3 + \lambda_1^3\lambda_2\lambda_3^2 + \lambda_1^2\lambda_2^3\lambda_3 + \lambda_1\lambda_2^3\lambda_3^2 + \lambda_1^2\lambda_2\lambda_3^3 + \lambda_1\lambda_2^2\lambda_3^3$.

We set $\Delta = (\lambda_1 - \lambda_2)^2(\lambda_1 - \lambda_3)^2(\lambda_2 - \lambda_3)^2$ and define the M_i, N_i for $1 \leqslant i \leqslant 5$ by

$$\begin{aligned}
M_1 &= 9m_{(4,2)} - 14m_{(4,1,1)} - 9m_{(3,3)} + 3m_{(3,2,1)} - 3m_{(2,2,2)},\\
M_2 &= 5m_{(4,2)} - 8m_{(4,1,1)} - 5m_{(3,3)} + 2m_{(3,2,1)} - 3m_{(2,2,2)},\\
M_3 &= 39m_{(4,2)} - 62m_{(4,1,1)} - 39m_{(3,3)} + 15m_{(3,2,1)} - 21m_{(2,2,2)},\\
M_4 &= 33m_{(4,2)} - 52m_{(4,1,1)} - 33m_{(3,3)} + 12m_{(3,2,1)} - 15m_{(2,2,2)},\\
M_5 &= 6m_{(4,2)} - 10m_{(4,1,1)} - 6m_{(3,3)} + 3m_{(3,2,1)} - 6m_{(2,2,2)},\\
N_1 &= -2m_{(5,2)} + 4m_{(5,1,1)} + 3m_{(4,3)} - 3m_{(4,2,1)} - 8m_{(3,3,1)} + 8m_{(3,2,2)},\\
N_2 &= 10m_{(9,4)} - 36m_{(9,3,1)} + 52m_{(9,2,2)} - 25m_{(8,5)} + 63m_{(8,4,1)} - 38m_{(8,3,2)} + 10m_{(7,6)}\\
&\quad + 30m_{(7,5,1)} - 146m_{(7,4,2)} + 216m_{(7,3,3)} - 60m_{(6,6,1)} + 70m_{(6,5,2)} - 32m_{(6,4,3)}\\
&\quad - 60m_{(5,5,3)} + 40m_{(5,4,4)},\\
N_3 &= -27m_{(4,4)} + 36m_{(4,3,1)} - 2m_{(4,2,2)} - 52m_{(3,3,2)},\\
N_4 &= -342m_{(8,6)} + 1006m_{(8,5,1)} - 1110m_{(8,4,2)} + 972m_{(8,3,3)} + 342m_{(7,7)} - 141m_{(7,6,1)}\\
&\quad - 1301m_{(7,5,2)} + 900m_{(7,4,3)} + 1178m_{(6,6,2)} - 15m_{(6,5,3)} - 724m_{(6,4,4)} + 238m_{(5,5,4)},\\
N_5 &= 10m_{(8,6)} - 30m_{(8,5,1)} + 34m_{(8,4,2)} - 30m_{(8,3,3)} - 10m_{(7,7)} + 5m_{(7,6,1)} + 37m_{(7,5,2)}\\
&\quad - 27m_{(7,4,3)} - 38m_{(6,6,2)} + 5m_{(6,5,3)} + 20m_{(6,4,4)} - 10m_{(5,5,4)}.
\end{aligned}$$

Set

$$\begin{aligned}
A_n &= n(n-1)(n-2)(M_1 n - 3M_2)\Delta,\\
B_n &= (n-1)(n-2)[(2M_1 n^2 - M_3 n)N_1 - 3N_2],
\end{aligned}$$

$$C_n = (n-2)[(M_1 n^3 - M_4 n^2)N_3 + 3nN_4 + 36N_5],$$
$$D_n = 8(2n-3)(4n-7)(4n-9)(M_1 n - M_5)m^3_{(1,1,1)}.$$

One could show the next result (using a computer!).

Lemma 2.10. *For any $n \geqslant 4$, we have*

$$A_n u_n + B_n u_{n-1} + C_n u_{n-2} + D_n u_{n-3} = 0.$$

The difference from the recurrence relations obtained in Lemma 2.8 is that the factor $M_1 n - 3M_2$ of A_n may suddenly vanish for a large value of n. Therefore, if we assume $u_k = u_{k+1} = u_{k+2} = 0$, then we do not succeed in showing $u_{k+3} = 0$. However, a closer analysis of the recurrence formula might probably be sufficient for proving $R(3)$.

3. *Preliminary results*

In Subsection 3.1, we recall a valuative criterion characterizing the elements of $\overline{f(V)}$ where $f\colon V \to W$ is a morphism of complex algebraic varieties. The only valuation ring we need is the ring of complex formal power series. In Subsection 3.2, after recalling equivalent definitions of an hsop, we give two results related to formal power series. In Subsection 3.3, we make some technical definitions which will allow us (in the next section) to prove Theorem B. Finally, in Subsection 3.4, we prove an easy lemma on the multidegree.

3.1. *Valuative criterion*

The valuative criterion given in Lemma 3.1 is familiar (for example, [**26**, Chapter 2, Section 1, pp. 52–54] and [**14**, Section 7]). We proved it in [**11**].

Let $\mathbb{C}[\![T]\!]$ be the algebra of complex formal power series and let $\mathbb{C}((T))$ be its quotient field. If V is a complex algebraic variety and A a complex algebra, then $V(A)$ will denote the points of V with values in A, that is, the set of morphisms $\operatorname{Spec} A \to V$. If v is a closed point of V and $\varphi \in V(\mathbb{C}((T)))$, then we will write $v = \lim_{T\to 0} \varphi(T)$ when the following conditions hold.

(i) The point $\varphi : \operatorname{Spec} \mathbb{C}((T)) \to V$ is a composition $\operatorname{Spec} \mathbb{C}((T)) \to \operatorname{Spec} \mathbb{C}[\![T]\!] \to V$.
(ii) The point v is the composition $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{C}[\![T]\!] \to V$.

For example, if $V = \mathbb{A}^1$ and $\varphi \in V(\mathbb{C}((T))) = \mathbb{C}((T))$, then we will write $v = \lim_{T\to 0} \varphi(T)$ when $\varphi \in \mathbb{C}[\![T]\!]$ and $v = \varphi(0)$.

Lemma 3.1. *Let $f : V \to W$ be a morphism of complex algebraic varieties and let w be a closed point of W. The two following assertions are equivalent*:

(i) *We have $w \in \overline{f(V)}$;*
(ii) *We have $w = \lim_{T\to 0} f(\varphi(T))$ for some $\varphi \in V(\mathbb{C}((T)))$.*

Remark 3.2. Note the analogy with the metric case where $w \in \overline{f(V)}$ if and only if there exists a sequence $(v_n)_{n\geqslant 1}$ of V such that $w = \lim_{n\to+\infty} f(v_n)$.

The following result is an easy consequence of Lemma 3.1 (see [**11**, Corollary 1.1]).

Corollary 3.3. *If* $d = (d_1, \ldots, d_k)$ *is a multidegree and* $f \in \mathcal{G}$, *then the following assertions are equivalent*:

(i) *We have* $f \in \bar{\mathcal{G}}_d$;
(ii) *We have* $f = \lim_{T\to 0} g_T$ *for some* $g \in \mathcal{G}_d(\mathbb{C}((T)))$.

Proof. For any integer $m \geqslant 2$, let $\mathcal{B}_m$ be the set of triangular automorphisms whose degree is exactly equal to m. Let $\mathcal{A}' = \mathcal{A} \setminus \mathcal{B}$ be the set of affine nontriangular automorphisms. It is enough to note that $\mathcal{G}_d$ is the image of the following morphism of algebraic varieties:

$$\mathcal{A}^2 \times (\mathcal{A}')^{k-1} \times \prod_{1\leqslant i\leqslant k} \mathcal{B}_{d_i} \longrightarrow \mathcal{G}$$

sending $((a_1, a_2), (a'_i)_i, (b_i)_i)$ to $a_1 \circ b_1 \circ a'_1 \circ b_2 \circ a'_2 \circ \cdots \circ a'_{k-1} \circ b_k \circ a_2$. The details are left to the reader. □

Remark 3.4. Since $\mathcal{G}_d$ is locally closed in $\mathcal{G}$ (see [**11**]), there is a natural identification between $\mathcal{G}_d(K)$ and the set of automorphisms of $\mathbb{A}^2_K$ whose multidegree is equal to d, for any field K containing $\mathbb{C}$.

3.2. *hsop and formal power series*

Let K be an algebraically closed field and let $\mathbb{A}^r_K$ be the affine r-space over K. Let us grade the polynomial algebra $R = K[z_1, \ldots, z_r]$ by assigning each z_k to be homogeneous of some strictly positive degree (depending on k). For each $m \geqslant 0$, the set of m-homogeneous polynomials is denoted by R_m. If $p = (p_1, \ldots, p_r) \in R^r$, let $\phi_p : \mathbb{A}^r_K \to \mathbb{A}^r_K$ be the morphism of algebraic varieties defined by $\phi_p(w) = (p_1(w), \ldots, p_r(w))$ for $w \in \mathbb{A}^r_K$. Also let I_p be the ideal of R generated by $p_1, \ldots, p_r$. Following the usual terminology (for example, [**34**, Chapter I, Definition 5.1, p. 33]), if each p_k is homogeneous of some strictly positive degree (depending on k) and if $K[z_1, \ldots, z_r]$ is a finitely generated module over $K[p_1, \ldots, p_r]$, then the sequence p is said to be an *hsop*. For the sake of completeness, we give the proof of the following classical lemma characterizing hsop.

Lemma 3.5. *Let* $p = (p_1, \ldots, p_r)$ *be a sequence of homogeneous polynomials of* $R = K[z_1, \ldots, z_r]$, *each* z_k *being homogeneous of some strictly positive degree* (*depending on* k). *The* r*-tuple* p *is an hsop of* R *if and only if the following equivalent assertions are satisfied.*

(i) *The morphism* ϕ_p *is finite.*
(ii) *The morphism* ϕ_p *is quasi-finite.*
(iii) *The fiber* $(\phi_p)^{-1}(0)$ *is finite.*
(iv) *We have* $\dim_K R/I_p < +\infty$.
(v) *We have* $(\phi_p)^{-1}(0) = \{0\}$.
(vi) *The morphism* ϕ_p *is proper.*
(vii) *The polynomial algebra* R *is a finitely generated and free module over* $K[p_1, \ldots, p_r]$.
(viii) *The morphism* ϕ_p *is flat.*
(ix) *For any* $d > \max_k \deg p_k$, *we have* $R_{dl} \subseteq I_p^l$ *when* l *is large enough.*
(x) *For some* $d > \max_k \deg p_k$, *we have* $R_{dl} \subseteq I_p^l$ *when* l *is large enough.*

Proof. Note that assertion (i) is a reformulation of the definition.
(i) ⇒ (ii) and (ii) ⇒ (iii). These results are obvious.
(iii) ⇔ (iv). This follows from Hilbert's Nullstellensatz.

(iv) $\Rightarrow$ (i). Let $h_1, \ldots, h_s$ be homogeneous elements of R which form a K-basis of R/I_p. Set $S := K[p_1, \ldots, p_r]$. It is enough to show $R = \sum_i h_i S$. Set $N = \sum_i h_i S$ and let $S_+ = p_1 S + \cdots + p_r S$ be the ideal of S generated by the p_i. Note that S is a subgraded ring of R and that R and N are both graded modules over S. Finally, since $R = N + S_+ R$, by the graded version of Nakayama's lemma, we get $R = N$.

(iii) $\Rightarrow$ (v). Let us show by contradiction that $(\phi_p)^{-1}(0) = \{0\}$. Otherwise $(\phi_p)^{-1}(0)$ would contain a nonzero element $w = (w_1, \ldots, w_r)$. If $d_i := \deg z_i$ and $e_i := \deg p_i$, then we have

$$\forall \lambda \in K, p_i(\lambda^{d_1} z_1, \ldots, \lambda^{d_r} z_r) = \lambda^{e_i} p_i(z_1, \ldots, z_r).$$

This proves that $(\lambda^{d_1} w_1, \ldots, \lambda^{d_r} w_r)$ belongs to $(\phi_p)^{-1}(0)$ for any λ, contradicting (iii).

(v) $\Rightarrow$ (iii). This is obvious.

(i) $\Leftrightarrow$ (vi). This follows from a theorem of Chevalley (see [**15**, EGA, III, 4.4.2, p. 136]) asserting that a morphism of algebraic varieties is finite if and only if it is proper and affine (see also [**27**, Lemma 3.5.1, p. 52] or [**2**, p. 296]). Alternatively, it is well known that a morphism is finite if and only if it is proper and quasi-finite (see the previous reference of Grothendieck!). Furthermore, it is clear that an affine proper morphism is quasi-finite. Indeed, each of its fibres being affine and complete, it has to be finite.

(i) $\Leftrightarrow$ (vii). This follows from [**34**, Chapter I, Theorem 5.9, p. 35]. Note that the implication (vii) $\Rightarrow$ (i) is obvious. The implication (i) $\Rightarrow$ (vii) also follows from [**3**, Section 4, no. 4, Corollary of Proposition 5, p. 58]. This reference claims that R is a projective module over $K[p_1, \ldots, p_r]$. In this graded situation, any projective module is free.

(vii) $\Rightarrow$ (viii). Any free module is flat.

(viii) $\Rightarrow$ (ii). This is clear, since a flat morphism is equidimensional (for example, [**17**, III, Corollary 9.6, p. 257]).

(vii) $\Rightarrow$ (ix). Set $S = K[p_1, \ldots, p_r]$ and let $h_1, \ldots, h_s$ be homogeneous elements of R such that $R = \sum_i h_i S$. Set $M = \max_i \deg h_i$, $m = \max_k \deg p_k$ and let $d > m$ be any integer. Let us check that for any $l \geqslant (M/(d-m))$, we have $R_{dl} \subseteq (I_p)^l$. Let f be any element of R_{dl}. There exist homogeneous elements u_i of S such that $f = \sum_i h_i u_i$. Furthermore, each u_i admits an expression $u_i = \sum_\alpha u_{i,\alpha} p^\alpha$, where the sum is over $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^r$, $p^\alpha := p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ and $u_{i,\alpha}$ belongs to K. Set $|\alpha| = \alpha_1 + \cdots + \alpha_r$. We have $f = \sum_{i,\alpha} u_{i,\alpha} h_i p^\alpha$. If $u_{i,\alpha} \neq 0$, then we may assume $\deg f = \deg(h_i p^\alpha)$. We get

$$m|\alpha| \geqslant \deg p^\alpha = \deg f - \deg h_i \geqslant dl - M \geqslant ml,$$

from which follows $|\alpha| \geqslant l$ and finally $f \in (I_p)^l$.

(ix) $\Rightarrow$ (x). This is obvious.

(x) $\Rightarrow$ (v). This is easy and left to the reader. □

Corollary 3.6. *If $p = (p_1, \ldots, p_r)$ is an hsop of $K[z_1, \ldots, z_r]$, then the map $\phi_p \colon \mathbb{A}_K^r \to \mathbb{A}_K^r$ is surjective.*

Proof. Since ϕ_p is proper, it is a closed morphism and in particular its image is closed. Since ϕ_p is flat, it is an open morphism (see [**16**, EGA, IV$_2$, 2.4.6, p. 20]) and in particular its image is open. □

Remark 3.7. Let $p = (p_1, \ldots, p_r)$ be a sequence of homogeneous polynomials of $R = K[z_1, \ldots, z_r]$. The surjectivity of the morphism $\phi_p : \mathbb{A}_K^r \to \mathbb{A}_K^r$ is not sufficient to ensure that p is an hsop. Consider the algebra $\mathbb{C}[x, y]$ with the usual grading. The morphism $\mathbb{A}^2 \to \mathbb{A}^2$, $(x, y) \mapsto (xy^2, x(x+y)^2)$ is surjective, but not finite.

Let $\mathrm{val}\colon \mathbb{C}((T)) \to \mathbb{Z} \cup \{+\infty\}$ be the valuation associated to the discrete valuation ring $\mathbb{C}[\![T]\!]$.

LEMMA 3.8. *If $(p_1, \ldots, p_r)$ is an hsop of $R := \mathbb{C}[z_1, \ldots, z_r]$ and $q \in R$ is homogeneous with $\deg q > \max_k \deg p_k$, then for any $b \in \mathbb{A}^r_{\mathbb{C}[\![T]\!]}$ satisfying $b(0) = 0$, we have*

$$\mathrm{val}\, q(b) \geqslant \min_k \mathrm{val}\, p_k(b) + 1.$$

Proof. By point (ix) of Lemma 3.5, there exists $l \geqslant 1$ such that $q^l \in (p_1, \ldots, p_r)^l$. For any $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^r$, set $p^\alpha = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_r$. If $A := \{\alpha \in \mathbb{N}^r, |\alpha| = l\}$, then we can write $q^l = \sum_{\alpha \in A} s_\alpha p^\alpha$ $(s_\alpha \in R)$. Furthermore, we may assume $s_\alpha \in R_{d_\alpha}$, where $d_\alpha := \deg q^l - \deg p^\alpha \geqslant l(\deg q - \max_k \deg p_k) \geqslant l \geqslant 1$. Evaluating at b and taking the valuation, we get $l\, \mathrm{val}\, q(b) \geqslant \min_{\alpha \in A} \mathrm{val}\, s_\alpha(b) p^\alpha(b)$. But $\mathrm{val}\, s_\alpha(b) \geqslant 1$ and $\mathrm{val}\, p^\alpha(b) \geqslant l\ \min_k\ \mathrm{val}\, p_k(b)$, so $l\ \mathrm{val}\, q(b) \geqslant l \min_k\ \mathrm{val}\, p_k(b) + 1$. □

The following lemma is a nice application of Corollary 3.6.

LEMMA 3.9. *If $p = (p_1, \ldots, p_r)$ is an hsop of $\mathbb{C}[z_1, \ldots, z_r]$ and $\gamma \in \mathbb{A}^r_{\mathbb{C}}$, there exist $q \geqslant 1$ and $b \in \mathbb{A}^r_{\mathbb{C}[\![T]\!]}$ such that $b(0) = 0$ and $p(b) = T^q \gamma$.*

Proof. Let $K := \bigcup_{q \geqslant 1} \mathbb{C}((T^{1/q}))$ be the quotient field of the ring $\bigcup_{q \geqslant 1} \mathbb{C}[\![T^{1/q}]\!]$ of all formal Puiseux series. By the Newton–Puiseux theorem (see, for example, [**31**, Proposition 4.4]), K is an algebraic closure of $\mathbb{C}((T))$. Let us note that p is an hsop of $R := K[z_1, \ldots, z_r]$. By Corollary 3.6, there exists an element a in the affine space $\mathbb{A}^r_K$ such that $p(a) = T\gamma$. Let $q \geqslant 1$ be such that $b := a(T^q) \in \mathbb{A}^r_{\mathbb{C}((T))}$. Replacing T by T^q, we get $p(b) = T^q \gamma$. Using the valuative criterion of properness (see, for example, [**17**, II, 4.7, p. 101]) to the proper morphism $\phi_p\colon \mathbb{A}^r \to \mathbb{A}^r$ (see Lemma 3.5), it is clear that $b \in \mathbb{A}^r_{\mathbb{C}[\![T]\!]}$. Since $p(b(0)) = 0$, by point (v) of Lemma 3.5, we get $b(0) = 0$. □

3.3. *Technical definitions*

Let $m, n \geqslant 1$ be fixed integers and set $N := (m+1)(n+1) - 1$. Let $A_0, \ldots, A_m, B_1, \ldots, B_n$ be indeterminates and let $\mathbb{C}[B]$ (respectively, $\mathbb{C}[A, B]$) be the polynomial algebra generated by the B_j (respectively, by the A_i, B_j). We grade these polynomial algebras by assigning A_i, B_i to be homogeneous of degree i.

We now successively define homogeneous polynomials:

(i) $C_i \in \mathbb{C}[A, B]$ of degree i for $1 \leqslant i \leqslant N$;
(ii) $U_{i,j} \in \mathbb{C}[B]$ of degree $i - j$ for $1 \leqslant i \leqslant N$, $0 \leqslant j \leqslant m$;
(iii) $W_{i,j} \in \mathbb{C}[B]$ of degree $i - j$ for $1 \leqslant i, j \leqslant m$;
(iv) $D_i \in \mathbb{C}[B]$, $E_i \in \mathbb{C}[A, B]$ of degree i for $1 \leqslant i \leqslant m$;
(v) $F_i \in \mathbb{C}[B]$, $G_i \in \mathbb{C}[A, B]$ of degree i for $1 \leqslant i \leqslant N$.

Set $A(X) := \sum_{0 \leqslant i \leqslant m} A_i X^{i+1}$ and $B(X) := X + \sum_{1 \leqslant i \leqslant n} B_i X^{i+1}$. Then the polynomials C_i, D_i, E_i, F_i, G_i, $U_{i,j}$, $W_{i,j}$ satisfy the following points:

(0) $\sum_{1 \leqslant i \leqslant N} C_i X^{i+1} = A \circ B(X) - A_0 X$;
(1) $A_i = A_0 D_i + E_i$ for $1 \leqslant i \leqslant m$;
(2) $C_i = A_0 F_i + G_i$ for $1 \leqslant i \leqslant N$;
(3) $E_i = \sum_{1 \leqslant j \leqslant m} W_{i,j} C_j$ for $1 \leqslant i \leqslant m$;

(4) $G_i := \sum_{1\leqslant j\leqslant m} U_{i,j}E_j$ for $1 \leqslant i \leqslant N$;
(5) the matrices $(U_{i,j})_{1\leqslant i,j\leqslant m}$ and $(W_{i,j})_{1\leqslant i,j\leqslant m}$ are inverses of one another.

Later on, we will only use the above points (0)–(5) so that the reader in a hurry can skip the following trivial, but technical, construction.

3.3.1. *Construction*
(i) The polynomials C_i of $\mathbb{C}[A, B]$ are uniquely defined by (0).
(ii) The polynomials $U_{i,j}$ of $\mathbb{C}[B]$ are uniquely defined by $C_i = \sum_{0\leqslant j\leqslant m} U_{i,j}A_j$ $(1 \leqslant i \leqslant N)$.
Matricially, this may be written $\mathsf{C} = \mathsf{U}\cdot\mathsf{A}$, where A (respectively, C) is the column vector $\mathsf{A} = (A_i)_{0\leqslant i\leqslant m}$ (respectively, $\mathsf{C} = (C_i)_{1\leqslant i\leqslant N}$) and U is the matrix $\mathsf{U} = (U_{i,j})_{\substack{1\leqslant i\leqslant N\\ 0\leqslant j\leqslant m}}$.
We have $\mathsf{U} = \begin{pmatrix} \mathsf{B} & \mathsf{V} \\ & * \end{pmatrix}$ where B is the column vector $\mathsf{B} = {}^t(B_1, \ldots, B_n, 0, \ldots, 0)$ and $\mathsf{V} := (U_{i,j})_{1\leqslant i,j\leqslant m}$ is a lower triangular square matrix whose diagonal entries are equal to 1.
(iii) The matrix $\mathsf{W} = (W_{i,j})_{1\leqslant i,j\leqslant m}$ is defined as the inverse of V.
(iv) The column vectors $\mathsf{D} = (D_i)_{1\leqslant i\leqslant m}$ and $\mathsf{E} = (E_i)_{1\leqslant i\leqslant m}$ are defined by $\mathsf{D} := -\mathsf{W}\cdot\tilde{\mathsf{B}}$ and $\mathsf{E} := \mathsf{W}\cdot\tilde{\mathsf{C}}$, where $\tilde{\mathsf{B}}$ (respectively, $\tilde{\mathsf{C}}$) denotes the column vector obtained from B (respectively, C) by keeping the first m rows. Since the column vector $\tilde{\mathsf{A}} := {}^t(A_1, \ldots, A_m)$ satisfies $\tilde{\mathsf{C}} = A_0\tilde{\mathsf{B}} + \mathsf{V}\cdot\tilde{\mathsf{A}}$, we get $\tilde{\mathsf{A}} = -A_0\mathsf{W}\cdot\tilde{\mathsf{B}} + \mathsf{W}\cdot\tilde{\mathsf{C}}$, that is, $\tilde{\mathsf{A}} = A_0\mathsf{D} + \mathsf{E}$, which is assertion (1).
(v) The polynomials F_i, G_i are defined by

$$F_i := U_{i,0} + \sum_{1\leqslant j\leqslant m} U_{i,j}D_j \quad \text{and} \quad G_i := \sum_{1\leqslant j\leqslant m} U_{i,j}E_j, \quad 1 \leqslant i \leqslant N.$$

Since $C_i = \sum_{0\leqslant j\leqslant m} U_{i,j}A_j$ for $1 \leqslant i \leqslant N$ and $A_j = A_0D_j + E_j$ for $1 \leqslant j \leqslant m$, we get $C_i = U_{i,0}A_0 + \sum_{1\leqslant j\leqslant m} U_{i,j}(A_0D_j + E_j)$ and (2) follows. Assertions (3)–(5) are obvious.

REMARK 3.10. We have $G_i = \sum_{1\leqslant j,k\leqslant m} U_{i,j}W_{j,k}C_k$, so that if $i \leqslant m$, then we get $G_i = C_i$ and $F_i = 0$.

We will always use the conjecture $R(m, n)$ through the next statement.

LEMMA 3.11. *If the conjecture $R(m, n)$ holds, then the family $(F_{m+i})_{1\leqslant i\leqslant n}$ is an hsop of $\mathbb{C}[B]$.*

Proof. If $\bar{b} = (\bar{b}_1, \ldots, \bar{b}_n) \in \mathbb{C}^n$ satisfies $F_{m+i}(\bar{b}) = 0$ for $1 \leqslant i \leqslant n$, then we want to show $\bar{b} = 0$. Let us set $\bar{a}_0 = 1$, $\bar{a}_i = D_i(\bar{b})$ for $1 \leqslant i \leqslant m$ and $\bar{a} = (\bar{a}_0, \ldots, \bar{a}_m)$. By assertion (1) evaluated at $(\bar{a}, \bar{b})$, we get $E_i(\bar{a}, \bar{b}) = 0$ for $1 \leqslant i \leqslant m$. By (3) and (5), we get $C_i(\bar{a}, \bar{b}) = 0$ for $1 \leqslant i \leqslant m$. Moreover, for any $m + 1 \leqslant i \leqslant m + n$, we get $G_i(\bar{a}, \bar{b}) = 0$ by (4), so $C_i(\bar{a}, \bar{b}) = 0$ by (2) and by Remark 3.10. For the moment, we have proved $C_i(\bar{a}, \bar{b}) = 0$ for $1 \leqslant i \leqslant m + n$.
Let us set $a := X(1 + \bar{a}_1X + \cdots + \bar{a}_mX^m)$, $b := X(1 + \bar{b}_1X + \cdots + \bar{b}_nX^n)$ and $c := a \circ b = X(1 + \bar{c}_1X + \cdots + \bar{c}_NX^N) \in \mathbb{C}[X]$, where the $\bar{c}_i \in \mathbb{C}$. Using relation (0), we get $\bar{c}_i = C_i(\bar{a}, \bar{b}) = 0$ for $1 \leqslant i \leqslant m + n$. By $R(m, n)$, we get $a = b = X$. □

We finish this subsection with a technical result to be used in Lemma 4.1. Even if the proof relies on the above technical definitions, the statement itself is self-contained.

LEMMA 3.12. *Set $a(X) = \sum_{0\leqslant i\leqslant m} a_iX^{i+1}$ and $b(X) = X + \sum_{1\leqslant i\leqslant n} b_iX^{i+1}$, where a_i, b_i are elements of $\mathbb{C}((T))$. Set $c(X) = a \circ b(X) - a_0X$. Note that $a(X), b(X)$ and $c(X)$ are* *elements of* $\mathbb{C}((T))[X]$*. Assume* $\lim_{T\to 0} b = X$ *and* $\lim_{T\to 0} c = p \in \mathbb{C}[X]$*. Then, if the conjecture* $R(m,n)$ *holds, we get* $\deg p \leqslant m+n+1$.

Proof. If $\mathsf{a} := (a_0, \dots, a_m)$ and $\mathsf{b} := (b_1, \dots, b_n)$, then we have $c(X) = \sum_{1\leqslant i\leqslant N} C_i(\mathsf{a},\mathsf{b}) X^{i+1}$ by relation (0). Since $\lim_{T\to 0} b = X$, all the b_i belong to $T\mathbb{C}[\![T]\!]$ and since $\lim_{T\to 0} c$ exists, all the $C_i(\mathsf{a},\mathsf{b})$ belong to $\mathbb{C}[\![T]\!]$.

Claim 3.13. If $m+1 \leqslant i \leqslant N$, then $G_i(\mathsf{a},\mathsf{b}) \in T\mathbb{C}[\![T]\!]$.
If $1 \leqslant j,k \leqslant m$, then we have the following facts.

(i) The element $U_{i,j}(\mathsf{b})$ belongs to $T\mathbb{C}[\![T]\!]$ since $U_{i,j} \in \mathbb{C}[B]$ is homogeneous of degree $i-j \geqslant m+1-m = 1$.
(ii) The element $W_{j,k}(\mathsf{b})$ belongs to $\mathbb{C}[\![T]\!]$ since $W_{j,k}$ is a polynomial.
(iii) The element $C_k(\mathsf{a},\mathsf{b})$ belongs to $\mathbb{C}[\![T]\!]$.

By (5) or Remark 3.10, we have $G_i = \sum_{1\leqslant j,k\leqslant m} U_{i,j}W_{j,k}C_k$ and the claim follows.
By (2), we have $C_i(\mathsf{a},\mathsf{b}) = a_0F_i(\mathsf{b}) + G_i(\mathsf{a},\mathsf{b})$.
If $i \geqslant m+1$, then we have $\operatorname{val} C_i(\mathsf{a},\mathsf{b}) \geqslant 0$ and $\operatorname{val} G_i(\mathsf{a},\mathsf{b}) \geqslant 1$, so $\operatorname{val} a_0F_i(\mathsf{b}) \geqslant 0$. We want to show $\operatorname{val} C_i(\mathsf{a},\mathsf{b}) \geqslant 1$, when $i > m+n$. For this, it is sufficient to show $\operatorname{val} a_0F_i(\mathsf{b}) \geqslant 1$. By Lemmas 3.8 and 3.11, if $i > m+n$, then we have

$$\operatorname{val} F_i(\mathsf{b}) \geqslant \min_{1\leqslant j\leqslant n} \operatorname{val} F_{m+j}(\mathsf{b}) + 1,$$

so $\operatorname{val} a_0F_i(\mathsf{b}) \geqslant \min_{1\leqslant j\leqslant n} \operatorname{val} a_0F_{m+j}(\mathsf{b}) + 1 \geqslant 1$. □

3.4. *An easy lemma on the multidegree*

Lemma 3.14. *Let* K *be any field containing* $\mathbb{C}$ *and let* $f = (f_1, f_2)$ *be an automorphism of* $\mathbb{A}^2_K$ *of multidegree* $(d_1, \dots, d_k)$ *with* $k \geqslant 1$*; then we have* $\deg f_1 = d_1 \cdots d_k$ *or* $d_2 \cdots d_k$ *(and the same holds for* f_2*). Furthermore, if* $\deg f_1 = d_1 \cdots d_k$*, then there exists a unique scalar* $\lambda \in K$ *such that* $\deg(f_2 - \lambda f_1) < d_1 \cdots d_k$*, or equivalently such that* $\deg(f_2 - \lambda f_1) = d_2 \cdots d_k$.

Proof. By definition of the multidegree, f admits a reduced expression

$$f = \alpha_1 \circ \beta_1 \circ \cdots \circ \alpha_k \circ \beta_k \circ \alpha_{k+1},$$

where each α_j is affine and each β_j is triangular of degree d_j. Set

$$g := \alpha_1^{-1} \circ f = \beta_1 \circ \alpha_2 \circ \cdots \circ \alpha_k \circ \beta_k \circ \alpha_{k+1}.$$

An easy induction would establish that $\deg g_1 = d_1 \cdots d_k$ and $\deg g_2 = d_2 \cdots d_k$ (for details, see [**8**, Proof of Proposition 1, p. 606]). The result follows. □

4. *Proof of Theorem* B

In this section, $m, n \geqslant 1$ are fixed integers and we assume that the conjecture $R(m,n)$ holds. We set $d = (d_1, d_2) = (m+1, n+1)$ and we want to show $\bar{\mathcal{G}}_d = \bigcup_{e\preceq d} \mathcal{G}_e$.

Subsection 4.1 is devoted to the proof of the first inclusion $\bar{\mathcal{G}}_d \subseteq \bigcup_{e\preceq d} \mathcal{G}_e$. It only relies on the self-contained Lemma 3.12. Subsection 4.2 is devoted to the proof of the second inclusion $\bigcup_{e\preceq d} \mathcal{G}_e \subseteq \bar{\mathcal{G}}_d$. It is a little more involved, since it uses the polynomials C_i, D_i, E_i, F_i, G_i defined in Subsection 3.3.

4.1. *The first inclusion*

If $f \in \bar{\mathcal{G}}_d$, let us show $f \in \bigcup_{e \preceq d} \mathcal{G}_e$. By [9], the length is a lower semicontinuous function on $\mathcal{G}$ so that the length of f satisfies $l \leqslant 2$. We will consider three cases.

(i) We have $l = 0$. There is nothing to show.
(ii) We have $l = 1$. We conclude by Lemma 4.1.
(iii) We have $l = 2$. We conclude by Lemma 4.2.

LEMMA 4.1. *If* $e \geqslant 2$ *and* $\mathcal{G}_{(e)} \cap \bar{\mathcal{G}}_{(d_1,d_2)} \neq \emptyset$, *then* $e < d_1 + d_2$.

Proof. If $f \in \mathcal{G}_{(e)} \cap \bar{\mathcal{G}}_d$, let us prove $e < d_1 + d_2$. Since $\mathcal{A} \circ f \circ \mathcal{A} \subseteq \bar{\mathcal{G}}_d$, we may assume $f = (X + p(Y), Y)$ with $\deg p = e$. If $e \leqslant d_2$, then there is nothing to prove. So, let us assume $e > d_2$. By Corollary 3.3, there exists $g = (g_1, g_2) \in \mathcal{G}_d(\mathbb{C}((T)))$ such that $f = \lim_{T\to 0} g_T$. By Lemma 3.14, we must have $\deg g_1 = d_1 d_2$.

First claim. We may assume $\deg g_2 = d_2$.

Indeed, since g is of multidegree (d_1, d_2), by Lemma 3.14, there exists a unique $\lambda \in \mathbb{C}((T))$ such that $\deg(g_2 - \lambda g_1) = d_2$. It is enough to show $\mathrm{val}(\lambda) > 0$, because we can then replace g by $(u_1, u_2) := (g_1, g_2 - \lambda g_1)$. Let μ (respectively, ν) $\in \mathbb{C}((T))$ be the Y^e-coefficient of g_2 (respectively, u_1). Applying the equality $g_2 = u_2 + \lambda u_1$ to the Y^e-coefficient, we get $\mu = \lambda\nu$ (since we have assumed $e > d_2$). However, we have $\mathrm{val}(\nu) = 0$ (since $\lim_{T\to 0} u_1(T) = X + p(Y)$ and $\deg p = e$), so $\mathrm{val}(\lambda) = \mathrm{val}(\mu) > 0$ and the claim is proved.

Since $\deg g_1 > \deg g_2$, it is well known (see, for example, [8, Theorem 1.i]) that we can write (in a unique way)

$$g = \tau \circ t_1 \circ \sigma \circ t_2 \circ l,$$

where $\tau = (X + a, Y + b)$ is a translation, $t_1 = (X + \sum_{0\leqslant i\leqslant m} a_i Y^{i+1}, Y)$, $t_2 = (X + \sum_{1\leqslant i\leqslant n} b_i Y^{i+1}, Y)$ are triangular automorphisms, $\sigma = (Y, X)$ and $l = (l_1, l_2) = (\alpha X + \beta Y, \gamma X + \delta Y)$ are linear automorphisms, with the coefficients a_i, b_i, $a, b, \alpha, \beta, \gamma, \delta$ belonging to $\mathbb{C}((T))$. By making the composition, we get

$$g = \left(l_2 + \sum_{0\leqslant i\leqslant m} a_i \left[l_1 + \sum_{1\leqslant j\leqslant n} b_j l_2^{j+1} \right]^{i+1} + a, l_1 + \sum_{1\leqslant j\leqslant n} b_j l_2^{j+1} + b \right).$$

Since $f = \lim_{T\to 0} g_T$, by looking at the constant terms, we get $a, b \in \mathbb{C}[\![T]\!]$ and there is no restriction to assume $a = b = 0$.

Second claim. We may assume $l = (l_1, l_2) = (Y, X + \rho Y)$ for some $\rho \in \mathbb{C}((T))$.

Note that $\lim_{T\to 0} \alpha = 0$, $\lim_{T\to 0} \beta = 1$ and $\lim_{T\to 0} \alpha\delta - \beta\gamma = -1$. The last relation comes from the Jacobian equality $\mathrm{Jac}\, g = \mathrm{Jac}\, \sigma \times \mathrm{Jac}\, l = -(\alpha\delta - \beta\gamma)$. Set $\rho := (\delta - \alpha)/\beta$. Since $(l_1, l_2) = (Y, X + \rho Y) \circ h_T$, where $h_T := (-\rho l_1 + l_2, l_1)$, it is enough to show $\lim_{T\to 0} h_T = (X, Y)$. For the second component, it is clear. For the first, we have $-\rho l_1 + l_2 = (\gamma - \rho\alpha)X + (\delta - \rho\beta)Y$. But $\gamma - \rho\alpha = \alpha^2/\beta - (\alpha\delta - \beta\gamma)/\beta$, so $\lim_{T\to 0} \gamma - \rho\alpha = 1$ and $\delta - \rho\beta = \alpha$, so $\lim_{T\to 0} \delta - \rho\beta = 0$ and the claim is proved.

So, we can now assume

$$g = \left(X + \rho Y + \sum_{0\leqslant i\leqslant m} a_i \left[Y + \sum_{1\leqslant j\leqslant n} b_j (X + \rho Y)^{j+1} \right]^{i+1}, Y + \sum_{1\leqslant j\leqslant n} b_j (X + \rho Y)^{j+1} \right).$$

Inspecting the Y-powers, the relation $\lim_{T\to 0} g_T = (X + p(Y), Y)$ gives us

$$\lim_{T\to 0} \rho Y + \sum_{0\leqslant i\leqslant m} a_i \left[Y + \sum_{1\leqslant j\leqslant n} b_j \rho^{j+1} Y^{j+1} \right]^{i+1} = p(Y) \quad \text{and} \quad \lim_{T\to 0} \sum_{1\leqslant j\leqslant n} b_j \rho^{j+1} Y^{j+1} = 0.$$

Setting $\tilde{b}_j := b_j \rho^{j+1}$, we get

$$\lim_{T\to 0} \rho Y + \sum_{0\leqslant i\leqslant m} a_i \left[Y + \sum_{1\leqslant j\leqslant n} \tilde{b}_j Y^{j+1} \right]^{i+1} = p(Y) \quad \text{and} \quad \lim_{T\to 0} \sum_{1\leqslant j\leqslant n} \tilde{b}_j Y^{j+1} = 0.$$

Looking at the Y-coefficient, the first relation shows us $\lim_{T\to 0} \rho + a_0 = p_1$, where p_1 is the Y-coefficient of $p(Y)$.

Therefore, $\lim_{T\to 0} \sum_{0\leqslant i\leqslant m} a_i[Y + \sum_{1\leqslant j\leqslant n} \tilde{b}_j Y^{j+1}]^{i+1} - a_0 Y = p(Y) - p_1 Y$ and Lemma 3.12 tells us $\deg(p(Y) - p_1 Y) \leqslant m + n + 1 = d_1 + d_2 - 1$. □

Lemma 4.2. *If $\mathcal{G}_{(e_1,e_2)} \cap \bar{\mathcal{G}}_{(d_1,d_2)} \neq \emptyset$, then $e_1 \leqslant d_1$ and $e_2 \leqslant d_2$.*

This lemma is a consequence of the following result, which is [11, Theorem C].

Theorem 4.3. *If $u = (u_1, \ldots, u_l)$, $v = (v_1, \ldots, v_l)$ are two multidegrees with the same length, then the following assertions are equivalent*:

(i) *We have $\mathcal{G}_u \subseteq \bar{\mathcal{G}}_v$;*
(ii) *We have $\mathcal{G}_u \cap \bar{\mathcal{G}}_v \neq \emptyset$;*
(iii) *We have $u_i \leqslant v_i$ for each i (that is, $u \preceq v$).*

However, here is a simple proof of Lemma 4.2.

Proof of Lemma 4.2. Let V be the polynomial algebra $\mathbb{C}[X, Y]$. Any element v of V admits a unique expression $v = \sum_{i\geqslant 0} v_i$, where v_i is homogeneous of degree i. Let $\Pi_{>k} \colon V \to V$ be the projection sending v to $\sum_{i>k} v_i$. Two polynomials $u, v \in V$ are linearly dependent if and only if $u \wedge v = 0$ in $\bigwedge^2 V$. The key point is the fact that for each $f = (f_1, f_2) \in \mathcal{G}_{(d_1,d_2)}$, we have

$$d_2 = \min\{k, \Pi_{>k}(f_1) \wedge \Pi_{>k}(f_2) = 0\}.$$

In particular, we have $\Pi_{>d_2}(f_1) \wedge \Pi_{>d_2}(f_2) = 0$ and this condition still holds if we only assume that f belongs to $\bar{\mathcal{G}}_{(d_1,d_2)}$. Therefore, if $f \in \mathcal{G}_{(e_1,e_2)} \cap \bar{\mathcal{G}}_{(d_1,d_2)}$, then we get $e_2 \leqslant d_2$. The map $g \mapsto g^{-1}$ being an automorphism of (the infinite-dimensional algebraic variety) $\mathcal{G}$ sending an automorphism of multidegree $(u_1, \ldots, u_l)$ to an automorphism of multidegree $(u_l, \ldots, u_1)$, we also have $f^{-1} \in \mathcal{G}_{(e_2,e_1)} \cap \bar{\mathcal{G}}_{(d_2,d_1)}$ so $e_1 \leqslant d_1$. □

4.2. *The second inclusion*

Let us show $\mathcal{G}_e \subseteq \bar{\mathcal{G}}_d$ for any $e \preceq d$.

If e is of length 0 or 2, then it is easy. For the sake of completeness, let us prove it by using [8, Section 4] (see also [11, Section 7.2]). Indeed, we define the partial order $\leqslant$ on the set of multidegrees in the following way. If $u = (u_1, \ldots, u_k)$ and $v = (v_1, \ldots, v_l)$ are multidegrees, then we say $u \leqslant v$ if $k \leqslant l$ and if there exists a finite sequence $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant l$ such that $u_j \leqslant v_{i_j}$ for $1 \leqslant j \leqslant k$. By [8, Section 4], the inequality $u \leqslant v$ implies the inclusion $\mathcal{G}_u \subseteq \bar{\mathcal{G}}_v$. If e is of length 2, then the inclusion $\mathcal{G}_e \subseteq \bar{\mathcal{G}}_d$ is also a consequence of Theorem 4.3. Therefore, for showing the second inclusion, it is enough to prove the next result.

LEMMA 4.4. *If $2 \leqslant e < d_1 + d_2$, then the following inclusion holds*: $\mathcal{G}_{(e)} \subseteq \bar{\mathcal{G}}_{(d_1,d_2)}$.

Proof. It is sufficient to show $(X + \sum_{1\leqslant i\leqslant m+n} \gamma_i Y^{i+1}, Y) \in \bar{\mathcal{G}}_d$ for any $\gamma_i \in \mathbb{C}$.

We take again the notation of Subsection 3.3. By Lemmas 3.9 and 3.11, there exist $q \geqslant 1$ and $\tilde{b} := (\tilde{b}_1, \dots, \tilde{b}_n) \in \mathbb{A}^n_{\mathbb{C}[\![T]\!]}$ such that $\tilde{b}(0) = 0$ and

$$T^{-q} F_{m+i}(\tilde{b}) = \gamma_{m+i}, \quad 1 \leqslant i \leqslant n.$$

Set $a_0 := T^{-q}$, $a_i := a_0 D_i(\tilde{b})$ for $1 \leqslant i \leqslant m$, $\rho := -T^{-q}$ and $b_j := \tilde{b}_j \rho^{-j-1}$ for $1 \leqslant j \leqslant n$. Set $g := t_1 \circ \sigma \circ t_2 \circ l$, where

$$t_1 = \left(X + \sum_{0\leqslant i\leqslant m} a_i Y^{i+1}, Y \right), \quad t_2 = \left(X + \sum_{1\leqslant i\leqslant n} b_i Y^{i+1}, Y \right)$$

are triangular automorphisms, $\sigma = (Y, X)$ and $l = (Y, X + \rho Y)$. We have

$$g = \left(X + \rho Y + \sum_{0\leqslant i\leqslant m} a_i \left[Y + \sum_{1\leqslant j\leqslant n} b_j (X + \rho Y)^{j+1} \right]^{i+1}, Y + \sum_{1\leqslant j\leqslant n} b_j (X + \rho Y)^{j+1} \right).$$

CLAIM 4.5. We have $\lim_{T\to 0} g_T = (X + \sum_{m+1\leqslant i\leqslant m+n} \gamma_i Y^{i+1}, Y)$.

Let us begin by showing $\lim_{T\to 0} g_2 = Y$.

For $1 \leqslant j \leqslant n$, we have $b_j (X + \rho Y)^{j+1} = \tilde{b}_j (\rho^{-1} X + Y)^{j+1}$, where $\lim_{T\to 0} \tilde{b}_j = \lim_{T\to 0} \rho^{-1} = 0$, so $\lim_{T\to 0} b_j (X + \rho Y)^{j+1} = 0$ and the result is clear.

Let us now deal with the first component $g_1 = X + \rho Y + \sum_{0\leqslant i\leqslant m} a_i g_2^{i+1}$.

First step. Let us show that in this last expression of g_1, the limit of g_1 is unchanged if we replace g_2 by

$$p := Y + \sum_{1\leqslant j\leqslant n} \tilde{b}_j Y^{j+1}.$$

It is sufficient to check $\lim_{T\to 0} a_i (g_2^{i+1} - p^{i+1}) = 0$.

As $\lim_{T\to 0} g_2 = \lim_{T\to 0} p = Y$ and $g_2^{i+1} - p^{i+1} = (g_2 - p)(g_2^i + \cdots + p^i)$, we will only check $\lim_{T\to 0} a_i (g_2 - p) = 0$.

Since $g_2 - p = \sum_{1\leqslant j\leqslant n} b_j [(X + \rho Y)^{j+1} - (\rho Y)^{j+1}]$, it is enough to show

$$\lim_{T\to 0} a_i b_j [(X + \rho Y)^{j+1} - (\rho Y)^{j+1}] = 0.$$

As $\lim_{T\to 0} ((X + \rho Y)^{j+1} - (\rho Y)^{j+1} / (j+1) X Y^j \rho^j) = 1$, we will only show $\lim_{T\to 0} a_i b_j \rho^j = 0$.

It is clear, because $a_i b_j \rho^j = -(a_i / a_0) \tilde{b}_j$ where $\lim_{T\to 0} \tilde{b}_j = 0$ and $\lim_{T\to 0} (a_i/a_0) = 1$ (respectively, 0) if $i = 0$ (respectively, $i \geqslant 1$).

Second step. Let us show $\lim_{T\to 0} c = \sum_{m+1\leqslant i\leqslant m+n} \gamma_i Y^{i+1}$, where

$$c := \rho Y + \sum_{0\leqslant i\leqslant m} a_i p^{i+1} = -a_0 Y + \sum_{0\leqslant i\leqslant m} a_i p^{i+1}.$$

If $\mathsf{a} := (a_0, \dots, a_m)$, then by the relation (0) we have $c = \sum_{1\leqslant i\leqslant N} C_i(\mathsf{a}, \tilde{b}) Y^{i+1}$.

We get $E_j(\mathsf{a}, \tilde{b}) = 0$ for $1 \leqslant j \leqslant m$ by (1), so $G_i(\mathsf{a}, \tilde{b}) = 0$ for $1 \leqslant i \leqslant N$ by (4) and $C_i(\mathsf{a}, \tilde{b}) = a_0 F_i(\tilde{b}) = T^{-q} F_i(\tilde{b})$ for $1 \leqslant i \leqslant N$ by (2). Therefore, we have the following results:

(i) $C_i(\mathsf{a}, \tilde{b}) = 0$ for $1 \leqslant i \leqslant m$ (see Remark 3.10);
(ii) $\lim_{T\to 0} C_{m+i}(\mathsf{a}, \tilde{b}) = \lim_{T\to 0} T^{-q} F_{m+i}(\tilde{b}) = \gamma_{m+i}$ for $1 \leqslant i \leqslant n$;

(iii) $\lim_{T\to 0} C_i(\mathsf{a},\tilde{b}) = 0$ for $i > m+n$, since $\operatorname{val} F_i(\tilde{b}) \geqslant \min_{1\leqslant j\leqslant n} \operatorname{val} F_{m+j}(\tilde{b}) + 1 \geqslant q+1$ (by Lemma 3.8).

This proves the second step and the claim follows.

If we now set $f := t \circ g$, where $t := (X + \sum_{1\leqslant i\leqslant m} \gamma_i Y^{i+1}, Y)$ is a triangular automorphism, then $f \in \mathcal{G}_d(\mathbb{C}((T)))$ and $\lim_{T\to 0} f_T = (X + \sum_{1\leqslant i\leqslant m+n} \gamma_i Y^{i+1}, Y)$. We conclude by Corollary 3.3. □

5. *Proof of Theorem* D

We begin with the following lemma.

LEMMA 5.1. *Let $d \geqslant 2$ be an integer. Then the subset $\mathcal{G}_{(d)}$ of the group $\mathcal{G}$ is a double coset modulo the affine subgroup $\mathcal{A}$ if and only if $d = 2$ or 3.*

Proof. Let us define the equivalence relation $\sim$ on $\mathcal{G}$ by
for all $f, g \in \mathcal{G}$, $f \sim g \Leftrightarrow f$ and g have the same double coset modulo $\mathcal{A}$, that is, $\mathcal{A}\circ f \circ \mathcal{A} = \mathcal{A} \circ g \circ \mathcal{A}$.

Let us note that any automorphism in the double coset $\mathcal{A}\circ f\circ\mathcal{A}$ has the same degree as f. Let $p(Y), q(Y) \in \mathbb{C}[Y]$ be any polynomials.

CLAIM 5.2. *The following assertions are equivalent.*
(i) *We have $(X + p(Y), Y) \sim (X + q(Y), Y)$.*
(ii) *There exists $\alpha, \beta, \gamma \in \mathbb{C}$ with $\alpha\beta \neq 0$ such that $q''(Y) = \alpha p''(\beta Y + \gamma)$.*

Proof. There is no restriction to assume $\deg p = \deg q \geqslant 2$.

(i) $\Rightarrow$ (ii). If $(X + p(Y), Y) \sim (X + q(Y), Y)$, then there exist affine automorphisms u and v such that $(X + q(Y), Y) = u \circ (X + p(Y), Y) \circ v$. If $u = (aX + bY + c, dX + eY + f)$ and $v = (\tilde{a}X + \tilde{b}Y + \tilde{c}, \tilde{d}X + \tilde{e}Y + \tilde{f})$, then we get

$$X + q(Y) = ap(\tilde{d}X + \tilde{e}Y + \tilde{f}) + a(\tilde{a}X + \tilde{b}Y + \tilde{c}) + b(\tilde{d}X + \tilde{e}Y + \tilde{f}) + c.$$

Setting $X = 0$ and differentiating twice with respect to Y, we get $q''(Y) = \alpha p''(\beta Y + \gamma)$, where we have set $\alpha := a(\tilde{e})^2$, $\beta := \tilde{e}$ and $\gamma := \tilde{f}$. Since $q'' \neq 0$, it is clear $\alpha \neq 0$. If $\deg p = \deg q \geqslant 3$, then we have $\beta \neq 0$ for reason of degrees. If $\deg p = \deg q = 2$, then we can assume $\beta \neq 0$.

(ii) $\Rightarrow$ (i). If $q''(Y) = \alpha p''(\beta Y + \gamma)$, then, by integrating twice, we get $aq(Y) + bY + c = p(dY + e)$, where $a, b, c, d, e \in \mathbb{C}$ and $ad \neq 0$. It follows that

$$(X + p(Y), Y) \circ (aX, dY + e) = (aX + bY + c, dY + e) \circ (X + q(Y), Y)$$

and finally $(X + p(Y), Y) \sim (X + q(Y), Y)$. The claim is proved. □

If $d = 2$ or 3 and $\deg p = \deg q = d$, then the claim implies that $(X + p(Y), Y) \sim (X + q(Y), Y)$. Therefore, in this case, $\mathcal{G}_{(d)}$ is a double coset modulo $\mathcal{A}$.

If $d \geqslant 4$, let us set $p(Y) = Y^d$ and $q(Y) = Y^d + Y^{d-1}$. There does not exist any $\alpha, \beta, \gamma \in \mathbb{C}$ such that $q''(Y) = \alpha p''(\beta Y + \gamma)$, because the polynomial $p''(\beta Y + \gamma)$ has a unique root and this is not the case for the polynomial $q''(Y)$. By the claim, it follows that the automorphisms $(X + p(Y), Y)$ and $(X + q(Y), Y)$ have distinct double coset modulo $\mathcal{A}$. In particular, $\mathcal{G}_{(d)}$ is not a double coset modulo $\mathcal{A}$. □

Let us now prove Theorem D.

If $\mathcal{H}$ is a subgroup of $\mathcal{G}$ as in Theorem D, let us show $\mathcal{H} = \mathcal{G}$. By the hypothesis, $\mathcal{H}$ contains a triangular automorphism $f = (X + p(Y), Y)$ with $\deg p \geqslant 2$. If we set $g_\alpha := (X, Y + \alpha) \in \mathcal{H}$ ($\alpha \in \mathbb{C}$), then the commutator $[f, g_\alpha] := f \circ g_\alpha \circ f^{-1} \circ g_\alpha^{-1} \in \mathcal{H}$ is equal to $(X + q(Y), Y)$, where $q(Y) := p(Y) - p(Y - \alpha)$. If α is well chosen, then one may assume $\deg q = \deg p - 1$. Therefore, by a decreasing induction, we see that $\mathcal{H}$ contains a triangular automorphism of degree 2.

Since $\mathcal{G}_{(2)}$ is a double coset modulo $\mathcal{A}$ by Lemma 5.1, we get $\mathcal{G}_{(2)} \subseteq \mathcal{H}$. By induction, we get $\mathcal{G}_{(d)} \subseteq \mathcal{H}$ for any $d \geqslant 2$. Indeed, if $\mathcal{G}_{(d)} \subseteq \mathcal{H}$, then we get $\mathcal{G}_{(d,2)} \subseteq \mathcal{H}$, so $\bar{\mathcal{G}}_{(d,2)} \subseteq \bar{\mathcal{H}} = \mathcal{H}$ and $\mathcal{G}_{(d+1)} \subseteq \bar{\mathcal{G}}_{(d,2)}$ by Theorem C. Since $\mathcal{H}$ contains all $\mathcal{G}_{(d)}$, $d \geqslant 2$, it is now clear that $\mathcal{H} = \mathcal{G}$.

Acknowledgements. I would like to thank Michel Brion and Laurent Manivel for their generous help.

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