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DEFINING RELATIONS FOR THE CREMONA GROUP OF THE PLANE

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ABSTRACT. By methods of the geometry of rational surfaces and the topology of graphs and cell complexes connected with them, the author establishes defining relations, connecting projectives, and quadratic transformations, for the group of birational transformations of the plane over an algebraically closed field.

Bibliography: 11 titles.

Introduction

At the end of the fourth chapter of the book [1] it is remarked that "Noether has proved a theorem to the effect that a birational automorphism of the plane can be represented as the product of quadratic transformations and a projective transformation.... The interesting question of the extent to which the representation of a birational automorphism in terms of quadratic ones is unique has not yet been investigated." The results of the present paper enable one to describe the defining relations of the group Cr of birational transformations of the plane (over an algebraically closed field), this group being generated by the union $\mathcal{P} \cup \mathcal{Q}$ of the set $\mathcal{P}$ of projective transformations and the set $\mathcal{Q}$ of quadratic transformations (i.e. those that map a generic line to a conic). It turns out that any relation is deducible from relations of the form $g_1 g_2 g_3 = 1$, where $\{g_1, g_2, g_3\} \subset \mathcal{P} \cup \mathcal{Q}$. This assertion, a reformulation of Theorem 10.7 proved in [10], is inferred from some properties of rational surfaces; more precisely, from simple topological properties (namely, the connectivity and 1-connectivity) of the cell complex $\Delta(V)$ associated to a rational surface V. The use of such complexes in my view accords well both with the essence of the problem and with the geometrical methods of combinatorial group theory (see Chapter 3 of [11]). Of course, with respect to the latter methods this work could have been done more systematic by examining the inductive limit of the complexes and the action of the Cremona group on the limit, we do not follow that course here, since in any approach to the problem the main work must be carried out at a finite (prelimit) level corresponding to an individual surface.

I began to study the groups of transformations of algebraic surfaces by conbinatorial-topological methods with V. I. Danilov in the process of a joint study of the groups of automorphisms of affine surfaces. I here extend to him my heartfelt gratitude.

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Plan of the paper. In §0 we list some of the notation and terms used. §§1–6 can be considered as supplements to §0, since the main theorems are proved in §§7–10, and in §§1–6 we give a preliminary description of the various constructions.

In §1 we present some information we need on curves contractible to a point. For the most part these are either well known or simple, and so we do not always give proofs. Assertions 1.8 and 1.9 are given in a somewhat stronger form than is used.

In §2 we introduce the so-called exceptional sets (see [2, Chapter IV, §4.7]), that yield birational morphisms of one surface onto another (up to an isomorphism of the dominated surface). Although we mainly need those morphisms whose image is a projective plane (given by the exceptional sets called basic in §3), even in the study of these we must deal with exceptional sets of general form.

In §3 we discuss basic sets (bases, for short). With each pair of bases of a surface there is associated a double coset of the Cremona group by the projective subgroup.

In §4 we associate to a rational surface V the graph $\Gamma_1(V)$ whose vertices are the bases of V and whose edges join two such bases when their associated double coset consists of quadratic transformation. In §4.4 we introduce the criterion (which we often use) for the existence of a basis $\mathbf{M}$ near a given bases $\mathbf{L}$ in $\Gamma_1(V)$ and having a given complement $\mathbf{L}\backslash\mathbf{M}$ in $\mathbf{L}$. In §§4.7–4.9 we classify the surfaces that are minimal in the set of surfaces having a triangle in Γ_1. This classification is not necessary for us to deduce the main results of the paper, but since the triangles in Γ_1 yield defining relations, we need the classification elsewhere for a detailed description of the set of defining relations.

In §5 we discuss the family of bases called de Jonquières sets (or J-sets). Any two representations of such a family determine a de Jonquières transformation (see [9]); more precisely, a double coset in $\mathcal{P}\backslash\mathrm{Cr}/\mathcal{P}$ consisting of such transformations. Essential use of J-sets is made only in §9, but many of the sets of bases considered in §§6–10 are J-sets, so it seemed reasonable to discuss them in §5.

In §6 we meet the surfaces V_n used to resolve triangular automorphisms of degree n of the affine plane. Using such automorphisms, we enlarge $\Gamma_1(V)$ to a graph $\Delta_1(V)$. We discuss the form of the graphs Δ_1 for surfaces in some special series that play an important role in the proof of the 1-connectivity of $\Delta(V)$, since the contraction of a loop $\Delta_1(V)$ to a point is accomplished along cells with one-dimensional skeletons in such graphs.

In §7 we prove Theorem 7.2 on the connectedness of the graph $\Delta_1(V)$. This theorem is slightly stronger than M. Noether's theorem on the generators of the Cremona group. Theorem 7.2 allows one to join any two vertices of $\Delta_1(V)$ by a so-called monotonic path. A monotonic path is an analog of the reduced (or canonical) decompositions used in group theory (see [10]) and in group-theoretical questions of birational geometry (see Chapter 5 of [2], whose last section poses the question of the relations among the generators in Noether's theorem). The proof of Theorem 7.2 makes essential use of ideas of M. Noether and D. Fano as clarified and made precise by V. A. Iskovskih and Ju. I. Manin (see [3] and [4]).

The central result of §8 and of the whole paper is Theorem 8.1 on the 1-connectivity of the complex $\Delta(V)$.

In §9 we prove that a simplicial complex Γ (gotten in §4 from the graph Γ_1 by filling in the one-dimensional skeletons of the simplices) is stably 1-connected in the sense that for any loop ζ in $\Gamma_1(V)$ one can determine a natural number N such that if $f\colon U \to V$ is a blow-up of N points in general position, then we can contract the loop $f^{-1}(\zeta)$ in the complex $\Gamma(U)$ to a point.

In §10 we describe a relation among the loops of the graph Γ_1 and the relations among the generators of the Cremona group. The result of the preceding section on the stable 1-connectivity of Γ enables us to show that all the relations can be deduced from those corresponding to the triangles of Γ_1.

§0. Notations and conventions

0.1. We shall denote the number of elements of a finite set A by $\#A$. For sets A and B, $A \setminus B$ will denote the set of elements of A that do not lie in B, and $A \Delta B$ will denote the symmetric difference of A and B; that is, $A \Delta B = (A \setminus B) \cup (B \setminus A)$. We put $A \setminus B \setminus C = (A \setminus B) \setminus C$.

0.2. Let A be a partially ordered set and B a subset of it. We call B maximal in A if $A \setminus B$ has no elements a with $a \geqslant b$ for some b in B. B is minimal if $A \setminus B$ is maximal.

0.3. Nonnegative integers will be called natural numbers. For natural numbers a and b, $[a, b]$ will denote the set of natural numbers n such that $a \leqslant n \leqslant b$. And δ will denote the function of two arguments defined on any set such that $\delta(x, y) = 0$ if $x \neq y$, $\delta(x, y) = 1$ if $x = y$.

0.4. All the graphs we consider are combinatorial; i.e. they have looped edges, and any two vertices are joined by at most one edge. Γ_0 is the set of vertices of the graph Γ, and $\Gamma_1 = \Gamma$. We also consider partially oriented graphs, i.e. graphs some of whose edges bear arrows. For a subset $A \subset \Gamma_0$, $\Gamma | A$ will denote the subgraph of the graph Γ generated by this subset. A partial orientation of a graph Γ induces a partial orientation of $\Gamma | A$. An edge joining vertices A and B will be denoted by $[A, B]$ or $[B, A]$, the first choice being made if there is an arrow from A to B. A path ζ of length n ($n \geqslant 0$) in a graph will be given by a sequence of vertices $\zeta = [A_0, A_1, \ldots, A_n]$, where for $0 \leqslant i < n$ the vertices A_i and A_{i+1} are joined by the edge $[A_i, A_{i+1}]$, which we shall call an edge of the path ζ. We put $\zeta^{-1}[A_n, \ldots, A_1, A_0]$. Moreover, if $\eta = [B_0, B_1, \ldots, B_m]$ and $A_n = B_0$, then we put

$$\zeta \circ \eta = [A_0, \ldots, A_n, B_1, \ldots, B_m].$$

We shall often consider edge-weighted graphs, i.e. graphs whose edges are endowed with integers. The weight of an edge $[A, B]$ will be denoted by $w[A, B]$.

0.5. A cell complex (or the scheme corresponding to it by 0.5.1 or 0.5.2) will be called a filling of a graph Γ if Γ coincides with the one-dimensional skeleton of the complex. We shall fill in graphs in the following two ways (though in this paper we shall need only the two-dimensional skeletons gotten by these fillings):

0.5.1. *Simplicial filling.* For a graph Γ we form the simplicial scheme whose set of vertices is Γ_0, a simplex being given by a set of vertices $\{A_0, \ldots, A_n\}$ any two of which are joined by an edge. Such a scheme (or its topological realization) is called a simplicial filling of the graph Γ.

0.5.2. *Prismatic filling.* A prism (more precisely, a simplicial prism) is a Cartesian product of simplices. By the one-dimensional skeleton of a prism we shall mean the graph whose set of vertices is the product $A_1 \times \cdots \times A_n$ of nonempty finite sets $A_1, \ldots, A_n$ ($n \geqslant 1$), two vertices $(a_1, \ldots, a_n)$ and $(a'_1, \ldots, a'_n)$ being joined by an edge if $a_k = a'_k$ for all values but one of k in $[1, n]$. Note that if a subgraph of such a graph is isomorphic to the one-dimensional skeleton of a prism, then this subgraph is generated by a subset of the form $B_1 \times \cdots \times B_n$, where $B_i \subset A_i$, $1 \leqslant i \leqslant n$. By the prismatic scheme of a graph Γ we shall mean the set Γ_0 and the collection of all the subsets A of Γ_0 for which $\Gamma | A$ is the one-dimensional skeleton of the prism. It is clear how to reduce a cell complex to the topological realization of the prismatic scheme of a graph. This realization (as well as the scheme itself) will be called a prismatic filling of the graph.

We define the concept of a homotopy (in a fixed filling) of paths of a graph in the standard way: paths ξ and η are simply homotopic if one can write $\xi = \zeta \circ \xi_0 \circ \zeta_2$ and $\eta = \zeta_1 \circ \eta_0 \circ \zeta_2$, where $\xi_0 \circ \eta_0^{-1}$ is a loop lying in the one-dimensional skeleton of some cell of the filling; homotopy is the equivalence relation on the set of paths generated by the relation of simple homotopy. A contractible loop is one that is homotopic to a point.

0.6. The ground field k will be assumed algebraically closed. All surfaces considered are nonsingular, irreducible, complete and *rational*. The symbols U, V and W, possibly with indices, will denote only surfaces. K_V is the canonical class (in $\mathrm{Pic}(V)$) on V. All morphisms of surfaces considered are birational, i.e. are (regular) morphisms inducing an isomorphism of the fields of rational function. In other words, "morphism" always means "birational morphism."

If $f\colon U \to V$ is a morphism, then

f^{-1} is the inverse image operation in the sense of the theory of schemes;

f' means taking the proper preimage;

$f_*(D)$, where D is a divisor on U, is the divisorial part of the image of D under f (that is, $f_*(D)$ is the image of D considered as a one-dimensional cycle); and

$f^*\colon \mathrm{Pic}(V) \to \mathrm{Pic}(U)$ is the morphism of groups of divisor classes (with respect to linear equivalence) induced by f.

We shall consider the group $\mathrm{Pic}(V)$ as a subgroup of $\mathrm{Pic}(V) \otimes \mathbf{Q}$. We shall often denote the linear equivalence class of a divisor by the same symbol as the divisor itself; therefore, to denote equality of elements of $\mathrm{Pic}(V)$ we shall sometimes use the symbol $\sim$.

A curve on V is a nonnegative divisor on V. An exceptional curve on a surface is an irreducible, nonsingular rational curve with selfintersection (-1). $\mathrm{Neg}(V)$ is the set of irreducible curves on V with negative selfintersection. $\mathrm{Comp}(C)$ is the set of (irreducible) components of the curve C. $\nu_E(D)$ is the multiplicity with which an irreducible curve E occurs in a divisor D.

For divisors D, D' on V (and for $D, D' \in \mathrm{Pic}(V)$),

$(D \cdot D')$ is the intersection index,

(D^2) is the selfintersection index (or, for short, the selfintersection),

$\pi(D) = (D \cdot (D + K_V))/2 + 1$ is the arithmetic genus of D, and

$|D| = \{C \mid C \geqslant 0,\ C \sim D\}$ is the complete linear system determined by D.

A class $D \in \mathrm{Pic}(V)$ will be called effective if $|D| = \neq \varnothing$. If $|D| = |D - C| + C$, where C is a curve on V, but there is no curve C' such that $C' > C$ and $|D| = |D - C'| + C'$, then $|D - C|$ will be called the movable part of the system $|D|$, and C will be called immovable or fixed. A system $|D|$ will be called movable if its fixed part is zero. A system $|D|$ will be called a fibering by curves of genus 0 if it is one-dimensional, movable, and each of its parts is connected and has selfintersection zero and arithmetic genus zero.

0.7. A set $R = \{P_1, \ldots, P_n\}$ of n pairwise distinct points of a surface V is said to be in general position if for any subset $S \subset R$ and any blowing up $\sigma_S\colon V_S \to V$ of S the preimage $\sigma_S^{-1}(R \setminus S)$ under σ_S of the remaining points does not meet any of the curves occurring in $\mathrm{Neg}(V_S)$. In particular, a single point $P \in V$ is in general position if it lies on no curve in $\mathrm{Neg}(V)$.

0.8. In some constructions there arises a series of surfaces, say V_n, that depend not only on a natural numerical parameter n but also on a continuous parameter (i.e. one that takes values in some connected algebraic variety of positive dimension). The existence of an isomorphism of a surface U with such a surface will be expressed by the phrase "U is a surface of type V_n" (with a possible indication of the section where the V_n are constructed).

0.9. The first component of n of a heading $n.p$ (or $n.p.q$) of a section (subsection or formula) is the number of the section containing it. Such a heading indicates not only the start of the formulations of the section (subsection) indicated, but also the conclusion or the omission of proofs of assertions of the preceding one, except for those cases where it has been stated earlier that some assertion is proved in §§$n.p$-$n.q$ or when some subsections form a series of formulations that may be proved further on.

§1. Contractible curves

1.1. A curve L on V will be called contractible if there is a morphism $\varphi\colon V \to U$ such that $\varphi(L)$ is a point and $L = \varphi^{-1}(\varphi(L))$. If φ induces an isomorphism of $V \setminus L$ onto $U \setminus \{\varphi(L)\}$, then φ will be called a contraction of L and denoted by φ_L. Two contractions $\varphi\colon V \to U$ and $\chi\colon V \to W$ of a curve L differ by an isomorphism $p\colon U \to W$; that is, $\chi = p \circ \varphi$.

1.2. A curve L is contractible if and only if it satisfies all four of the following conditions:

1.2.1. $(L^2) = -1$.

1.2.2. $(L \cdot K_V) = -1$, where $V \supset L$.

1.2.3. $(L \cdot C) \leqslant 0$ for all $C \in \operatorname{Comp}(L)$.

1.2.4. L is connected.

Note that

1.2.5. The matrix of the intersections of the components of a contractible curve is negative definite.

1.3. From 1.2 it follows that if L is a contractible curve, then $\operatorname{Comp}(L)$ contains one and only one E for which $(E \cdot L) = -1$, $\nu_E(L) = 1$, and $(C \cdot L) = 0$ for $C \in \operatorname{Comp}(L - E)$. We denote this component E by $[L]$. Under any decomposition of the morphism φ_L (see 1.1) as a composition of successive contractions of components of L, the contraction σ of the component $[L]$ (more precisely, its image $\psi_L([L])$ under the composition ψ_L of the preceding contractions) is carried out last, and

$$\varphi_L = \sigma \circ \psi_L, \qquad L = \psi_L^{-1}(\psi_L([L])). \tag{1.3.1}$$

1.4. If L is a contractible curve on V, $D \in \operatorname{Pic}(V)$, $(D \cdot [L]) = 0$ (respectively, $(D \cdot L) = 0$), and the system $|D - aL|$ is movable, then $a \geqslant 0$. Indeed, $0 \leqslant ([L] \cdot (D - aL)) = a$ $(0 \leqslant (L \cdot (D - aL)) = a)$.

1.5. *If L is a contractible curve (or, more generally, a curve with negative definite matrix of intersections of its components), and A is a curve whose support is contained in L, then $|A| = \{A\}$.*

PROOF. Let D be the fixed part of the system $|A|$, and $B = A - D$. Then $(B^2) \geqslant 0$ and $\operatorname{sup} B \subset L$. From 1.2.5 it follows that $B = 0$, so $D = A$ and $|A| = \{A\}$.

1.6. *For a subset A of a surface V let $\mathcal{C}(A)$ denote the set of all contractible curves L with* $\operatorname{supp} L \subset A$. *The inclusion relation on the supports (equivalent to the usual inequality for contractible curves, since if L and M are contractible and* $\operatorname{supp} L \supset \operatorname{supp} M$, *then $L \geqslant M$) makes $\mathcal{C}(A)$ into an ordered set. A morphism $f\colon U \to V$ yields an inclusion $\mathcal{C}(A) \to \mathcal{C}(f^{-1}(A))$ mapping a curve L to the curve $f^{-1}(L)$.*

If L and M are contractible and $M \leqslant L$, then

1.6.1. $(L \cdot M) = -\delta(L, M)$,

1.6.2. $M \neq L$ and $M \cap [L] \neq \varnothing$ imply $(M \cdot [L]) = 1$, and

1.6.3. for any exceptional component E of L, either $E \cap M = \varnothing$ or $E < M$.

From 1.6.3 it follows that

1.6.4. For a contractible curve L and curves M and N lying in $\mathcal{C}(L)$, $M \cap N$ either is empty or coincides with M or N.

1.7. For contractible L we distinguish the subset $\mathcal{C}^*(L)$ of $\mathcal{C}(L)$ consisting of all those $M \in \mathcal{C}(L)$ for which $(M \cdot [L]) = 1$ (see 1.6.2).

1.7.1. The transitive relation on $\mathcal{C}(V)$ generated by a relation $M \in C^*(L)$ is the inclusion relation; that is, if L_0, $L \in \mathcal{C}(V)$ and $L_0 < L$, then $\mathcal{C}(V)$ contains a sequence $L_0, L_1, \ldots, L_n$ such that $L_n = L$ and $L_i \in \mathcal{C}^*(L_{i+1})$ for $0 \leqslant i < n$.

1.7.2. From (1.3.1) it follows that a contractible L can be represented in the form $L = [L] + \Sigma_1^s L_i$, where $\{L_1, \ldots, L_s\} = \mathcal{C}^*(L)$.

1.7.3. *The intersection index of two distinct contractible curves is always nonnegative.*

Proof. Let L and M be such curves. We may assume that neither of the inequalities $L \geqslant M$ or $M \geqslant L$ holds (see 1.6.1). We induct on $\#\operatorname{Comp} L$. If L is irreducible, then clearly $(L \cdot M) \geqslant 0$. Suppose $\#\operatorname{Comp} L > 1$. By 1.7.2,

$$L = [L] + \sum_{i=1}^{s} L_i, \qquad (L \cdot M) = ([L] \cdot M) + \sum_{i=1}^{s} (L_i \cdot M).$$

By the induction hypothesis, $(L_i \cdot M) \geqslant 0$ for $1 \leqslant i \leqslant s$. The case $([L] \cdot M) \geqslant 0$ is clear. Suppose $([L] \cdot M) < 0$. Then $[L] = [M]$, $([L] \cdot M) = -1$, all components of the curve L other than $[L]$ have nonnegative intersection with M, and at least one such component has positive intersection with M since L and M are connected and intersect. Therefore, $(L_j \cdot M) \geqslant 1$ for some j in $[1, s]$; therefore, $(L \cdot M) \geqslant 0$.

As a supplement to 1.7.3 we prove

1.7.4. *Suppose neither of the contractible curves L and M of a surface V contains the other, $L \cap M \neq \varnothing$ and $(L \cdot M) = 0$. Then $\mathcal{C}(L)$ contains a curve N for which $(N \cdot M) > 0$.*

Proof. We induct on $\#\operatorname{Comp} L$. For exceptional L, 1.7.4 clearly holds, so that we may assume $\#\operatorname{Comp} L > 1$. Using 1.7.2 we can write $0 = ([L] \cdot M) + \Sigma_1^s (L_i \cdot M)$, where $\{L_1, \ldots, L_s\} = \mathcal{C}^*(L)$. By 1.7.3, $(L_i \cdot M) \geqslant 0$ for $i \in [1, s]$. The inequality $([L] \cdot M) > 0$ is impossible. If $([L] \cdot M) < 0$, then, as we have seen in the proof of 1.7.3, $(L_j \cdot M) > 0$ for some $j \in [1, s]$; therefore, we may take $N = L_j$. It remains to consider the possibility $([L] \cdot M) = 0$. Then $(L_i \cdot M) = 0$ for $i \in [1, s]$. If $[L] \subset M$, then all the curves L_i intersect M; and, since L is not contained in M, at least one of them, say L_j, is not contained in M; from this and the induction hypothesis it follows that $\mathcal{C}(L_j)$ contains a curve N for which $(N \cdot M) > 0$. If $\nu_{[L]}(M) = 0$, then $[L] \cap M = \varnothing$; therefore, none of the curves L_i occurs in M; applying the inductive hypothesis to L_j now gives the required N.

1.8. *Suppose L is a contractible curve, $L > M \geqslant 0$, where M is a divisor that is the sum of contractible curves, such that if $M = 0$, then L has a distinguished component C. Then there is a sequence of irreducible curves*

$$C_1, C_2, \ldots, C_n, \tag{1.8.1}$$

starting with the given curve $C_1 = C$ for $M = 0$, such that

$$L = M + \sum_{i=1}^{n} C_i, \tag{1.8.2}$$

and the inequality

$$\left(\left(\sum_{k=1}^{i-1} C_k + M\right) \cdot C_i\right) > 0 \tag{1.8.3}$$

holds for $1 \leqslant i \leqslant n$ if $M > 0$, and for $2 \leqslant i \leqslant n$ if $M = 0$.

PROOF. We induce on $\#\operatorname{Comp} L$. If L is exceptional, then 1.8 is evident. Suppose L is reducible, E is an exceptional component of L, $\nu = \nu_E(L)$, $\mu = \nu_E(M)$, $d = \nu - \mu$, τ: $V \to \overline{V}$ is a contradiction of the curve E, $\overline{L} = \tau_*(L)$, and $\overline{M} = \tau_*(M)$. Suppose $\overline{C} = \tau_*(C)$ if $M = 0$ and $\tau_*(C) \neq 0$; and in the cases where $M = 0$, $\tau_*(C) = 0$, or $M \neq 0$, $\overline{M} = 0$, suppose $\overline{C}$ is a component of the curve $\overline{L}$ passing through the point $\tau(E)$. We take a sequence

$$\overline{C}_1, \ldots, \overline{C}_{\bar{n}}, \tag{1.8.4}$$

satisfying the inductive hypothesis for $\overline{L}$ and $\overline{M}$ and starting with $\overline{C}$ for $\overline{M} = 0$. Let $\overline{C}_{i_1}, \ldots, \overline{C}_{i_d}$ $(i_1 < \cdots < i_d)$ are all the terms of (1.8.4) passing through the point $\tau(E)$. Note that the number d of such terms equals $\nu - \mu$ $(= \nu_E(L - M))$. We construct the sequence (1.8.1) by first taking the proper τ-preimages of the terms (1.8.4) and then placing after each curve $\tau'(\overline{C}_{i_\alpha})$ $(1 \leqslant \alpha \leqslant d)$ the curve E and, moreover, in the case $M = 0$, $C = E$, interchanging the first two terms. Let us verify conditions (1.8.2) and (1.8.3) for this sequence. (1.8.2) is a result of applying τ^{-1} to the terms of the analogous equation connecting $\overline{L}$, $\overline{M}$, and the terms of (1.8.4). Further, suppose $i \in [1, \bar{n}]$, $\alpha = \#([0, i-1] \cap \{i_1, \ldots, i_d\})$ and $\beta = \#(\{i\} \cap \{i_1, \ldots, i_d\})$. Then

$$\begin{aligned}\left(\tau'(\overline{C}_i) \cdot \left(M + \sum_{k=1}^{i-1} \tau'(\overline{C}_k) + \alpha E\right)\right) &= \left((\tau^*(\overline{C}_i) - \beta E) \cdot \left(\tau^*\left(\overline{M} + \sum_{k=1}^{i-1} \overline{C}_k - \mu E\right)\right)\right) \\ &= \left(\overline{C}_i \cdot \left(\overline{M} + \sum_{k=1}^{i-1} \overline{C}_k\right)\right) + \beta\mu > 0,\end{aligned}$$

and for $\beta = 1$ for the term E of the sequence (1.8.1) that follows right after $\tau'(\overline{C}_i)$ we have

$$\begin{aligned}\left(E \cdot \left(M + \sum_{k=1}^{i} \tau'(\overline{C}_k) + \alpha E\right)\right) &= \left(E \cdot \left(\tau^*\left(\overline{M} + \sum_{k=1}^{i-1} \overline{C}_k\right) + \tau'(\overline{C}_i) - \mu E\right)\right) \\ &= 1 + \mu > 0.\end{aligned}$$

1.9. Before deducing Corollary 1.9.1 from 1.8 we make a well-known and simple remark

1.9.0. If C is an irreducible curve on V, D is an effective class in $\operatorname{Pic}(V)$ (that is, $|D| \neq \varnothing$) and either $(D \cdot C) < 0$ or $(D \cdot C) = 0$ but $D' \cap C \neq \varnothing$ for some $D' \in |D|$, then $|D - C| \neq \varnothing$.

1.9.1. *Suppose L and M are curves on V, $L > M \geqslant 0$, M is the sum of contractible curves, $D \in \operatorname{Pic}(V)$, $|D - M| \neq \varnothing$, $(D \cdot E) \leqslant 0$ for all $E \in \operatorname{Comp}(L - M)$, and also that if $M = 0$, then*

1.9a) *$(D \cdot C) < 0$ for some $C \in \operatorname{Comp} L$,*

or

1.9b) *$D' \cap L \neq \varnothing$ for some $D' \in |D|$.*

Then $|D - L| \neq \varnothing$.

PROOF. We use the sequence (1.8.1), which for $M = 0$ starts with the component C either taken from 1.9a) or intersecting the divisor D^* indicated in 1.9b). By induction on t $(1 \leqslant i \leqslant n)$ we shall show that

$$\left| D - M - \sum_{k=1}^{i} C_k \right| \neq \varnothing . \tag{1.9.1.i}$$

From (1.8.3) with $i = 1$ and from 1.9.1 it follows that $(C_1 \cdot (D - M)) \leqslant 0$, whence, since $|D - M|$ is nonempty, by 1.9.0 we get (1.9.1.1). Suppose (1.9.1.i − 1) holds. Then from (1.8.3) and the conditions of 1.9.1 we get

$$\left(\left(D - M - \sum_{k=1}^{i-1} C_k \right) \cdot C_i \right) < 0,$$

whence, because of (1.9.1.i − 1), by 1.9.0 we get (1.9.1.i).

1.10. We shall represent the set $\mathcal{C}(V)$ of all contractible curves on V as the set of vertices of an edge-weighted graph $\mathcal{C}_1(V)$, considering two vertices L and M as joined by an edge of weight $(L \cdot M)$ if $(L \cdot M) > 0$.

1.11. To any graph Γ one can associate the opposite (dual) graph $\overline{\Gamma}$ with the same vertices such that in $\overline{\Gamma}$ two distinct vertices are joined by an edge if and only if they are not joined by an edge in Γ. We shall denote the graph dual to $\mathcal{C}_1(V)$ by $\overline{\mathcal{C}}(V)$. We shall denote by $\vec{\mathcal{C}}(V)$ the partially oriented graph whose unoriented support is $\overline{\mathcal{C}}(V)$ and in which an arrow goes from a vertex M to a vertex L if $M \in \mathcal{C}^*(L)$ (that is, $M < L$ and $([L] \cdot M) = 1$; see 1.7). To a morphism $f\colon U \to V$ there correspond three embeddings of graphs

$$\mathcal{C}_1(V) \to \mathcal{C}_1(U), \quad \overline{\mathcal{C}}(V) \to \overline{\mathcal{C}}(U), \quad \vec{\mathcal{C}}(V) \to \vec{\mathcal{C}}(U),$$

given by the assignment $L \mapsto f^{-1}(L)$.

§2. Exceptional sets

2.0. We recall some general concepts of the theory of graphs. A subset I of the set of vertices of a graph Γ is called independent (see [5], §13.3, or [6], p. 95) if not two vertices in I are joined by an edge. The opposite concept is the concept of a completely dependent set or clique (see [5], §13.3): a clique is a set of vertices any two of which are joined by an edge. A clique with three elements (or the subgraph generated by it) will be called a triangle. It is clear that a set I is independent in Γ if and only if it forms a clique in $\overline{\Gamma}$. For del Pezzo surfaces V the independent sets in $\mathcal{C}_1(V)$ have been used by Manin (see [2], Chapter IV, 4.7), who called these sets exceptional. We shall use his terms in 2.1.

2.1. DEFINITION. By an *exceptional set* on a surface V we shall mean a subject $\mathbf{I}$ of the set $\mathcal{C}(V)$ such that

2.1.1. $(L \cdot M) = -\delta(L, M)$ for $L, M \in \mathbf{I}$ (in particular, $\mathbf{I}$ is independent in $\mathcal{C}_1(V)$), and

2.1.2. $\mathbf{I}$ is minimal in $\mathcal{C}(V)$ (see 0.2); i.e., if L is a contractible curve contained in some curve in $\mathbf{I}$, then $L \in \mathbf{I}$ (in other words, if $M \in \mathbf{I}$ then $\mathcal{C}(M) \subset \mathbf{I}$).

The number $\#\mathbf{I}$ will be called the *length* of the exceptional set $\mathbf{I}$.

2.2. *Some remarks on Definition* 1.2.

2.2.1. From 2.1.2 and 1.11 it follows that no element of $\mathbf{I}$ can be the end to an arrow in $\vec{\mathcal{C}}(V)$ whose start is inside $\mathbf{I}$.

2.2.2. From 2.1 and 1.7.3–1.7.4 it follows that any two curves in $\mathbf{I}$ either do not intersect or one contains the other. Therefore, if $\{L_1,\dots,L_m\}$ is the set of all maximal curves in $\mathbf{I}$, then $L_i \cap L_j = \varnothing$ for $i \neq j$, and

$$\mathbf{I} = \bigcup_{i=1}^{m} \mathcal{C}(L_i). \tag{2.2.2.1}$$

Conversely, to any subset $\{L_1,\dots,L_m\} \subset \mathcal{C}(V)$ consisting of pairwise nonintersecting curves one can assign an exceptional set by means of (2.2.2.1).

2.3. *Any exceptional set* $\mathbf{I}$ *on a surface* V *yields a morphism* $\varphi_{\mathbf{I}}\colon V \to V_{\mathbf{I}}$ *(defined up to a change to* $p \circ \varphi_{\mathbf{I}}$*, where* p *is an isomorphism) such that*

2.3.1. $\varphi_{\mathbf{I}}(L)$ *is a point on* $V_{\mathbf{I}}$ *for all* L *in* $\mathbf{I}$*, and*

2.3.2. $\operatorname{rank}\operatorname{Pic}(V_{\mathbf{I}}) = \operatorname{rank}\operatorname{Pic}(V) - \#\mathbf{I}$.

For short we say that $\varphi_{\mathbf{I}}$ is the composition of contractible curves in $\mathbf{I}$ and the images of such curves under preceding contractions. Condition 2.3.2 ensures the absence of superfluous contractions. For example, if $L \in \mathcal{C}(V)$, then $\varphi_{\mathcal{C}(L)} = \varphi_L$ (see 1.1).

Conversely, to each morphism $\varphi\colon V \to W$ there corresponds a unique exceptional set $\mathbf{I}(\varphi) \subset \mathcal{C}(V)$ such that $\varphi = \varphi_{\mathbf{I}}$. This $\mathbf{I}(\varphi)$ consists of all contractible curves on V that pass through a point under φ.

2.4. *If* $\varphi\colon V \to W$ *is a morphism and* D *is a divisor on* W*, then*

$$\varphi^{-1}(D) = \varphi'(D) + \sum_{L \in \mathbf{I}(\varphi)} ([L]\cdot\varphi'(D))L. \tag{2.4.φ}$$

Indeed, (2.4.φ) is well known for a contraction φ of an exceptional curve, and a simple calculation enables one to deduce (2.4.$\chi \circ \psi$) from (2.4.ψ) and (2.4.χ).

2.5. If $\mathbf{I}$ and $\mathbf{J}$ are exceptional sets, $\mathbf{I} \subset \mathbf{J}$ and $\varphi = \varphi_{\mathbf{I}}$, then $\varphi(\mathbf{J})$ will denote the exceptional set consisting of the images $\varphi_*(L)$ of the curves $L \in \mathbf{J}$ for which these images are nonzero.

2.6. For a morphism $f\colon U \to V$ and an exceptional set $\mathbf{I}$ on V we put

$$f^{-1}(\mathbf{I}) = \mathbf{I}(f) \cup \{f^{-1}(L) \mid L \in \mathbf{I}\}; \tag{2.6.1}$$

that is, $f^{-1}(\mathbf{I})$ consists of the contractible curves passing through a point on V or curves in $\mathbf{I}$ under f. The set $f^{-1}(\mathbf{I})$ is exceptional, $\mathbf{I} = f(f^{-1}(\mathbf{I}))$.

2.7. *On some relations among exceptional sets on a single surface.*

Definition. Let $\mathbf{I}$ and $\mathbf{J}$ be two exceptional sets on V. We call $\mathbf{I}$ and $\mathbf{J}$ *s-contiguous* if

$$\#(\mathbf{I} \cap \mathbf{J}) = \#\mathbf{I} - s = \#\mathbf{J} - s$$

and for a suitable numbering of the elements of $\mathbf{I}\backslash\mathbf{J}$ and $\mathbf{J}\backslash\mathbf{I}$ we have

$$\begin{gathered} \mathbf{I}\backslash\mathbf{J} = \{L_1,\dots,L_s\}, \qquad \mathbf{J}\backslash\mathbf{I} = \{M_1,\dots,M_s\}, \\ (L_i\cdot M_j) = 1 - \delta(i,j) \quad \text{for } 1 \leqslant i,j \leqslant s. \end{gathered} \tag{2.7.1}$$

2.8. (2.7.1) is equivalent to the matrix equation

$$\begin{pmatrix} (L_1\cdot M_1) & (L_1\cdot M_2) & \cdots & (L_1\cdot M_s) \\ (L_2\cdot M_1) & (L_2\cdot M_2) & \cdots & (L_2\cdot M_s) \\ \vdots & \vdots & & \vdots \\ (L_s\cdot M_1) & (L_s\cdot M_2) & \cdots & (L_s\cdot M_s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \\ 1 & 1 & \cdots & 0 \end{pmatrix}.$$

A comparison of this equation with drawing 180 of the book [7] depicting the Schläfli double six shows that the sixes of this hexahedron, considered as exceptional sets on a cubic surface, are 6-contiguous.

If $\mathbf{I}$ and $\mathbf{J}$ are s-contiguous, then $\overline{\mathcal{C}}(V)|(\mathbf{I}\Delta\mathbf{J})$ is the one-dimensional skeleton of a simplicial prism that is the product of an $(s-1)$-dimensional simplex and a one-dimensional one.

2.9. We now recall some concepts of the theory of graphs such as that of a maximal independent set of vertices, i.e. an independent set that becomes dependent after addition to it of any vertex (see [5], §13.3). The opposite thing is a maximal clique. We shall be interested in maximal independent sets $\mathbf{I}$ ($\mathbf{I} \subset \mathcal{C}(V)$) of the graphs $\mathbf{C}_1(V)$ (for del Pezzo surfaces V such $\mathbf{I}$ are considered in Chapter IV, §4.8 of [2]). Note that for maximal $\mathbf{I}$, 2.1.2 holds automatically.

2.9.0. *If* $\mathbf{I}$ *is a maximal exceptional set in* $\mathcal{C}(V)$, *then either*

$$\#\mathbf{I} = \operatorname{rank}\operatorname{Pic}(V) - 2 \tag{2.9.1}$$

or

$$\#\mathbf{I} = \operatorname{rank}\operatorname{Pic}(V) - 1. \tag{2.9.2}$$

If (2.9.2) *holds for an exceptional set* $\mathbf{I}$, *then* $\mathbf{I}$ *is maximal and* $V \simeq \mathbf{P}_2$.

PROOF. For maximal $\mathbf{I}$ the image $V_{\mathbf{I}}$ of a morphism $\varphi_{\mathbf{I}}$ is a relatively minimal model of the field of rational functions in two variables. Therefore $V_{\mathbf{I}}$ is isomorphic either to one of the surfaces $\mathbf{F}_n$ ($n \geqslant 0$, $n \neq 1$) or to $\mathbf{P}_2$. Now 2.9.0 follows from 2.3.2 and the equations $\operatorname{rank}\operatorname{Pic}(\mathbf{F}_n) = 2$ and $\operatorname{rank}\operatorname{Pic}(\mathbf{P}_2) = 1$.

§3. Basic sets

3.1. DEFINITION. An exceptional set $\mathbf{L}$ on a surface V is called *basic* if it satisfies (2.9.2); that is, $\#\mathbf{L} = \operatorname{rank}\operatorname{Pic}(V) - 1$. The set of all bases on a surface V will be denoted by $\mathfrak{B}(V)$.

3.2. An exceptional set $\mathfrak{L} \subset \mathcal{C}(V)$ is basic if and only if $V_{\mathbf{L}} \simeq \mathbf{P}_2$ (see 2.9.0). Therefore, if V does not admit morphisms onto $\mathbf{P}_2$, then $\mathfrak{B}(V) = \varnothing$.

3.3. NOTATION. For a basic set $\mathfrak{L} \in \mathfrak{B}(V)$ the symbol L_0 (i.e. the letter denoting the basis but with the subscript 0 and not in boldface) will denote the class $\varphi_{\mathbf{L}}^*(l) \in \operatorname{Pic}(V)$, where l is the class of a line in $\operatorname{Pic}(V_{\mathbf{L}})$ (recall that $V_1 \simeq \mathbf{P}_2$).

Note that if for a morphism $f\colon U \to V$ and a basis $\mathbf{L} \in \mathfrak{B}(V)$ we put $\mathbf{M} = f^{-1}(\mathbf{L})$ (see 2.6.1), then $\mathbf{M} \in \mathfrak{B}(U)$ and $M_0 = f^*(L_0)$.

3.4. The class $L_0 \in \operatorname{Pic}(V)$ corresponding to a basis $\mathbf{L} \in \mathfrak{B}(V)$ is characterized by the conjunction of the following four properties.

3.4.1. $(L_0^2) = 1$,

3.4.2. $(K_V \cdot L_0) + 3 = 0$,

3.4.3. the system $|L_0|$ is two-dimensional and movable, and

3.4.4. $(L \cdot L_0) = 0$ for $L \in \mathbf{L}$.

3.5. From 3.4.4 and the maximality of $\mathbf{L}$ it follows that *the basis* $\mathbf{L}$ *is uniquely recoverable from the class* L_0:

$$\mathbf{L} = \{L \mid L \in \mathcal{C}(V), (L \cdot L_0) = 0\}.$$

3.6. Let us show that a basic set on V can be enlarged in a natural way to a basis of the group $\operatorname{Pic}(V)$, and establish some properties of the group basis thus obtained.

3.6.1. *If* $\mathbf{L} \in \mathfrak{B}(V)$, *then the set* $\{L_0\} \cup \mathbf{L}$ *(more precisely, the union of* $\{L_0\}$ *and the set of equivalence classes of curves in* $\mathbf{L}$*) is a basis of the abelian group* $\mathrm{Pic}(V)$.

3.6.2. $K_V \sim -3L_0 + \Sigma L$

(here and in 3.6.3, Σ is summation over the classes of the curves L in $\mathbf{L}$).

3.6.3. *The intersection index of elements* X *and* Y *in* $\mathrm{Pic}(Y)$, *expressed through the group basis* $\{L_0\} \cup \mathbf{L}$ *by the equation*

$$X = \chi L_0 - \sum \chi_L L, \qquad Y = yL_0 - \sum y_L L,$$

is given by the formula $(X \cdot Y) = xy - \Sigma x_L y_L$; *in particular,* $(x^2) = x^2 - \Sigma x_L^2$.

PROOF. 3.6.2 is a consequence of 3.4.2 and 1.2.2, and 3.6.3 is a consequence of 3.4.1 and 3.4.4.

3.7. *Suppose* $\mathbf{L} \in \mathfrak{B}(V)$, $L \in \mathfrak{L}$, *and* L' *is a maximal curve in* $\mathbf{L}$ *containing* L *(see the decomposition* (2.2.2.1)). *The system* $|L_0 - L|$

3.7.1. *is one-dimensional.*

3.7.2. *has as its fixed part the curve* $L' - L$,

3.7.3. *has as its movable part the system* $|L_0 - L|$, *making* V *a fibering by curves of genus* 0, *and*

3.7.4. *is movable if and only if* $L = L'$.

PROOF. The images under $\varphi_{\mathbf{L}}$ of the curves of the system $|L_0 - L|$ form the pencil of lines in $\mathbf{P}_2$ passing through the point $P = \varphi_{\mathbf{L}}(L)$. Hence 3.7.1 follows. Clearly $L' = \varphi_{\mathbf{L}}^{-1}(P)$; there the system $|L_0 - L'|$, whose generic fiber is the proper preimage of a generic line in the pencil referred to, is movable and is the movable part of the system $|L_0 - L|$. This proves 3.7.3. Let D be the fixed part of $|L_0 - L|$. Clearly $D \leqslant L'$ and $(L_0 - L') + D \sim L_0 - L$, whence $D \sim L' - L$; then, by 1.5, $D = L' - L$, which proves 3.7.2, and so also 3.7.4.

3.8. *Let* $\mathbf{L} \in \mathfrak{B}(V)$, L, $M \in \mathbf{L}$ *and* $L \neq M$. *Then*

$$|L_0 - L - M| \neq \varnothing. \tag{3.8.1}$$

3.8.2. *If at least one of the curves* L *or* M *is maximal in* $\mathbf{L}$, *then* $\dim |L_0 - L - M| = 0$; *that is,* $|L_0 - L - M| = \{N\}$, *where* N *is a curve.*

3.8.3. *If* L *is maximal in* $\mathbf{L}$ *and* $\nu_{[M]}(N) = 0$, *then the curve* N *in* 3.8.2 *is contractible.*

3.8.4. *If the set* $\{L, M\}$ *is maximal in* $\mathbf{L}$ *(see* 0.2), *then* N *is contractible.*

PROOF. As in 3.7 let L' be a maximal curve in $\mathbf{L}$ containing L, $L' \neq M$. According to 3.7 we decompose the system $|L_0 - L|$ into its fixed and movable parts $|L_0 - L| = (L' - L) + |L_0 - L'|$. Take a curve C in $|L_0 - L'|$ that intersects M. Since $(E \cdot C) = 0$ for all $E \in \mathrm{Comp}\, M$, by 1.9.1 we have $C - M \geqslant 0$; therefore, $(L' - L) + (C - M) \in |L_0 - L - M|$, which proves (3.8.1).

Now suppose L is maximal in $\mathbf{L}$, $C \in |L_0 - L|$, $C \cap M \neq \varnothing$, $|C - M| = N + |D|$ is the decomposition of the system $|C - M|$ into its fixed and movable parts, and $D \geqslant 0$. Then $(D^2) \geqslant 0$. The matrix of the intersections of the components of C occurring in the fibering by curves of genus 0, is negative semidefinite and has a one-dimensional isotropic space; therefore, $D \sim n(L_0 - L)$ for some natural number n, whence $N \sim (1 - n)L_0 + (n - 1)L - M$, which is possible when $n \geqslant 0$ and $N \geqslant 0$ only if $n = 0$. This means that $D = 0$, which proves 3.8.2.

Let us prove 3.8.3. Because $\nu_{[M]}(M) = 1$, the equation $\nu_{[M]}(N) = 0$ is equivalent to $\nu_{[M]}(C) = 1$. In proving that N is contractible we may assume that $M = [M]$ (otherwise we contract all the curves in $\mathcal{C}^*(M)$; these curves either occur in N or do not intersect N). But if we discard from the fiber C of the fibering by curves of genus 0 an exceptional component M occurring in C with multiplicity 1, we are left with a contractible curve $N = C - M$ (that N is contractible follows from conditions 1.2.1–1.2.4, which hold for it).

Assertion 3.8.4 follows from 3.8.3.

3.9. DEFINITION. If $\mathbf{L} = \{L_1, \dots, L_r\} \in \mathfrak{B}(V)$ and $\dim |L_0 - L_i - L_j| = 0$, then L_{ij} will denote a curve such that $\{L_{ij}\} = |L_0 - L_i - L_j|$, and l_{ij} will denote the line $(\varphi_{\mathbf{L}})_*(L_{ij}) \subset \mathbf{P}_2$.

3.10. DEFINITION. Let $D \in \operatorname{Pic}(V) \otimes \mathbf{Q}$. We shall say that the system $|D|$ is *rationally empty* if for any positive integer n with the property $nD \in \operatorname{Pic}(V)$ we have $|nD| = \varnothing$.

We shall show that *for any* $\mathbf{L} \in \mathfrak{B}(V)$ *and any rational* λ *with* $\lambda > 1/3$, *the system* $|L_0 + \lambda K_V|$ *is rationally empty.*

From 3.6.2 follows

$$|L_0 + \lambda K_V| = \left|(1 - 3\lambda)L_0 + \sum_{L \in \mathbf{L}} L\right|,$$

therefore, if a curve C lies in $|n(L_0 + \lambda K_V)|$, where $n > 0$, then the curve $(\varphi_{\mathbf{L}})_*(C)$ lies in $|n(1 - 3\lambda)l|$, where l is a line in $\mathbf{P}_2$; and for $\lambda > 1/3$ and $n > 0$ this is impossible.

3.11. We introduce a distance between bases $\mathbf{L}, \mathbf{M} \in \mathfrak{B}(V)$ by the formula

$$\rho(\mathbf{L}, \mathbf{M}) = \log_2(L_0 \cdot M_0). \tag{3.11.0}$$

To justify this we establish the following facts.

3.11.1. $(L_0 \cdot M_0) = 1$ *implies* $\mathbf{L} = \mathbf{M}$.

3.11.2. *For* $\mathbf{L}, \mathbf{M}, \mathbf{N} \in B(V)$

$$(L_0 \cdot N_0) \leqslant (L_0 \cdot M_0)(M_0 \cdot N_0).$$

PROOF OF 3.11.1. It suffices to deduce $L_0 = M_0$ from $(L_0 \cdot M_0) = 1$, since by 3.5 L_0 determines L uniquely. Suppose (see 3.6) $M_0 \sim aL_0 + \Sigma a_L L$. From $(M_0^2) = 1$, $(M_0 \cdot L_0) = 1$, and 3.6.3 it follows that $a = 1$ and $a_L = 0$ for all L in $\mathbf{L}$; that is, $M_0 = L_0$.

PROOF OF 3.11.2. In the expressions of L_0 and N_0 in terms of $\{M_0\} \cup \mathbf{M}$, i.e. $L_0 \sim aM_0 - \Sigma a_M M$ and $N_0 \sim bM_0 - \Sigma b_M M$, where Σ is the sum over $\mathbf{M}$, we have, by 1.4, $a_M \geqslant 0$ and $b_M \geqslant 0$. Then, by 3.6.3,

$$(L_0 \cdot N_0) = ab - \sum a_M b_M \leqslant ab = (L_0 \cdot M_0)(M_0 \cdot N_0).$$

3.12. NOTATION. For $\mathbf{L}, \mathbf{M} \in \mathfrak{B}(V)$, $\varphi(\mathbf{L}, \mathbf{M})$ will denote the double coset in

$$\mathcal{P} \backslash \mathrm{Cr} / \mathcal{P} \quad (\text{where } \mathcal{P} = \operatorname{Aut}(\mathbf{P}_2)), \tag{3.12.1}$$

represented by the transformation $\varphi_{\mathbf{L}} \circ \varphi_{\mathbf{M}}^{-1}$ (see 2.3 and 3.2; recall that the morphism $\varphi_{\mathbf{L}}$ is defined up to replacement by $p \circ \varphi_{\mathbf{L}}$, where $P \in \mathcal{P}$):

$$\begin{tikzcd} & V \arrow[dl, "\varphi_{\mathbf{M}}"'] \arrow[dr, "\varphi_{\mathbf{L}}"] & \\ \mathbf{P}_2 = V_{\mathbf{M}} \arrow[rr, dashed, "\varphi(\mathbf{L},\mathbf{M})"] & & V_L = \mathbf{P}_2 \end{tikzcd}$$

3.13. If $f\colon U \to V$ is a morphism, then

$$\varphi(f^{-1}(\mathbf{L}), f^{-1}(\mathbf{M})) = \varphi(\mathbf{L}, \mathbf{M})$$

(for the definition of $f^{-1}(\mathbf{I})$ see 2.6).

3.14. Any class in (3.12.1) can be represented as $\varphi(\mathbf{L},\mathbf{M})$. Indeed, if we take a representative of this call written in the form $\beta \circ \alpha^{-1}$, where α and β are morphisms of some V onto $\mathbf{P}_2$, then we may assume, by 2.3, that $\alpha = \varphi_{\mathbf{M}}$ and $\beta = \varphi_{\mathbf{L}}$ for basic sets $\mathbf{M} = \mathbf{I}(\alpha)$ and $\mathbf{L} = \mathbf{I}(\beta)$. Note further that in representing a given double coset in (3.12.1) in the form $\varphi(\mathbf{L},\mathbf{M})$ one can satisfy the requirement $\mathbf{L} \cap \mathbf{M} = \varnothing$.

3.15. To each transformation g in Cr we can assign a natural number $\deg(g)$, the degree of g (that is, $\deg(g) = (l \cdot g(l))$, where l is a generic line in $\mathbf{P}_2$ and $g(l)$ is its image under g). Multiplying a transformation on the left or right by a projective transformation does not change its degree; therefore, deg can be considered as a function on the set (3.12.1). We have

$$\deg \varphi(\mathbf{L},\mathbf{M}) = (L_0 \cdot M_0).$$

Therefore 3.11.1 simply says that a transformation of degree 1 is projective, and 3.11.2 says that the degree of a composition of transformations does not exceed the product of their degrees.

3.16. Let

$$\mathbf{L} = \{L_1,\ldots,L_r\} \in \mathfrak{B}(V), \quad \mathbf{M} = \{M_1,\ldots,M_r\} \in \mathfrak{B}(V), \qquad \mathbf{L} \cap \mathbf{M} = \varnothing.$$

We consider the passage from one group basis $\{M_0\} \cup \mathbf{M}$ to another, $\{L_0\} \cup \mathbf{L}$:

$$L_j \sim a_{0j}M_0 - \sum_{i=1}^{r} a_{ij}M_i \qquad (0 \leqslant j \leqslant r). \tag{3.16.1}$$

The numbers $a_{ij} = (L_j \cdot M_i)$ are nonnegative (by 1.4 and 1.7.3) and satisfy the following relations:

$$a_{0j}a_{0k} - \sum_{i=1}^{r} a_{ij}a_{ik} = a_{j0}a_{k0} - \sum_{i=1}^{r} a_{ji}a_{ki} = (2\delta(0,\, jk) - 1)\delta(j,k),$$
$$3a_{0j} - \sum_{i=1}^{r} a_{ij} = 3a_{j0} - \sum_{i=1}^{r} a_{ji} = 2\delta(0,\, j) + 1.$$

The relations in the first line are gotten by comparing the intersection indices of the members of (3.16.1), and those in the second by comparing the expression of K_V in terms of $\{M_0\} \cup \mathbf{M}$ and $\{L_0\} \cup \mathbf{L}$ indicated in 3.6.2.

3.17. Example. Take a blowing up $\varphi\colon V \to \mathbf{P}_2$ of five points $\{P_1,\ldots,P_5\}$ in general position in $\mathbf{P}_2$ and two bases $\mathbf{L} = \mathbf{I}(\varphi)$ (that is, $\mathbf{L} = \{L_1,\ldots,L_5\}$, where $L_i = \varphi^{-1}(P_i)$) and $\mathbf{M} = \{M_1,\ldots,M_5\}$, where M_1 is the proper φ-preimage of the conic passing through $P_1,\ldots,P_5$, and for $2 \leqslant i \leqslant 5$, $M_i = L_{1i}$ (see 3.9; that is, M_i is the proper φ-preimage of the line passing through P_1 and P_i). The edges of the graph $C_1(V)|\mathbf{L} \cup \mathbf{M}$ are $[L_1, M_i]$, $[M_1, L_i]$ and $[L_i, M_i]$ ($1 \leqslant i \leqslant 5$; there are 13 edges in all).

$\overline{C}(V)|\mathbf{L} \cup \mathbf{M}$ is the one-dimensional skeleton of a polyhedron in the four-dimensional space gotten from the simplicial antiprism (whose base if the three-dimensional simplices spanned by $\{L_2,\ldots,L_5\}$ and $\{M_2,\ldots,M_5\}$) of the superstructure pyramid on its bases. The matrix (a_{ij}) considered in 3.16 is this case is

$$\begin{pmatrix} 3 & 2 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.17.1}$$

3.18. EXAMPLE GENERALIZING 3.17. Suppose $d \geqslant 2$, $\varphi\colon V \to \mathbf{P}_2$ is the blowing up of a set $\{P_1,\ldots,P_{2d-1}\}$ of $2d-1$ points in general position, $\mathbf{L} = \mathbf{I}(\varphi) = \{L_1,\ldots,L_{2d-1}\}$, where $L_i = \varphi^{-1}(P_i)$, $1 \leqslant i \leqslant 2d-1$, $M = \{M_1,\ldots,M_{2d-1}\}$, where M_1 is the proper φ-preimage of the curve of degree $d-1$ passing through $P_2,\ldots,P_{2d-1}$ and having P_1 as a point of multiplicity $d-2$, and $M_i = L_{1i}$ for $2 \leqslant i \leqslant 2d-1$ (see 3.9). The graph $\mathcal{C}_1(V)\,|\,\mathbf{L} \cup \mathbf{M}$ has edges $[L_1, M_j]$, $[M_1, L_j]$ and $[L_j, M_j]$ $(2 \leqslant j \leqslant 2d-1)$ with unit weights, to which, if $d \geqslant 3$, one must add the edge $[L_1, M_1]$ with weight $d-2$. The graph $\overline{\mathcal{C}}(V)\,|\,\mathbf{L} \cup \mathbf{M}$ is a triangular prism for $d=2$; for $d \geqslant 3$ it is the one-dimensional skeleton of a polyhedron in the $2(d-1)$-dimensional space gotten from the simplicial antiprism of the superstructure pyramid over its two bases. The matrix (a_{ij}) in this case is

$$\begin{pmatrix} d & (d-1) & 1 \cdots 1 \\ (d-1) & (d-2) & 1 \cdots 1 \\ 1 & 1 & \\ \vdots & \vdots & I_{2d-2} \\ 1 & 1 & \end{pmatrix}, \tag{3.18.1}$$

where I_n is the $n \times n$ identity matrix.

§4. The graph $\Gamma_1(V)$

4.1. *If bases* $\mathbf{L}$ *and* $\mathbf{M}$ *in* $\mathfrak{B}(V)$ *are s-contiguous (in the sense of the definition given in* 2.7), *then either* $s=3$ *or* $s=6$.

PROOF. By contracting the part common to $\mathbf{L}$ and $\mathbf{M}$ ($\mathbf{L} \cap \mathbf{M}$ is exceptional), we may assume that $L \cap M = \varnothing$, $\mathbf{L} = \{L_1,\ldots,L_s\}$, $\mathbf{M} = \{M_1,\ldots,M_s\}$ and $(L_i \cdot M_j) = 1 - \delta(i,j)$. By the last equation, $M_s \sim mL_0 - \Sigma_1^{s-1} L_i$. Further, from 1.2 and 3.6 it follows that $m^2 - s + 2 = 0$ and $s - 3m = 0$, which is possible only *for* $m=1$, $s=3$ or $m=2$, $s=6$.

4.2. In the sequel, of two related bases with respect to 3-contiguity and 6-contiguity we shall be interested in the first only.

DEFINITION. We assign to a surface V the graph $\Gamma_1(V)$ whose vertices are the bases on V and whose edges join 3-contiguous bases. Bases connected by an edge will be called *contiguous* or *neighboring*. We shall also say that a basis $\mathbf{M}$ is *neighboring* (or *contiguous*) *with* $\mathbf{L}$ *along with set* $\{L_1, L_2, L_3\}$ if $\mathbf{L}\backslash\mathbf{M} = \{L_1, L_2, L_3\}$.

To a morphism $f\colon U \to Y$ there corresponds an embedding of graphs

$$\Gamma_1(f)\colon \Gamma_1(V) \to \Gamma_1(U),$$

mapping a vertex $\mathbf{L} \in \mathfrak{B}(V)$ to $f^{-1}(\mathbf{L})$ (see 2.6). We shall write f^{-1} in place of $\Gamma_1(f)$.

4.3. *Two distinct bases* $\mathbf{L}, \mathbf{M} \in \mathfrak{B}(V)$ *are neighbors in* $\Gamma_1(V)$ *if and only if each of the following conditions holds*:

4.3.1. $\#(\mathbf{L}\backslash\mathbf{M}) \leqslant 4$, *in other words (and more symmetrically in* $\mathbf{L}$ *and* $\mathbf{M}$), $\#(\mathbf{L} \cap \mathbf{M}) \geqslant r-4$, *where* $r = \operatorname{rank} \operatorname{Pic}(V) - 1$, *r being the length of the bases on V.*

4.3.2. $(L_0 \cdot M_0) = 2$; *that is* $\varphi(\mathbf{L},\mathbf{M})$ (*see* 3.12–3.15) *is contained in the set Q of quadratic transformations, or, in other words, the distance* (3.11.0) *between* $\mathbf{L}$ *and* $\mathbf{M}$ *is least possible (and equals* 1).

PROOF. If $\mathbf{L}$ and $\mathbf{M}$ are 3-contiguous, then $\#(\mathbf{L}\backslash\mathbf{M}) = \#(\mathbf{M}\backslash\mathbf{L}) = 3$, which proves the necessity of 4.3.1. Suppose $1 \leqslant s \leqslant 4$, $\mathbf{L}\backslash\mathbf{M} = \{L_1,\ldots,L_s\}$, and

$$M_0 \sim a_0 L_0 - a_1 I_1 - \cdots - a_s L_s. \tag{4.3.3}$$

Then $a_0 \geqslant 2$ by 3.11.1, and from 3.16 we get

$$a_0^2 - a_1^2 - \cdots - a_s^2 = 1, \tag{4.3.4}$$

$$3a_0 - a_1 - \cdots - a_s = 3. \tag{4.3.5}$$

Multiplying (4.3.4) by (-9) and (4.3.5) by $6a_0$ and adding, we get

$$\sum_{i=1}^{s} (3a_i - a_0)^2 = -a_0^2(9 - s) + 18a_0 - 9. \tag{4.3.6}$$

The quadratic trinomial in a_0 on the right side of (4.3.6) is nonnegative only on the real interval with endpoints $(9 - 3\sqrt{s})/(9 - s)$ and $(9 + 3\sqrt{s})/(9 - s)$. For $1 \leqslant s \leqslant 4$ this interval contains an integer a_0 greater than one only when $s \geqslant 3$ and $a_0 = 2$. The last equation establishes the necessity of condition 4.3.2. Let us establish its sufficiency (and with it finally 4.3.1). From $a_0 = 2$ and (4.3.3)–(4.3.5) it follows that exactly three of the numbers $a_1, \ldots$, say a_1, a_2, a_3, are equal to one, and the rest are zero. Therefore, $s = 3$ (if $s \geqslant 4$, then $(L_4 \cdot M_0) = a_4 = 0$, and then, by 3.5, $L_4 \in \mathbf{M} \cap \mathbf{L}$), and

$$M_0 \sim 2L_0 - L_1 - L_2 - L_3. \tag{4.3.7}$$

$\mathbf{M} \backslash \mathbf{L}$, as well as $\mathbf{L} \backslash \mathbf{M}$, consists of three curves. Suppose $M \in \mathbf{M} \backslash \mathbf{L}$, and $M \sim b_0 L_0 - b_1 L_1 - b_2 L_2 - b_3 L_3$. From $(M^2) = (M \cdot K_V) = -1$, $(M \cdot M_0) = 0$, and (4.3.7) it follows that

$$b_0^2 - \sum_{i=1}^{3} b_i^2 = -3b_0 + \sum_{i=1}^{3} b_i = -1, \qquad 2b_0 - \sum_{i=1}^{3} b_i = 0,$$

which is possible only when $b_0 = 1$ and (b_1, b_2, b_3) is a permutation of the sequence $(1, 1, 0)$. Therefore, we may assume that $\mathbf{M} \backslash \mathbf{L} = \{M_1, M_2, M_3\}$

$$M_1 \sim L_0 - L_2 - L_3, \quad M_2 \sim L_0 - L_1 - L_3, \quad M_3 \sim L_0 - L_1 - L_2, \tag{4.3.8}$$

whence follows (2.7.1) with $s = 3$, proving that $\mathbf{L}$ and $\mathbf{M}$ are contiguous.

4.4. *For a basis $\mathbf{L} \in \mathfrak{B}(V)$ and a three-element subset $\Lambda = \{L_1, L_2, L_3\}$ of it, there is a basis $\mathbf{M}$ neighboring $\mathbf{L}$ along Λ (that is, $\Lambda = \mathbf{L} \backslash \mathbf{M}$) if and only if the following three conditions hold*:

4.4.1. *Λ is a maximal subset of $\mathbf{L}$; that is, $\mathbf{L} \backslash \Lambda$ contains no curve containing a curve in Λ* (*see* 0.2).

4.4.2. $|L_i - L_j - L_k| = \varnothing$ *for any permutation (i, j, k) of the triple* $(1, 2, 3)$; *that is, none of the curves in Λ contains the sum of two other curves in Λ; in particular, the triangle $\mathcal{C}(V) \,|\, \Lambda$ does not contain two arrows with a common end* (*see* 1.11 *and* 1.7.2).

4.4.3. $|L_0 - L_1 - L_2 - L_3| = \varnothing$.

PROOF. The intersection $\mathbf{L} \cap \mathbf{M}$ of exceptional sets is exceptional and so minimal in $\mathcal{C}(V)$; therefore, $\Lambda = \mathbf{L} \backslash (\mathbf{L} \cap \mathbf{M})$ is maximal in $\mathbf{L}$, proving the necessity of 4.4.1. For neighboring $\mathbf{L}$ and $\mathbf{M}$ and for a curve C lying in the left part of 4.4.2 or 4.4.3, from (4.3.7) we obtain $(M_0 \cdot C) = -1$, contradicting the movability of $|M_0|$.

Let us prove the sufficiency of 4.4.1–4.4.3. From 4.4.1 it follows that the set $\mathbf{I} = \mathbf{L} \backslash \Lambda$ is exceptional. Put $\varphi = \varphi_{\mathbf{I}}$ (see 2.3). The basis $\varphi(\mathbf{L}) \in \mathfrak{B}(V_{\mathbf{I}})$ (see 2.5) and its (coincident) subset $\varphi(\Lambda)$ satisfy conditions 4.4.1–4.4.3. If $\mathfrak{B}(V_{\mathbf{I}})$ contains a basis different from $\varphi(\mathbf{L})$, by 4.3.1 it will neighbor $\varphi(\mathbf{L})$, and then its φ-preimage in the sense of 2.6 will neighbor $\mathbf{L}$. Therefore, in the sequel we may assume that $\mathbf{L} = \Lambda$; in particular, $\#(\mathbf{L}) = 3$ and $\operatorname{rank} \operatorname{Pic}(V) = 4$; i.e., the morphism $\varphi_{\mathbf{L}}\colon V \to \mathbf{P}_2$ is the composite of three successive contractions of exceptional curves. The graph $\vec{\mathcal{C}}(V)|\mathbf{L}$ (with a suitable numbering of its vertices L_1, L_2, L_3) can have, by 4.4.2, only one of the following forms T_0, T_1 or T_2:

T_0 is a triangle without arrows,

T_1 is a triangle with the single arrow $[L_3, L_1]$, and

T_2 is a triangle with the two arrows $[L_3, L_2]$ and $[L_2, L_1]$.

There is precisely one (up to isomorphism) surface V_α ($\alpha = 0, 1, 2$) gotten from $\mathbf{P}_2$ by three such blowings up of points such that the composition $\varphi_\alpha\colon V_\alpha \to \mathbf{P}_2$ of these blow ups defines a basis $\mathbf{L} = \mathbf{l}(\varphi_\alpha) = \{L_1, L_2, L_3\}$ satisfying 4.4.2 and 4.4.3 and generating in $\vec{\mathcal{C}}(V_\alpha)$ a subgraph $\vec{\mathcal{C}}(V_\alpha)|\mathbf{L}$ isomorphic to the graph T_α. Let us describe φ_α.

φ_0 is the blow up of a triple $\{P_1, P_2, P_3\} \subset \mathbf{P}_2$ in general position.

$\varphi_1 = \psi_1 \circ \chi_1$, where ψ_1 is the blow up of a pair $\{P_1, P_2\} \subset \mathbf{P}_2$, χ is the blow up of a point P_3 lying on $\psi_1^{-1}(P_1)\setminus\psi_1'(l_{12})$, and l_{12} is the line through P_1 and P_2.

$\varphi_2 = \psi_2 \circ \chi_2 \circ \xi_2$, where ψ_2 is the blow up of a point $P_1 \in \mathbf{P}_2$, χ_2 is the blow up of a point $P_2 \in \psi_1^{-1}(P_1)$, ξ_2 is the blow up of a point $P_3 \in \chi_2^{-1}(P_2)\setminus(\psi_2 \circ \chi_2)'(l_{12})$, and $l_{12} \subset \mathbf{P}_2$ is a line containing P_1 whose proper ψ_2-preimage contains the point P_2.

It is not hard to verify separately for each of the three surfaces constructed that for $1 \leqslant i < j \leqslant 3$ the system $|L_0 - L_i - L_j|$ is zero-dimensional, and the curves L_{ij} (see 3.9) are contractible and form a basis $\mathbf{M} = \{L_{23}, L_{13}, L_{12}\}$ on V_α.

4.5. *Some remarks on the surfaces V_α that arose in the preceding proof and on the elements of the set $\mathcal{B}(V_\alpha) = \{\mathbf{L}, \mathbf{M}\}$.*

4.5.1. $\mathcal{C}(C_\alpha) = \{L_1, L_2, L_3, L_{23}, L_{13}, L_{12}\} = \mathbf{L} \cup \mathbf{M}$, $\mathcal{C}_1(V_\alpha)$ is a hexagon, where L_i is joined by an edge to L_{ij} (we consider $L_{ij} = L_{ji}$), and $\bar{\mathcal{C}}(V_\alpha)$ is the one-dimensional skeleton of the triangular prism with bases $\bar{\mathcal{C}}(V_\alpha)|\mathbf{M}$ and $\bar{\mathcal{C}}(V_\alpha)|\mathbf{L}$. The vertex L_{ij} of the first base is joined by an edge to the vertex L_k of the second if $\{i, j, k\} = \{1, 2, 3\}$. The assignment $L_{ij} \mapsto L_k$ enables one to define an isomorphism of graphs $\vec{\mathcal{C}}(V_\alpha)|\mathbf{M} \simeq \vec{\mathcal{C}}(V_\alpha)|\mathbf{L}$ ($\simeq T_\alpha$; see 4.4.4) whose arrows map $\bar{\mathcal{C}}(V_\alpha)$ to $\vec{\mathcal{C}}(V_\alpha)$.

4.5.2. In the double coset $S_\alpha == \varphi(\mathbf{L}, \mathbf{M})$ (see 3.12) one can find a quadratic transformation s_α given, in a suitable choice of homogeneous coordinates (x_0, x_1, x_2) on $\mathbf{P}_2$, by the formulas

$$S_0\begin{cases} x_0' = x_1x_2, \\ x_1' = x_0x_2, \\ x_2' = x_0x_1, \end{cases} \qquad S_1\begin{cases} x_0' = x_1^2, \\ x_1' = x_0x_1, \\ x_2' = x_0x_2, \end{cases} \qquad S_2\begin{cases} x_0' = x_0^2, \\ x_1' = x_0x_1, \\ x_2' = x_1^2 - x_0x_2. \end{cases}$$

Note further that $\mathcal{Q} = S_0 \cup S_1 \cup S_2$, i.e., the set $\mathcal{Q}$ of all quadratic transformations falls into three double cosets $S_\alpha = \mathcal{P}s_2\mathcal{P}$, where $\alpha = 0, 1, 2$ and $\mathcal{P} = \mathrm{Aut}(\mathbf{P}_2)$.

4.5.3. Let us use the surfaces V_α to transform the graph $\Gamma_1(V)$ of an arbitrary surface V into an edge-weighted graph. To the edge $[\mathbf{L}, \mathbf{M}]$ of the graph $\Gamma_1(V)$ we assign the weight $w[\mathbf{L}, \mathbf{M}] = \alpha$ if $V_{\mathbf{L}\cap\mathbf{M}} \simeq V_\alpha$ ($\alpha = 0, 1$ or 2). For example,

$$\Gamma_1(V_\alpha) = \mathbf{L} \circ \overset{\alpha}{—} \circ \mathbf{M}.$$

4.6. We pass from edges to triangles (in the sense of 2.0). *For a three-element set $T = \{\mathbf{K}, \mathbf{L}, \mathbf{M}\} \subset \mathcal{B}(V)$ to be a triangle in $\Gamma_1(V)$ it is necessary and sufficient that the following three conditions hold*:

4.6.1. $\{\mathbf{K}, \mathbf{L}\}$, $\{\mathbf{L}, \mathbf{M}\}$ are pairs of neighboring vertices in $\Gamma_1(V)$,

$$\#((\mathbf{L}\setminus\mathbf{K}) \cap (\mathbf{L}\setminus\mathbf{M})) = 2.$$

4.6.2. $\#(\mathbf{K} \cap \mathbf{L} \cap \mathbf{M}) = r - 4 = \operatorname{rank} \operatorname{Pic}(V) - 5$, *where r is the length of the bases on V.*

4.6.3. *There exist a surface U, a morphism $f\colon V \to U$, and a subset $T' \subset \mathfrak{B}(U)$ such that* $\operatorname{rank} \operatorname{Pic}(U) = 5$ *and* $T = f^{-1}(T')$.

PROOF. Let us discuss conditions 4.6.1 and 4.6.2. Suppose

$$\mathbf{L} = \{L_1, \ldots, L_r\}, \quad \mathbf{L} \backslash \mathbf{K} = \{L_1, L_2, L_3\} \quad \mathbf{L} \backslash \mathbf{M} = \{L_i, L_{i+1}, L_{i+2}\},$$

where $i \in [2, 4]$. Then $\mathbf{K} \cap \mathbf{L} \cap \mathbf{M} = \{L_{i+3}, L_{i+4}, \ldots, L_r\}$.

We must show that $\mathbf{K}$ and $\mathbf{M}$ are neighbors if and only if $i = 2$. By (4.3.8) the classes in $\operatorname{Pic}(V)$ of the basis $\mathbf{M}$ (respectively $\mathbf{K}$) are

$$L_1, \ldots, L_{i-1}, \quad L_0 - L_{i+1} - L_{i+2}, \quad L_0 - L_i - L_{i+2}, \quad L_0 - L_i - L_{i+1}, \quad L_{i+3}, \ldots$$
$$(L_0 - L_2 - L_3, \quad L_0 - L_1 - L_3, \quad L_0 - L_1 - L_2, \quad L_4, \ldots).$$

From this and 1.5 it follows that $\#(\mathbf{K} \backslash \mathbf{M})$ is 3 when $i = 2$, and 6 when $i = 4$. By 4.3.1, $\mathbf{K}$ and $\mathbf{M}$ are contiguous if and only if $\#(\mathbf{K} \backslash \mathbf{M}) \leqslant 4$, i.e. when $i = 2$.

The necessity of 4.6.3 comes from the fact that for $\mathbf{I} = \mathbf{K} \cap \mathbf{L} \cap \mathbf{M}$, if we put $U = V_{\mathbf{I}}$, $f = \varphi_{\mathbf{I}}$ and $T' = f(T)$, we get, by 4.6.2 and 2.3.2, the required objects. The sufficiency of 4.6.3 follows from condition 4.6.2, guaranteeing that T' is a triangle, and from the fact that f, the preimage of a triangle, is a triangle.

4.7. To describe the triangles in Γ_1 and the surface minimal in this respect, by 4.6.3 we must enumerate the surfaces U for which $\#\mathfrak{B}(U) \geqslant 3$ and $\operatorname{rank} \operatorname{Pic}(U) = 5$. From the last equation and 4.3.1 it follows that in $\Gamma_1(U)$ any two vertices are joined by an edge. Here we construct five surfaces $U_0, \ldots, U_4$ and a one-parameter family of surfaces denoted by U_5. More precisely, we construct morphisms $\varphi_i\colon U_i \to \mathbf{P}_2$.

φ_0 is the blow up of four points $\{P_1, \ldots, P_4\} \subset \mathbf{P}_2$ in general position; that is, U_0 is a del Pezzo surface of degree 5.

φ_1 is the blow up of four points $\{P_1, \ldots, P_4\} \subset \mathbf{P}_2$ containing exactly one collinear triple $\{P_1, P_2, P_3\}$.

φ_2 (φ_3) is the composition $\psi \circ \chi$ of the blow up ψ of a triple $\{P_1, P_2, P_3\}$ of collinear (noncollinear) points and the blow up χ of a point P_4 on the curve $\psi^{-1}(P_3)$ but not on (on) the curve $\psi'(l_{13})$, where l_{13} is the line through P_1 and P_3.

φ_4 is the composition $\psi \circ \chi$ of the blowing up ψ of a pair $\{P_1, P_2\} \subset \mathbf{P}_2$ and the blowing up χ of a pair $\{P_3, P_4\} \subset \psi^{-1}(P_2) \backslash \psi^{-1}(l_{12})$, where l_{12} is the line through P_1 and P_2.

$\varphi_5 = \psi \circ \chi \circ \xi$, where ψ is the blowing up of a point $P_1 \in \mathbf{P}$, χ is the blowing up of $P_2 \in \psi^{-1}(P_1)$, ξ is the blowing up of a pair

$$\{P_3, P_4\} \subset \chi^{-1}(P_2) \backslash \chi'(\psi^{-1}(P_1) + \psi'(l_{12})),$$

and l_{12} is the line containing P_1 whose proper ψ-preimage passes through P_2.

4.8. To describe the properties of U_n we agree on some notation:

4.8.0. If $f\colon U \to V$ is a morphism, W is an intermediate surface between U and V (that is, $f = \beta \circ \alpha$, a composition $U \xrightarrow{\alpha} W \xrightarrow{\beta} V$), then for a point $P \in W$ we put $f^{-1}(P) = \alpha^{-1}(P)$.

On U_n ($0 \leqslant n \leqslant 5$) we take four contractible curves $L_K = \varphi_n^{-1}(P_k)$ (where $1 \leqslant k \leqslant 4$; P_k is the point occurring in the construction of φ_n; see 4.7) that form a basis $\mathbf{L} = \mathbf{I}(\varphi_n) = \{L_1, \ldots, L_4\}$. Below, using the notation of 3.9, we shall use curves of the form L_{kl}. A vertex L_k (L_{kl}) of a graph with zero-dimensional skeleton $\mathcal{C}(U_n)$ will be denoted by the symbol k (by kl).

For $0 \leqslant i \leqslant 2$

$$\mathcal{C}(U_i) = \{L_1, L_2, L_3, L_4, L_{12}, L_{13}, L_{14}, L_{23}, L_{24}, L_{34}\},$$

and for $3 \leqslant i \leqslant 5$ there is not well-defined curve L_{34} on U_i (since $|L_0 - L_3 - L_4|$ has as its movable part a fibering by curves of genus 0 and so is one-dimensional), and here

$$\mathcal{C}(U_i) = \{L_1, L_2, L_3, L_4, L_{12}, L_{13}, L_{14}, L_{23}, L_{24}\}.$$

For $0 \leqslant i \leqslant 2$ the graph $C_1(U_i)$ is a Petersen graph (see Figure 15.1.3 in [5] or Figure 9.6 in [6]), and for $3 \leqslant i \leqslant 5$ the graph $C_1(U_i)$ is gotten from the graph $C_1(U_0)$ by removing the vertex 34. The graph $\vec{\mathcal{C}}(U_i)$ is gotten from $\bar{\mathcal{C}}(U_i)$ by attaching the set of arrows indicated in the ith column in Table 4.8.1, where a, b is the arrow $[a, b]$.

TABLE 4.8.1

0	1	2	3	4	5
	1,23	1,23	3,14	3,2	3,2
	2,31	2,31	12,24	4,2	4,2
	3,12	3,12	4,1	4,13	4,13
		4,3	4,3	12,13	12,13
		13,14		12,14	12,14
		23,24		3,14	3,14
					2,1
					13,23
					14,24

In Chapter IV, §4.9 of [2] it is remarked that $\mathcal{C}_1(U_0)$ "cannot be drawn so that its many symmetries become evident". The fact is that the natural ambient space for the Petersen graph is the real projective plane $\mathbf{P}_2(\mathbf{R})$, for this graph is gotten by factoring the graph of a dodecahedron by a central involution (see in [7], Figure 167, a picture of the Petersen graph in $\mathbf{P}_2(\mathbf{R})$). On U_0 there are five bases; the bases different from $\mathbf{L}$ are determined by sets of vertices $\{a, bc, cd, db\}$ (such a set we denote for short by (a)), where $\{a, b, c, d\} = \{1, 2, 3, 4\}$. The bases on U_0 correspond to the cubes inscribed in a dodecahedron; more precisely, they are the images of the sets of vertices of these cubes under the factorization mentioned above. Using the notation (a) just introduced $(1 \leqslant a \leqslant 4)$ and the notation (0) for $\mathbf{L}$, $\mathfrak{B}(U_i)$ is described by the ith column of the Table 4.8.2.

TABLE 4.8.2

0	1	2	3	4	5
(0)	(0)	(0)	(0)	(0)	(0)
(1)	(1)	(1)	(3)	(3)	(3)
(2)	(2)	(2)	(4)	(4)	(4)
(3)	(3)				
(4)					

The graph $\Gamma_1(U_0)$ is the one-dimensional skeleton of a four-dimensional simplex all of whose edges have weight zero, and $\Gamma_1(U_1)$ is the skeleton of a triangular pyramid the edges of whose base have weight one and whose side edges have weight zero. The weights of the edges of the triangles $\Gamma_1(U_i)$ ($2 \leqslant i \leqslant 5$) are described by the triples (in order of increasing i) $(2, 1, 1)$, $(1, 1, 0)$, $(1, 1, 1)$ and $(2, 2, 2)$.

To complete the description of U_i we make some remarks on the group $\mathrm{Aut}(U_i)$ of (biregular) automorphisms of the surface U_i. The connected component of the identity $\mathrm{Aut}(U_i)_0$ of this algebraic group and the group $\mathrm{Aut}(\Gamma_1(U_i))$ of automorphisms of the edge-weighted graph $\Gamma_1(U_i)$ can be put into an exact sequence (where all the homomorphisms are naturally defined):

$$\{1\} \to \mathrm{Aut}(U_i)_0 \to \mathrm{Aut}(U_i) \to \mathrm{Aut}(\Gamma_i(U_i)) \to \{1\};$$

that is, $\pi_0(\mathrm{Aut}(U_i)) \simeq \mathrm{Aut}(\Gamma_1(U_i))$. The last group is isomorphic to the symmetric group $\mathfrak{S}_n$, where $n = 5$ for $i = 0$, $n = 3$ for $i = 1, 4, 5$, and $n = 2$ for $i = 2, 3$. The group $\mathrm{Aut}(U_i)$

for $i = 0$ is trivial,

for $i = 1$ is the one-dimensional torus $\mathbf{G}_m$,

for $i = 2$ or 4 is two-dimensional and isomorphic to $\mathrm{Aut}(\mathbf{A}_1)$,

for $i = 3$ is the two-dimensional torus $\mathbf{G}_m \times \mathbf{G}_m$, and

for $i = 5$ is four-dimensional and isomorphic to the group of affine transformation of the form $x' = \alpha + \beta x$, $y' = y + a + bx$, $\{a, b, \alpha, \beta\} \subset k$, $\beta \neq 0$.

Using the morphism φ_0: $U_0 \to \mathbf{P}_2$ and the isomorphism $\mathfrak{S}_5 \simeq \mathrm{Aut}(U_0)$, one can get a representation of $\mathfrak{S}_5$ in Cr. The description of this representation in coordinates can be accomplished as follows (see [8], p. 21). On the line $\mathbf{P}_1$ fix homogeneous coordinates (x_0, x_1); on it we take the five points

$$Q_1 = (0, 1), \quad Q_2 = (1, 0), \quad Q_3 = (1, 1), \quad Q_4 = (1, x), \quad Q_5 = (1, y).$$

For any permutation σ of these five points we introduce new coordinates $(x_0, x_1)_\sigma$ on $\mathbf{P}_1$ such that

$$\sigma^{-1}(Q_1) = (0, 1)_\sigma, \quad \sigma^{-1}(Q_2) = (1, 0)_\sigma, \quad \sigma^{-1}(Q_3) = (1, 1)_\sigma.$$

The permutation σ corresponds to the Cremona transformation

$$x' = f_\sigma(x, y), \qquad y' = g_\sigma(x, y),$$

where f_σ and g_σ are rational functions such that

$$\sigma^{-1}(Q_4) = (1, f_\sigma)_\sigma, \quad \sigma^{-1}(Q_5) = (1, g_\sigma)_\sigma.$$

4.9. *A surface U for which* $\operatorname{rank} \mathrm{Pic}(U) = 5$ *and* $\#\mathfrak{B}(U) \geqslant 3$ *is isomorphic to one of the surfaces $U_0, \dots, U_5$ considered in* 4.7 *and* 4.8.

PROOF. On U fix a basis $\mathbf{L} = \{L_1, \dots, L_4\}$, put $\varphi = \varphi_{\mathbf{L}}$ and $P_i = \varphi(L_i)$, $1 \leqslant i \leqslant 4$. We shall use the notation L_{ij} and l_{ij} from 3.9. From the fact that any $\mathbf{M} \in \mathfrak{B}(U) \backslash \{\mathbf{L}\}$ is uniquely determined by the triple $\mathbf{L} \backslash \mathbf{M}$, it follows that $\#\mathfrak{B}(U) \leqslant 5$. Let $\mathfrak{B}(U) = \{\mathbf{L}, \mathbf{L}^1, \dots, \mathbf{L}^r\}$, where $r = \#B(U) - 1 \in [2, 4]$. Let us agree that $\mathbf{L}^i \cap \mathbf{L} = \{L_i\}$ for $1 \leqslant i \leqslant r$; in particular, the curve L_i, being the unique element of the exceptional set $\mathbf{L}^i \cap \mathbf{L}$, is exceptional, the basis $\mathbf{L}^i$ neighbors $\mathbf{L}$ along the set $\Lambda_i = \mathbf{L} \backslash \{L_i\}$ and $\mathbf{L}^i = \{L_i, L_{kl}, L_{jl}, L_{jk}\}$, where $\{i, j, k, l\} = \{1, 2, 3, 4\}$, $1 \leqslant i \leqslant r$.

If $r = 4$, then all the curves in $\mathbf{L}$ are irreducible, and φ is the blowing up of the four points $\varphi(\mathbf{L}) \subset \mathbf{P}_2$, in which each triple is not collinear, in view of 4.4.3 for $\mathbf{L}$ and Λ_i; therefore $U \simeq U_0$.

Suppose $r = 3$. Then $\mathfrak{B}(U)$ contains a basis consisting of irreducible curves. Indeed, L_1, L_2 and L_3 are irreducible. In case L_4 is reducible, say, for $L_4 > L_1$, the basis $\mathbf{L}^1$ suits us, since for $2 \leqslant i, j \leqslant 4$ the curve L_{ij} coincides with $\varphi'(l_{ij})$, and so is irreducible. Thus we assume that $r = 3$ and $L_1, \ldots, L_4$ are irreducible. That $\mathcal{B}(U)$ lacks a basis neighboring $\mathbf{L}$ along Λ_4 is explained only by violating condition 4.4.3 for $\mathbf{L}$ and Λ_4, i.e. by the collinearity of the points P_1, P_2, P_3; hence $U \simeq U_1$.

The rest of the proof will be carried out in §§4.9.1–4.9.7; in it we shall assume $r = 2$; in particular, that L_1 and L_2 are irreducible.

4.9.1. *At least one of the curves L_3 or L_4 is reducible.*

Indeed, if all $L_i \in \mathbf{L}$ are irreducible, then we will have $U \simeq U_0$, $r = 4$, if $\varphi(\mathbf{L})$ lacks collinear triples; if there is exactly one such triple, then $U \simeq U_1$ and $r = 3$; and if there is more than one collinear triple in $\varphi(\mathbf{L})$, then all quadruples in $\varphi(\mathbf{L})$ are collinear and $\#\mathcal{B}(U) = \#\{\mathbf{L}\} = 1 < 3$.

In the sequel we shall assume L_3 reducible.

4.9.2. *If L_4 is irreducible and $L_3 > L_4$, then $U \simeq U_2$.*

Proof. Here $P_4 = P_3$ and $\varphi(\mathbf{L}) = \{P_1, P_2, P_3\}$. The triple $\varphi(\mathbf{L})$ is collinear, since otherwise, when φ is represented as $\psi \circ \chi$, where ψ is the blowing up of $\varphi(\mathbf{L})$ and χ is the blowing up of a point $Q \in \psi^{-1}(P_3)$, we see that Q cannot lie outside the curve $\psi'(l_{13} + l_{23})$, since then $U \simeq U_1$ and $r = 3$, nor on this curve, since then Λ_1 or Λ_2 would violate condition 4.4.3. From this condition it also follows that the intersection of L_4 with the proper φ-preimage of the line l containing $\varphi(\mathbf{L})$ is empty, whence it is not hard to deduce that $U \simeq U_2$.

4.9.3 *If L_4 is irreducible and* $\operatorname{Comp}(L_3) \cap \{L_1, L_2\} \neq \varnothing$, *then $U \simeq U_3$.*

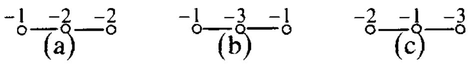
Proof. Suppose $L_3 > L_1$. The only cause of the lack of a basis joining $\mathbf{L}$ along Λ_4 is a violation of condition 4.4.3, i.e. the nonemptiness of $|L_0 - L_1 - L_2 - L_3|$. If $E \in |L_0 - L_1 - L_2 - L_3|$, then $E \cap L_4 = \varnothing$ (else $|L_0 - L_1 - L_3 - L_4| \neq \varnothing$); therefore, E is irreducible and $(E^2) = -2$. Hence it is clear that $U \simeq U_3$.

4.9.4. *If L_3 and L_4 are reducible and $L_3 \cap L_4 = \varnothing$, then $U \simeq U_3$.*

Proof. Here L_3 and L_4 have two components. We may assume that $L_3 > L_1$ and $L_4 > L_2$. Since Λ_2 and Λ_1 satisfy condition 4.4.3, it follows that $L_2 \cap L_{34} = L_1 \cap L_{34} = \varnothing$, so $U \simeq U_3$.

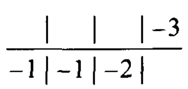
It remains to discuss the possibility that one of the curves L_3 or L_4 has more than two components. We make three remarks on such curves:

1) The dual graph of a contractible curve with three components can have only one of the following three forms:



2) From condition 4.4.2 for Λ_1 and Λ_2 it follows that neither L_3 nor L_4 can have the form (c).

3) From going through all ten graphs for contractible curves with four components it follows that if $\mathcal{C}(L) = \{L_1, L_2, M, L\}$, $\{L_1, L_2\} \subset \operatorname{Comp}(L)$, and neither of the inequalities $L \geqslant M + L_i$ $(i = 1, 2)$ holds, then L has the form



4.9.5. *If L_4 has the form* (a), *then $U \simeq U_2$.*

Proof. We may assume that $L_4 > L_3 > L_2$ and $L_1 \cap L_4 = \varnothing$. The system $|L_0 - L_4|$ (see 3.7) fibers U by curves of genus 0. Since the sets $L_1 \cap L_4$, $|L_0 - L_2 - L_3 - L_4|$ and

$|L_0 - L_1 - L_3 - L_4|$ are empty, it follows that this fibering has exactly two degenerate fibers one of which contains the curve L_1 and the other the curves L_2, L_3 and L_{34}. From this it is clear that $U \simeq U_2$.

4.9.6. *If L_3 has the form* (c) *and L_4 is irreducible, then $U \simeq U_4$.*

PROOF. L_4 does not occur in L_3; otherwise, either $L_3 \geqslant L_4 + L_1$ or $L_3 \geqslant L_4 + L_2$, contradicting 4.4.2 for Λ_1 and Λ_2. Since the sets $L_3 \cap L_4$, $|L_0 - L_2 - L_3 - L_4|$ and $|L_0 - L_1 - L_3 - L_4|$ are empty, it follows that the fibering of $|L_0 - L_3|$ contains exactly three degenerate fibers (one contains L_1, another L_2, and the third L_4). Hence it is clear that $U \simeq U_4$.

4.9.7. *If $\#\mathrm{Comp}(L_4) = 4$, then U is a surface of type U_5 (see* 0.8).

PROOF. By conditions 4.4.2 and 4.4.3 for Λ_1 and Λ_2, the curve L_4 has the form indicated in 3), and $L_1 \cap \varphi'(l_{34}) = L_2 \cap \varphi'(l_{34}) = \varnothing$, so $U \simeq U_5$.

§5. De Jonquières sets

5.1. DEFINITION. By a *de Jonquières set* (or *J-set*) on a surface V we mean a union $\mathfrak{L} \cup \mathfrak{L}'$ for which the following these conditions hold:

5.1.1. $\mathfrak{L} \subset \mathfrak{B}(V)$, $\mathfrak{L}' \subset \mathfrak{C}(V)$.

5.1.2. There is a bijection $\mathfrak{L} \to \mathfrak{L}'$ assigning to each basis $\mathbf{L}$ in $\mathfrak{L}$ a curve L_1 lying in this basis (the curve L_1 will be called the *distinguished curve* of the basis $\mathbf{L}$; we emphasize that the distinguished curve of a basis will be denoted by the same letter as the basis with the subscript 1, but not in boldface).

5.1.3. All classes of the form $L_0 - L_1$ in $\mathrm{Pic}(V)$ coincide; that is,

$$L_0 - L_1 \sim M_0 - M_1, \tag{5.1.3.1}$$

where $L, M \in \mathfrak{L}$, and L_1 and M_1 are distinguished curves in $\mathbf{L}$ and $\mathbf{M}$ respectively.

In 5.2 we shall show that all distinguished curves may be assumed *maximal* in the bases corresponding to them under the bijection of 5.1.2.

For bases $\mathbf{L}$ and $\mathbf{M}$ lying in a J-set let us agree to denote by $\mathbf{L(M)}$ the set $\mathbf{L}\backslash\mathbf{M}\backslash\{L_1\}$.

If for a set $\mathfrak{L} \subset \mathfrak{B}(V)$ there exists a subset $\mathfrak{L}' \subset \mathfrak{C}(V)$ which becomes a J-set after being united with some $\mathfrak{L}$, we shall say tha $\mathfrak{L}$ has a J-supplement (or, being somewhat careless, is a J-set), and $\mathfrak{L}'$ is a J-supplement to $\mathfrak{L}$.

5.1.4. Note that if on a surface V we have a family of J-sets $\{\mathfrak{L}_i \cup \mathfrak{L}_i' \mid i \in I\}$ such that the nerve of the sets $\{\mathfrak{L}_i \mid i \in I\}$ contained in $\mathfrak{B}(V)$ is connected and for all $i, j \in I$ the bijections $\mathfrak{L}_i \to \mathfrak{L}_i'$ and $\mathfrak{L}_j \to \mathfrak{L}_j'$ define identical mappings of the intersection $\mathfrak{L}_i \cap \mathfrak{L}_j$ onto $\mathfrak{C}(V)$, we can form the J-set $(\bigcup \mathfrak{L}_i) \cup (\bigcup \mathfrak{L}_i')$.

5.2. If for some basis $\mathfrak{L}$ in a J-set $\mathfrak{L}$ the distinguished curve L_1 is not maximal in $\mathbf{L}$, then by 5.1.3 and 3.7 in any other basis $\mathbf{M}$ in $\mathfrak{L}$ the distinguished curve M_1 is also not maximal because the system $|M_0 - M_1|$, coinciding with $|L_0 - L_1|$, will have a nonzero fixed part. Using 3.7 we can agree to change the distinguished curves in all the bases in $\mathfrak{L}$ so that the new distinguished curves become maximal in their corresponding bases. *In what follows we shall always assume this.*

In 5.3–5.7 we discuss the properties of J-sets consisting of two bases.

5.3. If $\{\mathbf{L}, \mathbf{M}\}$ is a J-set, then any transformation g in the class $\varphi(\mathbf{L}, \mathbf{M})$ (see 3.12) maps some pencil of lines in $\mathbf{P}_2$ to another such pencil; more precisely, if $g = \varphi_{\mathbf{L}} \circ \varphi_{\mathbf{M}}^{-1}$, then from (5.1.3.1) it follows that the generic member of the pencil of lines passing through $\varphi_{\mathbf{M}}(M_1)$ (where, we recall, M_1 is the distinguished curve of the basis $\mathbf{M}$) is mapped by the transformation g to a line passing through $\varphi_{\mathbf{L}}(L_1)$.

Conversely, if a Cremona transformation g is represented as $\varphi_1 \circ \varphi_{\mathbf{M}}^{-1}$, where $\{\mathbf{L}, \mathbf{M}\} \subset \mathfrak{B}(V)$, $\mathbf{L} \neq \mathbf{M}$, and g maps the pencil of lines passing through the point Q to the pencil of lines passing through P, then $\{\mathbf{L}, \mathbf{M}\}$ has a J-supplement, and the distinguished curves L_1 and M_1 can be chosen so that $\varphi_{\mathbf{L}}(L_1) = P$ and $\varphi_{\mathbf{M}}(M_1) = Q$. A Cremona transformation mapping some pencil of lines in $\mathbf{P}_2$ to another such pencil is called a *de Jonquières transformation* (see [9]). For an arbitrary point $Q \in \mathbf{P}_2$, J_Q will denote the set of de Jonquières transformations that map the pencil of lines through Q to this same pencil. It is clear that J_Q is a subgroup of Cr, $J_Q \cap \mathfrak{P} = \mathfrak{P}_Q$ is the stabilizer of the point Q in the projective group $\mathfrak{P}$, and $\mathfrak{P} J_Q \mathfrak{P}$ is the set of all de Jonquières transformations.

5.4. TWO EXAMPLES.

5.4.1. *Quadratic transformations are de Jonquières transformations.*

If $\mathbf{L}$ and $\mathbf{M}$ are neighbors in $\Gamma_1(V)$, $\mathbf{L} \backslash \mathbf{M} = \{L_1, L_2, L_3\}$, $\mathbf{M} \backslash \mathbf{L} = \{M_1, M_2, M_3\}$, and $M_i \sim L_{jk}$, where $\{i, j, k\} = \{1, 2, 3\}$ (see 3.9 and (4.3.8)), then as a J-supplement to $\{\mathbf{L}, \mathbf{M}\}$ we may take $\{L_i, M_i\}$, where L_i is maximal in $\mathbf{L}$ (and then, by 5.2, M_i is maximal in $\mathbf{M}$). Therefore, to the pair $\{\mathbf{L}, \mathbf{M}\}$, defining an edge of weight α ($0 \leqslant \alpha \leqslant 2$; see 4.5.3) in $\Gamma_1(V)$, we can choose a J-supplement in $(3 - \alpha)$ ways.

In 4.5.2 we noted the decomposition of the set Q into three double cosets with respect to $\mathfrak{P}$; here we note that $Q \cap J_Q$ (where $Q \in \mathbf{P}_2$; see 5.3) decomposes into four double cosets with respect to $\mathbf{P}_Q$; if we take coordinates (x_0, x_1, x_2) on $\mathfrak{P}_2$ in which $Q = (0, 0, 1)$, then these four classes can be represented by the transformations s_0, s_1, s_2 and $\varepsilon_{02} s_1 \varepsilon_{02}$; the first three are indicated in 4.5.2, and ε_{02} is the projective involution mapping x_0 to x_2.

5.4.2. *Cubic transformations are de Jonquières transformations.*

If $\mathbf{L}, \mathbf{M} \in \mathfrak{B}(V)$, $(L_0 \cdot M_0) = 3$ and $\mathbf{L} \cap \mathbf{M} = \varnothing$, then $M_0 \sim 3L_0 - \Sigma_1^r a_i L_i$, with $a_i > 0$ for $1 \leqslant i \leqslant r$. From $(M_0^2) = 1$ and $(M_0 \cdot K_V) = -3$ it follows that

$$\sum_{i=1}^{r} a_i^2 = 8, \qquad \sum_{i=1}^{r} a_i = 6;$$

therefore, one of the numbers a_i (say a_1) equals 2, and the others equal 1, $r = 5$; that is, $M_0 \sim 3L_0 - 2L_1 - L_2 - L_3 - L_4 - L_5$. Further, from $M \in \mathbf{M}$ it follows that $(M \cdot M_0) = 0$ and $(M^2) = (M \cdot K_V) = -1$; from this it is not hard to deduce that $M = \{M_1, L_{12}, L_{13}, L_{14}, L_{15}\}$, where M_1 is a curve whose class in Pic(V) is $2L_0 - \Sigma_1^5 L_i$. It is now clear that $L_0 - L_1 \sim M_0 - M_1$. Here the matrix (a_{ij}) introduced in 3.16 can have its columns put in the form 3.17.1 by some permutation of its rows. The pair $\{\mathbf{L}, \mathbf{M}\}$ from Example 3.17 is a de Jonquières set. Moreover, $\mathbf{L}$ and $\mathbf{M}$ in Example 3.18 also form a J-set since, by 3.18.1, for $i = 0, 1$,

$$M_i \sim (d - i)L_0 - (d - i - 1)L_1 - L_2 - \cdots - L_{2d-1},$$

whence $L_0 - L_1 \sim M_0 - M_1$.

5.5. The matrix 3.18.1 is the matrix (a_{ij}) for a J-supplement of a pair of bases without generic curves if we number the curves in these bases suitably. More precisely, *let* $\{\mathbf{L}, \mathbf{M}\}$ *be a J-set with two elements, L_1 and M_1 their distinguished curves, $d = (L_0 \cdot M_0)$, and Σ the summation over the elements of* $\mathbf{L}(\mathbf{M})$ (*see* 5.1). *Then*

$$M_0 \sim dL_0 - (d - 1)L_1 - \sum L, \tag{5.5.1}$$

$$M_1 \sim (d - 1)L_0 - (d - 2)L_1 - \sum L, \tag{5.5.2}$$

and

$$\#\mathbf{L}(\mathbf{M}) = \#\mathbf{M}(\mathbf{L}) = 2(d - 1) \tag{5.5.3}$$

and there is a bijection α: $\mathbf{M(L)} \to \mathbf{L(M)}$ *such that for any curve* M *in* $\mathbf{M(L)}$

$$M \sim L_0 - L_1 - \alpha(M). \tag{5.5.4}$$

PROOF. Since M_0 has zero intersection with all the curves in $\mathbf{M}$, in particular in $\mathbf{L} \cap \mathbf{M}$ (and, moreover, by 3.5, $\mathbf{L} \cap \mathbf{M} = \{L \mid L \in \mathbf{L},\ (L \cdot M_0) = 0\}$), M_0 can be expressed in terms of $\{L_0, L_1\} \cup \mathbf{L(M)}$, $M_0 \sim dL_0 - d_1L_1 - \Sigma d_L L$; in the last sum (over the elements of $\mathbf{L(M)}$) all the coefficients d_L are positive. From (5.1.3.1) we get

$$M_1 \sim (d-1)L_0 - (d_1 - 1)L_1 - \sum d_L L.$$

Since

$$1 = (M_0^2) = d^2 - d_1^2 - \sum d_L^2,$$
$$-1 = (M_1^2) = (d-1)^2 - (d_1 - 1)^2 - \sum d_L^2,$$
$$-3 = (K \cdot M_0) = -3d + d_1 + \sum d_L,$$

we have

$$d_1 = d - 1, \quad 2d - \sum d_L^2 = 2, \quad 2d - \sum d_L = 2.$$

From the last two equations and the positivity of the d_L it follows that $d_L = 1$ for all L in $\mathbf{L(M)}$. This proves (5.5.1)–(5.5.3).

Let $M \in \mathbf{M(L)}$ and $M \sim bL_0 - b_1L_1 - \Sigma b_L L$. From $(M \cdot (M_0 - M_1)) = 0$ and (5.1.3.1) it follows that $b = b_1$; then from $(M^2) = -1$ it follows that exactly one of the numbers b_L is 1, and the other b_L are zero. Put $L = \alpha(M)$; if $b_L = 1$ (i.e. if $(L \cdot M) = 1$), we get (5.5.4).

5.6. *If for a J-set* $\{\mathbf{L}, \mathbf{M}\}$ *we have* $(L_0 \cdot M_0) > 2$, *then the distinguished curve in the basis* $\mathbf{L}$ *is uniquely determined (for example, by the condition* $(L_1 \cdot M_0) = (L_0 \cdot M_0) - 1$ *or by the condition* $(L_1 \cdot M_0) > 1$).

This follows from (5.5.1).

5.7. The bijection α used in (5.5.4) reverses order; i.e., if $M', M'' \in \mathbf{M(L)}$ and $M' > M''$, then $\alpha(M') < \alpha(M'')$. Let us extend α to a one-to-one mapping, again denoted by α, $\{M_0\} \cup \mathbf{M} \to \{L_0\} \cup \mathbf{L}$, by mapping M_0 to L_0, M_1 to L_1, and having the rest of the elements of $\mathbf{L} \cap \mathbf{M}$ fixed. We shall also consider the automorphism $\alpha[\mathbf{L}, \mathbf{M}]$ of the quadratic module $\operatorname{Pic}(V)$ mapping the elements of the basis $\{M_0\} \cup \mathbf{M}$ into their images under α. The automorphism $\alpha[\mathbf{L}, \mathbf{M}]$ is an involution; more precisely, it is the symmetry with respect to the submodule of elements orthogonal to $\mathbf{M(L)}$ (or, what is the same, to $\mathbf{L(M)}$). We shall assume that $\alpha[\mathbf{L}, \mathbf{M}] = \alpha[\mathbf{M}, \mathbf{L}]$ and $\alpha[\mathbf{L}, \mathbf{L}] = \mathrm{id}$.

For any J-triple of bases $\{\mathbf{L}, \mathbf{M}, \mathbf{N}\}$ we have

$$\alpha[\mathbf{L}, \mathbf{N}] = \alpha[\mathbf{L}, \mathbf{M}]\alpha[\mathbf{M}, \mathbf{N}] = \alpha[\mathbf{N}, \mathbf{M}]\alpha[\mathbf{M}, \mathbf{L}]. \tag{5.7.1}$$

5.8. *For a J-triple* $\{L, M, N\}$ *we have*

$$\mathbf{L(M)} \Delta \mathbf{L(N)} = \alpha[\mathbf{L}, \mathbf{M}](\mathbf{M}, (\mathbf{N})). \tag{5.8.1}$$

PROOF. Changing the places of the bases $\mathbf{M}$ and $\mathbf{N}$ in (5.8.1) does not change its terms; this is evident for the left term, and for the right it follows from (5.7.1):

$$\alpha[\mathbf{L}, \mathbf{N}](\mathbf{N(M)}) = \alpha[\mathbf{L}, \mathbf{M}](\alpha[\mathbf{M}, \mathbf{N}](\mathbf{N(M)})) = \alpha[\mathbf{L}, \mathbf{M}](\mathbf{M(N)}).$$

Suppose that the curve L lies in the left-hand member of (5.8.1). By the symmetry in $\mathbf{M}$ and $\mathbf{N}$, we may assume that $L \in \mathbf{L(M)} \cap \mathbf{N}$ and prove that $\alpha[\mathbf{L}, \mathbf{M}](L) \in \mathbf{M(N)}$. If the last relation does not hold, then the elements $\alpha[\mathbf{L}, \mathbf{M}](L)$ lies in $\mathbf{M} \cap \mathbf{N}$ and is fixed under

$\alpha[\mathbf{N},\mathbf{M}]$; therefore, $\alpha[\mathbf{L},\mathbf{M}](L) \sim L_0 - L_1 - L$ implies $\alpha[\mathbf{N},\mathbf{M}](\alpha[\mathbf{M},\mathbf{L}](L)) \sim L_0 - L_1 - L$; from this and (5.7.1) it follows that $\alpha[\mathbf{L},\mathbf{N}](L) \sim L_0 - L_1 - L$, contradicting the hypothesis $L \in L(\mathbf{M}) \cap \mathbf{N} \subset \mathbf{L} \cap \mathbf{N}$.

Suppose the curve L lies in the right-hand member of (5.8.1). If L does not lie in the left member (that is, $L \in \mathbf{L}(\mathbf{M}) \cap \mathbf{L}(\mathbf{N})$), then, on the one hand, $\alpha[\mathbf{L},\mathbf{N}](L) \sim L_0 - L_1 - L$; on the other hand, by (5.7.1),

$$\begin{aligned}\alpha[\mathbf{L},\mathbf{N}](L) &= \alpha[\mathbf{N},\mathbf{M}](\alpha[\mathbf{M},\mathbf{L}](L))\\ &\sim M_0 - M_1 - \alpha[\mathbf{M},\mathbf{L}](\mathrm{L}) \sim L_0 - L_1 - \alpha[\mathbf{M},\mathbf{L}](L),\end{aligned}$$

whence it follows that $L = \alpha[\mathbf{L},\mathbf{M}](L)$; that is, $L \in \mathbf{L} \cap \mathbf{N}$, contradicting the fact that L belongs to $\mathbf{L}(\mathbf{N})$.

5.9. Suppose a J-set $\mathfrak{L}$ has a specially chosen element $\mathbf{L}$. To each pair $\{\mathbf{M},\mathbf{N}\}$ of elements of $\mathfrak{L}$ we assign the mark $\mu(\mathbf{M},\mathbf{N})$, a subset of $\mathbf{L}\backslash\{L_1\}$, by putting

$$\mu(\mathbf{M},\mathbf{N}) = \mu(\mathbf{N},\mathbf{M}) = \mathbf{L}(\mathbf{M})\Delta\mathbf{L}(\mathbf{N}). \tag{5.9.1}$$

From (5.5.3) and (5.8.1) we obtain

$$\#\mu(\mathbf{M},\mathbf{N}) = 2((M_0 \cdot N_0) - 1). \tag{5.9.2}$$

In particular,

5.9.3. *Two bases in $\mathfrak{L}$ are contiguous in $\Gamma_1(V)$ if and only if the pair generated by them is marked by a set with two elements.*

If $\mathbf{K}, \mathbf{M}$ and $\mathbf{N}$ are three bases from a J-set $\mathbf{L}$ pointed by $\mathbf{L}$, then

$$\mu(\mathbf{M},\mathbf{K}) = \mu(\mathbf{M},\mathbf{N})\Delta\mu(\mathbf{N},\mathbf{K}). \tag{5.9.4}$$

The mark of a pair of bases and one of the bases in this pair uniquely determine another basis and its distinguished curve, since if $\mathbf{M}$ and $\mu(\mathbf{M},\mathbf{N})$ are known, then, by (5.8.1),

$$\mathbf{M}(\mathbf{N}) = \alpha[\mathbf{L},\mathbf{M}](\mu(\mathbf{M},\mathbf{N})),$$

and the set $\mathbf{M}(\mathbf{N})$ is uniquely determined by the involution $\alpha[\mathbf{M},\mathbf{N}]$, the basis $\mathbf{N} = \alpha[\mathbf{M},\mathbf{N}](\mathbf{M})$, and the curve N_1, since

$$\{N_1\} = \mathbf{N}\backslash\alpha[\mathbf{M},\mathbf{N}](\mathbf{M}(\mathbf{N}) \cup (\mathbf{M} \cap \mathbf{N})).$$

5.10. If $\{\mathbf{L},\mathbf{M}\} \subset \mathfrak{B}(V)$ is a de Jonquières pair, then we can apply to the graphs $\mathcal{C}_1|(V)|\mathbf{L}\Delta\mathbf{M}$ and $\vec{\mathcal{C}}(V)|\mathbf{L}\Delta\mathbf{M}$ the description of the analogous pairs given in 3.18. But at one undistingusihed vertex of the graph $\vec{\mathcal{C}}(V)|\mathbf{L}\Delta\mathbf{M}$ no more than one arrow can enter, since if arrows going form L' and L'' enter at $L \in \mathbf{L}(\mathbf{M})$, then $L', L'' \in \mathbf{L}(\mathbf{M})$, the system $|L - L' - L''|$ is nonempty, but the intersection index of a member of this system with the members of the movable system $|M_0|$ is -1 by (5.5.1). For similar reasons, at a distinguished vertex (that is, at L_1 or M_1), no more than $(L_0 \cdot M_0) - 1$ arrows in $\vec{\mathcal{C}}(V)|\mathbf{L}\Delta\mathbf{M}$ can enter.

§6. The graph $\Delta_1(V)$

6.1. Below, in 6.2, we consider examples of surfaces V for which $\mathfrak{B}(V)$ has a J-supplement and consists of two bases without common curves. Later, in 7.14, we shall see that these surfaces V_0, V_1, V_2 and those indicated in 4.4 and 4.5 are exhausted by the surfaces V for which $\#\mathfrak{B}(V) = 2$ and $\mathfrak{B}(V) = \{\mathbf{L},\mathbf{M}\}$, $\mathbf{L} \cap \mathbf{M} = \varnothing$. Let us discuss the notation used to describe these examples. The surfaces occurring in the constructions will be the domains of definition of a composition of the form $\sigma_1 \circ \cdots \circ \sigma_r$ that take values in $\mathbf{P}_2$ and consist of blowings up σ_i with one-point centers P_i; $P_1 \in \mathbf{P}_2$. By E_i we shall denote both the curve $\sigma_i^{-1}(P_i)$ and its proper $(\sigma_{i+1} \circ \cdots \circ \sigma_j)$-preimages. On $\mathbf{P}_2$ we fix a line E_0 passing through P_1, and by E_0 we also denote the curves $(\sigma_1 \circ \cdots \circ \sigma_i)'(E_0)$. In the basis $\mathbf{L} = \mathbf{I}(\sigma_1 \circ \cdots \circ \sigma_r) = \{L_1, \ldots, L_r\}$, L_i denotes the curve $(\sigma_1 \circ \cdots \circ \sigma_r)^{-1}(P_i)$, $1 \leqslant i \leqslant r$.

The unique intersection point of curves A and B that intersect transversally will be denoted by $A \cdot B$.

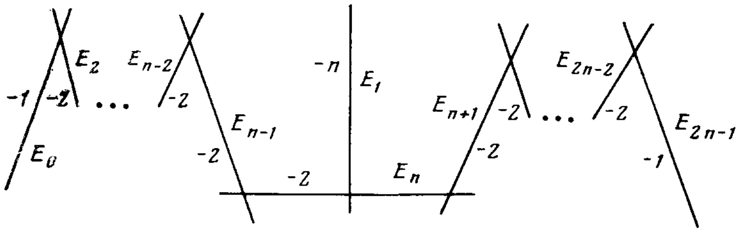


FIGURE 6.2.1

6.2. Suppose $n \geqslant 2$. Consider the surface V_n (more accurately called a "surface of type V_n"; see 0.8) gotten from $\mathbf{P}_2$ by successively blowing up points $P_1, \ldots, P_{2n-1}$, where $P_1 \in E_0$, $P_2 = E_0 \cdot E_1$, $P_i = E_i \cdot E_{i-1}$ for $3 \leqslant i \leqslant n$ and $P_j \in E_{j-1} \backslash (E_{j-2} + E_1)$ for $n < j < 2n$. Figure 6.2.1 shows the set of curves $\{E_0, E_1, \ldots, E_{2n-1}\}$ on V_n which coincide, as we have seen in 6.4, with $\operatorname{Neg}(V_n)$. Note that $\mathbf{L} = \mathcal{C}(L_1)$, $E_0 \sim L_0 - L_1 - L_2$, $E_i = L_i - L_{i+1}$ for $2 \leqslant i \leqslant 2n-2$, $E_i = L_1 - L_2 - \cdots - L_n$, $E_{2n-1} = L_{2n-1}$,

$$L_0 \sim E_0 + \sum_{i=1}^{n-1} iE_i + n \sum_{i=n}^{2n-1} E_i, \qquad L_1 = E_1 + \sum_{i=2}^{n-1} (i-1)E_i + (n-1) \sum_{i=n}^{2n-1} E_i,$$

and $L_k = \sum_{i=k}^{2n-1} E_i$ for $2 \leqslant k < 2n$. We take the following curve, which is symmetric with respect to L_1 (cf. the symmetry of Figure 6.2.1):

$$M_1 = E_1 + \sum_{i=n+1}^{2n-2} (2n-1-i)E_i + (n-1)\left(E_0 + \sum_{i=2}^{n} E_i\right),$$

and we put $\mathbf{M} = \mathcal{C}(M_1) = \{M_1, \ldots, M_{2n-1}\}$, where for $2 \leqslant i < 2n$

$$M_1 = E_1 + \sum_{i=n+1}^{2n-2} (2n-1-i)E_i + (n-1)\left(E_0 + \sum_{i=2}^{n} E_i\right).$$

For the class of M_0 we have

$$M_0 \sim E_{2n-1} + E_1 + \sum_{i=n+1}^{2n-2} (2n-i)E_i + n\left(E_0 + \sum_{i=2}^{r} E_i\right),$$

whence it follows that

$$M_0 \sim nL_0 - (n-1)L_1 - \sum_{i=2}^{2n-1} L_i,$$

$$M_1 \sim (n-1)L_0 - (n-2)L_1 - \sum_{i=2}^{2n-1} L_i, \qquad M_0 - M_1 \sim L_0 - L_1; \tag{6.2.2}$$

therefore $\{\mathbf{L}, \mathbf{M}\} \cup \{L_1, M_1\}$ is a de Jonquières set in the sense of Definition 5.1.

6.3. If we take morphisms $\varphi_{\mathbf{L}}$ and $\varphi_{\mathbf{M}}$ of a surface V_n to $\mathbf{P}_2$ and introduce coordinates (x_0, x_1, x_2) on $\mathbf{P}_2$ so that $\varphi_{\mathbf{L}}(L_1) = \varphi_{\mathbf{M}}(M_1) = (0,0,1)$ and $\varphi_{\mathbf{L}}(E_0) = \varphi_{\mathbf{M}}(E_{2n-1}) = [x_0]$, then the transformation $\varphi_{\mathbf{L}} \circ \varphi_{\mathbf{M}}^{-1}$ can be given by formulas of the form

$$x_0' = x_0^n, \quad x_1' = x_0^{n-1}(\alpha x_0 + \beta x_1), \quad x_2' = \gamma x_0^{n-1} x_2 + F_n(x_0, x_1), \tag{6.3.1}$$

where $\alpha, \beta, \gamma \in k$, $\beta\gamma \in k^*$, and $F_n(x_0, x_1)$ is a form in x_0 and x_1 of degree n with coefficients in k, with $F_n(0,1) \neq 0$. In the affine coordinates $x = x_1/x_0$, $y = x_2/x_0$, the transformation (6.3.1) can be rewritten as

$$x' = \alpha + \beta x, \qquad y' = \gamma y + f(x), \tag{6.3.2}$$

where $f(x) \in k[x]$ and $\deg f(x) = n$; (6.3.2) is called a triangular transformation of the affine plane $\mathbf{A}_2 = \mathbf{P}_2 \setminus [x_0]$.

Denote by T_n the group of triangular transformations (6.3.2) with $\deg f(x) \leqslant n$. In $\mathrm{Aut}(\mathbf{A}_2)$ we take the set Φ_n of all double cosets with respect to the group T_1 that have representatives in $T_n \setminus T_{n-1}$, and form the set Φ_n/θ gotten from Φ_n by factoring by the involution θ by factoring by the involution θ given by inversion in $\mathrm{Aut}(\mathbf{A}_2)$. The quotient set Φ_n/θ is in one-to-one correspondence with the set of surfaces of type V_n. Hence it is clear that up to isomorphism there is one surface V_2 and one surface V_3, while for $n \geqslant 4$ the surfaces of type V_n depend on continuous parameters.

6.4. $\mathrm{Neg}(V_n) = \{E_0, \ldots, E_{2n-1}\}$.

PROOF. Suppose a curve C is irreducible, different from all the E_i $(0 \leqslant i < 2n)$, and $(C^2) < 0$. For such a C we have

$$C \sim a_0 L_0 - \sum_{i=1}^{2n-1} a_i L_i, \qquad a_0 > 0, a_i \geqslant 0 \qquad (1 \leqslant i < 2n),$$

$a_0 - a_1 > 0$ (since $|L_0 - L_1|$ is a movable system whose degenerate member consists of curves E_i with $i \geqslant 2$),

$$\begin{gathered} a_0 - a_1 - a_2 = (C \cdot E_0) \geqslant 0, \\ a_1 - a_2 - \cdots - a_n = (C \cdot E_1) \geqslant 0, \\ a_i - a_{i+1} = (C \cdot E_i) \geqslant 0 \quad \text{for } 2 \leqslant i \leqslant 2n-2. \end{gathered}$$

The summands of the left side of the equation

$$3a_0 - \sum_{i=1}^{2n-1} a_i = 2 - 2\pi(C) + (C^2)$$

can be regrouped:

$$3(a_0 - a_1) + (a_1 - a_2 - \cdots - a_n) + (a_1 - a_{n+1} - \cdots - a_{2n-1}),$$

after which it is easy to see its incompatibility with the inequalities introduced above and with $\pi(C) \geqslant 0$.

6.5. From 6.4 follows

$$\mathcal{C}(V_n) = \mathcal{C}(E_0 + \cdots + E_{2n-1}) = \mathcal{C}(L_1) \cup \mathcal{C}(M_1) = \mathbf{L} \cup \mathbf{M};$$

therefore the graphs $\mathcal{C}_1(V_n)$ and $\bar{\mathcal{C}}(V_n)$ coincide with the graphs $\mathcal{C}_1(V) | \mathbf{L} \cup \mathfrak{M}$ and $\bar{\mathcal{C}}(V) | \mathbf{L} \cup \mathbf{M}$, respectively, described in 3.18, and the set of arrows of the graph $\vec{\mathcal{C}}(V_n) = \vec{\mathcal{C}}(V_n) | \mathbf{L} \Delta \mathbf{M}$ is shown in Figure 6.5.1. The last graph is extremal with respect to the restrictions on such graphs established in 5.10.

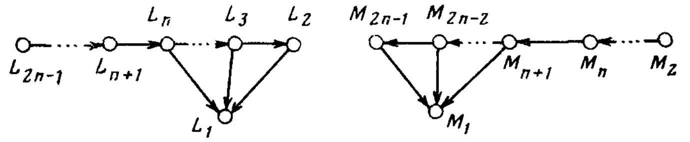


FIGURE 6.5.1

6.6. $\mathfrak{B}(V_n) = \{\mathbf{L}, \mathbf{M}\}$.

PROOF. Not all the curves of a basis on V_n can be contained in the set $\operatorname{Neg}(V_n)\setminus\{E_1\}$, since the maximal exceptional subsets of the latter are $\mathcal{C}(\Sigma_2^{2n-1}E_i)$ and

$$\mathcal{C}\left(E_0 + \sum_{i=2}^{j-1} E_i\right) \cup \mathcal{C}\left(\sum_{i=j+1}^{2n-1} E_i\right) \qquad (2 \leqslant j < 2n),$$

each of which contains $2n-2$ curves, whereas for a basis on V_n, $2n-1$ are required. Therefore, each basis on V_n contains a contractible curve containing E_1, which can be either L_1 or M_1; therefore, a basis on V_n coincides with either $\mathcal{C}(L_1)$ or $\mathcal{C}(M_1)$.

6.7. The set $\mathbf{L}$ and $\mathbf{M}$ are linearly ordered (see Figure 6.5.1). We shall show that *if on a surface V two bases $\mathbf{L}$ and $\mathbf{M}$ form a pair with a J-supplement and the sets $\mathbf{L}\setminus\mathbf{M}$ and $\mathbf{M}\setminus\mathbf{L}$ are linearly ordered by inclusion, then $V_{\mathbf{L}\cap\mathbf{M}} = \varphi_{\mathbf{L}\cap\mathbf{M}}(V)$ is a surface of type V_n, where* $n = (L_0 \cdot M_0)$.

PROOF. We may assume that $\mathbf{L} \cap \mathbf{M} = \varnothing$. Suppose that

$$\mathbf{L} = \{L_1, L_2, \ldots, L_{2n-1}\}, \qquad \mathbf{M} = \{M_1, M_2, \ldots, M_{2n-1}\},$$
$$L_1 > L_2 > \cdots > L_{2n-1}, \qquad M_1 > M_{2n-1} > \cdots > M_3 > M_2.$$

Because of their maximality the curves L_1 and M_1 are distinguised in the sense of 5.1; see 5.2. Further, for $2 \leqslant i < 2n$, by 5.7 and (5.5.4), $M_i \sim L_0 - L_1 - L_i$, and by (5.5.1),

$$M_0 \sim nL_0 - (n-1)L_1 - L_2 - \cdots - L_{2n-1}.$$

The sum of any two curves in the set $\mathbf{L}(\mathbf{M}) = \{L_2, \ldots, L_{2n-1}\}$ cannot occur in a curve belonging to it, since if $C = L_i - L_j - L_k \geqslant 0$, where $i, j, k \geqslant 2$, then the intersection index of C with the members of the movable system $|M_0|$ will be negative, a contradiction. Hence a maximal curve L_2 in $\mathbf{L}(\mathbf{M})$ has the form $E_2 + E_3 + \cdots + E_{2n-1}$, where the E_i are irreducible, $(E_{2n-1}^2) = -1$, $(E_i^2) = -2$ for $2 \leqslant i < 2n-1$, $E_i = L_i - L_{i+1}$, and the support of the sum $M_2 + L_1$ assumes the form indicated in Figure 6.2.1 if in this figure we replace E_0 by M_2 and n by k in the middle part of 6.2.1, where k is some number in $[2, 2n-1]$. We shall show that $k = n$. Since the divisor $L_1 - L_2 - \cdots - L_k$ is effective and its intersection index with M_0 is nonnegative, we have $k \leqslant n$; and from the now evident equation $\operatorname{supp} M_1 = M_2 + E_1 + \cdots + E_{2n-2}$ and symmetry considerations it follows that $k \geqslant n$.

6.8. For $n \geqslant 3$

$$\Gamma_1(V_n) = \mathbf{L} \circ \overset{n}{—} \circ \mathbf{M};$$

in particular, the graph $\Gamma_1(V_n)$ is not connected. We want to correct this deficiency of the functor Γ_1.

DEFINITION. To a surface V we assign the graph $\Gamma_1(V)$ whose set of vertices is $\mathfrak{B}(V)$, two vertices $\mathbf{L}$ and $\mathbf{M}$ being joined by an edge of weight n if and only if $V_{\mathbf{L}\cap\mathbf{M}}$ is a surface of type V_n, where V_0, V_1 and V_2 are the surfaces described in 4.4 and 4.5, and for $n \geqslant 2$ the surfaces of type V_n are described in 6.2–6.6. If $\mathbf{L}$ and $\mathbf{M}$ are joined by an edge in $\Delta_l(V)$, then we shall say the basis $\mathbf{M}$ is *contiguous with* (or that it *neighbors*) $\mathbf{L}$ along the set $\mathbf{L}\backslash\mathbf{M}$.

Clearly $\Gamma_l(V)$ is a subgraph of $\Delta_l(V)$.

6.9. *If* $\mathbf{L}$ *and* $\mathbf{M}$ *are contiguous in* $\Delta_l(V)$, *then*

6.9.1. $\{\mathbf{L}, \mathbf{M}\} = \varphi^{-1}_{\mathbf{L}\cap\mathbf{M}}(\mathfrak{B}(V_{\mathbf{L}\cap\mathbf{M}}))$,

6.9.2. *the pair* $\{\mathbf{L}, \mathbf{M}\}$ *has a J-supplement in the sense of* 5.1, *and for* $w[\mathbf{L}, \mathbf{M}] \geqslant 2$ *this J-supplement is uniquely determined and the sets* $\mathbf{L}\backslash\mathbf{M}$ *and* $\mathbf{M}\backslash\mathbf{L}$ *are linearly ordered, and*

6.9.3. *for any maximal curve* L_1 *of the set* $\mathbf{L}\backslash\mathbf{M}$ *there is precisely one J-supplement* $\{L_1, M_1\}$ *to* $\{\mathbf{L}, \mathbf{M}\}$ *containing it.*

6.10. SUPPLEMENT TO 4.4. *For* $\mathbf{L} \in \mathfrak{B}(V)$ *and for a given subset* $\Lambda \subset \mathbf{L}$ *there is a basis* $\mathbf{M}$ *contiguous with* $\mathbf{L}$ *along* Λ *such that* $w[\mathbf{L}, \mathbf{M}] = n \geqslant 3$ *if and only if the following conditions hold*:

6.10.0. $\#\Lambda = 2n - 1$.

6.10.1. Λ *is maximal in* $\mathbf{L}$.

6.10.2. Λ *is linearly ordered, i.e.* $\Lambda = \{L_1, \ldots, L_{2n-1}\}$ *with* $L_1 > L_2 > \cdots > L_{2n-1}$.

6.10.3. *In the notation of* 6.10.2

$$|L_1 - L_2 - \cdots - L_n| \neq \varnothing, \qquad |L_1 - L_2 - \cdots - L_{n+1}| = \varnothing.$$

6.10.4. $|L_i - L_j - L_k| = \varnothing$ *for* $i, j, k \geqslant 2$.

Conditions 6.10.2–6.10.4 can be replaced by the single condition

6.10.5. *The set of arrows of the graph* $\vec{\mathfrak{C}}(V)\,|\,\Lambda$ *has the form indicated in the left half of Figure* 6.5.1.

PROOF. The necessity follows from the construction of V_n; the sufficiency (in considering which one may assume $\mathbf{L} = \Lambda$, since, by 6.10.1, $\mathbf{L}\backslash\Lambda$ is exceptional) follows from the requirements on the points P_i indicated in 6.2 that are guaranteed by condition 6.10.5, $\varphi_{\mathbf{L}}$ consisting of blowings up of the P_i.

6.11. From 6.8 it follows that

$$\Delta_l(V_n) = \mathbf{L} \circ \overset{n}{—} \circ \mathbf{M}.$$

We shall discuss a surface with a somewhat more complicated graph Δ_l.

Suppose $n \geqslant 2$, σ: $W_n \to V_n$ *is the blowing up of a point* Q_1 *in general position* (*see* 0.7) *on the surface* V_n *described in* 6.2–6.6, *and* $\sigma^{-1}(\mathbf{L})$ (*respectively* $\sigma^{-1}(M)$) *is denoted by* $\mathbf{L}$ (*by* $\mathbf{M}$); *that is,* $\mathbf{L} = \{L_1, \ldots, L_{2n-1}, X\}$ *and* $\mathbf{M} = \{M_1, \ldots, M_{2n-1}, X\}$, *where* $X = \sigma^{-1}(Q_1)$ *and* L_i (M_i) *is the complete σ-preimage of the curve lying on* V_n *denoted by the same symbol. There exist bases* $\mathbf{A}, \mathbf{B}$ *in* $\mathfrak{B}(W_n)$ (*distinct for* $n \geqslant 3$ *and coincident for* $n = 2$) *such that* $(W_n)_{\mathbf{A}\cap\mathbf{B}}$ *is a surface of type* V_{n-1} *for* $n \geqslant 3$ *and is a projective plane for* $n = 2$, $A_0 \sim 2L_0 - L_1 - L_2 - X$,

$$B_0 \sim nL_0 - (n-1)L_1 - L_2 - \cdots - L_{2n-2} - X, \tag{6.11.0}$$

$\mathfrak{B}(W_n) = \{\mathbf{L}, \mathbf{M}, \mathbf{A}, \mathbf{B}\}$, $\mathfrak{B}(W_n)$ *has a J-supplement, the graph* $\Delta_l(W_n)$ ($n \geqslant 3$) *is*

$$\Delta_l(W_n) = \begin{array}{ccc} \mathbf{L} & \overset{n}{\circ\!—\!\circ} & \mathbf{M} \\ {\scriptstyle 1}\| & & |{\scriptstyle 1} \\ \mathbf{A} & \underset{n-1}{\circ\!—\!\circ} & \mathbf{B} \end{array} \tag{6.11.1}$$

and the surface W_2 *is isomorphic to the surface* U_2 *considered in* 4.7–4.8.

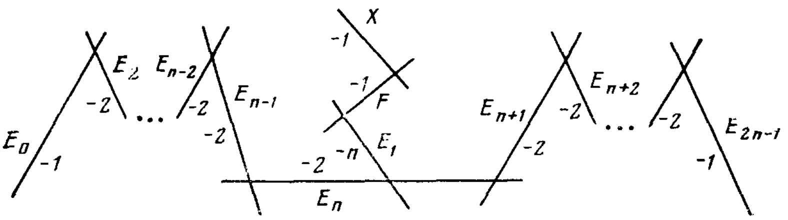


FIGURE 6.11.2

PROOF. Denote by E_i, $0 \leqslant i < 2n$, the proper σ-preimage of the curve on V_n denoted by the same symbol. W_n is fibered by the system $|L_0 - L_1| = |E_0 + \Sigma_2^{2n-1} E_i|$ of curves of genus 0, E_1 being a secant curve of this fibering. The fiber containing X is $X + F$, where F is an exceptional curve; see Figure 6.11.2. On W_n we take the curves

$$A_1 = F + E_1 + \sum_{i=3}^{n-1} (i-2)E_i + (n-2) \sum_{i=n}^{2n-1} E_i,$$

$$B_1 = F + E_1 + \sum_{i=n+1}^{2n-3} (2n-2-i)E_i + (n-2)\left(\sum_{i=2}^{n} E_i + E_0\right).$$

These curves are contractible,

$$A_1 \sim L_0 - L_2 - X, \qquad B_1 \sim (n-1)L_0 - (n-2)L_1 - \sum_{i=2}^{2n-2} L_i - X,$$

and the sets $\mathbf{A} = \{E_0\} \cup \mathcal{C}(A_1)$ and $\mathbf{B} = \{E_{2n-1}\} \cup \mathcal{C}(B_1)$ (coincident for $n = 2$) are bases on W_n. Clearly

$$\mathbf{L} \cap \mathbf{A} = \mathcal{C}\left(\sum_{i=3}^{2n-1} E_i\right), \qquad \mathbf{M} \cap \mathbf{B} = \mathcal{C}\left(\sum_{i=2}^{2n-3} E_i + E_0\right),$$

whence, by 4.3.1, we see that in $\Gamma_1(W_n)$ $\mathbf{L}$ is contiguous with $\mathbf{A}$, and $\mathbf{M}$ with $\mathbf{B}$. The contractions of the intersections $\mathbf{L} \cap \mathbf{A}$ and $\mathbf{M} \cap \mathbf{B}$ map W_n to V_1; therefore, $w[\mathbf{L}, \mathbf{A}] = w[\mathbf{M}, \mathbf{A}] = 1$. Further, $\mathbf{A} \cap \mathbf{B} = \{F, E_0, E_{2n-1}\}$; comparing Figures 6.11.2 and 6.2.1 shows that $(W_n)_{\mathbf{A} \cap \mathbf{B}}$ is a surface of type V_{n-1}. For the set $\{\mathbf{A}, \mathbf{B}, \mathbf{L}, \mathbf{M}\}$ a J-supplement is $\{A_1, B_1, L_1, M_1\}$. That $\mathfrak{B}(W_n)$ is exhausted by the bases constructed is proved like the analogous claim for $\mathfrak{B}(V_n)$ in 6.4–6.6; in particular, one must first verify that the whole set $\mathrm{Neg}(W_n)$ is depicted in Figure 6.11.2; we omit these details.

6.12. At the start of 6.8 we noted that the graph $\Gamma_1(V_n)$ is not connected when $n \geqslant 3$. However, after lifting two vertices of this graph to a suitable surface W dominating V_n, we can join them by a path in $\Gamma_1(W)$. More precisely, *if on the surface V_n we take $n - 2$ points $Q_1, \ldots, Q_{n-2}$ in general position (see 0.7) and blow up the set $R = \{Q_1, \ldots, Q_{n-2}\}$ by a morphism Σ: $W \to V_n$, then the bases $\Sigma^{-1}(\mathbf{L})$ and $\Sigma^{-1}(\mathbf{M})$ can be joined by a path in the graph $\Gamma_1(W)$ whose set of vertices has a J-supplement.*

PROOF. $\Sigma = \sigma \circ \mathrm{T}$, where σ is the blowing up of the point Q_1, and T is the blowing up of the set $\sigma^{-1}(R \backslash \{Q_1\})$. We shall use the notations and results of 6.11. Let $\varphi = \varphi_{\mathbf{A} \cap \mathbf{B}}$: $W_n \to V_{n-1}$. The composition $\varphi \circ \mathrm{T}$ is the blowing up of the set $\varphi(\mathbf{A} \cap \mathbf{B}) \cup R'$, where $R' = \varphi(\sigma^{-1}(R \backslash \{Q_1\}))$; therefore, it can be represented as $\Sigma_1 \circ \tau$, where Σ_1 is the blowing up of the set R', and τ is the blowing up of the set $\Sigma_1^{-1}(\varphi(\mathbf{A} \cap \mathbf{B}))$:

$$\begin{tikzcd}
W \arrow[r, "T"'] \arrow[rr, bend left, "\Sigma"] \arrow[d, "\tau"'] & W_n \arrow[r, "\sigma"'] \arrow[d, "\varphi"] & V_n \\
W' \arrow[r, "\Sigma_1"] & V_{n-1} &
\end{tikzcd}$$

The points of the set R' are in general position on V_{n-1}; therefore, inducting on n, in $\Gamma_1(W')$ we can pass a path ζ from $\Sigma_1^{-1}(\varphi(\mathbf{A}))$ to $\Sigma_1^{-1}(\varphi(\mathbf{B}))$; and then in $\Gamma(W)$ the bases $\Sigma^{-1}(\mathbf{L})$ and $\Sigma^{-1}(\mathbf{M})$ can be joined by the path

$$\left[\Sigma^{-1}(\mathbf{L}), T^{-1}(\mathbf{A})\right] \circ \tau^{-1}(\zeta) \circ \left[T^{-1}(\mathbf{B}), \Sigma^{-1}(\mathbf{M})\right].$$

The existence of a J-supplement for the set of vertices of ζ follows from the induction hypothesis and 5.1.4.

6.13. Here and in 6.14 we shall examine the conditions under which a chain of three bases in $\Delta_1(V)$ can be supplemented by a fourth to form a quadrangle induced by the quadrangle of $\Delta_1(W_n)$ in (6.11.1).

Suppose $[\mathbf{K}, \mathbf{L}, \mathbf{M}]$ *is a path in* $\Delta_1(V)$ *such that*

$$w[\mathbf{K},\mathbf{L}] = n \geqslant 3, \quad w[\mathbf{L},\mathbf{M}] = 1, \quad \#((\mathbf{L}\backslash\mathbf{K}) \cap (\mathbf{L}\backslash\mathbf{M})) = 2.$$

Then $V_{\mathbf{I}}$, *where* $\mathbf{I} = \mathbf{K} \cap \mathbf{L} \cap \mathbf{M}$, *is a surface of type* W_n; *therefore, there is a basis* $\mathbf{N}$ *on* V *such that*

$$\{\mathbf{K},\mathbf{L},\mathbf{M},\mathbf{N}\} = \varphi_{\mathbf{I}}^{-1}(\mathfrak{B}(V_{\mathbf{I}})),$$

$$\Delta_1(V)|\{\mathbf{K},\mathbf{L},\mathbf{M},\mathbf{N}\} = \begin{tikzcd}
\mathbf{K} \arrow[r, dash, "n"] \arrow[d, dash, "1"'] & \mathbf{L} \arrow[d, dash, "1"] \\
\mathbf{N} \arrow[r, dash, "n-1"'] & \mathbf{M}
\end{tikzcd}$$

$$N_0 \sim nL_0 - (n-1)L_1 - L_2 - \cdots - L_{2n-2} - X, \tag{6.13.1}$$

where $\mathbf{L}\backslash\mathbf{K} = \{L_1,\ldots,L_{2n-1}\}$, $L_1 > L_2 > \cdots > L_{2n-1}$ *and* $\mathbf{L}\backslash\mathbf{M} = \{L_1, L_2, X\}$.

Proof. We may assume $\mathbf{I} = \varnothing$ and prove that V is of type W_n. It is clear that $\mathbf{K} \cap \mathbf{L} = \{X\}$; therefore, the contraction φ_x maps V on a surface V_X of type V_n. Now we must only show that the point $\varphi_X(X)$ does not lie on curves in $\operatorname{Neg}(V_X)$. This follows from the fact that X is not contained in L_1 (since $w[\mathbf{L},\mathbf{M}] = 1$), and X is not contained in $L_{1,2}$ (see 3.9) since, by condition 4.4.3 for $\mathbf{L}\backslash\mathbf{M}$, we have $|L_0 - L_1 - L_2 - X| = \varnothing$.

6.14. *Suppose* $[\mathbf{K}, \mathbf{L}, \mathbf{M}]$ *is a path in* $\Delta_1(V)$ *such that the following conditions hold*:

a) $w[\mathbf{K},\mathbf{L}] = n \geqslant 2$ *and* $w[\mathbf{L},\mathbf{M}] = 1$.

b) $\#((\mathbf{L}\backslash\mathbf{K}) \cap (\mathbf{L}\backslash\mathbf{M})) = 1$.

c) $|L_0 - L_1 - L_2 - Y| = \varnothing$, *where* $\mathbf{L}\backslash\mathbf{K} = \{L_1,\ldots,L_{2n-1}\}$, $L_1 > L_2 > \cdots > L_{2n-1}$ *and* $\mathbf{L}\backslash\mathbf{M} = \{L_1, X, Y\}$, $L_1 > X$.

d) $\mathbf{L} \cap \mathbf{K}$ *contains a maximal curve* L_{2n} *for which* $|L_{2n-1} - L_{2n}| \neq \varnothing$ *and* $|L_{2n-2} - L_{2n-1} - L_{2n}| = \varnothing$.

Then the set $\mathbf{I} = (\mathbf{K} \cap \mathbf{L} \cap \mathbf{M}) \setminus \{L_{2n}\}$ *is exceptional, and* $V_{\mathbf{I}}$ *is a surface of type* W_{n+1}; *therefore, there is a basis* $\mathbf{N} \in \mathfrak{B}(V)$ *such that*

$$\{\mathbf{K}, \mathbf{L}, \mathbf{M}, \mathbf{N}\} = \varphi_{\mathbf{I}}^{-1}(\mathfrak{B}(V_{\mathbf{I}})),$$

$$\Delta_1(V) | \{\mathbf{K}, \mathbf{L}, \mathbf{M}, \mathbf{N}\} = \begin{array}{ccc} \mathbf{K} \circ & \overset{n}{\text{---}} & \circ \mathbf{L} \\ 1 | & & | 1 \\ \mathbf{N} \circ & \underset{n+1}{\text{---}} & \circ \mathbf{M} \end{array}$$

$$N_0 \sim (n+1)L_0 - nL_1 - \sum_{i=2}^{2n} L_i - X. \tag{6.14.1}$$

PROOF. Clearly, the curve L_{2n} lies in the intersection $\mathbf{K} \cap \mathbf{L} \cap \mathbf{M}$ and is maximal in it, so $\mathbf{I}$ is exceptional. We may assume $\mathbf{I} = \varnothing$ and prove that V is of type W_{n+1}. Suppose $\mathbf{J} = \mathbf{K} \cap \mathbf{L}$ and $\varphi = \varphi_{\mathbf{J}}$. Evidently $\mathbf{J} = \{X, Y, L_{2n}\}$, and $V_{\mathbf{J}}$ is a surface of type V_n. On $V_{\mathbf{J}}$ we take curves $E_0 = \varphi(L_{1,2})$ and $E_i = \varphi([L_i])$, $1 \leqslant i < 2n$, whose position is described in Figure 6.2.1. We must show that

$$\varphi(X) \in E_1 \setminus E_n, \quad \varphi(Y) \in E_0 \setminus E_2, \quad \varphi(L_{2n}) \in E_{2n-1} \setminus E_{2n-2}.$$

The first relation follows from the fact that, by c), L_1 contains X but L_2 does not (see b) and c)); the second follows from c) and the fact that L_2 does not contain Y (again, see b) and c)); and the third follows from d).

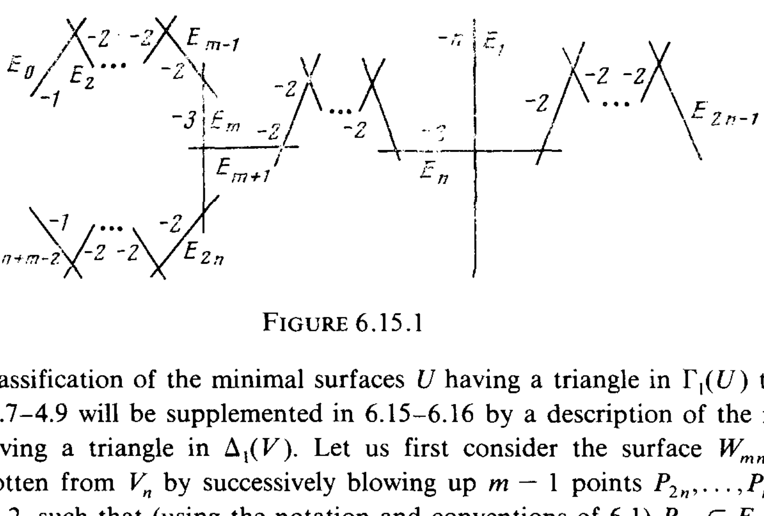


FIGURE 6.15.1

6.15. The classification of the minimal surfaces U having a triangle in $\Gamma_1(U)$ that was presented in 4.7–4.9 will be supplemented in 6.15–6.16 by a description of the minimal surfaces V having a triangle in $\Delta_1(V)$. Let us first consider the surface W_{mn} (where $2 \leqslant m \leqslant n$) gotten from V_n by successively blowing up $m-1$ points $P_{2n}, \ldots, P_r$, where $r = 2n + m - 2$, such that (using the notation and conventions of 6.1) $P_{2n} \in E_m \setminus (E_{m-1} + E_{m+1} + E_0 + E_1)$ and $P_i \in E_{i-1} \setminus (E_{i-2} + E_m)$ for $2n < i \leqslant r$. We argue as in 6.4, proving that $\mathrm{Neg}(W_{mn}) = \{E_0, \ldots, E_r\}$ (see Figure 6.15.1).

Note that $W_{2,2}$ is a surface of type U_5 in 4.7 and 4.8. If the curve $(\sigma_{2n} \circ \cdots \circ \sigma_r)^{-1}(L_1)$ is denoted by L_1, the curve $(\sigma_{2n} \circ \cdots \circ \sigma_r)^{-1}(M_1)$ by M_1, and on W_{mn} we take the curve

$$K_1 = E_1 + (n-1)\sum_{i=n}^{2n-1} E_i + (m-1)\left(E_0 + \sum_{i=2}^{m-1} E_i\right)$$
$$+ \sum_{i=2n}^{r} (r-i)E_i + \sum_{i=m}^{n-1} (i-1)E_i,$$

then $\mathbf{L} = \mathcal{C}(L_1)$, $\mathbf{K} = \mathcal{C}(K_2)$ and $\mathbf{M} = \mathcal{C}(M_1)$ are bases on W_{mn}, $\mathfrak{B}(W_{mn}) = \{\mathbf{K}, \mathbf{L}, \mathbf{M}\}$, $\Delta_1(W_{mn})$ is a triangle, and

$$w[\mathbf{K}, \mathbf{L}] = m, \qquad w[\mathbf{L}, \mathbf{M}] = w[\mathbf{M}, \mathbf{K}] = n.$$

A surface of type W_{mn} can also be gotten from V_n by blowings up with a different arrangement of the centers $P_{2n}, \ldots, P_r$ $(r = 2n + m - 2)$, namely,

$$P_{2n} \in E_{2n-m} \backslash (E_{2n-1} + E_{2n-m-1} + E_{2n-m-1} + E_{2n-m+1} + E_1),$$
$$P_i \in E_{i-1} \backslash (E_{i-2} + E_{2n-m}) \qquad (2n < i \leqslant r).$$

If we introduce L_1, M_1, $\mathbf{L}$ and $\mathbf{M}$ as above and put

$$K_1 = E_1 + (n-1)\left(E_0 + \sum_{i=2}^{n} E_i\right) + \sum_{i=n+1}^{2n-m} (2n-1)E_1$$
$$+ \sum_{i=2n}^{r} (r-i)E_i + (m-1) \sum_{i=2n-m+1}^{2n-1} E_i,$$

$\mathbf{K} = \mathcal{C}(K_1)$, then $\{\mathbf{K}, \mathbf{L}, \mathbf{M}\}$ is the set of all bases of the surface thus constructed in whose graph Δ_1 we have

$$w[\mathbf{K}, \mathbf{L}] = w[\mathbf{L}, \mathbf{M}] = n, \qquad w[\mathbf{M}, \mathbf{K}] = m.$$

In both variants of the construction of W_{mn} the triple $\{\mathbf{K}, \mathbf{L}, \mathbf{M}\}$ has J-supplement $\{K_1, L_1, M_1\}$.

6.16. We supplement 4.6 and 4.9 with the following assertions.

6.16.1. *If* $\{\mathbf{K}, \mathbf{L}\}$ *and* $\{\mathbf{L}, \mathbf{M}\}$ *are neighboring bases in* $\Delta_1(V)$, $w[\mathbf{K}, \mathbf{L}] \geqslant 2$, $w[\mathbf{L}, \mathbf{M}] \geqslant 2$ *and* $\#((\mathbf{L}\backslash\mathbf{K}) \cap (\mathbf{L}\backslash\mathbf{M})) \geqslant 2$, *then* $\Delta_1(V) \mid \{\mathbf{K}, \mathbf{L}, \mathbf{M}\}$ *is a triangle, and* $V_{\mathbf{I}}$ *is a surface of type* W_{ab}, *where* $\mathbf{I} = \mathbf{K} \cap \mathbf{L} \cap \mathbf{M}$ *and*

$$b = \max\{(L_0 \cdot K_0), (L_0 \cdot M_0)\}, \quad a = (L_0 \cdot K_0) + (L_0 \cdot M_0) - \#((\mathbf{L}\backslash\mathbf{K}) \cap (\mathbf{L}\backslash\mathbf{M})).$$

6.16.2. *If* $\Delta_1(V) \mid \{\mathbf{K}, \mathbf{L}, \mathbf{M}\}$ *is a triangle and* $\mathbf{I} = \mathbf{K} \cap \mathbf{L} \cap \mathbf{M}$, *then* $V_{\mathbf{I}}$ *is either isomorphic to one of the surfaces* $U_0, \ldots, U_4$ *in* 4.7 *and* 4.8 *or is a surface of type* W_{ab}, *where* a *and* b *are given in* 6.16.1.

PROOF OF 6.16.1. We may assume that $\mathbf{I} = \varnothing$, $n \geqslant m$, $\mathbf{L}\backslash\mathbf{K} = \{L_1, \ldots, L_{2n-1}\}$, $\mathbf{L}\backslash\mathbf{M} = \{X_1, \ldots, X_{2m-1}\}$, $L_1 > L_2 > \cdots > L_{2n-1}$ and $X_1 > X_2 > \cdots > X_{2m-1}$. These two chains of contractible curves cannot be enlarged; i.e., there is no contractible curve C with $L_i > C > L_{i+1}$ (or $X_i > C > X_{i+1}$), since the curve $\operatorname{supp} L_i - \operatorname{supp} L_{i+1}$ is irreducible. Since the sets $\mathbf{L}\backslash\mathbf{K}$ and $\mathbf{L}\backslash\mathbf{M}$ are maximal in $\mathbf{L}$, so is their intersection, which therefore has the form $\{L_1, \ldots, L_r\}$, where $L_1 = X_1, \ldots, L_r = X_r$. By hypothesis, $r \geqslant 2$.

We shall prove that $r \geqslant m$. If $r + 1 \leqslant m$, then both the curves X_{r+1} and L_{r+1} intersect both $[L_1]$ and $[L_r]$. From the absence of triple points in $\operatorname{supp} L_1$ it follows that $[L_1] \cap [L_r] \cap (X_{r+1} + L_{r+1})$ is empty, and from the 1-connectivity of the dual graph of the curve L_1 it follows that $X_{r+1} \cap L_{r+1} \neq \varnothing$; from this and 1.6.4 we get either $X_{r+1} \leqslant L_{r+1}$ or $L_{r+1} \leqslant X_{r+1}$. Now the fact that the chains $L_r > L_{r+1}$ and $X_r > X_{r+1}$ cannot be enlarged entails $X_{r+1} = L_{r+1}$, contradicting the definition of r.

Thus $r \geqslant m$. Clearly $r \leqslant 2m - 1$. We can write

$$\begin{aligned} \mathbf{L} &= \{L_1, L_2, \ldots, L_{2n-1}, X_{r+1}, \ldots, X_{2m-1}\}, \\ \mathbf{K} &= \{K_1, K_2, \ldots, K_{2n-1}, X_{r+1}, \ldots, X_{2m-1}\}, \\ \mathbf{M} &= \{L_{r+1}, \ldots, L_{2n-1}, M_1, M_2, \ldots, M_{2m-1}\}, \end{aligned} \tag{6.16.3}$$

where $K_i \sim L_0 - L_1 - L_i$ for $2 \leqslant i < 2n$ and $M_j \sim L_0 - L_1 - X_j$ for $2 \leqslant j < 2m$. From this and 1.5 it is evident that $K_i = M_i$ for $2 \leqslant i \leqslant r$, and

$$\mathbf{K} \cap \mathbf{M} = \{K_2, \ldots, K_r\},$$
$$\mathbf{K}\backslash\mathbf{M} = \{K_1, K_{2n-1}, \ldots, K_{r+1}, X_{r+1}, \ldots, X_{2m-1}\},$$
$$\mathbf{M}\backslash\mathbf{K} = \{M_1, M_{2m-1}, \ldots, M_{r+1}, L_{r+1}, \ldots, L_{2n-1}\}.$$

The triple $\{\mathbf{K}, \mathbf{L}, \mathbf{M}\}$ has the triple $\{K_1, L_1, M_1\}$ as a J-supplement. To prove that $\mathbf{K}$ and $\mathbf{M}$ are neighbors in $\Delta_1(V)$ it suffices, by 6.7, to establish that $\mathbf{K}\backslash\mathbf{M}$ and $\mathbf{M}\backslash\mathbf{K}$ are linearly ordered, for which it suffices to prove the inequalities $K_{r+1} > X_{r+1}$ and $M_{r+1} > L_{r+1}$, i.e. to prove the effectiveness of the classes $K_{r+1} - X_{r+1}$ and $M_{r+1} - L_{r+1}$, coinciding with $L_0 - L_1 - L_{r+1} - X_{r+1}$. But the class $L_0 - L_1 - L_2$ is effective ($\sim M_2$), and so are $L_2 - L_3, \ldots, L_{r-1} - L_r$, $L_r - L_{r+1} - X_{r+1}$ (the last follows from the nonemptiness of $[L_r] \cap L_{r+1}$ and $[L_r] \cap X_{r+1}$). By putting together classes enumerated, we get what is required.

Note that either $r = m$ or $r > m = n$, since $|L_1 - L_2 - \cdots - L_m| \neq \varnothing$ but $|L_1 - L_2 - \cdots - L_{m+1}| = \varnothing$. That V is a surface of type W_{ab} follows from (6.16.3) and 6.15.

PROOF OF 6.16.2. We may assume that the given triangle is different from those considered in 4.6–4.9 containing an edge of weight $n \geqslant 3$, say $n = w[\mathbf{L}, \mathbf{K}]$,

$$\mathbf{L}\backslash\mathbf{K} = \{L_1, L_2, \ldots, L_{2n-1}\}, \qquad \mathbf{K}\backslash\mathbf{L} = \{K_1, K_2, \ldots, K_{2n-1}\},$$
$$L_1 > L_2 > \cdots > L_{2n-1}, \qquad K_1 > K_{2n-1} > K_{2n-2} > \cdots > K_2.$$

Put $m = (L_0 \cdot M_0)$ and $\mathbf{L}\backslash\mathbf{M} = \{X_1, \ldots, X_{2m-1}\}$, where X_1 is a maximal curve in $\mathbf{L}\backslash\mathbf{M}$.

Let us show that $\mathbf{L}_1 \in \mathbf{L}\backslash\mathbf{M}$. If $\mathbf{L}\backslash\mathbf{M}$ does not contain L_1, then, since it is maximal in $\mathbf{L}$, it contains none of curves L_i, so $(\mathbf{L}\backslash\mathbf{M}) \cap (\mathbf{L}\backslash\mathbf{K}) = \varnothing$. From 5.5.1–5.5.4 we get

$$M_0 \sim mnK_0 - m(n-1)k_1 - (m-1)X_1 - m\sum_{i=2}^{2n-1} K_i - \sum_{i=2}^{2m-1} X_i,$$

whence it is clear that $\{\mathbf{K}, \mathbf{M}\}$ has no J-supplement.

Thus $L_1 \in \mathbf{L}\backslash\mathbf{M}$, and L_1 is maximal in $\mathbf{L}$. By 6.9.3, $\mathbf{M}$ has a curve M_1 such that $M_0 - M_1 \sim L_0 - L_1$; from this and 5.1.4 it follows that $\{K_1, L_1, M_1\}$ is a J-supplement to $\{\mathbf{K}, \mathbf{L}, \mathbf{M}\}$.

To prove that V_1 is of type W_{ab}, by 6.16.1 it suffices to show that $\mathbf{L}(\mathbf{K}) \cap \mathbf{L}(\mathbf{M})$ is not empty and $w[\mathbf{L}, \mathbf{M}] \geqslant 2$. But if this intersection is empty or $w[\mathbf{L}, \mathbf{M}] \leqslant 1$, then, by (5.9.1) and (5.9.2),

1) $(K_0 \cdot M_0) = \#(\mathbf{L}(\mathbf{K})\Delta\mathbf{L}(\mathbf{M}))/2 + 1 \geqslant n \geqslant 3$,

2) the set $\mathbf{L}(\mathbf{K})\Delta\mathbf{L}(\mathbf{M})$ is not linearly ordered by inclusion, and

3) the set $\mathbf{M}(\mathbf{K})$, which by (5.8.1) consists of curves whose classes are $L_0 - L_1 - L$, where $L \in \mathbf{L}(\mathbf{K})\Delta\mathbf{L}(\mathbf{M})$, is also not linearly ordered.

It remains to note that 1) and 3) contradict $\mathbf{K}$ being contiguous to $\mathbf{M}$.

6.17. An *ordered* quadruple of bases $\{\mathbf{K}, \mathbf{L}, \mathbf{M}, \mathbf{N}\}$ will be called a *quadrangle* if $\Delta_1(V)$ contains edges $[\mathbf{K}, \mathbf{L}]$, $[\mathbf{L}, \mathbf{M}]$, $[\mathbf{M}, \mathbf{N}]$ and $[\mathbf{N}, \mathbf{K}]$. Clearly, if in such a quadruple we reverse the order or change it cyclically, the quadruple becomes a quadrangle. If there exists an edge joining opposite vertices of a quadrangle, this edge will be called a *diagonal*. Here and in 6.18 and 6.19 we construct some surfaces V with quadrangles without diagonals in $\Delta_1(V)$.

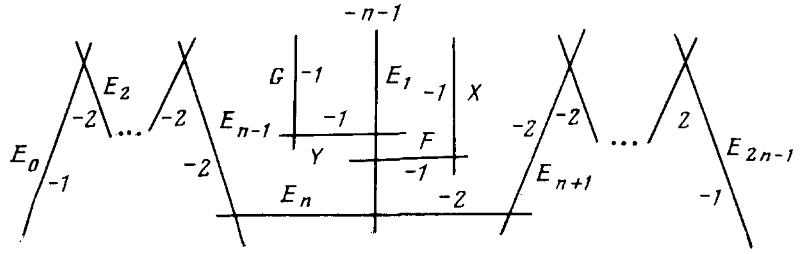


FIGURE 6.17.1

We take the surface W_n in 6.11 and the blowing up $\sigma\colon W_n^0 \to W_n$ of a point Q lying in $E_1 \setminus (F + E_n)$ (see Figure 6.11.2); denote by Y the curve $\sigma^{-1}(Q)$, by G the exceptional curve on W_n^0 which, together with Y, forms the fiber $Y + G$ of the fibering $|F + X|$ by curves of genus 0, and by $F, X, E_0, \ldots, E_{2n-1}$ the proper σ-preimages of these curves on W_n; we shall write $\mathbf{L}, \mathbf{M}, \mathbf{A}, \mathbf{B}$ instead of $\sigma^{-1}(\mathbf{L}), \ldots, \sigma^{-1}(\mathbf{B})$ respectively. As in 6.4 we can show that (see Figure 6.17.1)

$$\operatorname{Neg}(W_n^0) = \{F, G, X, Y, E_0, \ldots, E_{2n-1}\}. \tag{6.17.0}$$

The contraction τ of the curve F maps W_n^0 on a surface of type W_n; it turns out that bases $\mathbf{A}$ and $\mathbf{B}$ in $\mathfrak{B}(W_n^0)$ can be considered as the τ-preimages of the bases in $\mathfrak{B}(\tau(W_n^0))$ denoted in the same way. We take bases $\mathbf{K}$ and $\mathbf{N}$ on W_n^0 such that $\{\mathbf{K}, \mathbf{A}\}$ and $\{\mathbf{N}, \mathbf{B}\}$ are pairs of neighboring bases, $\{\mathbf{K}, \mathbf{N}, \mathbf{A}, \mathbf{B}\} = \tau^{-1}(\mathfrak{B}(\tau(W_n^0)))$. More explicitly, $\mathbf{K} = \{G\} \cup \mathcal{C}(K_1)$ and $\mathbf{N} = \{G\} \cup \mathcal{C}(N_1)$, where

$$K_1 = F + E_1 + \sum_{i=2}^{n-1} (i-1)E_i + (n-1)\sum_{i=n}^{2n-1} E_i,$$

$$N_1 = F + E_1 + (n-1)\left(E_0 + \sum_{i=2}^{n} E_i\right) + \sum_{i=n+1}^{2n-1} (2n-1-i)E_i.$$

Note that

$$K_0 \sim 2L_0 - L_1 - X - Y, \qquad N_0 \sim (n+1)L_0 - nL_1 - \sum_{i=2}^{2n-1} L_i - X - Y. \tag{6.17.2}$$

(6.17.0) implies that each basis on W_n^0 contains a curve with component E_1, whence $B(W_n^0) = \{\mathbf{K}, \mathbf{L}, \mathbf{M}, \mathbf{N}, \mathbf{A}, \mathbf{B}\}$.

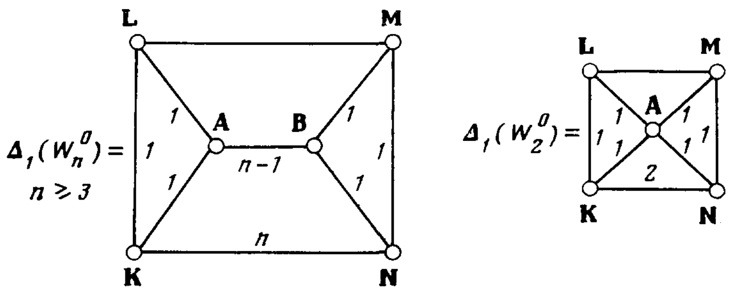


FIGURE 6.17.3

6.18. Suppose $n, m \geqslant 2$ and $r = 2n + 2m - 3$. Consider the surface V_{mn} gotten from V_n (see 6.2) by successive blowings up $\sigma_{2n}, \ldots, \sigma_r$ of $2(m-1)$ points $P_{2n}, \ldots, P_r$, where (see 6.1 in connection with the notation used) $P_{2n} \in E_1 \setminus E_n$, $P_i = E_i \cdot E_{i-1}$ for $2n < i \leqslant 2n + m - 2$, and $P_i \in E_{i-1} \setminus (E_1 + E_{i-2})$ for $2n + m - 1 \leqslant i \leqslant r$. Suppose F is an irreducible rational curve with selfintersection on V_n that intersects E_1 transversally in the point P_{2n}, and let F_0 be its proper preimage on V_{mn}. As in 6.4, $\mathrm{Neg}(V_{mn}) = \{F_0, E_0, \ldots, E_r\}$ (see Figure 6.18.1). Let

$$E = E_0 + \sum_{i=0}^{r} E_i, \quad \mathbf{L} = \mathcal{C}(L_1), \quad \mathbf{K} = \mathcal{C}(K_1), \quad \mathbf{M} = \mathcal{C}(M_1), \quad \mathbf{N} = \mathcal{C}(N_1)$$

where

$$\begin{aligned} \operatorname{supp} L_1 &= E - (E_0 + F_0), & \operatorname{supp} M_1 &= E - (F_0 + E_{2n-1}), \\ \operatorname{supp} K_1 &= E - (E_0 + E_r), & \operatorname{supp} N_1 &= E - (E_r + E_{2n-1}). \end{aligned}$$

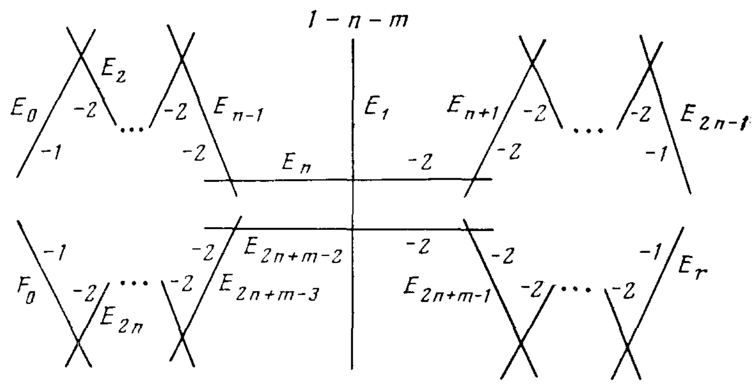


FIGURE 6.18.1

Note that

$$\begin{aligned} K_0 &\sim mL_0 - (m-1)L_1 - \sum_{i=2n}^{r} L_i, \\ N_0 &\sim (n+m-1)L_0 - (n+m-2)L_1 - \sum_{i=2} L_i. \end{aligned} \tag{6.18.2}$$

From the description of the set $\mathrm{Neg}(V_{mn})$ it follows that

$$\mathcal{B}(V_{mn}) = \{\mathbf{K}, \mathbf{L}, \mathbf{M}, \mathbf{N}\},$$

$$\Delta_1(V_{mn}) = \begin{array}{ccc} \mathbf{K} \circ & \overset{m}{\text{—}} & \circ \mathbf{L} \\ n | & & | n \\ \mathbf{N} \circ & \underset{m}{\text{—}} & \circ \mathbf{M} \end{array} \tag{6.18.3}$$

6.19. *If* $[\mathbf{K}, \mathbf{L}, \mathbf{M}]$ *is a path in* $\Delta_1(V)$, $w[\mathbf{K}, \mathbf{L}] \geqslant 2$, $w[\mathbf{L}, \mathbf{M}] \geqslant 2$,

$$\#((\mathbf{L} \setminus \mathbf{K}) \cap (\mathbf{L} \setminus \mathbf{M})) = 1, \tag{6.19.1}$$

and $\mathbf{I} = \mathbf{K} \cap \mathbf{L} \cap \mathbf{M}$, *then* $V_{\mathbf{I}}$ *is a surface of type* V_{mn} *from* 6.18; *in particular,* V *has a basis* $\mathbf{N}$ *such that the graph* $\Delta_1(V) \mid \{\mathbf{K}, \mathbf{L}, \mathbf{M}, \mathbf{N}\}$ *is the quadrangle depicted in Figure* 6.18.3, *and, according to* 6.18.2,

$$N_0 \sim (n+m-1)L_0 - (n+m-2)L_1 - \sum_{i=2}^{2m-1} L_i - \sum_{i=2}^{2n-1} X_i, \qquad (6.19.2)$$

where $\mathbf{L}\backslash\mathbf{K} = \{L_1, \ldots, L_{2m-1}\}$ *and* $\mathbf{L}\backslash\mathbf{M} = \{L_1, \ldots, X_{2m-1}\}$.

PROOF. We may assume $\mathbf{I} = \varnothing$, $\mathbf{L} = (\mathbf{L}\backslash\mathbf{K}) \cup (\mathbf{L}\backslash\mathbf{M})$ and

$$L_1 > L_2 > \cdots > L_{2m-1}, \qquad L_1 > X_2 > \cdots > X_{2n-1}.$$

From the maximality of $\mathbf{L}$ in $\mathbf{L}\backslash\mathbf{M}$ it follows that the set $\mathbf{J} = \mathbf{L}\backslash\mathbf{K}\backslash\{L_1\}$ is exceptional and that the point $\varphi_{\mathbf{J}}(L_2)$ of the surface $V_{\mathbf{J}}$ (of type V_n) does not lie on the curve $\varphi_{\mathbf{J}}(X_2)$.

§7. Connectivity of the graph $\Delta_1(V)$

7.0. In this section we obtain the possibility of passing from one basis on V to any other basis on V along on edge of the graph $\Delta_1(V)$. Moreover, in §8 we shall need the existence of a sufficiently good (*monotonic*; see 7.1) path in $\Delta_1(V)$ joining two given vertices.

7.1. In the definition below we shall denote by A the class of A_0 (see 3.3) corresponding to the path at the origin in a basis $\mathbf{A}$.

DEFINITION. A path $\zeta = [\mathbf{A}, \ldots, \mathbf{K}, \mathbf{L}, \ldots]$ in the graph $\Delta_1(V)$ is called *monotonic* if, for any edge $[\mathbf{K}, \mathbf{L}]$ of this path and for curves C_1, C_2 in $\mathbf{L}\backslash\mathbf{K}$ satisfying the additional requirements that the pair $\{C_1, C_2\}$ is maximal in $\mathbf{L}\backslash\mathbf{K}$ and the pair $\{(C_1 \cdot A), (C_2 \cdot A)\}$ is maximal in $\{(L \cdot A) \mid L \in \mathbf{L}\backslash\mathbf{K}\}$, the following conditions hold:

7.1.0. $((L_0 - K_0) \cdot A) > 0$.

7.1.1. $\max\{(C_1 \cdot A), (C_2 \cdot A)\} = \max\{(L \cdot A) \mid L \in \mathbf{L}\}$.

7.1.2. *If neither of the curves* C_1 *and* C_2 *is contained in the other, then for* $L \in \mathbf{L}\backslash\mathbf{K}$ *and* $L' \in \mathbf{L}\backslash\mathbf{K}$

$$((L - L') \cdot A) \geqslant 0.$$

7.1.3. *If* C_2 *is contained in* C_1, *then for any curve* L *in* $\mathbf{L}\backslash\mathbf{K}$

$$((2L + C_1 - L_0) \cdot A) > 0. \qquad (7.1.3.1)$$

7.1.4. *If* C_2 *is contained in* C_1, $w[\mathbf{L}, \mathbf{K}] \geqslant 2$, *and* $\mathbf{L} \cap \mathbf{K}$ *has curves* X *and* Y *that* $\{C_1, X, Y\}$ *is maximal in* $\mathbf{L}$, $C_1 > X$, $|L_0 - C_1 - C_2 - Y| \neq \varnothing$, *and*

$$((L_0 - C_1 - X - Y) \cdot A) \leqslant 0, \qquad (7.1.4.1)$$

then either

7.1.4.2. $\mathbf{L} \cap \mathbf{K}$ *contains a curve* L *that is contained in all the curves in* $\mathbf{L}\backslash\mathbf{K}$ *and satisfies* (7.1.3.1),

or

7.1.4.3. $\mathfrak{B}(Y)$ *contains bases* $\mathbf{K}'$ *and* $\mathbf{K}''$ *forming, together with* $\mathbf{K}$ *and* $\mathbf{L}$, *a quadrangle without diagonals* $\{\mathbf{K}', \mathbf{K}, \mathbf{L}, \mathbf{K}''\}$ (*in the sense of* 6.17) *of the form considered in* 6.18 *and* 6.19, *and* $\{C_1, X\} \subset \mathbf{L}\backslash\mathbf{K}''$ *and any curve* L *in* $\mathbf{L}\backslash\mathbf{K}''$ *satisfies* (7.1.3.1).

7.2. THEOREM. *For any two bases* $\mathbf{A}, \mathbf{L} \in \mathfrak{B}(V)$, $\Delta_1(V)$ *contains a monotonic path starting at* $\mathbf{A}$ *and ending at* $\mathbf{L}$.

In 7.4 we shall deduce this theorem from Proposition 7.3.

7.3. PROPOSITION. *Suppose* $\mathbf{L} \in \mathfrak{B}(V)$ *and* A *is an element of* $\operatorname{Pic}(V)$ *such that*

7.3.1. $A \neq L_0$,

7.3.2. $(A^2) > 0$,

7.3.3. *the system* $|A|$ *is movable, and*

7.3.4. *the system* $|A + \lambda K_V|$ *is rationally empty for all* λ *in* $\mathbf{Q} \cap]1/3, \infty[$ (*see* 3.10).

Then there exists a basis $\mathbf{K}$ *neighboring* $\mathbf{L}$ *in* $\Delta_1(V)$ *that satisfies the requirements* 7.10–7.1.4.

7.4. *Deduction of Theorem* 7.2 *from Proposition* 7.3. Let $\mathbf{A}$ and $\mathbf{L}$ be two distinct bases on V. By 3.4, 3.5 and 3.10, for $A = A_0$ all the conditions 7.3.1–7.3.4 hold. Let $\mathbf{K}$ be the basis guaranteed by the proposition. If $\mathbf{K} = \mathbf{A}$, the required path is $[\mathbf{A}, \mathbf{L}]$. If $\mathbf{K} \neq \mathbf{A}$, then, inducting on $(A \cdot L_0)$ and using 7.1.0, we can extend a monotonic path ζ from $\mathbf{A}$ to $\mathbf{K}$. Then the path $\zeta \circ [\mathbf{K}, \mathbf{K}]$ is the required one.

We shall prove Proposition 7.3 in 7.5–7.13.

7.5. Let the curves of the basis $\mathbf{L} = \{L_1, \ldots, L_r\}$ be numbered so that

7.5.0. $L_i > L_j$ implies $i < j$,

in particular,

7.5.1. each subset $\{L_1, \ldots, L_s\}$ is maximal in $\mathbf{L}$, where $1 \leqslant s \leqslant r$,

and

7.5.2. in the expression $A = l_0 L_0 - l_1 L_1 - \cdots - l_r L_r$ of the class of A in terms of the group basis $\{L_0\} \cup \mathbf{L}$, the coefficients l_i satisfy the inequalities

$$l_1 \geqslant l_2 \geqslant \cdots \geqslant l_r. \tag{7.5.3}$$

We put

$$\rho = \max(\{i \mid l_i > 0\} \cup \{0\}). \tag{7.5.4}$$

In the process of the proof we shall often use both the decomposition of $A + \lambda K_V$ in terms of the basis $\{L_0\} \cup \mathbf{L}$,

$$A + \lambda K_V = (l_0 - 3\lambda) L_0 + \sum_{i=1}^{r} (\lambda - l_i) L_i, \tag{7.5.5}$$

and the decomposition of $A + \lambda K_V$ in terms of other bases consisting of effective classes. Note the effectiveness of the classes

$$\Lambda_0 = L_0 - L_1, \qquad \Lambda_i = L_0 - L_1 - L_i \quad (2 \leqslant i \leqslant r) \tag{7.5.6}$$

(see 3.7 and 3.8). From 7.3.1–7.3.3 follows

$$l_0 \geqslant 2; \qquad l_0 > l_i \quad \text{for } 1 \leqslant i \leqslant r. \tag{7.5.7}$$

Further,

$$3 l_1 > l_0 \tag{7.5.8}$$

(in particular, $\delta \geqslant 1$), since otherwise after substituting the number $\lambda = l_0/3$ in (7.5.5) (which, by (7.5.7), is greater than $1/3$) all the coefficients in (7.5.5) become nonnegative, contradicting 7.3.4.

7.6. We show that

$$\rho \geqslant 2, \tag{7.6.1}$$

$$2 l_2 + l_1 > l_0. \tag{7.6.2}$$

Consider the expression of the class $A + \lambda K_V$ in terms of the group basis $\{\Lambda_0\} \cup \mathbf{L}$ (Λ_0 was introduced in (7.5.6)):

$$A + \lambda K_V = (l_0 - 3\lambda)\Lambda_0 + (l_0 - l_1 - 2\lambda)L_1 + \sum_{i=2}^{r} (\lambda - l_i)L_i. \qquad (7.6.3)$$

If (7.6.1) or (7.6.2) does not hold (that is, $\rho = 1$ or $l_2 \leqslant (l_0 - l_1)/2$), then putting $\lambda = (l_0 - l_1)/2$ in (7.6.3) produces nonnegative coefficients in its right-hand side.

7.7. We show that

$$l_0 \geqslant l_1 + l_2. \qquad (7.7.1)$$

and if equality holds in (7.7.1), then $\rho \geqslant 3$.

The introduction index of the members of the movable system $|A|$ with the effective class Λ_2 introduced in (7.5.6) is nonnegative and equals the difference between the left and right sides of (7.7.1). If this difference is zero and $\rho = 2$, then (7.5.5), taken with $\lambda = 1/2$, can be rewritten as

$$A + K_V/2 = (l_2 - 1)L_0 + (l_1 - 1/2)\Lambda_2 + (l_1 - l_2)L_2.$$

In the last equation all the coefficients are nonnegative, contradicting 7.3.4.

7.8. We shall prove the inequalities

$$\rho \geqslant 3, \qquad (7.8.1)$$

$$l_1 + l_2 + l_3 > l_0. \qquad (7.8.2)$$

If $l_1 + l_2 = 1_0$, then 7.7 implies $\rho \geqslant 3$, and then $l_3 > 0$ by (7.5.4); that is, (7.8.1) and (7.8.2) hold. Therefore, we assume (7.7.1) to be a strict inequality. Let us decompose $A = \lambda K_V$ in terms of the group basis $\{\Lambda_0, \lambda_1\} \cup (\mathbf{L}\backslash\{\mathbf{L}\})$ (where $\Lambda_1 = \Lambda_2 + L_1$, and Λ_0 and Λ_2 are given in (7.5.6)):

$$\begin{aligned} A + \lambda K_V = {} & (l_1 - \lambda)\Lambda_0 + (l_0 - l_1 - 2\lambda)\Lambda_1 \\ & + (l_0 - l_1 - l_2 - \lambda)L_2 + \sum_{i=3}^{r} (\lambda - l_i)L_i. \end{aligned}$$

If (7.8.1) or (7.8.2) does not hold, then replacing λ by the number $l_0 - l_1 - l_2$ in the latter (which, by the strictness of (7.7.1), is not less than one), we get, by (7.5.3), (7.5.8) and (7.6.2), nonnegative coefficients in the right-hand side, and this is impossible by 7.3.4.

7.8.3. We shall show that if L_2 is not contained in L_1, then there is a basis $\mathbf{K} \in \mathfrak{B}(V)$ for which 7.1.0–7.1.4 hold and

$$\mathbf{L}\backslash\mathbf{K} = \{L_1, L_2, L_3\}.$$

Let us check that conditions 4.4.1–4.4.3 hold for $\mathbf{L}$ and $\{L_1, L_2, L_3\}$. Condition 4.4.1 follows from 7.5.1, 4.4.2 from 7.5.0 and the assumption that L_1 does not contain L_2, and 4.4.3 from 7.3.3 and (7.8.2).

Let $\mathbf{K}$ be a basis neighboring $\mathbf{L}$ along $\{L_1, L_2, L_3\}$. Since $K_0 \sim 2L_0 - L_1 - L_2 - L_3$, 7.1.0 is equivalent to (7.8.2). If $\{C_1, C_2\} = \{L_1, L_2\}$, then 7.1.2 and 7.1.3 follow from (7.5.3) and (7.8.2); and the inequality $C_2 < C_1$ is impossible here. In case $\{C_1, C_2\} \neq \{L_1, L_2\}$, we have $l_2 = l_3$ and again 7.1.2 and 7.1.3 follow from (7.5.3) and (7.8.2), and the inequality $C_2 < C_1$ and $w[\mathbf{L}, \mathbf{K}] \geqslant 2$ are impossible.

In the following parts of the proof of Proposition 7.3 we shall assume that $L_2 < L_1$.

7.9. Let $\mathfrak{L}(\mathfrak{M})$ be the subset of $\mathbf{L}\backslash\{L_1\}$ ($\mathbf{L}\backslash\{L_1, L_2\}$) consisting of those L for which

$$2(A \cdot L) + l_1 > l_0 \tag{7.9.1}$$

$$\left(|L_1 - L_2 - L| = \varnothing, |L_2 - L| = \varnothing\right). \tag{7.9.2}$$

Note that:

7.9.3. $L_2 \in \mathfrak{L}$ (see (7.6.2)).

7.9.4. $\mathfrak{M} \supset \mathbf{L}\backslash\mathcal{C}(L_1)$.

7.9.5. $\mathfrak{L}\backslash\{L_2\}$ and $\mathfrak{M}$ are maximal in the set $\mathfrak{L}\backslash\{L_1, L_2\}$.

Therefore we have

7.9.6. The intersection $\mathfrak{L} \cap \mathfrak{M}$ is maximal in $\mathbf{L}\backslash\{L_1, L_2\}$.

Further,

7.9.7. For any three curves L, L', L'' in $\mathfrak{L}$ the system $|L - L' - L''|$ is empty.

In fact, otherwise (7.9.1) and 7.3.3 would imply

$$(A \cdot L) \geqslant (A \cdot (L' + L'')) > l_0 - l_1,$$

contradicting (7.6.2) and (7.5.3).

7.10. We shall show that if $\mathfrak{L} \cap \mathfrak{M} \neq \varnothing$ and L is a maximal curve in $\mathfrak{L} \cap \mathfrak{M}$ for which $(L \cdot A) \geqslant (C \cdot A)$ when $C \in \mathfrak{L} \cap \mathfrak{M}$, then there is a basis $\mathbf{K}$ that neighbors $\mathbf{L}$ along the set $\Lambda = \{L_1, L_2, L\}$ and satisfies 7.1.0–7.1.4.

We first verify 4.4.1–4.4.3 for Λ. The maximality of Λ in $\mathbf{L}$ follows from 7.9.6 and 7.5.1; that 4.4.2 holds follows from (7.9.2) and 7.5.1; and the emptiness of $|L_0 - L_1 - L_2 - L|$ follows from (7.9.1) and (7.5.3).

Let $\mathbf{K}$ be a basis neighboring $\mathbf{L}$ along Λ. Inequality 7.1.0 follows from (7.9.1), and 7.1.1 follows from

$$\{(C_1 \cdot A), (C_2 \cdot A)\} = \{l_1, l_2\}.$$

If $C_1 = L_1$ and $C_2 = L_2$, then, since $C_1 > C_2$, it makes no sense to verify 7.1.2; but if $C_1 = L_1$ and $C_2 = L$, then $(L \cdot A) = l_2$, and 7.1.3 follows from 7.6.2. Since $|L_2 - L| = \varnothing$ (see (7.9.2)), we must have $w[\mathbf{K}, \mathbf{L}] = 1$, and so it does not make sense to verify 7.1.4.

In the remaining parts of the proof of the proposition we shall assume that $L_2 < L_1$, $\mathfrak{L} \cap \mathfrak{M} = \varnothing$, *and in particular* (*see* 7.9.3 *and* 7.9.4), $\mathfrak{L} \subset \mathcal{C}(L_1)$.

7.10.1. From 7.9.7, 1.7.2 and 1.11 it follows that the set of arrows of the graph $\vec{\mathcal{C}}(V)|\mathfrak{L}$ is the disjoint union of oriented paths $\zeta_1, \dots, \zeta_l$ (some of which may have length zero) of the form

$$\bigcirc \leftarrow \bigcirc \leftarrow \bigcirc \leftarrow \cdots - \bigcirc.$$

Let L^i be the set of vertices of the path ζ_i, $t(i) = \#\mathbf{L}^i$, and $\mathbf{L}^i = \{L_1^i, \dots, L_{t(i)}^i\}$, where the curves in $\mathbf{L}^i$ are numbered so that $L_\alpha^i > L_\beta^i$ for $1 \leqslant \alpha < \beta \leqslant t(i)$. Suppose that, for $i \in [1, l]$, $s(i)$ is the largest possible number in $[1, t(i)]$ such that $\{L_1^i, \dots, L_{s(i)}^i\} \subset \mathcal{C}^*(L_1)$; i.e., the graph $\vec{\mathcal{C}}(V)$ has arrows ending at L_1 and starting from the vertices $L_1^i, \dots, L_{s(i)}^i$ but none from the other vertices $L_j^i \in \mathbf{L}^i$ ($j > s(i)$) (see 1.7 and 1.11). The set of arrows of the graph $\vec{\mathcal{C}}(V)|\mathfrak{L} \cup \{L_1\}$ is shown in Figure 7.10.2.

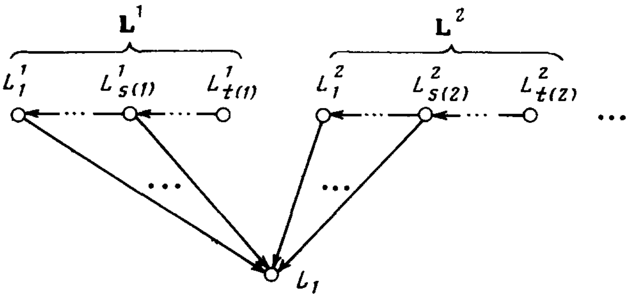


FIGURE 7.10.2

7.11. We shall prove that there exists $i \in [1, l]$ such that

$$t(i) \geqslant 2s(i). \tag{7.11.1}$$

Put

$$\sigma = 1 + \sum_{i=1}^{l} s(i), \qquad \tau = 1 + \sum_{i=1}^{l} t(i). \tag{7.11.2}$$

Consider the basis of the group $\operatorname{Pic}(V)$ consisting of the following effective classes: $\Lambda_0 = L_0 - L_1$ (see (7.5.6)), $\Lambda_1 = L_1 - \Sigma_{i=1}^{l}\Sigma_{j=1}^{s(i)} L_j^i$ (the effectiveness of Λ_1 follows from 1.7.2, 1.11 and Figure 7.10.2), $\Lambda_j^i = L_0 - L_1 - L_j^i$, where $1 \leqslant j \leqslant t(i)$ and $1 \leqslant i \leqslant l$ (compare (7.5.6)), and L_K, where $\tau \leqslant k \leqslant r$.

Since the curves $L_\tau, \dots, L_r$ do not lie in $\mathfrak{L}$, we have

$$2l_k \leqslant l_0 - l_1 \quad \text{for } \tau \leqslant k \leqslant r. \tag{7.11.3}$$

Further, the elements of the old basis $\{L_0\} \cup \mathbf{L}$ lying in $\{L_0, L_1\} \cup (\bigcup_1^l \mathbf{L}^i)$ are expressed in terms of the new basis as follows:

$$L_0 = \sigma\Lambda_0 + \Lambda_1 - \sum_{i=1}^{l}\sum_{j=1}^{s(i)} \Lambda_j^i, \qquad L_1 = (\sigma - 1)\Lambda_0 + \Lambda_1 - \sum_{i=1}^{l}\sum_{j=1}^{s(i)} \Lambda_j^i,$$

$$L_j^i = \Lambda_0 - \Lambda_j^i \quad \text{for } 1 \leqslant i \leqslant l,\ 1 \leqslant j \leqslant t(i).$$

Therefore

$$K_V = (\tau - 2\sigma - 2)\Lambda_0 - 2\Lambda_1 + \sum_{i=1}^{l}\left(\sum_{j=1}^{s(i)} \Lambda_j^i - \sum_{j=s(i)+1}^{t(i)} \Lambda_j^i\right) + \sum_{k=\tau}^{r} L_k,$$

and if the class A is expressed in terms of the old basis by the equation

$$A = l_0 L_0 - l_1 L_1 - \sum_{i=1}^{l}\left(\sum_{j=1}^{t(i)} l_j^i L_j^i\right) - \sum_{k=\tau}^{r} l_k L_k,$$

then its expression in terms of the new one is

$$\left(\sigma l_0 - (\sigma - 1)l_1 - \sum_{i=1}^{l}\sum_{j=1}^{t(i)} l_j^i\right)\Lambda_0 + (l_0 - l_1)\Lambda_1$$
$$+ \sum_{i=1}^{l}\left[\sum_{j=1}^{s(i)} \left(l_1 - l_0 + l_j^i\right)\Lambda_j^i + \sum_{j=s(i)+1}^{t(i)} l_j^i \Lambda_j^i\right] - \sum_{k=\tau}^{r} l_k L_k.$$

Hence for the class $A + \lambda K_V$ adjoint to A we get the following expression:

$$\left[\sigma l_0 - (\sigma - 1)l_1 - \sum_{i=1}^{l}\sum_{j=1}^{t(i)} l_j^i + \lambda(\tau - 2\sigma - 2)\right]\Lambda_0 + (l_0 - l_1 - 2\lambda)\Lambda_1$$

$$+ \sum_{i=1}^{l}\left[\sum_{j=1}^{s(i)}\left(l_1 - l_0 + l_j^i + \lambda\right)\Lambda_j^i + \sum_{j=s(i)+1}^{t(i)}\left(l_j^i - \lambda\right)\Lambda_j^i\right] + \sum_{k=\tau}^{r}(\lambda - l_k)L_k. \qquad (7.11.4)$$

Inequality (7.11.1) for some $i \in [1, l]$ will be proved by contradiction; i.e., assume that for all i in $[1, l]$

$$t(i) \leqslant 2s(i) - 1; \qquad (7.11.5)$$

we shall derive a contradiction to 7.3.4 by showing that the coefficients in (7.11.4) are nonnegative for $\lambda = (l_0 - l_1)/2$. We go through these coefficients from right to left.

From (7.11.3) follows $\lambda - l_k \geqslant 0$ for $\tau \leqslant k \leqslant r$. Further, the nonnegativity of the coefficients $l_j^i - \lambda$ and $l_1 - l_0 + l_j^i + \lambda$ follows from $L_j^i \in \mathfrak{L}$ and (7.9.1). The coefficient of Λ_1 is zero; and the doubled coefficient of Λ_0 in (7.11.4) (recall that $\lambda = (l_0 - l_1)/2$) is easily verified to equal the expression

$$\sum_{i=1}^{l}(t(i) - 1)(l_0 - l_1 - l_1^i) + 2\left(l_1 - \sum_{i=1}^{l}\sum_{j=1}^{s(i)} l_j^i\right) + (l-1)(l_0 - l_1)$$

$$+ \sum_{i=1}^{l}(2s(i) - t(i) - 1)l_1^i + 2\sum_{i=1}^{l}\sum_{j=s(i)+1}^{t(i)}\left(l_1^i - l_j^i\right), \qquad (7.11.6)$$

in which, because of (7.11.5) and the effectiveness of the classes Λ_1 and $L_1^i - L_j^i$, each difference in parentheses is nonnegative.

7.12. Suppose the natural number i whose existence was proved in 7.11 (see (7.11.1)) equals 1; i.e.,

$$t(l) \geqslant 2s(1). \qquad (7.12.1)$$

Assume that $\mathbf{L} \cap \mathbf{K}$ contains no curves X and Y with the properties indicated in 7.1.4 if we take $C_1 = L_1$ and $C_2 = L_1^1$ in 7.1.4. Putting

$$n = s(1) + 1, \qquad \Lambda = \left\{L_1, L_1^1, \ldots, L_{2n-2}^1\right\},$$

we see that $\mathbf{L}$ and Λ satisfy 6.10.0–6.10.5; therefore, there is a basis $\mathbf{K}$ neighboring $\mathbf{L}$ along Λ for which all the requirements 7.1.0–7.1.4 hold. Indeed, 7.1.0 is a consequence of (6.2.2) and (7.9.1); further, we must have $C_1 = L_1$ and $C_2 = L_1^1$, so 7.1.1 is evident, and 7.1.2 and 7.1.4 need not be checked, while 7.1.3 is a consequence of (7.9.1)

7.13. Now assume that for the basis $\mathbf{K}$ given in 7.12, for the basis $\mathbf{L}$ and the curves $\mathbf{C}_1 = \mathbf{L}_1$ and $C_2 = L_1^1$, 7.1.4.2 does not hold, which means that equality holds in (7.12.1), since if strict inequality held in (7.12.1), the curve $L = L_{2s(1)+1}^1$ would satisfy 7.1.4.2. It is now evident that to complete the proof of the proposition it suffices to show that

$$l \geqslant 2 \qquad (7.13.1)$$

and that there exists $i \in [2, l]$ for which either the inequality (7.11.1) is strict, i.e.

$$t(i) > 2s(i), \qquad (7.13.2)$$

or

$$t(i) = 2s(i) \quad \text{and} \quad X \in \mathbf{L}^i. \qquad (7.13.3)$$

To prove (7.13.1) it suffices to note that $X \in \mathbf{L}$, since combining the inequalities

$$l_0 - l_1 - l_1^1 - (Y \cdot A) \geqslant 0, \qquad -l_0 + l_1 + (X \cdot A) + (Y \cdot A) \geqslant 0$$

that follows from condition 7.1.4 yields $(X \cdot A) - l_1^1 \geqslant 0$, whence $(X \cdot A) \geqslant l_1^1 > (l_0 - l_1)/2$. Assume that, contrary to (7.13.2), for all $i \in [2, l]$

$$t(i) \leqslant 2s(i). \tag{7.13.4}$$

If at least one of the inequalities (7.13.4) is strict, say $t(l) + 1 \leqslant 2s(l)$, then the group of summands

$$(l-1)(l_0 - l_1) + \sum_{i=1}^{l} (2s(i) - t(i) - 1)l_1^i$$

occurring in (7.11.6) can be rewritten as

$$\sum_{i=1}^{l-1} (l_0 - l_1 - l_1^i) + \sum_{i=1}^{l-1} (2s(i) - t(i))l_1^i + (2s(l) - t(l) - 1)l_1^l$$

and we see that the sums as well as the whole expression (7.11.6) are nonnegative, contradicting 7.3.4. But if for all i in $[2, l]$ we have equality in (7.13.4), then, taking an $i \in [2, l]$ for which $X \in \mathbf{L}^i$, we get (7.13.3). It remains to note that if we assume (7.13.3) to hold with $i = 2$, then for bases $\mathbf{K}, \mathbf{L}$ and $\mathbf{K}''$, where $\mathbf{L} \backslash \mathbf{K}'' = \{L_1, L_1^2, \ldots, L_{t(2)}^2\}$ and $X = L_1^2$, we can see that the hypotheses of assertion 6.19 are satisfied (if we replace $\mathbf{M}$ by $\mathbf{K}''$ in it); and then, denoting by $\mathbf{K}'$ the basis $\mathbf{N}$ of 6.19, we get the required quadrangle $\{\mathbf{K}', \mathbf{K}, \mathbf{L}, \mathbf{K}''\}$.

Proposition 7.3 and Theorem 7.2 have been proved.

7.13.5 Remark. From the proof of the proposition and from 5.1.4 and (5.5.1) it is evident that if $\{\mathbf{A}, \mathbf{L}\} \cup \{A_1, L_1\}$ is a de Jonquières set, then the set of vertices of the monotonic path from $\mathbf{A}$ to $\mathbf{L}$ constructed in the proof has a J-supplement, and the J-supplement may be assumed to contain $\{A_1, L_1\}$.

In 7.14–7.17 we present some consequences of the connectedness of $\Delta_1(V)$.

7.14. *If $\mathfrak{B}(V)$ consists of precisely two bases $\mathbf{L}$ and $\mathbf{M}$ for which $\mathbf{L} \cap \mathbf{M} = \varnothing$, then V is a surface of type V_n, $n \geqslant 0$ (see 4.4, 4.5 and 6.2–6.8).*

7.15. *If a class $A \in \operatorname{Pic}(V)$ satisfies 7.3.2–7.3.4 and $\mathfrak{B}(V) \neq \varnothing$, then there is a basis $\mathbf{A} \in \mathfrak{B}(V)$ such that $A = A_0$.*

Proof. It suffices to show that

$$\min\{(L_0 \cdot A) | \mathbf{L} \in \mathfrak{B}(V)\} = 1,$$

since if $\mathbf{L}$ is a basis for which $(L_0 \cdot A) = 1$, then 7.3.3 implies (compare the proof of 3.11.1) $A = L_0$. Assume that the minimum considered equals μ, $\mu > 1$, $\mu = (L_0 \cdot A)$ and $\mathbf{L} \in \mathfrak{B}(V)$. Then for $\mathbf{L}$ and $\mathbf{A}$ all the conditions 7.3.1–7.3.4 hold. If $\mathbf{K}$ is the basis given in Proposition 7.3, then, by 7.1.0, $(K_0 \cdot A) < (L_0 \cdot A) = \mu$, which is impossible.

7.16. *The graph Γ_1 is stably connected in the sense that for any two bases $\mathbf{L}$ and $\mathbf{M}$ on the surface V one can find a natural number N such that any blowing up $f\colon U \to V$ of a set consisting of N points in general position on V guarantees the possibility of joining by a path in the graph $\Gamma_1(U)$ the vertices $f^{-1}(\mathbf{L})$ and $f^{-1}(\mathbf{M})$ of this graph.*

Proof. Let $\zeta = [\mathbf{L}_0, \ldots, \mathbf{L}^l]$ be a path in $\Delta_1(V)$ starting at $\mathbf{L}^0 = \mathbf{L}$ and ending at $\mathbf{L}^l = \mathbf{M}$. Let $n_i = w[\mathbf{L}^{i-1}, \mathbf{L}^i]$ $(1 \leqslant i \leqslant l)$, and $N = \max\{0, n_1 - 2, \ldots, n_l - 2\}$. We take a blowing up $f\colon U \to V$ of a set of N points of V in general position. From 6.12 it follows that the vertices $f^{-1}(\mathbf{L}^{i-1})$ and $f^{-1}(\mathbf{L}^i)$ are joined by a path in $\Gamma_1(U)$, and this proves 7.16.

7.17. From 7.16 follows M. Noether's theorem: *Any birational transformation of the plane (over an algebraicaly closed field) is a composition of quadratic and projective transformations.*

PROOF. Any birational transformation of $\mathbf{P}_2$ can be represented as $\varphi_{\mathbf{L}} \circ \varphi_{\mathbf{M}}^{-1}$ (see 3.12–3.14), where $\mathbf{L}$ and $\mathbf{M}$ are two bases on some V. We may assume that $\mathbf{L}$ and $\mathbf{M}$ are joined by a path in $\Gamma_1(V)$ (otherwise, using 3.13, 7.16 and the morphism $f\colon U \to V$, we could lift to the surface U, where $f^{-1}(\mathbf{L})$ and $f^{-1}(\mathbf{M})$ are already joined by a path in $\Gamma_1(U)$). If $\mathbf{L}_0, \ldots, \mathbf{L}_n$ are the vertices of this path, $\mathbf{L}_0 = \mathbf{L}, \ldots, \mathbf{L}_n = \mathbf{M}$, then

$$\varphi_{\mathbf{L}} \circ \varphi_{\mathbf{M}}^{-1} = \left(\varphi_{\mathbf{L}_0} \circ \varphi_{\mathbf{L}_1}^{-1}\right) \circ \cdots \circ \left(\varphi_{\mathbf{L}_{n-1}} \circ \varphi_{\mathbf{L}_n}^{-1}\right)$$

and the birational transformations of the plane in parentheses are quadratic by 4.2 and 4.3.

§8. 1-connectivity of the complex $\Delta(V)$

8.0. Here we introduce the complexes $\Gamma(V)$ and $\Delta(V)$ for an arbitrary surface V.

DEFINITION. $\Gamma(V)$ is the simplicial complex gotten as the simplicial filling of the graph $\Gamma_1(V)$ (see 0.5.1), and $\Delta(V)$ is the cell complex gotten as the prismatic filling of the graph $\Delta_1(V)$ (see 0.5.2).

We need only the one-dimensional skeletons of these complexes. In $\Delta(V)$ the two-dimensional cells are triangles and quadrangles without diagonals (in the sense of 6.17). Theorem 7.2 implies that the complex $\Delta(V)$ is connected. As is evident from considering the surface $V_{2,2}$ (see Figure 6.18.3 with $m = n = 2$), the complex $\Gamma(V)$ can be connected, but need not be. The simplicial fillings of the graphs $\Delta_1(V_{mn})$ and $\Delta_1(W_n)$ when $n \geqslant 3$ (see Figures 6.18.3 and 6.11.1) coincide with these graphs, so they are not 1-connected. This matter is simpler for a prismatic filling.

8.1. THEOREM. *The complex $\Delta(V)$ is 1-connected.*

The proof occupies the rest of this section, i.e. 8.2–8.15.

8.2. Suppose ζ is a loop in $\Delta(V)$ (i.e., ζ is a closed path in the graph $\Delta_1(V)$). $\mathbf{A} \in \mathfrak{B}(V)$, and $\mathbf{A}$ is the origin of the loop ζ. Using Theorem 7.3 we may join the origin of this loop to any vertex of it by a monotonic path; therefore, to prove Theorem 8.1 it suffices to prove the contractibility of a loop of the form

$$\zeta = \xi \circ [\mathbf{L}, \mathbf{M}] \circ \eta^{-1}, \tag{8.2.1}$$

where ξ and η are monotonic paths with origin $\mathbf{A}$, $\mathbf{L}$ is the end of ξ, $\mathbf{M}$ is the end of η, and $[\mathbf{L}, \mathbf{M}]$ is an edge of $\Delta_1(V)$.

We introduce some notation.

8.3. For a loop ζ of the form (8.2.1) we put

$$h(\zeta) = \max\{(A_0 \cdot L_0), (A_0 \cdot M_0)\} \tag{8.3.1}$$

($h(\zeta)$ is the height of the loop ζ) and

$$\delta(\zeta) = \delta((A_0 \cdot L_0), (A_0 \cdot M_0)) \tag{8.3.2}$$

(that is, $\delta(\zeta) = 1$ when $((L_0 - M_0) \cdot A_0) = 0$, and $\delta(\zeta) = 0$ otherwise); $\delta(\zeta)$ characterizes the inclination of the edge $[\mathbf{L}, \mathbf{M}]$ with respect to the vertex $\mathbf{A}$.

8.4. The proof that the loop (8.2.1) is contractible will be carried out by induction on the pairs $(h(\zeta), \delta(\zeta))$ in lexicographic order (see 8.3). In case $h(\zeta) \leqslant 2$, the loop lies on a simplex, and so is contractible. In what follows we shall assume the inequality

$$((L_0 - M_0) \cdot A_0) \geqslant 0 \tag{8.4.0}$$

(if this does not hold, we replace ζ by its opposite). Denote by $[\mathbf{K}, \mathbf{L}]$ the last edge of the path ξ; that is, $\xi = \xi_0 \circ [\mathbf{K}, \mathbf{L}]$, where ξ_0 is a monotonic path (possibly of length zero) with origin $\mathbf{A}$ and end $\mathbf{K}$. In 8.6 we shall reduce Theorem 8.1 to the following proposition.

8.5. PROPOSITION. *Suppose* $[\mathbf{K}, \mathbf{L}, \mathbf{M}]$ *is a path in the graph* $\Delta_1(V)$, $\mathbf{I} = \mathbf{K} \cap \mathbf{L} \cap \mathbf{M}$, A *is an element of* $\operatorname{Pic}(V)$, *where the system* $|A|$ *is movable*,

$$((L_0 - M_0) \cdot A) \geqslant 0, \tag{8.5.a}$$

and for any curves C_1 *and* C_2 *in* $\mathbf{L} \backslash \mathbf{K}$ *the pair* $\{C_1, C_2\}$ *is maximal in* $\mathbf{L} \backslash \mathbf{K}$, *the pair* $\{(C_1 \cdot A), (C_2 \cdot A)\}$ *is maximal in* $\{(L \cdot A) \mid L \in \mathbf{L} \backslash \mathbf{K}\}$, *and the following conditions* (*almost coinciding with* 7.1.0–7.1.4) *hold*:

8.5.0. $((L_0 - K_0) \cdot A) > 0$,

8.5.1. $\max\{(C_1 \cdot A), (C_2 \cdot A)\} = \max\{(L \cdot A) \mid L \in \mathbf{L} \backslash \mathbf{I}\}$.

8.5.2. *If neither of the curves* C_1 *or* C_2 *contains the other*, *then for* $L \in \mathbf{L} \backslash \mathbf{K}$ *and* $L' \in ((\mathbf{L} \cap \mathbf{K}) \backslash \mathbf{I})$

$$((L - L') \cdot A) \geqslant 0.$$

8.5.3. *If* C_2 *is contained in* C_1, *then for any* $L \in \mathbf{L} \backslash \mathbf{K}$

$$((2L + C_1 - L_0) \cdot A) > 0. \tag{8.5.3.1}$$

8.5.4. *If* $C_2 < C_1$, $w[\mathbf{L}, \mathbf{K}] \geqslant 2$, *and* $(\mathbf{L} \cap \mathbf{K}) \backslash \mathbf{M}$ *contains curves* X *and* Y *such that* $\{C_1, X, Y\}$ *is maximal in* $\mathbf{L}$, $C_1 > X$ *and* $|L_0 - C_1 - C_2 - Y| \neq \varnothing$, $((L_0 - C_1 - X - Y) \cdot A) \leqslant 0$, *then either*

8.5.4.1. $\mathbf{L} \cap \mathbf{K}$ *contains a curve* L *that is contained in all the curves in* $\mathbf{L} \backslash \mathbf{K}$ *and satisfies* (8.5.3.1),

or

8.5.4.2. $\mathfrak{B}(V)$ *contains bases* $\mathbf{K}'$ *and* $\mathbf{K}''$ *which*, *together with* $\mathbf{K}$ *and* $\mathbf{L}$, *form a quadrangle* $\{\mathbf{K}', \mathbf{K}, \mathbf{L}, \mathbf{K}''\}$ *of the form considered in* 6.18 *and* 6.19, *and* $\{C_1, X\} \subset \mathbf{L} \backslash \mathbf{K}''$ *and any curve* L *in* $\mathbf{L} \backslash \mathbf{K}''$ *satisfies* (8.5.3.1).

Then the graph $\Delta_1(V)$ *contains a path* $\kappa = [\mathbf{K}^0, \mathbf{K}^1, \ldots, \mathbf{K}^r]$ *for which the following conditions hold*:

8.5.5. $\mathbf{K}^0 = \mathbf{K}$ *and* $\mathbf{K}^r = \mathbf{M}$.

8.5.6.*i*. $((L_0 - K_0^i) \cdot A) > 0$, *where* $0 \leqslant i < r$.

8.5.7. *The loop* $[\mathbf{K}, \mathbf{L}, \mathbf{M}] \circ \kappa^{-1}$ *is contractible in the complex* $\Delta(V)$.

8.5.8. *A J-supplement* (*if one exists*) *to the triple* $\{\mathbf{K}, \mathbf{L}, \mathbf{M}\}$ *can be enlarged to a J-supplement to the set of vertices of the loop* $[\mathbf{K}, \mathbf{L}, \mathbf{M}] \circ \kappa^{-1}$.

8.6. *Deduction of Theorem* 8.1 *from Proposition* 8.5. Let us apply 8.5 to the situation where ζ is a loop of the form (8.2.1), $\xi = \xi_0 \circ [\mathbf{K}, \mathbf{L}]$ and $A = A_0$, where $\mathbf{A}$ is the origin of ζ. The movability of $|A|$ was proved in 3.4.3, condition (8.5.a) coincides with (8.4.0), and requirements 8.5.0–8.5.4 follow from the properties of the monotonic path ξ. Join the vertex $\mathbf{A}$ to the vertices $\mathbf{K}$ $(1 \leqslant i < r)$ by monotonic paths ξ_i, and denote the path η by ξ_r. For $i \in [1, r]$ put

$$\zeta_i = \xi_{i-1} \circ [\mathbf{K}^{i-1}, \mathbf{K}^i] \circ \xi_i^{-1}.$$

By 8.5.7 the loop ζ is homotopic in $\Delta(V)$ to the composition $\zeta_1 \circ \cdots \circ \zeta_r$. From the induction hypothesis (recall that we are inducting on the lexicographically ordered pairs $(h(\zeta), \delta(\zeta))$, where h and δ are defined by (8.3.1) and (8.3.2) respectively) it follows that each of the paths ζ_i $(1 \leqslant i \leqslant r)$ is contractible, since for $i < r$ we have $h(\zeta_i) < h(\zeta)$, and for ζ_r we have

$$(h(\zeta_r), \delta(\zeta_r)) < (h(\zeta), \delta(\zeta)).$$

Proposition 8.5 will be proved in 8.6–8.15. After constructing the path κ the main point will be to check that the conditions 8.5.6.i $(1 \leqslant i < r)$ hold, and the truth of 8.5.5, 8.5.7 and 8.5.8 will be an evident consequence of the construction.

8.7. We introduce the following notation:

$$\begin{aligned} n &= (L_0 \cdot K_0), & m &= (L_0 \cdot M_0), \\ \mathbf{L}\backslash\mathbf{K} &= \{L_1, \ldots, L_{2n-1}\}, & \mathbf{K}\backslash\mathbf{L} &= \{K_1, \ldots, K_{2n-1}\}, \\ \mathbf{L}\backslash\mathbf{M} &= \{X_1, \ldots, X_{2m-1}\}, & \mathbf{M}\backslash\mathbf{L} &= \{M_1, \ldots, M_{2m-1}\}, \\ l_i &= (L_i \cdot A) \quad \text{for } 0 \leqslant i < 2n, & x_i &= (X_i \cdot A) \quad \text{for } 1 \leqslant i < 2m. \end{aligned} \tag{8.7.1}$$

We shall assume that $L_i > L_j$ or $X_i > X_j$ implies $i < j$; in particular, if $w[\mathbf{L}, \mathbf{K}] \geqslant 2$ (respectively, $w[\mathbf{L}, \mathbf{M}] \geqslant 2$), then $L_1 > L_2 > \cdots > L_{2n-1}$ and so

$$l_1 \geqslant l_2 \geqslant \cdots \geqslant l_{2n-1} \tag{8.7.2}$$

(respectively, $X_1 > X_2 > \cdots > X_{2m-1}$ and so

$$x_1 \geqslant x_2 \geqslant \cdots \geqslant x_{2m-1}. \qquad (8.7.3))$$

If we consider a supplement for the pairs $\{\mathbf{K}, \mathbf{L}\}$ ($\{\mathbf{L}, \mathbf{M}\}$), we shall consider it to be $\{K_1, L_1\}$ ($\{X_1, M_1\}$) and then, by 5.5, we have

$$\begin{aligned} K_0 &= nL_0 - (n-1)L_1 - L_2 - \cdots - L_{2n-1}, \\ M_0 &= mL_0 - (m-1)X_1 - X_2 - \cdots - X_{2m-1}. \end{aligned}$$

From this and 8.5.0, (8.5.a) follows:

$$(n-1)(l_0 - l_1) - l_2 - \cdots - l_{2n-1} < 0, \tag{8.7.4}$$

$$(m-1)(l_0 - x_1) - x_2 - \cdots - x_{2m-1} \leqslant 0. \tag{8.7.5}$$

From (8.7.4) we obtain

$$2\max\{l_2, \ldots, l_{2n-1}\} > l_0 - l_1. \tag{8.7.6}$$

From the assumption $L_1 > L_2$ follows the stronger assertion (see (8.5.3.1)) that

$$2l_i > l_0 - l_1 \quad \text{for all } i \in [1, 2n-1]. \tag{8.7.7}$$

From (8.7.5) we get

$$2\max\{x_2, \ldots, x_{2m-1}\} \geqslant l_0 - x_1. \tag{8.7.8}$$

8.8. Here we examine the possibility

$$\#((\mathbf{L}\backslash\mathbf{K}) \cap (\mathbf{L}\backslash\mathbf{M})) \geqslant 2. \tag{8.8.1}$$

Since $\mathbf{L}\backslash\mathbf{K}$ and $\mathbf{L}\backslash\mathbf{M}$ are maximal subsets in $\mathbf{L}$, we may assume that $L_1 = X_1$ and $L_2 = X_2$. If either $w[\mathbf{L}, \mathbf{K}] \geqslant 2$ and $w[\mathbf{L}, \mathbf{M}] \geqslant 2$, or $w[\mathbf{L}, \mathbf{K}] \leqslant 2$ and $w[\mathbf{L}, \mathbf{M}] \leqslant 2$, then, by 6.16.1 and 4.6.1, $\Delta_1(V) | \{\mathbf{K}, \mathbf{L}, \mathbf{M}\}$ is a triangle whose edge $[\mathbf{K}, \mathbf{M}]$ can be taken to be the required path κ.

We must still consider the cases

$$w[\mathbf{L},\mathbf{K}] \geqslant 3, \qquad w[\mathbf{L},\mathbf{M}] \leqslant 1 \tag{8.8.1.a}$$

and

$$w[\mathbf{L},\mathbf{K}] \leqslant 1, \qquad w[\mathbf{L},\mathbf{M}] \geqslant 3. \tag{8.8.1.b}$$

Let us begin with (a). From (8.8.1) and (8.8.1.a) it follows that $w[\mathbf{L},\mathbf{M}] = 1$, $\mathbf{L}\backslash\mathbf{M} = \{L_1, L_2, X_3\}$, $L_1 > L_2$, $\mathcal{C}(X_3)\backslash\{X_3\} \subset \mathbf{I}$ and

$$X_3 \in ((\mathbf{L}\backslash\mathbf{M}) \cap \mathbf{K})\backslash\mathcal{C}(L_1). \tag{8.8.2}$$

The set $\mathbf{J} = \mathbf{I} \cup \{X_3\}$ is exceptional; $V_\mathbf{J}$ is a surface of type V_n (see 6.2). The point $Q = \varphi_\mathbf{J}(X_3)$ cannot lie on curves in $\mathrm{Neg}(V_\mathbf{J})$, since if $Q \in \varphi_\mathbf{J}(L_i)$, $1 \leqslant i < 2n$, then $X_3 < L_1$, contrary to (8.8.2); but if $Q \in \varphi_\mathbf{J}(E)$, where $E \in |L_0 - L_1 - L_2|$, then $|L_0 - L_1 - L_2 - X_3| \neq \varnothing$, contrary to condition 4.4.3 for $\mathbf{L}\backslash\mathbf{M}$. Thus the blow up of the point Q leads us to a surface $V_\mathbf{I}$ of type W_n (see 6.11). Taking a basis $\mathbf{K}^1$ on V such that

$$\{\mathbf{K}^1, \mathbf{K}, \mathbf{L}, \mathbf{M}\} = \varphi_1^{-1}(\mathfrak{B}(V_1)),$$

we get in $\Delta_1(V)$ a quadrangle $\{\mathbf{K}^1, \mathbf{K}, \mathbf{L}, \mathbf{M}\}$ of the form indicated in Figure 6.11.1. Put $\kappa = [\mathbf{K}, \mathbf{K}^1, \mathbf{M}]$. We prove 8.5.6.1, which by 6.11.0 is equivalent to the inequality

$$(n-1)(l_0 - l_1) - l_2 - \cdots - l_{2n-2} - x_3 < 0.$$

If the opposite inequality

$$(n-1)(l_0 - l_1) - l_2 - \cdots - l_{2n-2} - x_3 \geqslant 0,$$

were true, then, combining it with the inequality $-l_0 + l_1 + l_2 + x_3 \geqslant 0$, which in our case is a reformulation of (8.7.5), we get

$$(n-2)(l_0 - l_1) - l_3 - \cdots - l_{2n-2} \geqslant 0,$$

which is impossible by (8.7.7).

Let us discuss (8.8.1.b). Here $w[\mathbf{L},\mathbf{K}] = 1$, $\mathbf{L}\backslash\mathbf{K} = \{L_1, L_2, L_3\}$, $\mathbf{L}\backslash\mathbf{M} = \{L_1, L_2, X_3, \ldots, X_{2m-1}\}$, $L_3 \in \mathbf{L}\backslash\mathcal{C}(L_1)$, and $\mathcal{C}(L_3)\backslash\{L_3\} \subset \mathbf{I}$. The set $\mathbf{J} = \mathbf{I} \cup \{L_3\}$ is exceptional, and $V_\mathbf{J}$ is a surface of type V_m (see 6.2). The point $Q = \varphi_\mathbf{J}(L_3)$ does not lie on curves in $\mathrm{Neg}(V_\mathbf{J})$, since L_3 is not contained in L_1 and $|L_0 - L_1 - L_2 - L_3| = \varnothing$. Therefore, the surface $V_\mathbf{I}$ gotten from $V_\mathbf{J}$ by blowing up the point Q is of type W_m (see 6.11). Let us define a basis $\mathbf{K}^1 \in \mathfrak{B}(V)$ and a path κ as we did in discussing (8.8.1.a). The inequality 8.5.6.1 in this case can be written as

$$(m-1)(l_0 - l_1) - l_2 - x_3 - \cdots - x_{2m-2} - l_3 < 0. \tag{8.8.3}$$

If $x_{2m-1} - l_3 < 0$, then, using (8.7.5), we get (8.8.3). Suppose $x_{2m-1} \geqslant l_3$. Then, since

$$l_3 > (l_0 - l_1)/2 \tag{8.8.4}$$

(see (8.7.7.)) and (8.7.3), it follows that

$$x_i > (l_0 - l_1)/2 \quad \text{for } 2 \leqslant i \leqslant 2m-2. \tag{8.8.5}$$

Combining (8.8.5) and (8.8.4), we get (8.8.3).

8.9. In 8.9–8.14 we discuss the possibility

$$\#((\mathbf{L}\backslash\mathbf{K}) \cap (\mathbf{L}\backslash\mathbf{M})) = 1,$$

which we call the *J-case* since here, by the maximality in $\mathbf{L}$ of the intersection considered in (8.9.1), we may assume that $(\mathbf{L}\backslash\mathbf{K}) \cap (\mathbf{L}\backslash\mathbf{M}) = \{L_1\} = \{X_1\}$, and, according to 6.9.3, the curves K_1 and M_1 may be assumed to satisfy

$$K_0 - K_1 \sim L_0 - L_1 \sim M_0 - M_1;$$

that is, $\{K_1, L_1, M_1\}$ is a J-supplement to $\{\mathbf{K}, \mathbf{L}, \mathbf{M}\}$. Moreover, we shall assume that (see (5.5.4)) $K_i \sim L_0 - L_1 - L_i$ and $M_j \sim L_0 - L_1 - X_j$ for $2 \leqslant i < 2n$ and $2 \leqslant j < 2m$. Put

$$\lambda = \max\{(L \cdot A) | L \in \mathbf{L} \backslash \mathbf{K}\} = \max\{l_1, \ldots, l_{2n-1}\}. \tag{8.9.2}$$

8.10. Assume that we are in the J-case; let us prove the proposition under the additional assumption

$$l_1 < \lambda, \tag{8.10.1}$$

where λ is taken from (8.9.2).

From (8.10.1) it follows that $\mathbf{L} \backslash \mathbf{K}$ contains at least two maximal curves; therefore, $w[\mathbf{L}, \mathbf{K}] \leqslant 1$, $n = 2$ and $\mathbf{L} \backslash \mathbf{K} = \{L_1, L_2, L_3\}$. Suppose $l_1 = \lambda$ and L_2, as well as L_1, is a maximal curve in $\mathbf{L} \backslash \mathbf{K}$. We shall assume that $x_2 \geqslant x_3 \geqslant \cdots \geqslant x_{2n-1}$ and that X_2 is maximal in $\mathbf{L} \backslash \mathbf{M} \backslash \{L_1\}$. Let us prove the inequality

$$l_0 - l_1 - l_2 - x_2 < 0. \tag{8.10.2}$$

If the opposite inequality

$$l_0 - l_1 - l_2 - x_2 \geqslant 0, \tag{8.10.3}$$

holds, then combining it with the inequality $-l_0 + l_1 + 2x_2 \geqslant 0$ (see (8.7.8)), we get $x_2 \geqslant l_2$; but, by 8.5.2, $l_2 \geqslant x_2$; therefore, $l_2 = x_2$, and then (8.10.3) contradicts (8.7.6).

We now prove the existence of a basis neighboring $\mathbf{L}$ along $\Lambda = \{L_1, L_2, X_2\}$; i.e., we prove that $\mathbf{L}$ and Λ satisfy 4.4.1–4.4.3. Conditions 4.4.1 and 4.4.2 follow from the maximality in $\mathbf{L}$ of the subsets $\mathbf{K} \backslash \mathbf{M}$, $\{L_1\}$, $\{L_2\}$ and $\{L_1, X_2\}$. Further, if the system $|L_0 - L_1 - L_2 - X_2|$ is not empty, then by intersecting its members with the members of the movable system $|A|$ we find that (8.10.3) is impossible.

Let $\mathbf{K}^1$ be a basis neighboring $\mathbf{L}$ along Λ. By 4.6.1, $\mathbf{K}^1$ also neighbors $\mathbf{K}$, since

$$(\mathbf{L} \backslash \mathbf{K}^1) \cap (\mathbf{L} \backslash \mathbf{K}) = \{L_1, L_2\}.$$

This means that $\Delta_1(V) | \{\mathbf{K}^1, \mathbf{K}, \mathbf{L}\}$ is a triangle. Because of (8.10.2), 8.5.6.1 holds. Further, conditions 8.8.1 and 8.5.0–8.5.4 also hold for $\mathbf{K}^1$, $\mathbf{L}$ and $\mathbf{M}$ if $\mathbf{K}$ is replaced by $\mathbf{K}^1$ in them; therefore, by 8.8, the path κ^1 from $\mathbf{K}^1$ to $\mathbf{M}$ required by Proposition 8.5 (with $\mathbf{K}^1$ replacing $\mathbf{K}$) does exist. It remains to put $\kappa = [\mathbf{K}, \mathbf{K}^1] \circ \kappa^1$.

8.11. In further discussion of the J-case we shall assume that $l_1 = \max\{l_1, \ldots, l_{2n-1}\}$, and we may then assume that inequalities (8.7.2) hold.

Suppose that in the J-case we have

$$w[\mathbf{L}, \mathbf{K}] \geqslant 2, \qquad w[\mathbf{L}, \mathbf{M}] \geqslant 2. \tag{8.11.1}$$

Then, according to 6.19, V_{I} is a surface of type V_{mn}, and, taking in $\mathfrak{B}(V)$ the basis $\mathbf{K}^1$, denoted by $\mathbf{N}$ in 6.19, we get the quadrangle $\Delta_1(V) | \{\mathbf{K}^1, \mathbf{K}, \mathbf{L}, \mathbf{M}\}$ depicted in (6.18.3) if in the latter we replace m by n, and $\mathbf{N}$ by $\mathbf{K}^1$. Further, by (6.19.2), inequality 8.5.6.1 is equivalent to

$$(m + n - 2)(l_0 - l_1) - \sum_{i=2}^{2n-1} l_i - \sum_{i=2}^{2m-1} x_i < 0,$$

and this is indeed true, as one sees by combining (8.7.4) and (8.7.5). Therefore, in the J-case, in (8.11.1) we may take $\kappa = [\mathbf{K}, \mathbf{K}^1, \mathbf{M}]$.

8.12. We shall show that in the J-case we can assume

$$L_1 > L_2, \qquad L_1 > X_2, \tag{8.12.1}$$

where X_2 is a maximal curve in $\mathbf{L}\backslash\mathbf{M}\{L_1\}$ such that

$$x_2 = \max\{x_2, \ldots, x_{2m-1}\}.$$

Suppose that at least one of the curves L_2 or X_2 is not contained in L_1. Let us verify conditions 4.4.1–4.4.3 for the set $\Lambda = \{L_1, L_2, X_2\}$. The set Λ is maximal in $\mathbf{L}$, and none of the curves in Λ can contain the sum of two other curves in Λ; all this is evident. By combining (8.7.6) and (8.7.8) we get

$$l_0 - l_1 - l_2 - x_2 < 0, \tag{8.12.2}$$

whence $|L_0 - L_1 - L_2 - X_2| = \varnothing$. Suppose $\mathbf{K}^1$ is a basis on V such that $\mathbf{L}\backslash\mathbf{K}^1 = \Lambda$. That 8.5.6.1 holds is guaranteed by (8.12.2), and now the final reasoning of 8.10 allows us to conclude the discussion of the J-case with a violation of condition (8.12.1).

From what has been proved here and in 8.11 it follows that in further investigating the J-case we may assume that either

$$L_1 > L_2, \quad L_1 > X_2, \quad w[\mathbf{L}, \mathbf{K}] = 1, \tag{8.12.a}$$

or

$$L_1 > L_2, \quad L_1 > X_2, \quad w[\mathbf{L}, \mathbf{K}] = 2, \quad w[\mathbf{L}, \mathbf{M}] = 1, \tag{8.12.b}$$

where X_2 is a maximal curve in $\mathbf{L}(\mathbf{M})$ and $x_2 = \max\{x_2, \ldots, x_{2m-1}\}$.

8.13. We shall prove the proposition under condition (8.12.a). From (8.7.7) and (8.7.8) it follows that $l_0 - l_1 - l_3 - x_2 < 0$. We now reduce the matter to the final two arguments of 8.10, where it is only necessary to replace L_2 by L_3, and to replace the reference to the impossibility of inequality (8.10.3) by a reference to the inequality contrary to the one just adduced.

8.14. To complete our work with the J-case it remains to examine the possibility (8.12.b). Here $\mathbf{L}\backslash\mathbf{M} = \{L_1, X_2, X_3\}$ and $L_1 > X_2$.

Let us first assume that, besides (8.12.b),

$$|L_0 - L_1 - L_2 - X_3| = \varnothing. \tag{8.14.1}$$

Put $\mathbf{J} = \mathbf{I} \cup \{X_2, X_3\}$. The set $\mathbf{J}$ is exceptional, $V_\mathbf{J}$ is a surface of type V_n (in 6.2), and $\varphi_\mathbf{J}(X_3)$ is a point in general position on $V_\mathbf{J}$ (see 0.7). The last follows from (8.14.1) and (8.12.b), which entails the impossibility of the inequality $L_1 > X_3$. In the notation of 6.2, the point $\varphi_\mathbf{J}(X_2)$ lies on the complement $E_1 \backslash E_n$. Therefore, the surface $V_\mathbf{I}$ gotten from $V_\mathbf{J}$ by blowing up on the pair $\varphi_\mathbf{J}(X_2 \cup X_3)$ is a surface of type W_n^0 in 6.17, and we may assume that the bases $\mathbf{K}, \mathbf{L}, \mathbf{M} \in \mathfrak{B}(V_\mathbf{I})$ introduced in 6.17 induce, by means of $\varphi_\mathbf{I}^{-1}$, bases on V denoted by the same letters respectively. If we put $\mathbf{K}^1 = \varphi_\mathbf{I}^{-1}(\mathbf{N})$, where $\mathbf{N}$ is as in 6.17, then the path $\kappa = [\mathbf{K}, \mathbf{K}^1, \mathbf{M}]$ is the required one. Indeed, we need only verify that 8.5.6.1 holds; and by 6.17.2 it is equivalent to the inequality

$$n(l_0 - l_1) - l_2 - \cdots - l_{2n-1} - x_2 - x_3 < 0.$$

But this follows by combining (8.7.4) and (8.7.5).

Now assume that (8.12.b) holds and

$$|L_0 - L_1 - L_2 - X_3| \neq \varnothing. \tag{8.14.2}$$

Here we need condition 8.5.4, applied to the situation where

$$C_1 = L_1, \quad C_2 = L_2, \quad X = X_2, \quad Y = X_3.$$

The inequality $((L_0 - C_1 - X - Y) \cdot A) \leqslant 0$ from 8.5.4 is equivalent to (8.7.5) if we take $m = 2$ and $C_1 = L_1 = X_1$.

Assume that 8.5.4.1 is true; that is, there is a curve $L \in \mathbf{L} \cap \mathbf{K}$ such that

$$L < L_{2n-1}, \qquad ((2L + L_1 - L_0) \cdot A) > 0. \tag{8.14.3}$$

Take a maximal possible L with these properties and denote it by L_{2n}. Clearly, $L_{2n} \in \mathcal{C}^*(L_{2n-1})$ (see 1.7.1). Putting, in harmony with the notation of (8.7.1), $l_{2n} = (A \cdot L_{2n})$, we can rewrite (8.14.3) in the form

$$2l_{2n} > l_0 - l_1. \tag{8.14.4}$$

All the conditions a)–d) of assertion 6.14 hold here. Therefore, $\mathfrak{B}(V)$ contains the basis $\mathbf{N}$ indicated in 6.14; here we denote it by $\mathbf{K}^1$. To prove that the path $\kappa[\mathbf{K}, \mathbf{K}^1, \mathbf{M}]$ is correct, we must verify 8.5.6.1, which, by (6.14.1), is equivalent to the inequality

$$n(l_0 - l_1) - l_2 - \cdots - l_{2n} - x_2 < 0.$$

This evidently follows from (8.14.4), (8.7.4), (8.7.8) and the equation $x_1 = l_1$.

Finally, assume that the conclusion of 8.5.4.2 holds. Denote the bases $\mathbf{K}'$ and $\mathbf{K}''$ given in it by $\mathbf{K}^1$ and $\mathbf{K}^2$ respectively. Put

$$\mathbf{L}\backslash\mathbf{K}^2 = \{L_1, X_2, Y_3, \ldots, Y_{2s-1}\}, \qquad y_i = (Y_i \cdot A),$$

where $3 \leqslant i < 2s$ and $s = w[\mathbf{L}, \mathbf{K}^2]$. Note that

$$x_2 > (l_0 - l_1)/2, \qquad y_i > (l_0 - l_1)/2, \quad \text{where } i \in [3, 2s-1]. \tag{8.14.5}$$

Combining all these inequalities, we get

$$(s-1)(l_0 - l_1) + x_2 + y_2 + \cdots + y_{2s-1} < 0, \tag{8.14.6}$$

which is equivalent to 8.5.6.2. By (6.19.2), the inequality 8.5.6.1 is gotten by combining (8.7.4) and (8.14.6). Further, $\mathbf{K}^2$, $\mathbf{L}$ and $\mathbf{M}$ satisfy

$$\begin{tikzcd} \mathbf{K} \arrow[r, dash] \arrow[d, dash] & \mathbf{L} \arrow[r, dash] \arrow[d, dash] & \mathbf{M} \\ \mathbf{K}^1 \arrow[r, dash] & \mathbf{K}^2 \end{tikzcd}$$

and also (8.8.1) and 8.5.0–8.5.4, if we replace $\mathbf{K}$ in them by $\mathbf{K}^2$; therefore, according to 8.8 the path κ^2 from $\mathbf{K}^2$ to $\mathbf{M}$ required by Propostion 8.5 (with $\mathbf{K}^2$ replacing $\mathbf{K}$) does exist. It remains to put $\kappa = [\mathbf{K}, \mathbf{K}^1, \mathbf{K}^2] \circ \kappa^2$; this completes our examination of the J-case.

8.15. Suppose $(\mathbf{L}\backslash\mathbf{K}) \cap (\mathbf{L}\backslash\mathbf{M}) = \varnothing$. We may assume that

$$l_1 \geqslant l_2, \text{ and the pair } \{l_1, l_2\} \text{ is maximal in } \{l_1, \ldots, l_{2n-1}\}, \tag{8.15.1}$$

$$x_1 = \max\{x_1, \ldots, x_{2m-1}\}. \tag{8.15.2}$$

We shall prove that

$$l_0 - l_1 - l_2 - x_1 < 0. \tag{8.15.3}$$

If the contrary inequality $x_1 \leqslant l_0 - l_1 - l_2$ holds, then $x_1 \leqslant (l_0 - l_1)/2$ by (8.7.6) and (8.15.1); from this and (8.15.2) and (8.7.5) we get

$$(m-1)(l_0 - x_1) + (2m-2)(l_0 - l_1)/2 \leqslant 0.$$

Dividing out by $m-1$ we get $3l_1 < l_0$, which, by (8.15.1) and (8.7.6), is impossible. Thus (8.15.3) is proved, and we have incidentally established that

$$x_1 > (l_0 - l_1)/2. \tag{8.15.4}$$

There exists a basis **N** neighboring **L** along $\Lambda = \{L_1, L_2, X_1\}$, since conditions 4.4.1 and 4.4.2 for **L** and Λ follows from the maximality in **L** of the subsets $\{L_1, L_2\}$ and $\{X_1\}$; and 4.4.3 follows from (8.15.3). From (8.15.3) it also follows that

$$(N_0 \cdot A) < (L_0 \cdot A). \tag{8.15.5}$$

The triple **K, L, N** (**N, L, M**) satisfies conditions 8.5.0–8.5.4 if in them we replace **M** (**K**) by **N**; moreover,

$$(\mathbf{L}\backslash\mathbf{K}) \cap (\mathbf{L}\backslash\mathbf{N}) = \{L_1, L_2\}, \tag{8.15.6}$$

$$(\mathbf{L}\backslash\mathbf{N}) \cap (\mathbf{L}\backslash\mathbf{M}) = \{X_1\}. \tag{8.15.7}$$

From (8.15.5), (8.15.6) and 8.8 we deduce the existence of a path from **K** to **N** satisfying the requirements of Proposition 8.5, where **M** has been changed to **N**; and from (8.15.5), (8.15.7) and the previous examination of the J-case (8.9.1) it follows that there is a path κ^2 from **N** to **M** that satisfies the requirements of Proposition 8.5, where **K** has been replaced by **N**. From κ^1 and κ^2 we can compose the needed path $\kappa = \kappa^1 \circ \kappa^2$.

Proposition 8.5 and Theorem 8.1 have been proved.

§9. The stable simple connectedness of the complex Γ

9.1. THEOREM. *For any surface V and any loop ζ in the graph $\Gamma_1(V)$ there is a natural number N such that, for the blowing up $f\colon U \to V$ of a set of N points in general position on V, the loop $f^{-1}(\zeta)$ in the complex $\Gamma(U)$ (see* 8.0) *is contractible.*

This theorem will be proved in 9.2–9.5; subsections 9.2–9.4 are devoted to de Jonquières paths, i.e., paths $\zeta = [\mathbf{A}, \mathbf{B}, \ldots]$, the set of whose vertices $\zeta_0 = \{\mathbf{A}, \mathbf{B}, \ldots\}$ has a J-supplement in the sense of 5.1. By ζ_1 we shall denote the one-dimensional complex whose support is the loop ζ. For a subcomplex $\mathfrak{L}$ of $\Gamma(V)$ (or in $\Delta(V)$), $\mathfrak{L}_i$ will denote its i-dimensional skeleton.

9.2. PROPOSITION. *Let ζ be a J-loop of the graph $\Gamma_1(V)$. There is a natural number N such that for any blowing up $f\colon U \to V$ of a set of N points in general position $\Gamma(U)$ contains a simplicial subcomplex $\mathfrak{L}$ such that*

a) *$\mathfrak{L}$ is connected and of dimension* $\leqslant 2$.

b) $\mathfrak{L} \supset f^{-1}(\zeta_1)$, $\mathfrak{L}_0 \cap f^{-1}(\mathfrak{B}(V)\backslash\zeta_0) = \varnothing$, *and*

c) *$\mathfrak{L}_0$ can be J-supplemented to a set $\mathfrak{L}'_0$ containing $f^{-1}(\zeta'_0)$, where ζ'_0 is a given J-supplement to ζ_0.*

Proposition 9.2 will be proved in 9.3 and 9.4.

9.3. Here we introduce some notation. Let $\mathbf{L} = \mathbf{L}^0 = \mathbf{L}^n$ be the origin of the loop $\zeta = [\mathbf{L}^0, \mathbf{L}^1, \ldots, \mathbf{L}^n]$; this **L** we assume distinguished in ζ_0 in the applications of 5.9; suppose $\zeta'_0 = \{L_1^0, \ldots, L_1^n\}$ is a J-supplement to ζ_0. The superscript i of $\mathbf{L}^i$ and L_1^i we shall consider as the residue class modulo n. For $i \in \mathbf{Z}/n\mathbf{Z}$ we denote by (i) the pair $(i, [\mathbf{L}^i, \mathbf{L}^{i+1}])$, which we shall call the ith edge of the loop ζ. As the cyclic distance between edges (i) and (j) we take the minimum of the pairs of smallest natural numbers (see 0.3) that are residues modulo n of the numbers $i-j$ and $j-i$. By 5.9 we can assign to each edge $[\mathbf{L}^i, \mathbf{L}^{i+1}]$ (and then to (i) too) the mark $\mu(i) = \mu(\mathbf{L}^i, \mathbf{L}^{i+1})$, a two-element (see (5.9.3) subset of $\mathbf{L}\backslash\{L_1\}$. For a curve L in $\mathbf{L}\backslash\{L_1\}$ we shall denote by $s_L(\zeta)$ the least possible cyclic distance between the various edges of ζ whose marks contain L. If L does not occur in the marks, we take $s_L(\zeta) = \infty$. Note that (5.9.4) implies that the symmetric difference $\mu(1)\Delta\mu(2)\Delta \cdots \Delta\mu(n-1)$ is empty; therefore, any L meets the marks of the edges of loops an even number of times.

Put $s(\zeta) = \min\{s_L(\zeta) \mid L \in \mathbf{L}\backslash\{L_1\}\}$, let $t(\zeta)$ be the number of edges in ζ_1, and call the pair $(t(\zeta), s(\zeta))$ the characteristic of the loop ζ. Note that $t(\zeta) \leqslant n$ and $s(\zeta) \leqslant [n/2]$, where n is the length of ζ, and [ ] is the integral part.

9.4. We prove 9.2 by induction on the lexicographically ordered set of characteristics.

Note that what is essential in the hypothesis and conclusion of 9.2 is not the loop ζ itself but its support ζ_1. In particular, the order of traversing its vertices is inessential; an additional wandering over parts lying in ζ_1 has no essential effect on 9.2.

Assume that ζ has no point of selfintersection. Then we can choose loops ξ and η in the graph $\Gamma_1(V)$ such that

9.4.0. $\zeta_1 = \xi_1 \cup \eta_1$, $t(\xi) < t(\zeta)$, $t(\eta) < t(\zeta)$, and $\xi_1 \cap \eta_1$ is a connected and simply connected graph.

For ξ let us take the number N_1 guaranteed by the induction hypothesis. Let $f_1\colon U_1 \to V$ be the blowing up of N_1 points in general position, and $\mathfrak{L}^1 \subset \Gamma(U_1)$ a subcomplex satisfying the requirements of 9.2 (where we have replaced ζ by ξ, f by f_1, $\mathfrak{L}$ by $\mathfrak{L}^1$, and ζ_0' by the subset $\xi_0' \subset \zeta_0'$). For the loop $f_1^{-1}(\eta)$ let us take the number N_2 and put $N = N_1 + N_2$. If $f\colon U \to V$ is the blowing up of N points in general position, then we may assume that $f = f_1 \circ f_2$, where $f_2\colon U \to U_1$ is the blowing up of N_2 points. If $\mathfrak{L}^2 \subset \Gamma(U)$ is the subcomplex for the loop $f_1^{-1}(\eta)$, we put

$$\mathfrak{L} = f_2^{-1}(\mathfrak{L}^1) \cup \mathfrak{L}^2, \qquad \mathfrak{L}_0' = f_2^{-1}\big((\mathfrak{L}_0^1)'\big) \cup \big(\mathfrak{L}_0^2\big)'.$$

The connectedness and simple connectedness of $\mathfrak{L}$ follows from $f_2^{-1}(\mathfrak{L}^1) \cap \mathfrak{L}^2 = f^{-1}(\xi_1 \cap \eta_1)$ and from the fact that $\mathfrak{L}^1$ and $\mathfrak{L}^2$ have these properties.

In what follows we shall assume that ζ has no selfintersections and $t(\zeta) = n$.

Assume that $s(\zeta) = 1$; i.e., that $\mathbf{L}\backslash\{L_1\}$ contains a curve L that meets the marks of two successive edges of the loop ζ, say for the edges (0) and (1), $\mu(0) = \{L, A\}$, $\mu(1) = \{L, B\}$. Since $\mathbf{L}^0 \neq \mathbf{L}^2$, we have $A \neq B$ and, by (5.9.4),

$$\mu(\mathbf{L}^0, \mathbf{L}^2) = \{L, A\}\Delta\{L, B\} = \{A, B\};$$

from this and (5.9.3) it follows that the bases $\mathbf{L}^0$ and $\mathbf{L}^2$ are neighbors in $\Gamma_1(V)$. For the loops $\xi = [\mathbf{L}^0, \mathbf{L}^1, \mathbf{L}^2, \mathbf{L}^0]$ and $\eta = [\mathbf{L}^0, \mathbf{L}^2, \mathbf{L}^3, \ldots, \mathbf{L}^n]$ we have (compare 9.4.0) $\zeta_1 \subset \xi_1 \cup \eta_1$, $t(\xi) = 3$, $t(\eta) < t(\zeta)$, and $\xi_1 \cap \eta_1 = [\mathbf{L}^0, \mathbf{L}^2]$, and we conclude the discussion of the case $s(\zeta) = 1$ by following the argument of 9.4.0 if in the latter we take $N_1 = 0$, $f_1 = 1_V$, and $\mathfrak{L}^1$ a triangular cell spanned by $\{\mathbf{L}^0, \mathbf{L}^1, \mathbf{L}^2\}$.

Now suppose $s(\zeta) = s > 1$. Take the blowing up $\varphi\colon W \to V$ of a point Q in general position and put $X = \varphi^{-1}(Q)$; we shall write $\mathbf{L}^i$, L_0^i, L_1^i, ζ instead of $\varphi^{-1}(\mathbf{L}^i), \ldots, \varphi^{-1(\zeta)}$ respectively. The plan of what follows is that in $\Gamma_1(W)$ we construct over the polygon ζ as a base the one-dimensional skeleton of an antiprism (see Figure 9.4.1) whose upper base (on the inside in Figure 9.4.1) turns out to lie outside $\varphi^{-1}(\Gamma_1(V))$ and with characteristic less than the characteristic of the lower base; and we apply to the upper polygon the induction hypothesis and include in the subcomplex the lateral surface of the antiprism.

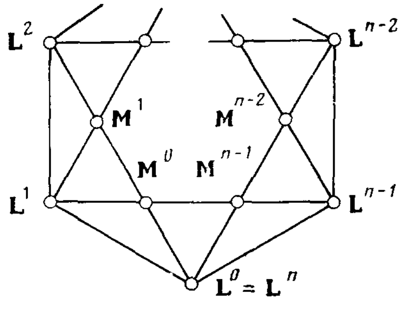


FIGURE 9.4.1

Recall that the superscripts in $\mathbf{L}^i$, L_0^i and L_1^i are considered as elements of $\mathbf{Z}/n\mathbf{Z}$. The transformation $\alpha[\mathbf{L}^0, \mathbf{L}^i]$ (see 5.7 and 5.8) mapping $\mathbf{L}^0$ to $\mathbf{L}^i$ bijectively will be denoted by α_i. The marks $\mu(i) = \{A_i, B_i\}$ of the edges $[\mathbf{L}^i, \mathbf{L}^{i+1}]$ will be assumed ordered so that the element $\alpha_i(A_i) \in \mathbf{L}^i$ is maximal in the pair $\alpha_i(\mu(i)) = \{\alpha_i(A_i), \alpha_i(B_i)\}$.

For any $i \in \mathbf{Z}/n\mathbf{Z}$ we shall prove the existence of a basis $\mathbf{M}^i \in \mathfrak{B}(W)$ such that

$$\mathbf{L}^i \backslash \mathbf{M}^i = \{L_1^i, \alpha_i(A_i), X\}; \tag{9.4.2}$$

that is, we shall verify 4.4.1–4.4.3 for $\mathbf{L}^i$ and for the set on the right-hand side of (9.4.2). This set is maximal in $\mathbf{L}^i$, since in $\mathbf{L}^i$ the subsets

$$\{X\}, \{L_1^i\}, \qquad \mathbf{L}^i \backslash \mathbf{L}^{i+1} = \{L_1^i, \alpha_i(A_i), \alpha_i(B_i)\},$$

are maximal and the curve $\alpha_i(A_i)$ is maximal in $\{\alpha_i(A_i), \alpha_i(B_i)\}$. For the same reason 4.4.2 holds. Further, if $|L_0^i - L_1^i - \alpha_i(A_i) - X| \neq \varnothing$, then the point $Q = \mathfrak{Z}(X)$ lies on a curve of the system $|L_0^i - L_1^i - \alpha_i(A_i)|$ (on V; here we violate our last notational convention for the moment) consisting, by 3.8.3, of one contractible curve, contradicting the fact that Q is in general position (see 0.7).

From (9.4.2), 5.4.1 and 5.1.4 it follows that $\zeta_0 \cup (\cup_1^{n-1}\{\mathbf{M}^i\})$ can be J-supplemented by the set $\varphi^{-1}(\zeta_0') \cup (\cup_1^{n-1}\{M_1^i\}$, where $\{M_1^i\} = |L_0^i - \alpha_i(A_i) - X|$. For the edges of the antiprism we then have the following marks (that these are actually edges in $\Gamma_1(W)$ follows from (5.9.3) and the fact that the marks have two elements):

$$\mu(\mathbf{L}^i, \mathbf{L}^{i+1}) = \{A_i, B_i\}, \qquad \mu(\mathbf{L}^i, \mathbf{M}^i) = \{X, A_i\},$$
$$\mu(\mathbf{L}^{i+1}, \mathbf{M}^i) = \mu(\mathbf{L}^i, \mathbf{L}^{i+1})\Delta\mu(\mathbf{L}^i, \mathbf{M}^i) = \{X, B_i\},$$
$$\mu(\mathbf{M}^i, \mathbf{M}^{i+1}) = \mu(\mathbf{L}^{i+1}, \mathbf{M}^i)\Delta\mu(\mathbf{L}^{i+1}, \mathbf{M}^{i+1}) = \{B_i, A_{i+1}\}.$$

If $s(\zeta) = s = s_B(\zeta)$, where $B \in \mathbf{L}\backslash\{L_1\}$, and B coincides with some B_i in the marks considered, then, by the definition of $s_B(\zeta)$ in 9.3, on ζ_1 there is a path consisting of the edges $(i), (i+1), \ldots, (i+s)$ endowed with the marks

$$\{A_i, B_i\}, \{A_{i+1}, B_{i+1}\}, \ldots, \{A_{i+s}, B_{i+s}\},$$

where $B = B_i = A_{i+s}$. (If B is not second in $\mu(i)$ (i.e. the curve $\alpha_i(B)$ is not minimal in $\alpha_i(\mu(i))$, then we can replace the loop ζ by its opposite.) Then the segment $[\mathbf{M}^i, \ldots, \mathbf{M}^{i+s-1}]$ of the loop $\eta = [\mathbf{M}^0, \mathbf{M}^1, \ldots, \mathbf{M}^n]$ will have the marks

$$\{B_i, A_{i+1}\}, \{B_{i+1}, A_{i+2}\}, \ldots, \{B_{i+s-1}, A_{i+s}\},$$

whence it is clear that $s_B(\eta) \leqslant s-1$; since $t(\eta) \leqslant t(\zeta)$ we have

$$(t(\eta), s(\eta)) < (t(\zeta), s(\zeta)).$$

We apply the induction hypothesis to η, take the corresponding N_1, and put $N = N_1 + 1$. If $f\colon U \to V$ is the blowing up of N points in general position, we may assume $f = \varphi \circ f_1$, where $f_1\colon U \to W$ is the blowing up of N_1 such points. If $\mathfrak{L}^1 \subset \Gamma(U)$ is the subcomplex corresponding to η, we can take $\mathfrak{L}$ to be the union of $\mathfrak{L}^1$ and the f_1-preimage of the simplicial filling of the graph in Figure 9.4.1, and complete the proof of the proposition.

9.4.3. Remark. The N in 9.2 can be assumed to depend only on the length n of the loop ζ; for example, $N = (n!)^{n!}$.

9.5. *Deduction of* 9.1 *from* 9.2, 8.1 *and* 6.12. Let ζ be a loop in $\Gamma_1(V)$. By 8.1, ζ is homotopic in $\Delta(V)$ to a point. Let us take the subcomplex $\mathcal{K} \subset \Delta(V)$ along whose cells the loop ζ passes to a point by successive replacements by a simply homotopic loop (see 0.5) to be finite and of dimension $\leqslant 2$.

In what follows we lift to a surface U dominating V, then partition the edges of $\mathcal{K}$ into smaller ones according to the weights of the edges in $\Gamma_1(U)$, and contract each two-dimensional cell of $\mathcal{K}$ to a simply connected triangulated film by triangles in $\Gamma(U)$.

Let A be the set of all edges α of the complex $\mathcal{K}$ for which $w(\alpha) > 2$, $N_1 = \Sigma_{\alpha \in A} w(\alpha) - 2\#A$, and $f_1\colon U_1 \to V$ is the blowing up of N_1 points in general position. By 6.12, for all pairs of vertices $\mathbf{L}, \mathbf{M} \in f_1^{-1}(\mathcal{K}_0)$ joined by an edge in A, we can choose J-paths $\xi(\mathbf{L},\mathbf{M})$ in the graph $\Gamma_1(U_1)$ going from $\mathbf{L}$ to $\mathbf{M}$ such that $\xi(\mathbf{M},\mathbf{L}) = \xi(\mathbf{L},\mathbf{M})^{-1}$ and the common vertices of any distinct such paths can only be their boundary, if the paths are not inverses (this last is evident from the construction 6.11, 6.12 and the choice of N_1). If vertices $\mathbf{L}$ and $\mathbf{M}$ in $f_1^{-1}(K_0)$ are joined by and edge in $f_1^{-1}(\mathcal{K}) \cap \Gamma_1(U_1)$, then $\xi[\mathbf{L},\mathbf{M}]$ will denote the path $[\mathbf{L},\mathbf{M}]$. If $[\mathbf{L}^0,\ldots,\mathbf{L}^3]$ ($[\mathbf{L}^0,\ldots,\mathbf{L}^4]$) is a path describing a triangular (quadrangular) cell in $f_1^{-1}(\mathcal{K})$, then the path

$$\xi(\mathbf{L}^0,\mathbf{L}^1) \circ \cdots \circ \xi(\mathbf{L}^2,\mathbf{L}^3) \qquad (\xi(\mathbf{L}^0,\mathbf{L}^1) \circ \cdots \circ \xi(\mathbf{L}^3,\mathbf{L}^4))$$

is, by 6.12 and 5.1.4., a de Jonquières loop in $\Gamma_1(U_1)$. Let $B = \{\beta_1,\ldots,\beta_n\}$ be a set of such paths taken one from each two-dimensional cell of $f^{-1}(\mathcal{K})$; let N_i' be the number corresponding to the loop β_i by 9.2, $N_2 = \Sigma_1^n N_i'$ (of course, N_2 also corresponds to each β_i), and $N = N_1 + N_2$. If $f\colon U \to V$ is the blowing up of N points in general position, then we may assume $f = f_1 \circ f_2$, where f_1 has been considered above. By 9.2 we can find n connected, simply connected subcomplexes $\mathfrak{L}^i \subset \Gamma(U)$ of dimension $\leqslant 2$ such that

$$\mathfrak{L}^i \supset f_2^{-1}(\beta_i), \qquad \mathfrak{L}^i_0 \cap f_2^{-1}(\mathfrak{B}(U_1) \backslash (\beta_i)_0) = \varnothing,$$
$$\mathfrak{L}^i \cap \mathfrak{L}^j \subset (\beta_i)_1 \cap (\beta_j)_1 \quad \text{for } i \neq j.$$

It is now clear that the loop $f^{-1}(\zeta)$ can be deformed to a point by successively moving its links along the cells of the subcomplex $\bigcup_1^n \mathfrak{L}^i$.

Theorem 9.1 is proved.

§10. Relations between projective and quadratic transformations

10.0. The goal of this section is to prove the assertion on the relations that was stated in the Introduction.

We use the notation $\mathcal{P}$ and $\mathcal{Q}$ given there. Here and in 10.1 we introduce a series of definitions analogous to those in the introductory chapter of [10], only here we shall consider as "trivially equal to unity" not only gg^{-1} but also (see 10.1.3 and 10.1.4) the words that arise from the multiplication law in $\mathcal{P}$ (i.e. the words $p_1p_2p_3$, where $p_i \in \mathcal{P}$, $p_3 = (p_1 \circ p_2)^{-1}$ and from the actions of group $\mathcal{P}$ on $\mathcal{Q}$ from both the left and the right (i.e. the words pq_1q_1 and q_4q_3p, where $p \in \mathcal{P}$, $q_i \in \mathcal{Q}$, $q_2 = (p \circ q_1)^{-1}$ and $q_4 = (q_3 \circ p)^{-1}$).

By a *$\mathcal{P}\mathcal{Q}$-word* (*$\mathcal{Q}$-word*) we shall mean a finite sequence

$$g_1, g_2, \ldots, g_n, \tag{10.0.1}$$

where $n \geqslant 1$; n is called the *length* of the word, and $g_i \in \mathcal{P} \cup \mathcal{Q}$ ($g_i \in \mathcal{Q}$) for $1 \leqslant i \leqslant n$. We shall consider $i \in \mathcal{P}$ as the word of length zero. The sequence (10.0.1) will be written without commas, i.e. in the form $g_1g_2 \cdots g_n$, and the composition of the birational transformations of the sequence (10.0.1) will be written as

$$g_1 \circ g_2 \circ \cdots \circ g_n. \tag{10.0.2}$$

For a word w (of the form (10.0.1)) the *inverse word* is $w^{-1} = g_n^{-1} \cdots g_1^{-1}$. The word (10.0.1) is called a *relation* if the composition (10.0.2) equals 1; in this case we shall also write the relation as $g_1g_2 \cdots g_n = 1$.

10.1. Let F be a set of $\mathcal{P}\mathcal{Q}$-words. A word w is called *simply deducible from F* if w is gotten as a result of applying one of the following operations to some word v in F:

10.1.1. inserting one of the words u, u^{-1}, gg^{-1} or 1 (where $u \in F$ and $g \in \mathcal{P} \cup \mathcal{Q}$) either between two neighboring terms of v, or before, v, or after v,

10.1.2. removing from v parts that coincide with one of the words given in 10.1.1,

10.1.3. changing a pair g, h of neighboring terms in v, of which at least one is in $\mathcal{P}$, to the term $g \circ h$, or

10.4. by changing a term f in v to two successive terms gh, where g and h are such that at least one is in $\mathcal{P}$ and $g \circ h = f$.

A word w is called *deducible form F* if there is a sequence of words $w_0, w_1, \ldots, w_l$ such that $w_0 = 1$, $w_l = w$, and each word w_i ($1 \leqslant i \leqslant l$) is simply deducible from $\{w_0, \ldots, w_{i-1}\} \cup F$.

Two sets of words F and F' are called *equivalent* (written $F \sim F'$) if each word in F' is deducible from F and vice versa. Words w and w' are called *equivalent* if $\{w\} \sim \{w'\}$.

10.2. *Three remarks on the definitions in* 10.1 *and* 10.1:

10.2.1. If F consists of relations, then any word deducible from F is a relation.

10.2.2. Two mutually inverse words or two words gotten from each other by a cyclic permutation of terms are equivalent.

10.2.3. Any $\mathcal{P}\mathcal{Q}$-word w is equivalent to either some $\mathcal{Q}$-word gotten from w by applying the operation 10.1.3 or a $\mathcal{P}$-word of length $\leqslant 1$.

10.3. *To each $\mathcal{Q}$-word*

$$w = q_1q_2 \cdots q_n \tag{10.3.1}$$

let us assign a surface V and a path

$$\zeta = [\mathbf{L}^0, \mathbf{L}^1, \ldots, \mathbf{L}^n] \tag{10.3.2}$$

in the graph $\Gamma_1(V)$ such that

10.3.3. *to any set of morphisms ψ_i: $V \to \mathbf{P}_2$ with the property $\mathbf{I}(\psi_i) = \mathbf{L}^i$ ($0 \leqslant i \leqslant n$) (see* 2.3) *there corresponds a sequence of projective transformations $p_0, p_1, \ldots, p_n$ satisfying the equations*

$$q_i \circ q_{i+1} \circ \cdots \circ q_l = p_{i-1} \circ \psi_{i-1} \circ \psi_l^{-1} \circ p_l^{-1}, \tag{10.3.3.1}$$

for $1 \leqslant i \leqslant l \leqslant n$.

For this we take surfaces X_i and morphisms u_i: $X_i \to \mathbf{P}_2$ and v_i: $X_i \to \mathbf{P}_2$ such that

$$q_i = v_i \circ u_i^{-1} \qquad (1 \leqslant i \leqslant n). \tag{10.3.4}$$

By induction on n it is not hard to establish the existence of a surface V and morphisms f_i: $V \to X_i$ satisfying the equations

$$v_{i+1} f_{i+1} = u_i f_i, \tag{10.3.5}$$

i.e. making the following diagram commutative:

$$\begin{tikzcd}
 & & & & V \arrow[dlll, "f_n"'] \arrow[dl, "f_{n-1}"'] \arrow[dr, "f_2"] \arrow[drrr, "f_1"] & & & & \\
 & X_n \arrow[dl, "u_n"'] \arrow[dr, "v_n"] & & X_{n-1} \arrow[dl, "u_{n-1}"] & \cdots & X_2 \arrow[dr, "v_2"'] & & X_1 \arrow[dl, "u_1"'] \arrow[dr, "v_1"] & \\
\mathbf{P}_2 \arrow[rr, dashed] & & \mathbf{P}_2 \arrow[rr, dashed] & & \cdots \arrow[rr, dashed] & & \mathbf{P}_2 \arrow[rr, dashed] & & \mathbf{P}_2
\end{tikzcd} \tag{10.3.6}$$

Put

$$\varphi_0 = v_1 f_1, \qquad \varphi_i = u_i f_i \quad \text{for } 1 \leqslant i \leqslant n, \tag{10.3.7}$$

$$\mathbf{L}^i = \mathbf{I}(\varphi_i) \qquad (0 \leqslant i \leqslant n). \tag{10.3.8}$$

From (10.3.5)–(10.3.7) it follows that

$$\varphi_{i-1} = v_i \circ f_i \quad \text{for } 1 \leqslant i \leqslant n, \tag{10.3.9}$$

and from (10.3.7) and (10.3.9)

$$q_i = \varphi_{i-1} \circ \varphi_i^{-1} \qquad (1 \leqslant i \leqslant n), \tag{10.3.10}$$

whence in turn we get

$$q_i \circ q_{i+1} \circ \cdots \circ q_l = \varphi_{i-1} \circ \varphi_l^{-1} \qquad (1 \leqslant i \leqslant l \leqslant n). \tag{10.3.11}$$

Let $\psi_0, \ldots, \psi_n$ be the morphisms in 10.3.3. Then by (10.3.8) and 2.3, for some $p_0, \ldots, p_n$ in $\mathcal{P}$ we get

$$\varphi_i = p_i \psi_i \qquad (0 \leqslant i \leqslant n), \tag{10.3.12}$$

which, together with (10.3.11), implies (10.3.3.1).

We make two remarks

10.3.3. If the word (10.3.1) is a relation, then (10.3.3.1) with $i = 1$ and $l = n$ yields $\psi_0 \psi_n^{-1} \in \mathcal{P}$; that is, $\mathbf{L}^0 = \mathbf{L}^n$, and ζ is a loop.

10.3.14. If ζ is the path in $\Gamma_1(V)$ constructed from the word (10.3.1) and f: $U \to V$ is a morphism, then the path $f^{-1}(\zeta)$ is also suitable, i.e. satisfies 10.3.3.

10.4. *To any triple* (ζ, φ, ψ), *where* $\zeta = [\mathbf{L}^0, \mathbf{L}^1, \ldots, \mathbf{L}^n]$ *is a path of positive length n in* $\Gamma_1(V)$ *and* φ *and* ψ *are morphisms from* V *to* $\mathbf{P}_2$ *for which* $\mathbf{I}(\varphi) = \mathbf{L}^0$ *and* $\mathbf{I}(\psi) = \mathbf{L}^n$, *we assign the family* $F(\zeta, \varphi, \psi)$ *consisting of the* $\mathcal{Q}$*-words of length n that are equivalent* (*as* $\mathcal{P}\mathcal{Q}$*-words*) *to each other. Moreover*:

10.4.1. *If* $w \in F(\zeta, \varphi, \psi)$, *then among all possible paths constructed for the path w by the process in* 10.3, *the path* ζ *is found.*

10.4.2. *If* (ζ, Φ, Ψ) *is another such triple with the same path* ζ *and*

$$\psi^{-1}\varphi = \Psi^{-1}\Phi, \tag{10.4.2.1}$$

then $F(\zeta, \varphi, \psi) = F(\zeta, \Phi, \Psi)$.

In particular,

10.4.3. *If ζ is a loop in $\Gamma_1(V)$, then the family $F(\zeta, \varphi, \varphi)$ does not depend on the choice of φ (so we shall denote such a family by $F(\zeta)$).*

Furthermore,

10.4.4. *The family $F(\zeta)$ in* 10.4.3 *consists of $\mathcal{Q}$-relations.* We also agree that *if $\zeta = [\mathbf{L}^0]$ is a path of length zero and $\mathbf{I}(\varphi) = \mathbf{L}^0$, then $F(\zeta, \varphi, \varphi) = F(\varphi) = \{1\}$.*

To construct $F(\zeta, \varphi, \psi)$ we take morphisms $\varphi_i\colon V \to \mathbf{P}_2$ $(0 \leqslant i \leqslant n)$ satisfying (10.3.8) and the equation

$$\varphi_n^{-1}\varphi_0 = \psi^{-1}\varphi. \tag{10.4.5}$$

We define quadratic transformations q_i using (10.3.10) and form the word (10.3.1). Thus for all possible sets $\{\varphi_i \mid 0 \leqslant i \leqslant n\}$ satisfying (10.3.8) and (10.4.5), words of the family $F(\zeta, \varphi, \psi)$ are also defined.

Let us take another such set $\{\psi_i\}$ and from it construct the words $r_1, \ldots, r_n$, where $r_i = \psi_{i-1}\psi_i^{-1}$ $(1 \leqslant i \leqslant n)$. The p_i indicated in 10.3.3 satisfy (10.3.12) and, by (10.4.5), the equation $\psi_n^{-1}\psi_0 = \varphi_n^{-1}\varphi_0$, whence $p_0 = p_n$. Therefore, by (10.3.3.1) with $1 \leqslant i = l \leqslant n$, the sequence $r_1, \ldots, r_n$ coincides with the sequence

$$p_0^{-1}q_1 p_1, p_1^{-1}q_2 p_2, \ldots, p_{n-1}^{-1}q_n p_0,$$

and so it is clear that the words $r_1 \cdots r_n$ and $q_1 \cdots q_n$ are equivalent. Then assertions 10.4.1–10.4.4 are evident.

10.5. *Some supplements to* 10.4.

10.5.0. If for paths ξ, η and ζ of the graph $\Gamma_1(V)$ we have $\zeta = \xi \circ \eta$ and α, β and γ are morphisms from V to $\mathbf{P}_2$ for which $\mathbf{I}(\alpha)$ is the origin of ξ, $\mathbf{I}(\beta)$ is the end of ξ = origin of η, and $\mathbf{I}(\gamma)$ is the end of η, then we can choose words $u \in F(\xi, \alpha, \beta)$ and $v \in F(\eta, \beta, \gamma)$ such that the word uv lies in $F(\zeta, \alpha, \gamma)$.

For this we can construct a word u (v) from the set $\{\varphi_i\}$ beginning with α (β) and ending with β (γ).

10.5.1. If two loop η and ζ are gotten from each other by a cyclic permutation of the vertices or by changing the direction of traversal, then the families $F(\eta)$ and $F(\zeta)$ introduced in 10.4.3 are equivalent.

This follows from 10.2.2.

10.5.2. If a loop ζ has length 2 (i.e. is gotten by going back and forth along some edge), then $F(\zeta) \sim \{1\}$.

Indeed, in $F(\zeta)$ there is a word of the form qq^{-1} which is equivalent both to the other words in $F(\zeta)$ and to 1.

10.5.3. For a loop ζ in $\Gamma_1(V)$ and a morphism $f\colon U \to V$ we have

$$F(f^{-1}(\zeta)) = F(\zeta).$$

10.6. Here we discuss the set T of three-termed (or triangular) relations.

To any relation of the form $q_1q_2q_3 = 1$, where $q_i \in \mathcal{Q}$, $i = 1, 2, 3$, there corresponds, by 10.3 and 10.3.13, a triangular loop $\theta = [\mathbf{K}, \mathbf{L}, \mathbf{M}, \mathbf{K}]$ in the graph $\Gamma_1(V)$ of some surface V. Conversely, to any triangular loop θ there corresponds, by 10.4, a family $F(\theta)$ of three-termed $\mathcal{Q}$-relations. Therefore

$$T = \bigcup_{V, \theta} F(\theta), \tag{10.6.1}$$

where V runs through the set of all surfaces with triangles in $\Gamma_1(V)$ and θ runs through the set of all triangular loop in $\Gamma_1(V)$. By 10.5.3, in (10.6.1) we can take not all of V but just the minimal surfaces among those containing triangles in Γ_1, i.e. the surfaces U_0, U_1, U_2, U_3, U_4 and the surfaces of type U_5 in 4.7–4.9. Further, if we are interested in making a list of representatives of the equivalence classes of triangular relations, then in (10.6.1) we can choose one triangle in each $\Gamma_1(U_\alpha)$ (the group $\mathrm{Aut}(U_\alpha)$ acts transitively on the set of triangles in $\Gamma_1(U_\alpha)$; see 4.8); for each such triangle we can choose a loop traversing it (by 10.5.1 any change in the order of the vertices in θ does not change the equivalence class of the family $F(\theta)$); finally, for each θ chosen we can choose one word in $F(\theta)$. We shall not do all this here.

10.7. THEOREM. *Any $\mathcal{P}\mathcal{Q}$-relation is deducible (in the sense of Definition* 10.1) *from the set* T (*see* 10.6) *of triangular $\mathcal{Q}$-relations.*

PROOF. By 10.2.3 any $\mathcal{P}\mathcal{Q}$-relation is equivalent to some $\mathcal{Q}$-relation, which, by 10.3.13, 10.4.3 and 10.4.4, can be assumed to lie in $\mathcal{F}(\zeta)$, where ζ is a loop in $\Gamma_1(V)$ for some V. Replacing V, if necessary, by a surface U gotten from V by a morphisms $f\colon U \to V$ and ζ by $f^{-1}(\zeta)$, and using 9.1 and 10.5.3, we may assume that ζ is contractible in the complex $\Gamma(V)$; i.e., that there exists a sequence $\zeta^0, \zeta^1, \ldots, \zeta^n$ of loops in $\Gamma_1(V)$ such that (see 0.5) for $0 \leqslant i < n$ the loops ζ^i and ζ^{i+1} are simply homotopic in $\Gamma(V)$, $\zeta^n = \zeta$, and ζ^0 is a path of length zero. Hence it is clear that to prove the theorem it suffices to establish the following fact.

10.7.1 *If paths ξ and η of the graph $\Gamma_1(V)$ are simply homotopic in the complex $\Gamma(V)$ (in particular, they have the same origin* $\mathbf{M}$ *and end* $\mathbf{N}$) *and there are morphisms $\varphi\colon V \to \mathbf{P}_2$ and $\psi\colon V \to \mathbf{P}_2$ such that $\mathbf{I}(\varphi) = \mathbf{M}$, $\mathbf{I}(\psi) = \mathbf{N}$, and $\varphi = \psi$ if at least one of the paths has length zero, then the sets of words $\mathrm{T} \cup F(\xi, \varphi, \psi)$ and $\mathcal{T} \cup F(\eta, \varphi, \psi)$ (see* 10.4–10.6) *are equivalent in the sense of* 10.1.

PROOF OF 10.7.1. Put $F = F(\xi, \varphi, \psi)$ and $F' = F(\eta, \varphi, \psi)$. Since any member of the family F (or F') is equivalent to the rest of its members, it suffices to find words $w \in F$ and $w' \in F'$ such that $\mathrm{T} \cup \{w\} \sim \mathrm{T} \cup \{w'\}$. Let $\xi = \zeta \circ \xi_0 \circ \zeta_2$ and $\eta = \zeta_1 \circ \eta_0 \circ \zeta_2$, where the loop $\zeta = \xi_0 \circ \eta_0^{-1}$ lies in the one-dimensional skeleton of some simplex s of the complex $\Gamma(V)$ (see 0.5). We may assume that $1 \leqslant \dim s \leqslant 2$ and $\mathrm{length}(\zeta) \leqslant 3$. Denote by $\mathbf{K}$ the common origin of the paths ξ_0 and η_0, and by $\mathbf{L}$ their common end; choose morphisms $\alpha\colon V \to \mathbf{P}_2$ and $\beta\colon V \to \mathbf{P}_2$ such that $\mathbf{I}(\alpha) = \mathbf{K}$ and $\mathbf{I}(\beta) = \mathbf{L}$, and such that if ζ_1, ζ_2, ξ_0 or η_0 has length zero, then, respectively, $\alpha = \varphi$, $\beta = \psi$, $\alpha = \beta$ or $\alpha = \beta$. By 10.5.0 we may choose the four words

$$w_1 \in F(\zeta_1, \varphi, \alpha), \qquad w_2 \in F(\zeta_2, \beta, \psi),$$
$$u \in F(\xi_0, \alpha, \beta), \qquad u \in F(\eta_0, \alpha, \beta)$$

so that

$$w = w_1 u w_2 \in F, \qquad w' = w_1 v w_2 \in F'.$$

Note that $uv^{-1} \in F(\zeta)$. If $\dim s = 1$, then, by 10.5.2 and 10.1.1, $\{w\} \sim \{w'\}$. If $\dim s = 2$, then $F(\zeta) \subset \mathrm{T}$ (that is, uv^{-1} is a triangular relation); it is now clear that

$$\mathrm{T} \cup \{w\} \supset \{uv^{-1}, w\} \sim \{uv^{-1}, w'\} \subset \mathrm{T} \cup \{w'\},$$

which was to be proved.

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