On some tensor representations of the Cremona group of the projective plane

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Intellectus est universalium et non singularium.
Thomas Aquinas, from *Summa contra gentiles* (1264)

0 Introduction

The Cremona group $\mathrm{Cr} = \mathrm{Cr}(2, K)$ is the group of birational automorphisms $\mathrm{Bir}\,\mathbb{P}_2$ of the projective plane; it is (anti-)isomorphic to the automorphism group $\mathrm{Aut}\,K(x, y)$ of the rational function field in two variables.

Let W be a 3-dimensional vector space (over an algebraically closed field K of characteristic zero), and $\mathbb{P}_2 = \mathbb{P}(W)$ its projectivization; $\mathcal{P} = \mathrm{Aut}(\mathbb{P}_2) = \mathrm{PGL}(W) = \mathrm{PGL}(3, K)$ is the collineation group of $\mathbb{P}_2$, that is, the group of projective linear transformations. Thus $\mathcal{P} \subset \mathrm{Cr}$ is a subgroup of the Cremona group. For a linear representation $r\colon \mathrm{GL}(W) \to \mathrm{GL}(V)$, consider the projectivization

$$\rho = \mathbb{P}(r)\colon \mathcal{P} = \mathrm{PGL}(W) \to \mathrm{PGL}(V) = \mathrm{Aut}\,\mathbb{P}(V).$$

An *extension* of ρ is a homomorphism

$$\widetilde{\rho}\colon\ \mathrm{Cr} \to \mathrm{Bir}\,\mathbb{P}(V)$$

which restricts to ρ on $\mathcal{P}$, that is, $\widetilde{\rho}_{|\mathcal{P}} = \rho\colon \mathcal{P} \to \mathrm{Aut}\,\mathbb{P}(V)$; in other words, there is a commutative diagram

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\rho} & \mathrm{Aut}\,\mathbb{P}(V) \\ \downarrow & & \downarrow \\ \mathrm{Cr} & \xrightarrow{\widetilde{\rho}} & \mathrm{Bir}\,\mathbb{P}(V) \end{array}$$

where the vertical arrows are the natural inclusions.

Question *Given the projectivization ρ of a linear representation r, does there exist an extension $\widetilde{\rho}$ of ρ to the whole Cremona group* Cr*?*

We shall see that the answer is yes if $r = S^m(r_0)$ is the mth symmetric power of the natural representation r_0 of $\mathrm{GL}(W)$ in the vector space W^* of linear forms, and $m = 2, 3, 4$. In other words, the Cremona group of the plane has an action on the spaces of plane conics, cubics or quartics, extending the actions of the group of plane collineations.

A first approach to the above question was proposed by Igor Artamkin in his thesis [1], [2], where he constructed an action of the Cremona group of the plane on moduli spaces of stable vector bundles over the projective plane, and deduced an action on the curves of jumping lines of the bundles. A drawback of his approach is that the generic curve of degree > 3 is not realized as the curve of jumping lines of a vector bundle. Moreover, Artamkin's action applies to curves with the additional structure of an even theta characteristic.

Our approach is more algebraic, although we believe that at a deeper level, the reasons underlying Artamkin's constructions and ours are the same. A rough outline of our constructions is as follows.

The group $\mathrm{Cr}(2, K)$ is known to be generated by the collineations $\mathcal{P}$ and the standard quadratic transformation. Given a variety and an action of $\mathcal{P}$, we can obtain the required extension by choosing an action of the standard quadratic transformation with the lucky property that all the relations holding between the collineations and the standard quadratic transformation are satisfied. Of course, to realize this approach, one needs to find a handy and explicit way to verify the list of relations. Section 1 of this paper carries out this program. In a sense, this section complements the main theorem of [11]; it was omitted from [11] in view of the length of the paper.

Section 2 contains a series of general definitions of some objects connected with natural actions of the group $\mathrm{Cr}(n, K)$, or of a more general group $\mathrm{UCr}(n, K)$ (Definition 2.7), which we call the *universal Cremona group.* Our definitions are perhaps too general for applications, but we hope that this philosophy will clarify our constructions. Section 2 ends with a series of verifications of relations as just explained.

Section 3 describes actions of the Cremona group of the plane on the spaces of curves of degrees 2, 3 and 4. We present the first two actions in some detail, but only sketch the treatment for quartics; we hope to return to this case on another occasion.

As an introduction to these ideas, we describe the effect of the standard quadratic transformation s_0 on a generic conic C, following Artamkin [1]. We write $\widehat{C}$ for the dual conic of C; let P_0, P_1, P_2 be the three fundamental points of s_0 and $Q'_0, Q''_0, Q'_1, Q''_1, Q'_2, Q''_2$ the six points of intersection of $\widehat{C}$ with the sides of the fundamental triangle, with Q'_i, Q''_i on the side L_i opposite the vertex P_i for $0 \leq i \leq 2$. Write R'_i and R''_i for the intersection points of L_i and the proper transforms of the lines $P_iQ'_i$ and $P_iQ''_i$ under s_0; then all six points R'_i, R''_i lie on a conic D, and the dual conic $\widehat{D}$ is the image of C by the action of the standard quadratic transformation on the space of conics.

About the same time, a letter from Dolgachev [7] contained the formulas

$$x_0' = x_1x_2,\ x_1' = x_2x_0,\ x_2' = x_0x_1,\ x_3' = x_3x_0,\ x_4' = x_4x_1,\ x_5' = x_5x_2$$

for a quadratic transformation of $\mathbb{P}_5$, which he considered as an analog for $\mathbb{P}_5$ of the standard quadratic transformation s_0. These formulas express the relation between the coefficients of $\widehat{C}$ and of D in Artamkin's construction (compare (3.0) below).

Acknowledgments

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1 Generators and defining relations for Cr

1.1 Generators of the Cremona group

We suppose that the ground field K is algebraically closed; let $(x_0 : x_1 : x_2)$ be homogeneous coordinates on the projective plane $\mathbb{P}_2$ over K. A rational transformation of $\mathbb{P}_2$ can be written

$$x_0' = f_0(x_0, x_1, x_2),\ x_1' = f_1(x_0, x_1, x_2),\ x_2' = f_2(x_0, x_1, x_2), \tag{1.0}$$

where f_0, f_1, f_2 are either homogeneous polynomials of the same degree, or quotients of homogeneous polynomials having the same degrees of homogeneity. The image of a point $(a_0 : a_1 : a_2) \in \mathbb{P}_2$ under such a transformation is

$$\big(f_0(a_0, a_1, a_2) : f_1(a_0, a_1, a_2) : f_2(a_0, a_1, a_2)\big).$$

Let $\mathrm{Cr} = \mathrm{Cr}(2, K)$ denote the set of all invertible rational transformations of $\mathbb{P}_2$ over K.

Let $\mathcal{P} = \mathrm{Aut}\,\mathbb{P}_2 = \mathrm{PGL}(3, K)$ be the set of all projective transformations

$$\begin{aligned} x_0' &= c_{00}x_0 + c_{01}x_1 + c_{02}x_2, \\ x_1' &= c_{10}x_0 + c_{11}x_1 + c_{12}x_2, \quad \text{with } c_{ij} \in K \text{ and } \det(c_{ij}) \neq 0. \\ x_2' &= c_{20}x_0 + c_{21}x_1 + c_{22}x_2, \end{aligned} \tag{1.1}$$

We write $\mathcal{Q}$ for the set of all quadratic transformations, that is, invertible rational transformations of the form (1.0), where f_0, f_1, f_2 are homogeneous polynomials of degree 2 with no common linear factor. The set of quadratic transformations splits into three double cosets under the group of collineations (more precisely, with respect to the natural two-sided action of $\mathcal{P} \times \mathcal{P}$ on $\mathcal{Q}$):

$$\mathcal{Q} = \mathcal{P}s_0\mathcal{P} \sqcup \mathcal{P}s_1\mathcal{P} \sqcup \mathcal{P}s_2\mathcal{P}, \tag{1.2}$$

(here $\sqcup$ means disjoint union), where s_0 is the so-called standard quadratic transformation, given by

$$s_0\colon x_0' = x_1x_2, \quad x_1' = x_0x_2, \quad x_2' = x_0x_1, \tag{1.3}$$

$$\text{or} \qquad x_0' = x_0^{-1}, \quad x_1' = x_1^{-1}, \quad x_2' = x_2^{-1}. \tag{1.4}$$

Next, s_1 is the first degeneration of the standard quadratic transformation, where two of the three fundamental points of s_0 come together, and is given by

$$s_1\colon x_0' = x_1^2, \quad x_1' = x_0x_1, \quad x_2' = x_0x_2, \tag{1.5}$$

$$\text{or} \qquad x_0' = x_1x_0^{-1}x_1, \quad x_1' = x_1, \quad x_2' = x_2. \tag{1.6}$$

Finally, s_2 is the second degeneration of s_0 (or a further degeneration of s_1), where all the fundamental points of s_0 come together to one point, and is given in formulas by

$$s_2\colon x_0' = x_0^2, \quad x_1' = x_0x_1, \quad x_2' = x_1^2 - x_0x_2, \tag{1.7}$$

$$\text{or} \qquad x_0' = x_0, \quad x_1' = x_1, \quad x_2' = x_1x_0^{-1}x_1 - x_2. \tag{1.8}$$

Note that (1.4), (1.6), (1.8) are destined for future noncommutative generalizations (see (2.9)).

Remark 1.1 The third double coset $\mathcal{P}s_2\mathcal{P}$ of (1.2) contains all nonunit elements of the following one-parameter subgroup σ_t (with parameter $t \in K$)

$$\sigma_t\colon x_0' = x_0^2, \quad x_1' = x_0x_1, \quad x_2' = x_0x_2 + tx_1^2$$

$$\text{or} \qquad x_0' = x_0, \quad x_1' = x_1, \quad x_2' = x_2 + tx_1x_0^{-1}x_1.$$

We can write these elements as composites $\sigma_t = p_2 \circ s_2 \circ p_1$ in terms of s_2 and the collineations $p_1, p_2 \in \mathcal{P}$ given by

$$\begin{aligned} &p_1\colon x_0' = x_0, \quad x_1' = -tx_1, \quad x_2' = -tx_2, \\ \text{and} \quad &p_2\colon x_0' = x_0, \quad x_1' = -t^{-1}x_1, \quad x_2' = t^{-1}x_2. \end{aligned}$$

Theorem 1.1 (Max Noether) *The Cremona group* $\mathrm{Cr}(2, K)$ *is generated by* $\mathcal{P} \cup \mathcal{Q}$.

Note that one can, as usual, replace $\mathcal{P} \cup \mathcal{Q}$ by the more economic set of generators $\mathcal{P} \cup \{s_0\}$, writing the transformations s_1, s_2 in terms of s_0 and collineations. More precisely,

$$s_1 = g_0 \circ s_0 \circ g_0 \circ s_0 \circ g_0, \tag{1.9}$$

$$\text{where} \quad g_0\colon x_0' = x_1 - x_0, \quad x_1' = x_1, \quad x_2' = x_2; \tag{1.10}$$

and

$$s_2 = s_1 \circ g_1 \circ s_1, \tag{1.11}$$

$$\text{where} \quad g_1\colon x_0' = x_0, \quad x_1' = x_1, \quad x_2' = x_0 - x_2 \tag{1.12}$$

Remark 1.2 The identities (1.9) and (1.11) have interesting analogs in the Cremona group $\mathrm{Cr}(3, K)$ of 3-space. The involution

$$S_0\colon x_0' = x_0^{-1}, \quad x_1' = x_1^{-1}, \quad x_2' = x_2^{-1}, \quad x_3' = x_3^{-1} \tag{1.4a}$$

is the standard *cubic* transformation of $\mathbb{P}_3$. Take the projective transformation

$$G_0\colon x_0' = x_1 - x_0, \quad x_1' = x_1, \quad x_2' = x_2 \quad x_3' = x_3 \tag{1.10a}$$

as an analog of (1.10). Then the composite

$$S_1 = G_0 \circ S_0 \circ G_0 \circ S_0 \circ G_0, \tag{1.9a}$$

is the *quadratic* space transformation

$$S_1\colon x_0' = x_1^2, \quad x_1' = x_0x_1, \quad x_2' = x_0x_2, \quad x_3' = x_0x_3. \tag{1.5a}$$

Moreover, if we take the collineation

$$G_1\colon x_0' = x_0, \quad x_1' = x_1, \quad x_2' = x_2, \quad x_3' = x_0 - x_3 \tag{1.12a}$$

as an analog of g_1 in (1.12), then the composite

$$S_2 = S_1 \circ G_1 \circ S_1, \tag{1.11a}$$

is the *quadratic* transformation

$$S_2\colon x_0' = x_0^2,\ \ x_1' = x_0x_1,\ \ x_2' = x_0x_2,\ \ x_3' = x_1^2 - x_0x_3 \tag{1.7a}$$

or (in affine coordinates $x = x_1/x_0, y = x_2/x_0, z = x_3/x_0$)

$$S_2\colon x' = x, \quad y' = y, \quad z' = x^2 - z.$$

The composites (1.9a) and (1.11a) contradict some propositions of Dolgachev and Ortland [9], p. 93 (which are comparatively lucid paraphrases of some claims of S. Kantor [13], A. Coble [6], H. Hudson [12], and P. Du Val [10]).

More precisely, write $\mathrm{Cr}_{\mathrm{reg}}(3, K)$ for the subgroup of $\mathrm{Cr}(3, K)$ generated by S_0 and the subgroup of collineations; the elements of $\mathrm{Cr}_{\mathrm{reg}}(3, K)$ are the "regular" transformations in the sense of Coble. Let $\mathrm{Punct}(3, K)$ be the set of Cremona transformations of $\mathbb{P}_3$ without fundamental curves of the first kind, that is, transformations without curves whose proper image in the projective space is a surface, see [12], [9]. The authors listed above start by asserting (sometimes with some provisos) that "one can prove that all punctual transformations form a subgroup of the Cremona group". This is false, because each factor of the right-hand side of (1.9a) is a punctual transformation, whereas the composite S_1 has $x_0 = 0, x_1 = 0$ as a fundamental line of the first kind (maybe, more precisely, a curve infinitely near to this line is a fundamental curve of the first kind); at any rate, no blowups of $\mathbb{P}_3$ at a finite sets of points can reduce the transformations S_1 and S_2 to pseudoisomorphisms in the sense of Dolgachev and Ortland [9]. The identity (1.9a) also refutes the conjectured equality $\mathrm{Punct}(n, K) = \mathrm{Cr}_{\mathrm{reg}}(n, K)$, or even the inclusion $\supset$. Note that the fact that the composites (1.9a) and (1.11a) have even degree also contradicts Coble's formulas, according to which the degree of a "regular" transformation of $\mathbb{P}_d$ is of the form $(d-1)m+1$.

1.2 Defining relations between the generators $\mathcal{P} \cup \mathcal{Q}$

We now reproduce and comment on the main theorem of [11], Theorem 10.7, with some changes of formulation. If $a, b, c, \ldots$ are finitely many elements of the set $\mathcal{P} \cup \mathcal{Q}$ of generators of $\mathrm{Cr}(2, K)$ (see Theorem 1.1), we write $abc\cdots$ to mean a *word* over the *alphabet* $\mathcal{P} \cup \mathcal{Q}$, whereas the expression $a \circ b \circ c \circ \cdots$ means the ordinary *composite* in Cr, that is, a birational transformation of $\mathbb{P}_2$. The theorem on relations is as follows.

Theorem 1.2 *Every relation between the generators $\mathcal{P} \cup \mathcal{Q}$ that holds in $\mathrm{Cr}(2, K)$ is a consequence of the 3-term relations of the form*

$$g_1g_2g_3 = 1, \tag{1.13}$$

where $\{g_1, g_2, g_3\}$ is an ordered triple of elements of $\mathcal{P} \cup \mathcal{Q}$ for which the corresponding composite $g_1 \circ g_2 \circ g_3$ of rational maps equals the identity transformation of $\mathbb{P}_2$.

We pull out some special relations from the above large family (1.13), and each relation of the family will be a consequence of the marked special ones.

1.2.1 The first family of special relations: the multiplication law of the projective group

They are relations of the form

$$p_1 p_2 p_3 = 1, \tag{1.14}$$

where $p_1, p_2, p_3 \in \mathcal{P}$ are collineations, and $p_1^{-1} = p_2 \circ p_3$.

1.2.2 Generalities on edge relations

The *edge relations* arise from the two-sided action of the collineations on the set of quadratic transformations, and are of the form

$$p_1 q_1 p_2 = q_2, \tag{1.15}$$

where $p_1, p_2 \in \mathcal{P}$, $q_1, q_2 \in \mathcal{Q}$, and $p_1 \circ q_1 \circ p_2 = q_2$. More precisely, each relation (1.15) gives three 3-term relations for use in Theorem 1.2:

$$\begin{gathered} p_1 q_1 (p_2 \circ q_2^{-1}) = 1, \quad p_1 (q_1 \circ p_2) q_2^{-1} = 1, \\ \text{and} \quad (p_1 \circ q_1) p_2 q_2^{-1} = 1. \end{gathered} \tag{1.15a}$$

We picture a relation (1.15) as follows:

$$p_1 \ \bigcirc \underset{q_2^{-1}}{\overset{q_1}{\rightleftarrows}} \bigcirc \ p_2$$

This describes a relation (1.15) as a loop of length 2 going out along an edge and back along the same edge; our term "edge relation" arises from this. The family of all relations (1.15) is still too large and cumbersome, but in 1.2.3, 1.2.4, 1.2.5 below, we distinguish three special edge relations which, together with (1.14), imply all the edge relations. Note that in any relation (1.15), the quadratic transformations q_1 and q_2 both belong to the same double coset of (1.2); this leads us to separate and classify the edge relations according to the subscript n of the representative s_n of the double coset $\mathcal{P}s_n\mathcal{P}$ for $n \in \{0, 1, 2\}$. We call the corresponding relation an (n)*-edge relation.*

1.2.3 The (0)-edge relations

Our second family of special relations are edge relations arising from a loop of length 2 obtained by going out and back along an edge corresponding to the standard quadratic transformation s_0. The loop in question is a marked

$$\begin{tikzcd} g \arrow[r, shift left, "s_0"] & (\overline{g})^{-1} \arrow[l, shift left, "s_0"] \end{tikzcd}$$

Figure 1: (0)-edge relation

path in the graph Γ_1 (see Figure 1 and compare [11], 4.1, 4.4, 4.5 and 10.5.3).

Let G_0 be the collineation group consisting of the transformations

$$g\colon x_0' = t_0 x_i, \quad x_1' = t_1 x_j, \quad x_2' = t_2 x_k, \tag{1.16}$$

where $\{i, j, k\}$ is a permutation of $\{0, 1, 2\}$ and $t_0, t_1, t_2 \in K^*$. In other words, $G_0 = \operatorname{Aut} V_0$, where $V_0 \to \mathbb{P}_2$ is the blowup of $\mathbb{P}_2$ in the three points

$$\{(1:0:0), (0:1:0), (0:0:1)\}.$$

Let $g \mapsto \overline{g}$ be the involutive automorphism of G_0 taking (1.16) to

$$\overline{g}\colon x_0' = t_0^{-1} x_i, \quad x_1' = t_1^{-1} x_j, \quad x_2' = t_2^{-1} x_k. \tag{1.17}$$

Our second family consists of the relations of the form

$$s_0 g s_0 = \overline{g} \quad \text{for } g \in G_0 \subset \mathcal{P}. \tag{1.18}$$

More precisely, each relation (1.18) provides three 3-term relations for use in Theorem 1.2:

$$\begin{gathered} s_0 g (s_0 \circ (\overline{g})^{-1}) = 1, \quad s_0 (g \circ s_0)(\overline{g})^{-1} = 1, \\ \text{and} \quad (s_0 \circ g) s_0 (\overline{g})^{-1} = 1; \end{gathered} \tag{1.18a}$$

compare (1.15) and (1.15a). Note that one of the simplest consequences of (1.18) is $s_0^2 = 1$ (take $g = 1$ in (1.18)).

1.2.4 The (1)-edge relations

Our third family of special relations are edge relations arising from a loop of length 2 obtained by going out and back along an edge corresponding to the first degeneration s_1 of the standard quadratic transformation (see (1.5), (1.6)). See Figure 2, where we omit arrows that can be deduced by analogy with Figure 1.

$$g \ \bigcirc \overset{s_1}{\underset{s_1}{\text{———}}} \bigcirc \ (\overline{g})^{-1}$$

Figure 2: (1)-edge relation

Let G_1 be the collineation group consisting of the transformations

$$g\colon x'_0 = t_0 x_0, \quad x'_1 = t_1 x_1, \quad x'_2 = t_2 x_2 + r x_0, \tag{1.19}$$

where $t_0, t_1, t_2 \in K^*$ and $r \in K$. The group G_1 is $\operatorname{Aut} V_1$, where $V_1 \to \mathbb{P}_2$ is the minimal resolution of the indeterminacy of the rational map s_1 of (1.5). Let $g \mapsto \overline{g}$ be the involutive automorphism of G_1 sending (1.19) to

$$\overline{g}\colon x'_0 = t_1^2 t_0^{-1} x_1, \quad x'_1 = t_1 x_1, \quad x'_2 = t_2 x_2 + r x_1. \tag{1.20}$$

Our third family consists of the relations of the form

$$s_1 g s_1 = \overline{g}, \tag{1.21}$$

where $g \in G_1$. More precisely, as in (1.15a) and (1.18a), (1.21) provides three 3-term relations. As before, $s_1^2 = 1$ is a consequence of (1.18).

1.2.5 The (2)-edge relations

Our fourth family of special relations are edge relations arising from a loop of length 2 obtained by going out and back along an edge corresponding to the second degeneration s_2 of the standard quadratic transformation (see (1.7), (1.8)). See Figure 3, where we omit arrows that can be deduced by analogy with Figure 1, and, here and below, we label the edges by n in place of s_n.

$$g \ \bigcirc \overset{2}{\underset{2}{\text{———}}} \bigcirc \ (\overline{g})^{-1}$$

Figure 3: (2)-edge relation

Let G_2 be the collineation group consisting of the transformations

$$g\colon x'_0 = x_0, \quad x'_1 = t x_1, \quad x'_2 = t^2 x_2 + r x_1 + s x_0, \tag{1.22}$$

where $t \in K^*$ and $r, s \in K$. The group G_2 is isomorphic to the group $\operatorname{Aut} V_2$ of automorphisms of the surface V_2, the minimal resolution of the indeterminacy of the rational map s_2. Let $g \mapsto \overline{g}$ be the involutive automorphism of G_1 sending (1.22) to

$$\overline{g}\colon x'_0 = x_0, \quad x'_1 = t x_1, \quad x'_2 = t^2 x_2 - r x_1 - s x_0. \tag{1.23}$$

Our fourth family consists of the relations of the form

$$s_2 g s_2 = \overline{g}, \tag{1.24}$$

where $g \in G_2$. As in (1.15)–(1.15a), (1.18)– (1.18a), (1.24) provides three 3-term relations. As before, $s_2^2 = 1$ is a consequence of (1.24).

1.2.6 Generalities on triangular relations

The relations (1.18), (1.21) and (1.24) were pictured as walks around the edge in Figures 1, 2 and 3. Our remaining special relations are pictured as marked loops around the triangle of Figure 4 (clockwise, as for the above edge relations). Here the vertices are marked by collineations $p_1, p_2, p_3 \in \mathcal{P}$, and

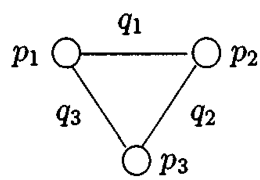


Figure 4: Triangular relations

the edges by quadratic transformations $q_1, q_2, q_3 \in \mathcal{Q}$. The marked triangular loop of Figure 4 gives a relation of the form

$$p_1 q_1 p_2 q_2 p_3 q_3 = 1, \tag{1.25}$$

whenever the composite of rational maps $p_1 \circ q_1 \circ p_2 \circ q_2 \circ p_3 \circ q_3$ is the identity. Although as it stands (1.25) has six terms, it actually reduces to a three-term relation (1.13) if we set $g_i = p_i \circ q_i$. We call (1.25) a *triangular relation.*

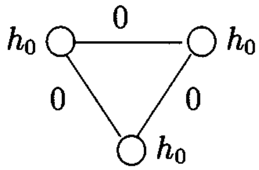
All the triangular relations follow from the special triangular relations written down in 1.2.7–1.2.12 below, together with the special relations already listed above. Each special triangular relation is of the form (1.25) with q_1, q_2, q_3 taken from the quadratic involutions s_0, s_1 or s_2 of (1.3)–(1.8). If (1.25) holds with

$$q_1 = s_{n(1)}, \quad q_2 = s_{n(2)} \quad \text{and} \quad q_3 = s_{n(3)}, \tag{1.26}$$

we say that Figure 4 is an $(n(1), n(2), n(3))$*-triangle* and that the relation (1.25) is an $(n(1), n(2), n(3))$*-triangular relation.* As in 1.2.5, we label edges with the number n instead of s_n.

1.2.7 The special (0,0,0)-triangular relation

This is the following relation:



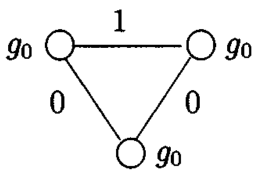
$$h_0 s_0 h_0 s_0 h_0 s_0 = 1, \tag{1.27}$$

where s_0 is the standard quadratic transformation, and h_0 the involutive collineation given by

$$h_0\colon x_0' = x_0, \quad x_1' = x_0 - x_1, \quad x_2' = x_0 - x_2. \tag{1.28}$$

1.2.8 The special (1,0,0)-triangular relation

This is the identity (1.9) written down as the relation

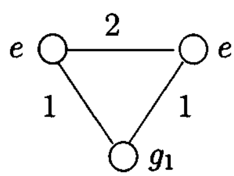


$$g_0 s_1 g_0 s_0 g_0 s_0 = 1, \tag{1.29}$$

where g_0 is the projective involution (1.10).

1.2.9 The special (2,1,1)-triangular relation

This is the identity (1.11) written down as the relation:

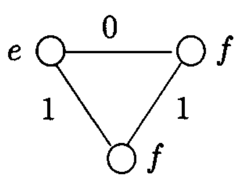


$$s_2 s_1 g_1 s_1 = 1, \tag{1.30}$$

where g_1 is the projective involution (1.12) and $e \in \mathcal{P}$ the identity.

1.2.10 The special (0,1,1)-triangular relation

This is the following relation:



$$s_0 f s_1 f s_1 = 1, \tag{1.31}$$

where f is the collineation

$$f\colon \quad x_0' = x_2, \quad x_1' = x_1, \quad x_2' = x_0.$$

Remark 1.3 It is interesting to note in passing that the relation (1.31) yields as a corollary:

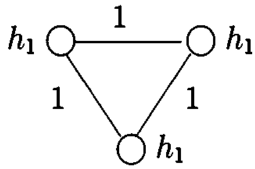
the set $\mathcal{P} \cup \{s_1\}$ *generates the group* $\mathrm{Cr}(2, K)$

(if K is an algebraically closed field, of course).

In contrast, the set $\mathcal{P} \cup \{s_2\}$ does not generate $\mathrm{Cr}(2, K)$. Indeed, $\mathcal{P} \cup \{s_2\}$ is contained in the subgroup $\mathrm{Cr}^{(3)}(2, K) \subset \mathrm{Cr}(2, K)$ consisting of Cremona transformations (f_0, f_1, f_2) (in the notation of (1.0)) having Jacobian determinant a perfect cube; this is a proper subgroup because, for example, $s_0 \notin \mathrm{Cr}^{(3)}(2, K)$.

1.2.11 The special (1,1,1)-triangular relation

This is the relation:



$$h_1 s_1 h_1 s_1 h_1 s_1 = 1, \tag{1.32}$$

where h_1 is the projective transformation

$$h_1 \colon x_0' = x_1 - x_0, \quad x_1' = x_1, \quad x_2' = x_2.$$

1.2.12 The special (2,2,2)-triangular relations

Our final family of relations depends on a parameter $t \in K$, with $t \neq 0, 1$. Write p_t for the projective transformation:

$$p_t \colon x_0' = x_0, \quad x_1' = -tx_1, \quad x_2' = tx_2,$$

and s_2 for the second degeneration of the standard quadratic transformation as in (1.7). Then our final special triangular relations are:

$$p_{t''} s_2 p_{t'} s_2 p_t s_2 = 1, \quad \text{where } t' = 1 - \frac{1}{t} \text{ and } t'' = \frac{1}{1-t}. \tag{1.33}$$

Remark 1.4 There is a more natural and convenient form of (1.33), namely, the multiplication law for the one-parameter group σ_t of Remark 1.1, that is, the relation

$$\sigma_t \sigma_s = \sigma_{t+s},$$

where $s, t \in K$, with $s, t \neq 0$ and $s + t \neq 0$. Note that the last equality is of the form presented by (1.25) with $p_1 = p_2 = p_3 = 1$, $q_1 = \sigma_{t+s}$, $q_2 = \sigma_{-s}$, $q_3 = \sigma_{-t}$.

1.2.13 Theorem 1.2 revisited

The more detailed statement of the theorem on relations is as follows.

Theorem 1.3 *Every relation holding between the generators* $\mathcal{P} \cup \{s_0, s_1, s_2\}$ *of the Cremona group* $\mathrm{Cr}(2, K)$ *is a consequence of the special relations (1.14), (1.18), (1.21), (1.24), (1.27), (1.29), (1.30), (1.31), (1.32), (1.33).*

For the proof, see [11], 10.6–10.7.

2 The universal Cremona group

2.1 Admissible triples, their spaces and maps

Definition 2.1 An *admissible triple* is a triple (R, A, M), where:

1. R is a commutative ring with a unit.
2. A is an R-algebra, not necessarily commutative or associative, but at least *alternative*; this means that R is contained in the centre of A and the subring of A generated by *any two* elements is associative.
3. $M \subset A$ is an R-submodule such that
$$mMm \subset M \quad \text{for every } m \in M.$$
4. If $m \in M$ has a total inverse m^{-1} in A, then $m^{-1} \in M$; here a *total inverse* of a (Mal'tsev [15], Chap. II, 4.3) means an element a^{-1} such that
$$a^{-1}(ax) = (xa)a^{-1} = x \quad \text{for every } x \in A.$$

Let $\mathbb{G}(M)$ denote the set of *units* or totally invertible elements of M.

Definition 2.2 Let R be a commutative ring having an involutive automorphism $r \mapsto \overline{r}$; by default, $^{-}$ is the identity map if no involution is specified.

An *R-algebra with involution* is an R-algebra A with a semilinear involutive anti-automorphism $a \mapsto a^*$; that is, $*$ is an involution satisfying the identities
$$(ra + sb)^* = \overline{r}a^* + \overline{s}b^* \quad \text{and} \quad (ab)^* = b^*a^*.$$

We write
$$A^+ = \{a \in A \,|\, a^* = a\}, \quad A^- = \{a \in A \,|\, a^* = -a\}$$
for the set of $*$-invariant (respectively $*$-anti-invariant) elements of A.

For an R-algebra A with involution, a triple (R, A, M) is *admissible* if it is admissible in the sense of Definition 2.1, and $M^* = M$.

Remark 2.1 If A is an R-algebra with involution, then both (R, A, A^+) and (R, A, A^-) are admissible triples in the sense of Definition 2.2.

Let R be a commutative ring and $n \geq 0$ an integer. We construct a functor $\mathbb{S}_n$ from the category of admissible triples to the category of sets.

Definition 2.3 If (R, A, M) is an admissible triple, we say that an $(n+1)$-tuple $\mathbf{m} = (m_0, \ldots, m_n) \in M^{n+1}$ is *invertible* if $\lambda_0 m_0 + \cdots + \lambda_n m_n$ is invertible in A for some $\lambda_0, \ldots, \lambda_n \in R$; in other words, if the components of $\mathbf{m}$ generate an R-submodule of M having nonempty intersection with $\mathbb{G}(M)$.

On invertible $(n+1)$-tuples, we introduce the equivalence relation $\sim$ which is generated by the elementary relation

$$\exists g \in \mathbb{G}(M) \quad \text{such that} \quad (m'_0, \ldots, m'_n) = (gm_0g, \ldots, gm_ng).$$

(This is the point at which we need A to be alternative.) In other words, two $(n+1)$-tuples $\mathbf{m}$ and $\mathbf{m}'$ are equivalent if and only if there are elements $g_1, \ldots, g_k \in \mathbb{G}(M)$ such that

$$m'_i = g_1(\cdots(g_{k-1}(g_k(m_i)g_k)g_{k-1})\cdots)g_1 \quad \text{for each } 0 \leq i \leq n.$$

We write $(m_0 : \cdots : m_n)$ (or sometimes simply $\mathbf{m}$) for the equivalence class of $\mathbf{m} = (m_0, \ldots, m_n)$, and define $\mathbb{S}_n(R, A, M)$ as the set of equivalence classes of invertible $(n+1)$-tuples under $\sim$. We define the functors $\mathbb{S}_n^+$ and $\mathbb{S}_n^-$ on the category of R-algebras with involution by

$$\mathbb{S}_n^+(A) = \mathbb{S}_n(R, A, A^+) \quad \text{and} \quad \mathbb{S}_n^-(A) = \mathbb{S}_n(R, A, A^-).$$

Remark 2.2 The main example in what follows is the functor $\mathbb{S}_n^+$, especially its value $\mathbb{S}_n^+(\mathrm{Mat}_p(K))$ on the K-algebra $\mathrm{Mat}_p(K)$ of $p \times p$ matrices with entries in an algebraically closed field K, where the involution $*$ is matrix transposition.

In general, $\mathbb{S}_n^+(A)$ is the set of $(n+1)$-tuples $(a_0, \ldots, a_n)$ with $a_i \in A^+$ and with an invertible R-linear combination $\sum \lambda_i a_i \in \mathbb{G}(A)$, modulo the equivalence relation:

$$(a_0, \ldots, a_n) \sim (a'_0, \ldots, a'_n) \iff ba_0b = a'_0, \ldots, ba_nb = a'_n,$$

where $b \in \mathbb{G}(A)$ is a product of elements of $\mathbb{G}(A^+)$. $\mathbb{S}_n^+(A)$ is called the *spherical n-space* or the *n-sphere* over A. This is partly justified by the fact that if A is an algebra with involution $*$ over $\mathbb{R}$ or over $\mathbb{C}$, such that $A^+ = \mathbb{R}$, then $\mathbb{S}_n^+(A)$ is in natural one-to-one correspondence with the unit sphere $S_n \subset \mathbb{R}^{n+1}$.

If K is an algebraically closed field, then the set $\mathbb{G}(\mathrm{Mat}_p(K)^+)$ of invertible symmetric matrices generates the whole group $\mathrm{GL}(p, K)$, hence the spherical space $\mathbb{S}_n^+(\mathrm{Mat}_p(K))$ coincides with the "noncommutative projective space" of Tyurin and Tyurin [17].

Under certain conditions, we define the polynomial $\Delta(K, A, M)(\mathbf{m})$ and some other polynomials $\Gamma(\Delta)$, usually considered up to proportionality. These polynomials depend on dual variables $(u_0, \dots, u_n)$ and $(x_0, \dots, x_n)$, and define hypersurfaces in $\mathbb{P}_n$ and the dual $\check{\mathbb{P}}_n$.

Let A be a finite dimensional associative algebra over the ground field K, and $N_{A/K}: A \to K$ its norm (see Bourbaki, Algèbre, [4], Livre II, Chap. VII; for our purposes, we can use the so-called principal norm in the sense of the exercise in [4], *loc. cit.*, or the reduced norm if A is a semisimple algebra). Write p for the degree of the characteristic polynomial of A ([4], *loc. cit.*), and let $K[u_0, \dots, u_n]_d$ be the vector space of homogeneous polynomials of degree d. We assume that the restriction of the norm $N_{A/K}$ to the subspace $M \subset A$ is the exact qth power of a polynomial: $N_{A/K|M} = (N^0)^q$, where $N^0_{A/K} \in K[M]$.

Definition 2.4 For a fixed $\mathbf{m} = (m_0 : \dots : m_n) \in \mathbb{S}_n(M)$, we set

$$\Delta(K, A, M)(\mathbf{m})(u_0, \dots, u_n) = N^0_{A/K}(u_0 m_0 + \dots + u_n m_n);$$

thus $\Delta(K, A, M) \in K[u_0, \dots, u_n]$ is a homogeneous polynomial of degree p/q.

If $\Gamma\colon K[u_0, \dots, u_n]_{p/q} \to K[x_0, \dots, x_n]$ is a contravariant and has nonzero value at $\Delta(K, A, M)(\mathbf{m})$, then $\Gamma(\Delta(K, A, M)(\mathbf{m}))$ is an equivariant like defined polynomial in the sense of Remark 2.5 below.

Remark 2.3 If $A = \mathrm{Mat}_p(K)$ is a matrix algebra over a field K, with involution matrix transposition, then $\Delta(K, A, A^+)(a) = \det(a)$.

Remark 2.4 If $\dim_K A < \infty$, the hypersurface $\Delta(M)(\mathbf{m})(u_0, \dots, u_n) = 0$ coincides with the so-called *spectrum set* of $\mathbf{m}$, that is, with the set of all points $(y_0 : \dots : y_n) \in \mathbb{P}_n(K)$ for which the linear combination $y_0 m_0 + \dots + y_n m_n$ is a noninvertible element of M. Indeed, by [4], Chap. VII, Proposition 12, an element $x \in A$ is invertible in A if and only if its norm $N_{A/K}(x)$ is invertible in K.

Remark 2.5 The group $\mathrm{PGL}(n+1, K)$ of projective transformations

$$g\colon x'_i = \sum_{j=0}^{n} g_{ij} x_j \quad \text{for } i = 0, \dots, n$$

acts (on the left) on $\mathbb{P}_n$ and (on the right) on $\check{\mathbb{P}}_n$ by the transpose map

$$g^{\mathrm{T}}\colon u'_i = \sum_{j=0}^{n} g_{ji} u_j \quad \text{for } i = 0, \dots, n.$$

This group acts (on the right) on $\mathbb{P}(K[u_0, \ldots, u_m]_d)$: if $F(u) \in K[u]_d$, then $g(F)(u) = F(g^{\mathrm{T}}(u))$. It also acts (on the left) on $\mathbb{S}_n(K, M, A)$:

$$g(\mathbf{m}) = (m'_0 : \cdots : m'_n), \quad \text{where} \quad m'_i = \sum_{j=0}^{n} g_{ij} m_j.$$

Note that

$$\Delta(K, A, M)(g(\mathbf{m})) = g(\Delta(K, A, M)(\mathbf{m})),$$
$$\Gamma(\Delta(K, A, M)(g(\mathbf{m}))) = g^{-1}(\Gamma(\Delta(K, A, M)(\mathbf{m}))).$$

The last equality means that the map

$$\Gamma(\Delta(K, A, M)) \colon \mathbb{S}_n(K, A, M) \to \mathbb{P}(K[x_0, \ldots, x_n])$$

is equivariant with respect to $\mathrm{PGL}(n+1, K)$, because the correspondence $g \mapsto (g^{-1})^{\mathrm{T}}$ is an automorphism of this group.

In the following two definitions, we now construct an analog of homogeneous rational functions, specially adapted to the noncommutative case; these are certain expressions in variables which are either letters, or elements of a K-algebra A. The pattern of our construction follows that of the well-formed formulas in the calculus of mathematical logic (for example, see Church's book [5]); our functions are always derived from well-formed rational expressions. Moreover, for a well-formed homogeneous rational expression f, we define at the same time its *domain of definition* $\operatorname{dom} f \subset M^{n+1}$ (here M is the third component of an admissible triple (K, A, M)), its *value* $f(\mathbf{m}) \in M$ at point $\mathbf{m} = (m_0, \ldots, m_n) \in \operatorname{dom} f$, and the notion of the *domain of invertibility* $\operatorname{dom} f^{-1} \subset M^{n+1}$ of such an expression. The degrees of homogeneity of our functions $f(x_0, \ldots, x_n)$ are the numbers $+1$ and -1; in what follows, ε stands for an element $\varepsilon \in \{+1, -1\}$. The ground field K is fixed, and its elements are called constants.

Definition 2.5 (i) For each $i \in \{0, \ldots, n\}$, we define the coordinate function $f(x_0, \ldots, x_n) = x_i$ to be an expression of degree 1. Its domain of definition $\operatorname{dom} x_i$ is the whole of M^{n+1}, its value at $\mathbf{m} = (m_0, \ldots, m_n)$ is equal to m_i, its domain of invertibility is

$$\{\mathbf{m} = (m_0, \ldots, m_i, \ldots, m_n) \mid m_i \in \mathbb{G}(M)\}.$$

(ii) If f is an expression of degree $\varepsilon = \pm 1$ and λ a nonzero constant, λf is an expression of degree ε by definition. Its domain of definition (or invertibility) coincides with that of f, and its value at $\mathbf{m}$ is equal to $\lambda f(\mathbf{m})$.

(iii) If f and g are expressions of the same degree $\varepsilon = \pm 1$, the sum $f + g$ is an expression of degree ε. Its domain of definition is the intersection of the corresponding domains for f and g, its value $(f + g)(\mathbf{m})$ equals $f(\mathbf{m}) + g(\mathbf{m})$ and its domain of invertibility is

$$\{\mathbf{m} \mid \mathbf{m} \in \operatorname{dom}(f + g) \text{ and } (f + g)(\mathbf{m}) \in \mathbb{G}(M)\}.$$

(iv) If f is an expression of degree ε, then $f^{-1} = 1/f$ is an expression of degree $-\varepsilon$ by definition. The domain of definition and the domain of invertibility of the expression f^{-1} coincide with the domain of invertibility of f. The value $f^{-1}(\mathbf{m})$ is equal to $(f(\mathbf{m}))^{-1}$.

(v) If f, g are expressions of degree ε then $fg^{-1}f$ and $f^{-1}gf^{-1}$, are expressions of degree ε and $-\varepsilon$ respectively by definition. The domain of definition or invertibility of each product is the intersection of the corresponding domains for all the three factors, and

$$(fg^{-1}f)(\mathbf{m}) = f(\mathbf{m})g^{-1}(\mathbf{m})f(\mathbf{m}),$$
$$(f^{-1}gf^{-1})(\mathbf{m}) = f^{-1}(\mathbf{m})g(\mathbf{m})f^{-1}(\mathbf{m}).$$

The smallest set of expressions satisfying the above conditions (i)–(v) is the set of well-formed homogeneous K-rational expressions of variables $(x_0, \dots, x_n)$. Every K-rational expression f has a definite degree $\deg f \in \{+1, -1\}$.

Remark 2.6 If $f(\mathbf{x}) = f(x_0, \dots, x_n)$ is a K-rational expression of degree δ and $g_0(\mathbf{y}), \dots, g_n(\mathbf{y})$ are n expressions in variables $\mathbf{y} = (y_0, \dots, y_m)$ of degree ε, then the composite $f(g_0, \dots, g_n)$ is obviously a rational expression in $\mathbf{y}$ of degree $\varepsilon\delta$.

If f is a well-formed expression of degree ε, and $\mathbf{m} \in M^{n+1}$ belongs to $\operatorname{dom} f$, then $b\mathbf{m}b \in \operatorname{dom} f$ for any $b \in \mathbb{G}(M)$, and $f(b\mathbf{m}b) = b^{\varepsilon} f(\mathbf{m}) b^{\varepsilon}$.

Definition 2.6 If f and g are two well-formed homogeneous expressions in $(n + 1)$ variables, each with nonempty domain of definition in some M^{n+1}, we say that f and g are equivalent (and write $f \equiv g$) if for every admissible triple (K, A, M) and for every element $\mathbf{m} \in \operatorname{dom} f \cap \operatorname{dom} g \subset M^{n+1}$, the equality $f(\mathbf{m}) = g(\mathbf{m})$ holds. A *homogeneous K-rational function* is defined as an equivalence class of well-formed homogeneous K-rational expressions $f(x_0, \dots, x_n)$ with a nonempty domain of definition in some M^{n+1}; the set of these is denoted by $\operatorname{Rat}(n+1, K)$. Note that nonzero homogeneous K-rational functions fall into two sets according to their degree.

Remark 2.7 We have the following identities in two variables x, y:

$$\begin{aligned}(x^{-1})^{-1} &\equiv x, \qquad (x^{-1}yx^{-1})^{-1} \equiv xy^{-1}x,\\ (x^{-1}-y^{-1})^{-1} &\equiv x(x-xy^{-1}x)^{-1}x\\ xy^{-1}x &\equiv x-(x^{-1}-(x-y)^{-1})^{-1}, \qquad (2.1)\\ x(x-xy^{-1}x)^{-1}x &\equiv x-x(x-y)^{-1}x \qquad (2.2)\\ (x^{-1}-y^{-1})^{-1} &\equiv x-x(x-y)^{-1}x. \qquad (2.3)\end{aligned}$$

See Mal'tsev [16], Chap. 2, 4.3 for (2.1); (2.3) follows from (2.1) on substituting $y \mapsto x - y$, and (2.2) is similar.

Definition 2.7 A well-formed K-rational map from projective n-space to projective p-space is given by a $(p+1)$-tuple of K-rational functions of the same degree in variables $(x_0, \dots, x_n)$:

$$\widetilde{f}(\mathbf{x}) = (f_0(\mathbf{x}) : \cdots : f_p(\mathbf{x})); \qquad (2.4)$$

$$\text{or} \quad \widetilde{f}\colon x'_0 = f_0(x_0, \dots, x_n), \dots, x'_p = f_p(x_0, \dots, x_n).$$

Two $(p+1)$-tuples give the same map if they are equivalent under the equivalence relation generated by the following primitive relation: another $(p+1)$-tuple $(g_0(\mathbf{x}) : \cdots : g_p(\mathbf{x}))$ is equivalent to (2.4) if there exists a K-rational function $h(\mathbf{x})$ with $\deg(h) = -\deg(g_i)$ and with nonempty domain of invertibility, such that we have equivalences (in the sense of Definition 2.6) $f_i \equiv hg_ih$ for each $0 \le i \le p$.

The identity map is the transformation given by $x'_0 = x_0, \dots, x'_n = x_n$.

The map $\widetilde{f}(\mathbf{x})$ (2.4) induces a family of partially defined maps $\mathbb{S}_n(M) \to \mathbb{S}_p(M)$, one for every admissible triple (K, A, M); it follows from Remark 2.6 that these maps are well defined. The domain of definition of a $(p+1)$-tuple (2.4) consisting of rational expressions f_i is

$$\{\mathbf{m} \mid \mathbf{m} \in \operatorname{dom} f_i, \text{ for } 0 \le i \le p\},$$

and on it $\widetilde{f}(\mathbf{m})$ defines a point of $\mathbb{S}_p(M)$ in the sense of Definition 2.3.

A well-formed K-birational transformation of projective n-space is a well-formed rational map F of this space to itself such that there is an inverse map G with the property that both composites $F \circ G$ and $G \circ F$ are equal (more precisely, equivalent) to the identity map. We call such a map F a *universal Cremona transformation*; the group of all these is called the *universal Cremona group*, and is denoted UCr(n, K).

A partially defined map $f\colon \mathbb{S}_n(M) \to \mathbb{S}_n(M)$ with a nonempty domain of definition, where M is the third component of an admissible triple (K, A, M), is a *Cremona transformation* if there is an element $F \in \mathrm{UCr}(n, K)$ inducing f. We identify two such maps if they coincide on some nonempty intersection of the domains of definition of some well-formed representatives for both maps. The group of these maps will be denoted by $\mathrm{Cr}(n, M)$.

Thus the universal Cremona group $\mathrm{UCr}(n, K)$ is endowed with a family of epimorphisms

$$\Pi(n, M)\colon\ \mathrm{UCr}(n, K) \to \mathrm{Cr}(n, M). \tag{2.5}$$

Our immediate goal is to construct (for the case of an algebraically closed ground field K) a section $\Sigma(2, K)$ of the epimorphism $\Pi(2, K)$.

Remark 2.8 If A is a finite dimensional associative algebra with involution over an algebraically closed field K and (K, A, M) an admissible triple such that the set $\mathbb{G}(M)$ generates a semisimple algebraic subgroup G of $\mathbb{G}(A)$, then, at least birationally, one may view $\mathbb{S}_n(K, A, M)$ as a geometric quotient of M^{n+1} with respect to the two-sided diagonal action of G. Thus a generic element of $\mathbb{S}_n(K, A, M)$ may be viewed as a generic point of some algebraic variety over K; and moreover, we may view the transformations of the $\mathbb{S}_n(K, A, M)$ induced by elements $\mathrm{UCr}(n, K)$ as birational transformations of the variety.

2.2 An action of the Cremona group of the plane on the 2-spaces $\mathbb{S}_2(K, A, M)$

Let A be a K-algebra over an algebraically closed field K and (K, A, M) an admissible triple. The collineation group $\mathcal{P} = \mathrm{PGL}(3, K)$ acts on the set $\mathbb{S}_2(K, A, M)$. Our goal is to extend the action to the whole Cremona group (see the Introduction), making it act on $\mathbb{S}_2(K, A, M)$ by birational transformations. The group $\mathrm{Cr}(2, M)$ acts on $\mathbb{S}_2(K, A, M)$ and, according to equation (2.5) (see (2.6) below), we have the epimorphism $\Pi(2, M)$ of the universal Cremona group $\mathrm{UCr}(2, K)$ onto $\mathrm{Cr}(2, M)$, hence this universal group acts on $\mathbb{S}_2(M)$. A special case of (2.5) is

$$\Pi(2, K)\colon\ \mathrm{UCr}(2, K) \to \mathrm{Cr}(2, K). \tag{2.6}$$

If the homomorphism (2.6) admits a section

$$\Sigma(2, K)\colon\ \mathrm{Cr}(2, K) \to \mathrm{UCr}(2, K), \tag{2.7}$$

(of course, by definition, so that the composite $\Pi(2, K) \circ \Sigma(2, K)$ is the identity of $\mathrm{Cr}(2, K)$), then this section provides the required extension.

In the rest of this section, our plan is as follows. First, we already have a natural inclusion

$$\Sigma_{\mathcal{P}}\colon \mathcal{P} \to \mathrm{UCr}(2, K). \tag{2.8}$$

of the collineation group $\mathcal{P}$ into the universal Cremona group.

Next, we find three universal birational maps $S_0, S_1, S_2 \in \mathrm{UCr}(2, K)$ that map to the quadratic transformations s_0, s_1, s_2 under $\Pi(2, K)$. Finally, we check that all the relations mentioned in Theorem 1.3 hold in $\mathrm{UCr}(2, K)$, or more precisely, the relations obtained from those by substituting S_0, S_1, S_2 respectively for s_0, s_1, s_2.

Let $\mathbf{x} = (x_0 : x_1 : x_2)$. We define the effect of the action on $\mathbf{x}$ of the quadratic maps S_0, S_1, S_2 (compare (1.4), (1.6), (1.8)) in the following natural way:

$$\begin{aligned} S_0(\mathbf{x}) &= (x_0^{-1} : x_1^{-1} : x_2^{-1}), \\ S_1(\mathbf{x}) &= (x_1 x_0^{-1} x_1 : x_1 : x_2), \\ S_2(\mathbf{x}) &= (x_0 : x_1 : x_1 x_0^{-1} x_1 - x_2). \end{aligned}$$

Note that, to be correct, we should perhaps write "$\equiv$" instead of "$=$" in all the verifications below, but we neglect to do it.

2.2.1 Verifying the relations (1.14)

These relations hold because the natural inclusion (2.8) of the collineation group of the plane into the universal Cremona group is a homomorphism.

2.2.2 Verifying the relations (1.18)

If G is the collineation $G(\mathbf{x}) = (t_0 x_i : t_1 x_j : t_2 x_k)$ (more precisely, the image of the collineation (1.16) under the inclusion (2.8)), then

$$GS_0(\mathbf{x}) = (t_0 x_i^{-1} : t_1 x_j^{-1} : t_2 x_k^{-1}), \quad S_0 G S_0(\mathbf{x}) = (t_0^{-1} x_i : t_1^{-1} x_j : t_2^{-1} x_k),$$

that is, $S_0 G S_0 = \overline{G}$, where $\overline{G} = \Sigma_{\mathcal{P}}(\overline{g})$. Thus (1.18) is satisfied here.

2.2.3 Verifying the relations (1.21)

Set $G = \Sigma_{\mathcal{P}}(g)$, where g is the collineation (1.19) and $\overline{G} = \Sigma_{\mathcal{P}}(\overline{g}))$, where $\overline{g}$ is (1.20); then

$$\begin{aligned} GS_1(\mathbf{x}) &= (t_0 x_1 x_0^{-1} x_1 : t_1 x_1 : t_2 x_2 + r x_0), \\ S_1 G S_1(\mathbf{x}) &= (t_1 x_1 (t_0 x_1 x_0^{-1} x_1)^{-1} t_1 x_1 : t_1 x_1 : t_2 x_2 + r x_0) \\ &= (t_1^2 t_0^{-1} x_0 : t_0 t_1 x_1 : t_2 x_2 + r x_0) = \overline{G}(\mathbf{x}). \end{aligned}$$

Thus (1.21) is satisfied here.

2.2.4 Verifying the relations (1.24)

Similarly, set $G = \Sigma_{\mathcal{P}}(g)$, where g is the collineation (1.22), and $\overline{G} = \Sigma_{\mathcal{P}}(\overline{g})$, where $\overline{g}$ is (1.23); then

$$\begin{aligned} GS_2(\mathbf{x}) &= (x_0 : t_1x_1 : t^2(x_1x_0^{-1}x_1 - x_2) + rx_1 + sx_0), \\ S_2GS_2(\mathbf{x}) &= (x_0 : tx_1 : -t^2(x_1x_0^{-1}x_1 - x_2) - rx_1 - sx_0 + tx_1x_0^{-1}tx_1) \\ &= (x_0 : tx_1 : t^2x_2 - rx_1 - sx_0) = \overline{G}(\mathbf{x}). \end{aligned}$$

Thus (1.24) is satisfied here.

2.2.5 Verifying the relation (1.27)

Let $H_0 = \Sigma_{\mathcal{P}}(h_0)$, where h_0 is the collineation (1.28). We have to check that

$$S_0H_0S_0(\mathbf{x}) = H_0S_0H_0(\mathbf{x}).$$

First, on the left-hand side,

$$\begin{aligned} S_0(\mathbf{x}) &= (x_0^{-1} : x_1^{-1} : x_2^{-1}), \\ H_0S_0(\mathbf{x}) &= (x_0^{-1} : x_0^{-1} - x_1^{-1} : x_0^{-1} - x_2^{-1}), \\ S_0H_0S_0(\mathbf{x}) &= (x_0 : (x_0^{-1} - x_1^{-1})^{-1} : (x_0^{-1} - x_2^{-1})^{-1}). \end{aligned}$$

Similarly, on the right-hand side,

$$\begin{aligned} S_0H_0(\mathbf{x}) &= (x_0^{-1} : (x_0 - x_1)^{-1} : (x_0 - x_2)^{-1}), \\ H_0S_0H_0(\mathbf{x}) &= (x_0^{-1} : x_0^{-1} - (x_0 - x_1)^{-1} : x_0^{-1} - (x_0 - x_2)^{-1}) \\ &= (x_0 : x_0 - x_0(x_0 - x_1)^{-1}x_0 : x_0 - x_0(x_0 - x_2)^{-1}x_0). \end{aligned}$$

Thus the required relation follows from the identity (2.3).

2.2.6 Verifying the relation (1.29)

Let $G_0 = \Sigma_{\mathcal{P}}(g_0)$, where g_0 is the collineation (1.10). We have to check that

$$S_1(\mathbf{x}) = G_0S_0G_0S_0G_0(\mathbf{x}).$$

We build the following pyramid of equivalences:

$$\begin{aligned} G_0(\mathbf{x}) &= (x_1 - x_0 : x_1 : x_2), \\ S_0G_0(\mathbf{x}) &= ((x_1 - x_0)^{-1} : x_1^{-1} : x_2^{-1}), \\ G_0S_0G_0(\mathbf{x}) &= (x_1^{-1} - (x_1 - x_0)^{-1} : x_1^{-1} : x_2^{-1}), \\ S_0G_0S_0G_0(\mathbf{x}) &= ((x_1^{-1} - (x_1 - x_0)^{-1})^{-1} : x_1 : x_2), \\ G_0S_0G_0S_0G_0(\mathbf{x}) &= (x_1 - (x_1^{-1} - (x_1 - x_0)^{-1})^{-1} : x_1 : x_2). \end{aligned}$$

By virtue of the identity (2.1), the first component of the last triple coincides with $x_1x_0^{-1}x_1$. Hence the required relation is established.

2.2.7 Verifying the relation (1.30)

Let $G_1 = \Sigma_{\mathcal{P}}(g_1)$, where g_1 is the collineation (1.12). We have to check that

$$S_2(\mathbf{x}) = S_1 G_1 S_1(\mathbf{x}).$$

This is easy; indeed,

$$\begin{aligned}
S_1(\mathbf{x}) &= (x_1 x_0^{-1} x_1 : x_1 : x_2),\\
G_1 S_1(\mathbf{x}) &= (x_1 x_0^{-1} x_1 : x_1 : x_1 x_0^{-1} x_1 - x_2),\\
S_1 G_1 S_1(\mathbf{x}) &= (x_1 (x_1 x_0^{-1} x_1)^{-1} x_1 : x_1 : x_1 x_0^{-1} x_1 - x_2)\\
&= (x_0 : x_1 : x_1 x_0^{-1} x_1 - x_2) = S_2(\mathbf{x}).
\end{aligned}$$

2.2.8 Verifying the relation (1.31)

Let $F = \Sigma_{\mathcal{P}}(f)$, where f is the collineation of (1.31). We have to check that

$$S_0(\mathbf{x}) = F S_1 F S_1(\mathbf{x}).$$

As before, this is easy; indeed,

$$\begin{aligned}
S_1(\mathbf{x}) &= (x_1 x_0^{-1} x_1 : x_1 : x_2),\\
F S_1(\mathbf{x}) &= (x_2 : x_1 : x_1 x_0^{-1} x_1),\\
S_1 F S_1(\mathbf{x}) &= (x_1 x_2^{-1} x_1 : x_1 : x_1 x_0^{-1} x_1) = (x_2^{-1} : x_1^{-1} : x_0^{-1}),\\
F S_1 F S_1(\mathbf{x}) &= (x_0^{-1} : x_1^{-1} : x_2^{-1}).
\end{aligned}$$

2.2.9 Verifying the relation (1.32)

Let $H_1 = \Sigma_{\mathcal{P}}(h_1)$, where h_1 is the collineation participating in (1.32). We have to check that

$$S_1 H_1 S_1(\mathbf{x}) = H_1 S_1 H_1(\mathbf{x}).$$

First, on the left-hand side,

$$\begin{aligned}
S_1(\mathbf{x}) &= (x_1 x_0^{-1} x_1 : x_1 : x_2),\\
H_1 S_1(\mathbf{x}) &= (x_1 - x_1 x_0^{-1} x_1 : x_1 : x_2),\\
S_1 H_1 S_1(\mathbf{x}) &= (x_1 (x_1 - x_1 x_0^{-1} x_1)^{-1} x_1 : x_1 : x_2).
\end{aligned}$$

Next, on the right-hand side,

$$\begin{aligned}
H_1(\mathbf{x}) &= (x_1 - x_0 : x_1 : x_2),\\
S_1 H_1(\mathbf{x}) &= (x_1 (x_1 - x_0)^{-1} x_1 : x_1 : x_2),\\
H_1 S_1 H_1(\mathbf{x}) &= (x_1 - x_1 (x_1 - x_0)^{-1} x_1 : x_1 : x_2).
\end{aligned}$$

Thus the required relation now follows from the identity (2.2).

2.2.10 Verifying the relation (1.33)

Let p_t be the collineation of 1.2.12, and $P_t = \Sigma_{\mathcal{P}}(p_t)$ its image; as in (1.33), write $t' = 1 - \frac{1}{t}$ and $t'' = \frac{1}{1-t}$. Then

$$\begin{aligned}
S_2(\mathbf{x}) &= (x_0 : x_1 : x_1 x_0^{-1} x_1 - x_2),\\
P_t S_2(\mathbf{x}) &= (x_0 : -t x_1 : t(x_1 x_0^{-1} x_1 - x_2)),\\
S_2 P_t S_2(\mathbf{x}) &= (x_0 : -t x_1 : (t^2 - t)(x_1 x_0^{-1} x_1) + t x_2),\\
P_{t'} S_2 P_t S_2(\mathbf{x}) &= (x_0 : (t-1) x_1 : (t-1)^2 (x_1 x_0^{-1} x_1) + (t-1) x_2),\\
S_2 P_{t'} S_2 P_t S_2(\mathbf{x}) &= (x_0 : (t-1) x_1 : -(t-1) x_2),\\
P_{t''} S_2 P_{t'} S_2 P_t S_2(\mathbf{x}) &= (x_0 : x_1 : x_2).
\end{aligned}$$

All our verifications are now completed. Q.E.D.

3 Conics, cubics and quartics

3.1 Left and right actions of the Cremona group of the plane on the space of plane conics

Let $A = \mathrm{Mat}_2(K)$ be the 2×2 matrix algebra over an algebraically closed field K of characteristic $\neq 2$, with involution given by transposition $* = {}^{\mathrm{T}}$; thus A^+ is the set of symmetric matrices. We write $D(P, Q)$ for the mixed determinant of two 2×2-matrices P, Q; in other words, if

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}, \quad Q = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix},$$

then

$$D(P, Q) = \frac{1}{2}\left(\begin{vmatrix} p_{11} & q_{12} \\ p_{21} & q_{22} \end{vmatrix} + \begin{vmatrix} q_{11} & p_{12} \\ q_{21} & p_{22} \end{vmatrix} \right).$$

The spherical 2-space $\mathbb{S}_2^+(A)$ consists of triples (m_0, m_1, m_2) of symmetric matrices $m_i \in A^+$, such that some K-linear combination $\lambda_0 m_0 + \lambda_1 m_1 + \lambda_2 m_2$ is invertible, modulo the equivalence relation: $(m_0, m_1, m_2) \sim (n_0, n_1, n_2)$ if $n_i = C m_i C^{\mathrm{T}}$ for some invertible matrix $C \in A$. Let $\mathbf{m} = (m_0 : m_1 : m_2)$ denote the equivalence class of (m_0, m_1, m_2). The collineation group $\mathcal{P}$ acts (on the left) on $\mathbb{S}_2^+(A)$. The $\mathcal{P}$-anti-equivariant map

$$\Delta \colon \mathbb{S}_2^+(\mathrm{Mat}_2(K)) \to \mathbb{P}_5(K) = \mathbb{P}(K[u_0, u_1, u_2]_2)$$

(compare Definition 2.4) sends each triple $\mathbf{m} = (m_0 : m_1 : m_2)$ to the ternary quadratic form

$$\Delta(\mathbf{m})(u_0, u_1, u_2) = \det(u_0 m_0 + u_1 m_1 + u_2 m_2) = \sum_{ij} a_{ij}(\mathbf{m}) u_i u_j,$$

considered up to proportionality; here $a_{ij}(\mathbf{m}) = D(m_i, m_j)$. The Cremona group of the plane acts (on the left) on $\mathbb{S}_2^+(A)$. It is possible to define a natural (right) action of this group on the space of conics in such a way that Δ is a Cr$(2, K)$-anti-equivariant map. Indeed, we can use the identities

$$\begin{aligned}
\det(P^{-1}) &= (\det(P))^{-1}, \quad D(P, P) = \det(P), \\
D(P^{-1}, Q^{-1}) &= \frac{D(P, Q)}{\det(P)\det(Q)}, \\
D(QP^{-1}Q, Q) &= \frac{D(P, Q)\det(Q)}{\det(P)}, \\
D(QP^{-1}Q, P) &= 2D(P, Q)^2(\det(P))^{-1} - \det(Q), \\
D(QP^{-1}Q, R) &= \frac{2D(P, Q)D(Q, R) - D(P, R)\det(Q)}{\det(P)}.
\end{aligned}$$

For s_0, we get

$$a_{ij}(S_0(\mathbf{m})) = a_{ij}(\mathbf{m})(a_{ii}(\mathbf{m})a_{jj}(\mathbf{m}))^{-1} \quad \text{for } 0 \leq i, j \leq 2;$$

to write down explicit formulas for the actions of the three quadratic transformations s_0, s_1, s_2 on conics. That is, in other expressions, the right action of the standard quadratic transformation on the space of plane conics is described by the formulas

$$\begin{aligned}
a'_{00} &= a_{11}a_{22}, & a'_{11} &= a_{22}a_{00}, & a'_{22} &= a_{00}a_{11}, \\
a'_{12} &= a_{12}a_{00}, & a'_{02} &= a_{02}a_{11}, & a'_{01} &= a_{01}a_{22}.
\end{aligned} \tag{3.0}$$

Similarly, we get the following formulas for s_1 and s_2:

$$\begin{aligned}
&a_{00}(S_1(\mathbf{m})) = (a_{00}(\mathbf{m}))^{-1}(a_{11}(\mathbf{m}))^2, \qquad a_{11}(S_1(\mathbf{m})) = a_{11}(\mathbf{m}), \\
&a_{22}(S_1(\mathbf{m})) = a_{22}(\mathbf{m}), \qquad\qquad\qquad\qquad\; a_{12}(S_1(\mathbf{m})) = a_{12}(\mathbf{m}), \\
&a_{01}(S_1(\mathbf{m})) = a_{01}(\mathbf{m})a_{11}(\mathbf{m})(a_{00}(\mathbf{m}))^{-1}, \\
&a_{02}(S_1(\mathbf{m})) = (a_{00}(\mathbf{m}))^{-1}(2a_{01}(\mathbf{m})a_{12}(\mathbf{m}) - a_{02}(\mathbf{m})a_{11}(\mathbf{m}))
\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}
&a_{00}(S_2(\mathbf{m})) = a_{00}(\mathbf{m}), \quad a_{11}(S_2(\mathbf{m})) = a_{11}(\mathbf{m}), \quad a_{01}(S_2(\mathbf{m})) = a_{01}(\mathbf{m}), \\
&a_{12}(S_2(\mathbf{m})) = a_{11}(\mathbf{m})a_{01}(\mathbf{m})((a_{00}(\mathbf{m}))^{-1} - a_{12}(\mathbf{m}), \\
&a_{02}(S_2(\mathbf{m})) = 2(a_{01}(\mathbf{m}))^2(a_{00}(\mathbf{m}))^{-1} - a_{02}(\mathbf{m}) - a_{11}(\mathbf{m}), \\
&a_{22}(S_2(\mathbf{m})) = a_{22}(\mathbf{m}) + (a_{00}(\mathbf{m}))^{-1}((a_{11}(\mathbf{m}))^2 \\
&\qquad\qquad + 2a_{02}(\mathbf{m})a_{11}(\mathbf{m}) - 4a_{01}(\mathbf{m})a_{12}(\mathbf{m})).
\end{aligned} \tag{3.2}$$

An alternative way of writing the action of S_1 is as follows:

$$\begin{aligned} a'_{00} &= a_{11}^2, & a'_{11} &= a_{11}a_{00}, & a'_{22} &= a_{22}a_{00}, \\ a'_{12} &= a_{12}a_{00}, & a'_{02} &= 2a_{01}a_{12} - a_{02}a_{11}, & a'_{01} &= a_{01}a_{11}, \end{aligned} \tag{3.1a}$$

and similarly for the action of S_2:

$$\begin{aligned} a'_{00} &= a_{00}^2, & a'_{11} &= a_{11}a_{00}, & a'_{01} &= a_{01}a_{00}, \\ a'_{12} &= a_{11}a_{01} - a_{12}a_{00}, & a'_{02} &= 2a_{01}^2 - (a_{02} + a_{11})a_{00}, \\ a'_{22} &= a_{22}a_{00} + a_{11}^2 + 2a_{02}a_{11} - 4a_{01}a_{12}. \end{aligned} \tag{3.2a}$$

All the special relations (where, of course, we replace each collineation g by its transpose g^{T}, and reverse the order of terms in products) are satisfied here.

If we want a left action of the Cremona group of the plane on the space of plane conics, then we must pass to the dual conic. Let $(A_{ij})_{0 \le i,j \le 2}$ be the adjoint matrix of (a_{ij}); then the left action of s_0 is given by the formulas

$$a'_{ij} = a_{ii}a_{jj}A_{ij} \tag{3.0b}$$

The left action of s_1 is defined by

$$\begin{aligned} a'_{00} &= a_{00}^2 A_{00}, & a'_{11} &= a_{11}^2 A_{11} - 4a_{02}a_{12}A_{02}, \\ a'_{22} &= a_{11}^2 A_{22}, & a'_{12} &= a_{01}a_{11}A_{02} - a_{11}a_{12}A_{22}, \\ a'_{02} &= -a_{00}a_{11}A_{02}, & a'_{01} &= a_{00}a_{12}A_{02} - a_{00}a_{01}A_{00}. \end{aligned} \tag{3.1b}$$

The left action of s_2 is defined by

$$\begin{aligned} a'_{00} &= a_{00}^2 A_{00} - 2a_{00}a_{11}A_{02} + a_{11}^2 A_{22}, \\ a'_{11} &= a_{00}^2 A_{11} + 4a_{00}a_{01}A_{12} + 4a_{01}^2 A_{22}, \qquad a'_{22} = a_{00}^2 A_{22}, \\ a'_{01} &= a_{00}^2 A_{01} + 2a_{00}a_{01}A_{02} - a_{00}a_{11}A_{12} - 2a_{01}a_{11}A_{22}, \\ a'_{02} &= -a_{00}^2 A_{02} + a_{00}a_{11}A_{22}, \qquad a'_{12} = -a_{00}^2 A_{12} - 2a_{00}a_{01}A_{22}. \end{aligned} \tag{3.2b}$$

More precisely, if a matrix $g \in \mathrm{PGL}(3, K) = \mathcal{P}$, viewed up to scalar multiples, acts on the space of symmetric 3×3 matrices (also viewed up to scalar multiples)

$$a = (a_{ij}) \in \check{\mathbb{P}}_5(K) = \mathbb{P}(S^2(W^*))$$

according to the rule

$$g(a) = (g^{-1})^{\mathrm{T}} a (g^{-1}),$$

and the quadratic transformations s_0, s_1, s_2 act on $\check{\mathbb{P}}_5(K)$ according to formulas (3.0b), (3.1b), (3.2b) respectively, then we have a well-defined (left) action of $\mathrm{Cr}(2, K)$ on the space of plane conics.

Note that our good luck in the case of conics is based on the fact that the map Δ is a birational isomorphism (for some analogs of this fact see below, 3.2.5, Theorem 3.3 and 3.5, formulas (3.39)-(3.40)).

3.2 Left and right actions of the Cremona group of the plane on the space of plane cubics

Now let $A = \mathrm{Mat}_3(K)$ be the 3×3 matrix algebra over an algebraically closed field K of characteristic $\neq 2, 3$, with the involution given by transposition $* = {}^{\mathrm{T}}$; thus A^+ is again the set of symmetric 3×3 matrices. Let $D(P, Q, R)$ denote the mixed determinant of three 3×3 matrices; that is, if

$$P = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}, \quad Q = \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{pmatrix}, \quad R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix},$$

then $6D(P, Q, R)$ equals

$$\begin{vmatrix} p_{11} & q_{12} & r_{13} \\ p_{21} & q_{22} & r_{23} \\ p_{31} & q_{32} & r_{33} \end{vmatrix} + \begin{vmatrix} p_{11} & r_{12} & q_{13} \\ p_{21} & r_{22} & q_{23} \\ p_{31} & r_{32} & q_{33} \end{vmatrix} + \begin{vmatrix} q_{11} & p_{12} & r_{13} \\ q_{21} & p_{22} & r_{23} \\ q_{31} & p_{32} & r_{33} \end{vmatrix}$$

$$+ \begin{vmatrix} q_{11} & r_{12} & p_{13} \\ q_{21} & r_{22} & p_{23} \\ q_{31} & r_{32} & p_{33} \end{vmatrix} + \begin{vmatrix} r_{11} & p_{12} & q_{13} \\ r_{21} & p_{22} & q_{23} \\ r_{31} & p_{32} & q_{33} \end{vmatrix} + \begin{vmatrix} r_{11} & q_{12} & p_{13} \\ r_{21} & q_{22} & p_{23} \\ r_{31} & q_{32} & p_{33} \end{vmatrix}.$$

The spherical 2-space $\mathbb{S}_2^+(A)$ consists of triples of matrices (m_0, m_1, m_2) in A^+ for which some K-linear combination $\lambda_0 m_0 + \lambda_1 m_1 + \lambda_2 m_2$ is invertible, and $(m_0, m_1, m_2) \sim (n_0, n_1, n_2)$ if there exists an invertible matrix $C \in A$ such that $n_i = C m_i C^{\mathrm{T}}$. We write $\mathbf{m} = (m_0 : m_1 : m_2)$ for the equivalence class. The collineation group $\mathcal{P}$ acts (on the left) on $\mathbb{S}_2^+(A)$. The $\mathcal{P}$-anti-equivariant map (see Definition 2.4)

$$\Delta\colon \mathbb{S}_2^+(\mathrm{Mat}_3(K)) \to \mathbb{P}_9(K) = \mathbb{P}(K[u_0, u_1, u_2]_3)$$

associates with each triple $\mathbf{m} = (m_0 : m_1 : m_2)$ the ternary cubic form

$$\Delta(\mathbf{m})(u_0, u_1, u_2) = \det(u_0 m_0 + u_1 m_1 + u_2 m_2) = \sum_{ij} a_{ijk}(\mathbf{m}) u_i u_j u_k$$

(up to proportionality), where $a_{ijk}(\mathbf{m}) = D(m_i, m_j, m_k)$. The cubic curve $\Delta(\mathbf{m})(u_0, u_1, u_2) = 0$ inherits an additional structure from the matrix triple $\mathbf{m}$, namely, an even theta characteristic, that is, a nonzero 2-torsion point; we now treat these relations more explicitly.

3.2.1 Invariants, covariants, contravariants and 2-torsion of plane cubic curves

We take the three variables x_0, x_1, x_2 to be homogeneous coordinates on $\mathbb{P}_2$, and normalize the coefficients a_{ijk} of $x_i x_j x_k$ in a cubic form F as follows:

$$\begin{aligned} F = {} & a_{000}x_0^3 + a_{111}x_1^3 + a_{222}x_2^3 + 6a_{012}x_0x_1x_2 \\ & + 3a_{001}x_0^2x_1 + 3a_{002}x_0^2x_2 + 3a_{110}x_0x_1^2 \\ & + 3a_{112}x_1^2x_2 + 3a_{220}x_0x_2^2 + 3a_{221}x_1x_2^2. \end{aligned} \tag{3.3}$$

Let u_0, u_1, u_2 be dual homogeneous coordinates on the projective plane $\check{\mathbb{P}}_2$. The *Hessian form* $\mathrm{He}(F)$ of the cubic (3.3) is defined by

$$\mathrm{He}(F) = \frac{1}{36}\det(\mathrm{HE}(F)),$$

where $\mathrm{HE}(F)$ is the Hessian matrix

$$\mathrm{HE}(F) = \begin{pmatrix} \dfrac{\partial^2 F}{\partial x_0^2} & \dfrac{\partial^2 F}{\partial x_0 \partial x_1} & \dfrac{\partial^2 F}{\partial x_0 \partial x_2} \\ \dfrac{\partial^2 F}{\partial x_1 \partial x_0} & \dfrac{\partial^2 F}{\partial x_1^2} & \dfrac{\partial^2 F}{\partial x_1 \partial x_2} \\ \dfrac{\partial^2 F}{\partial x_2 \partial x_0} & \dfrac{\partial^2 F}{\partial x_2 \partial x_1} & \dfrac{\partial^2 F}{\partial x_2^2} \end{pmatrix}. \tag{3.4}$$

Note that our Hessian form differs slightly from that of Salmon's book [16] or Dolgachev and Kanev [8] (ours is multiplied by 6). Normalized coefficients by monomials $x_i x_j x_k$ of the Hessian are written down in [16], N^o 218. The *Cayley form* $\mathrm{Ca}(F)$ of F is

$$\mathrm{Ca}(F) = 3 \times \begin{vmatrix} a_{000} & a_{110} & a_{220} & a_{012} & a_{002} & a_{001} \\ a_{001} & a_{111} & a_{221} & a_{112} & a_{012} & a_{011} \\ a_{002} & a_{112} & a_{222} & a_{122} & a_{022} & a_{012} \\ 2u_0 & 0 & 0 & 0 & u_2 & u_1 \\ 0 & 2u_1 & 0 & u_2 & 0 & u_0 \\ 0 & 0 & 2u_2 & u_1 & u_0 & 0 \end{vmatrix}.$$

There is a well-defined natural scalar product (or convolution) (F, G) of two ternary forms $F(x_0, x_1, x_2)$ and $G(u_0, u_1, u_2)$ of the same degree in dual variables. For example, if F and G are ternary cubic forms (where F is (3.3), and G has normalized coefficients b_{ijk}), then

$$\begin{aligned}(F, G) = a_{000}b_{000} + a_{111}b_{111} + a_{222}b_{222} + 6a_{012}b_{012} \\ + 3a_{001}b_{001} + 3a_{002}b_{002} + 3a_{110}b_{110} \\ + 3a_{112}b_{112} + 3a_{220}b_{220} + 3a_{221}b_{221}.\end{aligned} \tag{3.5}$$

The Aronhold invariants $S = S(F), T = T(F)$ and $R = R(F)$ of a cubic form F are defined by

$$S(F) = -(F, \mathrm{Ca}(F)), \quad T(F) = -(\mathrm{He}(F), \mathrm{Ca}(F)), \quad R(F) = T(F)^2 - S(F)^3$$

(compare [16], N^{o}s 220–221). It is convenient to use the following contravariant cubic form

$$D(F) = \frac{1}{3}\big(T(F)\,\mathrm{Ca}(F) - \mathrm{Ca}(\mathrm{He}(F))\big). \tag{3.6}$$

The operation D is an analog of passing to the dual of a quadratic form. Indeed,

$$D(D(F)) = -32R(F)^6 S(F)^2 F, \tag{3.7}$$

or in other words, D iterated twice yields the initial cubic form (up to a factor). We may consider D as a "birational null-correlation", because the contravariant $D(F)$ defines a hyperplane in the space of cubic curves, and F belongs to this hyperplane: $(F, D(F)) = 0$, where $(\ ,\ )$ is the scalar product (3.5). The operation D interchanges the Hessian and the Cayley forms up to a factor, in the sense that

$$\begin{aligned}\mathrm{He}(D(F)) &= 2R(F)^2\,\mathrm{Ca}(F) \\ \text{and}\quad \mathrm{Ca}(D(F)) &= -4R(F)^2\,\mathrm{He}(F).\end{aligned} \tag{3.8}$$

We refer to the pencil of cubic forms

$$uF(x_0, x_1, x_2) + v\,\mathrm{He}(F)(x_0, x_1, x_2),$$

as the *Hessian pencil* (an alternative term *syzygetic pencil* is due to L. Cremona), and

$$u\,\mathrm{Ca}(F)(u_0, u_1, u_2) + vD(F)(u_0, u_1, u_2)$$

as the *Cayley pencil.* The Hessian operation preserves both these pencils, giving rise to the following actions (compare [16], N^o 225). On the Hessian pencil:

$$\begin{aligned}\mathrm{He}(uF + v\,\mathrm{He}(F)) = 3v\Big(u^2S + 2uvT + v^2S^2\Big)\cdot F \\ + \Big(u^3 - 3Suv^2 - 2Tv^3\Big)\cdot \mathrm{He}(F),\end{aligned} \tag{3.9}$$

$$\text{in particular}\quad \mathrm{He}(\mathrm{He}(F)) = 3S^2F - 2T\,\mathrm{He}(F). \tag{3.10}$$

(Here and below, we write $S = S(F), T = T(F), R = R(F)$.) On the Cayley pencil:

$$\begin{aligned} \mathrm{He}(u\,\mathrm{Ca}(F) + vD(F)) = 6u\Big(u^2 - 2Tuv + Rv^2\Big) \cdot D(F) \qquad\qquad (3.11)\\ + 2\Big(2Tu^3 - 3Ru^2v + R^2v^3\Big) \cdot \mathrm{Ca}(F), \end{aligned}$$

$$\text{in particular} \quad \mathrm{He}(\mathrm{Ca}(F)) = 6T(F)\,\mathrm{Ca}(F) - 2\,\mathrm{Ca}(\mathrm{He}(F)). \qquad (3.12)$$

The Cayley operation takes the Hessian pencil into the Cayley pencil; namely,

$$\begin{aligned} S(F)\,\mathrm{Ca}(uF + v\,\mathrm{He}(F)) &= 3v\Big(u^2 - S(F)v^2\Big) \cdot D(F)\\ &+ \Big(S(F)u^3 + 3T(F)u^2v + 3S(F)^2uv^2 + T(F)S(F)v^3\Big) \cdot \mathrm{Ca}(F); \end{aligned}$$

The operation D acts in a similar way. Furthermore,

$$\begin{aligned} \mathrm{Ca}(u\,\mathrm{Ca}(F) + vD(F)) &= 12S(F)^2u\Big(Rv^2 - u^2\Big) \cdot F\\ &+ 4\Big(T(F)u^3 - 3R(F)u^2v + 3R(F)T(F)uv^2 - R(F)^2v^3\Big) \cdot \mathrm{He}(F), \qquad (3.13) \end{aligned}$$

and

$$D(u\,\mathrm{Ca}(F) + vD(F)) = 16S^2R\Phi(u,v)^2[2uS\,\mathrm{He}(F) - (Tu + 2Rv)F],$$

where

$$\Phi(u,v) = 9u^4 - 8T(F)u^3v + 6R(F)u^2v^2 - v^4R(F)^2.$$

In particular,

$$D(\mathrm{Ca}(F)) = 288R(F)S(F)^2\big(T(F)F - S(F)\,\mathrm{He}(F)\big). \qquad (3.14)$$

Evaluating the Aronhold operations $S(\cdot), T(\cdot)$, and $R(\cdot)$ on our two pencils gives the following: on the Hessian pencil,

$$\begin{aligned} S(uF + v\,\mathrm{He}(F)) &= u^4S + 4u^3vT + 6u^2v^2S^2\\ &+ 4uv^3ST + v^4(4T^2 - 3S^3), \qquad (3.15) \end{aligned}$$

$$\text{in particular} \quad S(\mathrm{He}(F)) = 4T^2 - 3S^3 = T^2 + 3R. \qquad (3.16)$$

Also,

$$\begin{aligned} T(uF + v\,\mathrm{He}(F)) &= u^6T + 6u^5vS^2 + 15u^4v^2ST\\ &+ 20u^3v^3T^2 + 15u^2v^4S^2T \qquad (3.17)\\ &+ 6uv^5(3S^3 - 2T^2)S + v^6(9S^3 - 8T^2)T, \end{aligned}$$

$$\text{in particular} \quad T(\mathrm{He}(F)) = (9S^3 - 8T^2)T = T^3 - 9RT. \qquad (3.18)$$

Further, (3.15) and (3.17) give

$$\begin{aligned} R(uF + v\,\mathrm{He}(F)) &= (u^4 - 6Su^2v^2 - 8Tuv^3 - 3S^2v^4)^3 R, \\ \text{in particular} \quad R(\mathrm{He}(F)) &= -27S^6 R(F). \end{aligned}$$

On the Cayley pencil, we get

$$\begin{aligned} S(u\,\mathrm{Ca}(F) + vD(F)) = 4 \times \Big(&(4T^2 - 3R)u^4 - 4RTu^3v \\ &+ 6RT^2u^2v^2 - 4R^2Tuv^3 + R^3v \Big), \end{aligned} \tag{3.19}$$

$$\text{in particular,} \quad S(\mathrm{Ca}(F)) = 4(4T^2 - 3R) \quad \text{and} \quad S(D(F)) = 4R(F)^3.$$

Also,

$$\begin{aligned} T(u\,\mathrm{Ca}(F) + vD(F)) = 8 \times \Big(&-T(9R - 8T^2)u^6 + 6R(3R - 2T^2)u^5v \\ &- 15R^2Tu^4v^2 + 20R^2T^2u^3v^3 - 15R^3Tu^2v^4 + 6R^4uv^5 - TR^4v^6 \Big), \end{aligned}$$

in particular

$$T(\mathrm{Ca}(F)) = 8T(8T^2 - 9R) \text{ and } T(D(F)) = -8R(F)^4 T(F). \tag{3.20}$$

Finally,

$$\begin{aligned} R(\mathrm{Ca}(F)) &= (-12S(F))^3 \big(T(F)^2 - S(F)^3\big)^2 = -12^3 S(F)^3 R(F)^2, \\ R(D(F)) &= 64R(F)^8 S(F)^3. \end{aligned}$$

3.2.2 The space of marked cubics

An even theta characteristic of a nonsingular plane cubic curve is a nonzero 2-torsion point on the Jacobian curve of this cubic. The right parameter space for marked cubics (that is, cubics with a marked 2-torsion point) is the weighted projective space $\mathbb{P}(1^{10}; 2)$ with coordinates

$$(F; \theta) = (a_{000}, a_{111}, a_{222}, a_{001}, a_{002}, a_{110}, a_{112}, a_{220}, a_{221}, a_{012}; \theta).$$

A similar statement holds for the spherical 2-space $\mathbb{S}_2^+(\mathrm{Mat}_3(K))$, compare Theorem 3.3 below.

Definition 3.1 The space of marked cubics is the hypersurface $V \subset \mathbb{P}(1^{10}; 2)$ defined by the equation

$$\theta^3 - 3S(F)\theta - 2T(F) = 0. \tag{3.21}$$

Remark 3.1 The affine equation

$$\frac{3}{2}y^2 + x^3 - 3S(F)x - 2T(F) = 0 \tag{3.22}$$

defines the Jacobian curve of the generic cubic curve $F = 0$, where F is the form (3.3); hence a 2-torsion point of the Jacobian corresponds to a zero of the left-hand side of (3.22) of the form $(x, 0)$; this justifies the above definition. Here θ is an "irrational invariant" of a ternary cubic form, and its degree equals 2. The fact that θ is invariant ensures that the action of $\mathcal{P}$ on the space of cubic forms extends to V. A point of V is a cubic curve with a marked 2-torsion point. The hypersurface V is birationally equivalent to the projective space of bare (unmarked) plane cubics (compare Dolgachev and Kanev [8], who attribute this result to G. Salmon [16]). We give two constructive proofs of the Salmon–Dolgachev–Kanev theorem (see Theorem 3.1, Claims (A) and (B) below).

Example 3.1 Let F be a generic cubic form and $\mathrm{He}(F)$ its Hessian; then twice the value of the Aronhold T-invariant of F defines a 2-torsion point of $\mathrm{He}(F)$. That is, $\theta = 2T(F)$ is a root of the equation

$$\theta^3 - 3S(\mathrm{He}(F))\theta - 2T(\mathrm{He}(F)) = 0.$$

This follows from (3.18) and (3.16). Hence we get a map

$$\mathrm{he}\colon \mathbb{P}_9 \to V \quad \text{defined by} \quad \mathrm{he}(F) = \big(\mathrm{He}(F); 2T(F)\big) \tag{3.23}$$

from the space $\mathbb{P}_9$ of ternary cubic forms to the space V of cubics with a marked 2-torsion point.

Example 3.2 Let F be a generic cubic form and $\mathrm{Ca}(F)$ its Cayley form; then $-4T(F)$ defines a 2-torsion point of $\mathrm{Ca}(F)$. That is, $\theta = -4T(F)$ is a root of the equation

$$\theta^3 - 3S(\mathrm{Ca}(F))\theta - 2T(\mathrm{Ca}(F)) = 0.$$

This follows from (3.20) and (3.19). Hence we get a map

$$\mathrm{ca}\colon \mathbb{P}_9 \to V \quad \text{defined by} \quad \mathrm{ca}(F) = \big(\mathrm{Ca}(F); -4T(F)\big) \tag{3.24}$$

from the space $\mathbb{P}_9$ of ternary cubic forms to the space V of cubics with a marked 2-torsion point.

The next theorem shows that each of (3.23) and (3.24) is a birational equivalence.

Theorem 3.1 *(A) The map* $g\colon V \to \mathbb{P}_9$ *defined by*

$$g((F;\theta)) = \theta F + \mathrm{He}(F) \tag{3.25}$$

is a birational inverse of the map he *of (3.23).*

(B) The map $d\colon V \to \mathbb{P}_9$ *defined by*

$$d((F;\theta)) = R(F)\,\mathrm{Ca}(F) + (\theta S(F) + T(F))D(F)$$

is a birational inverse of the map ca *of (3.24).*

Proof of (A) This follows from (3.9), (3.10) and (3.25):

$$g(\mathrm{he}(F)) = g\big((\mathrm{He}(F); 2T(F))\big) = 2T(F)\,\mathrm{He}(F) + \mathrm{He}(\mathrm{He}(F)) = 3S^2(F)F,$$

hence $g \circ \mathrm{he} = \mathrm{id}_{\mathbb{P}_9}$. The right hand side of (3.17) equals:

$$\frac{9}{2}(Su^2 + 2Tuv + S^2v^2)^2uv + T(u^3 - 3Suv^2 - 2Tv^3)^2 + \\ + \frac{3}{2}(u^3 - 3Suv^2 - 2Tv^3)((Su + Tv)^2 + 3Rv^2).$$

Using this, together with (3.15), (3.17), (3.23), (3.25), we get

$$\begin{aligned} \mathrm{he}(g((F;\theta))) &= \mathrm{he}(\theta F + \mathrm{He}(F)) \\ &= \Big(\mathrm{He}(\theta F + \mathrm{He}(F));\ 2T(\theta F + \mathrm{He}(F)\Big) \\ &= \Big(3(S(F)\theta^2 + 2T(F)\theta + S(F)^2)F;\ 9(S(F)\theta^2 + T(F)\theta + S(F)^2)^2\theta\Big). \end{aligned}$$

This point of $\mathbb{P}(1^{10}; 2)$ coincides with $(F;\theta)$, hence $\mathrm{he} \circ g = \mathrm{id}_V$.

Proof of (B) Substituting from (3.14), (3.20), (3.19) gives

$$\begin{aligned} d(\mathrm{ca}(F)) &= d((\mathrm{Ca}(F); -4T(F))) \\ &= R(\mathrm{Ca}(F))\,\mathrm{Ca}(\mathrm{Ca}(F)) + (T(\mathrm{Ca}(F)) - 4T(F)S(\mathrm{Ca}(F)))D(\mathrm{Ca}(F)) \\ &= -12^3 R(F)^2 S(F)^2 S(\mathrm{Ca}(F))F, \end{aligned}$$

that is, $d(\mathrm{ca}(F))$ is proportional to F, hence $d \circ \mathrm{ca} = \mathrm{id}_{\mathbb{P}_9}$.

To study the inverse composite $\mathrm{ca} \circ d$, and for some further comments on the theorem (Remark 3.2), we need an additional series of identities.

Lemma 3.1 *Suppose that θ satisfies (3.21), and set*

$$\tau = S(F)\theta + T(F). \tag{3.26}$$

Then the coefficient of He(F) *in (3.13) vanishes at $u = R(F)$, $v = \tau$:*

$$\tau^3 - 3T(F)\tau^2 + 3R(F)\tau - T(F)R(F) = 0. \tag{3.27}$$

Furthermore,

$$\begin{aligned} T(R\,\mathrm{Ca}(F) + \tau D(F)) &= 72R^4S^6(-9T\tau^2 + 8R\tau - 3TR), \\ (\tau^2 - R)^2\theta &= -2S^2(-9T\tau^2 + 8R\tau - 3TR), \\ -4T(R\,\mathrm{Ca}(F) + \tau D(F)) &= (12R^2S^2(\tau^2 - R))^2\theta. \end{aligned} \tag{3.28}$$

Moreover, if we set

$$\Lambda(\tau) = T(F)\tau^2 - 2R(F)\tau + T(F)R(F), \tag{3.29}$$

then

$$\Lambda(\tau)^3 = S^6(9T\tau^2 - 8R\tau - 3TR)\tau^2, \tag{3.30}$$

$$\mathrm{He}(R\,\mathrm{Ca}(F) + \tau D(F)) = 6R^2\Lambda(\tau)\tau^{-1}(\tau\,\mathrm{Ca}(F) + D(F)). \tag{3.31}$$

These identities can be checked directly, but we omit the details.

Proof of (B), continued Applying (3.14), (3.27) and (3.28) yields:

$$\begin{aligned} \mathrm{ca}(d(F)) &= \mathrm{Ca}\Big(R\,\mathrm{Ca}(F) + \tau D(F);\ -4T(\mathrm{Ca}(R\,\mathrm{Ca}(F) + \tau D(F))\Big) \\ &= \Big(12S^2R^2(\tau^2 - R)F;\ (12S^2R^2)^2(\tau^2 - R)^2\theta)\Big). \end{aligned}$$

This point of $\mathbb{P}(1^{10}; 2)$ coincides with $(F;\theta)$, hence $\mathrm{ca} \circ d = \mathrm{id}$.

Remark 3.2 Comparing the two assertions (A) and (B) gives new information concerning two birational transformations: (1) the transformation $D\colon \mathbb{P}_9 \dashrightarrow \mathbb{P}_9$ of (3.6) of the projective space of plane cubics, and (2) an involutive transformation $E\colon V \dashrightarrow V$ described below of the space of marked cubics.

First, D equals the composite $g \circ \mathrm{ca}$ (and the composite $d \circ \mathrm{he}$): for

$$g(\mathrm{ca}(F)) = g(\mathrm{Ca}(F); -4T(F)) = \mathrm{He}(\mathrm{Ca}(F)) - 4T(F)\,\mathrm{Ca}(F) = 6D(F),$$

by (3.12) and (3.6). Our second map E is the composite $\mathrm{he} \circ d$ (which is equal to $\mathrm{ca} \circ g$). We claim that

$$E(F;\theta) = \Big((\theta S + T)\,\mathrm{Ca}(F) + D(F);\ -4S^2(T\theta^2 + 2S^2\theta + TS)\Big),$$

or in terms of the notation (3.26), (3.29),

$$E(F;\theta) = (\tau\,\mathrm{Ca}(F) + D(F); -4\Lambda(\tau)).$$

For, applying (3.28), (3.30), (3.11), (3.31), we get

$$\begin{aligned}\mathrm{he}(d(F)) &= \mathrm{he}\Big(R\,\mathrm{Ca}(F) + \tau D(F);\, 2T(R\,\mathrm{Ca}(F) + \tau D(F)\Big)\\ &= \Big(6R^2\Lambda(\tau)\tau^{-1}(\tau\,\mathrm{Ca}(F) + D(F);\, -4(6R^2\Lambda(\tau)\tau^{-1})^2\Lambda(\tau)\Big)\\ &= \Big(\tau\,\mathrm{Ca}(F) + D(F); -4\Lambda(\tau)\Big).\end{aligned}$$

3.2.3 A birational transformation of the space of marked cubics

We describe a birational transformation Σ_0 of the variety V and of the ambient weighted projective space $\mathbb{P}(1^{10};2)$. This transformation is an analog of the action of the standard quadratic transformation on the space of conics. It is convenient to make a coordinate change $(F;\theta) \mapsto (F;\eta)$ in $\mathbb{P}(1^{10};2)$, replacing the final coordinate θ by

$$\eta = -\frac{1}{4}(\theta + 2P), \quad (\text{so that } \theta = -4\eta - 2P),$$

$$\text{where} \quad P = a_{012}^2 - G, \quad \text{and} \quad G = a_{110}a_{220} + a_{001}a_{221} + a_{002}a_{112}.$$

In the new variables, the hypersurface $V \subset \mathbb{P}(1^{10};2)$ of (3.21) is now defined by the equation:

$$32\eta^3 + 48P\eta^2 + 6(4P^2 - S)\eta + T + 4P^3 - 3SP = 0. \tag{3.32}$$

We introduce the monomial birational transformation Σ_0 of $\mathbb{P}(1^{10};2)$, given by $(a;\eta) \mapsto (a^*,\eta^*)$, where:

$$\begin{array}{lll} a^*_{000} = a_{111}a_{222}, & a^*_{111} = a_{000}a_{222}, & a^*_{222} = a_{000}a_{111},\\ a^*_{001} = a_{110}a_{222}, & a^*_{002} = a_{220}a_{111}, & a^*_{110} = a_{001}a_{222},\\ a^*_{112} = a_{221}a_{000}, & a^*_{220} = a_{002}a_{111}, & a^*_{221} = a_{112}a_{000},\\ a^*_{012} = \eta; & & \eta^* = a_{000}a_{111}a_{222}a_{012}. \end{array} \tag{3.33}$$

Theorem 3.2 *(A) The map Σ_0 is an involutive birational transformation of $\mathbb{P}(1^{10};2)$.*

(B) It preserves the hypersurface V.

Proof It is obvious that Σ_0 is an involution, because on double application of Σ_0, each weight 1 coordinate is multiplied by

$$M = a_{000}a_{111}a_{222}, \tag{3.34}$$

and the final weight 2 coordinate η is multiplied by M^2.

Let $A = K[a_{000}, a_{111}, a_{222}, a_{001}, a_{002}, a_{110}, a_{112}, a_{220}, a_{221}]$ be the polynomial ring generated by all the coefficients of cubics except a_{012}. The first nine equations of (3.33) define an endomorphism $*$ of A; write f^* for the image of $f \in A$ under $*$. Expanding the terms in the defining equation (3.32) of V in powers of a_{012} gives:

$$\begin{aligned} 4P^2 - S &= 4Ba_{012} + 4C, \\ T + 4P^3 - 3SP &= 32Ma_{012}^3 - 48G^*a_{012}^2 + 24Ea_{012} + D, \end{aligned}$$

where $M, B, C, D, E \in A$ (here M is the multiplier spelt out in (3.34)),

$$\begin{matrix} M^* = M^2, & B^* = MB, & D^* = M^2D, \\ C^* = ME, & E^* = M^2C, & G^{**} = M^2G. \end{matrix} \tag{3.35}$$

We can rewrite (3.32) as

$$\begin{aligned} 32(\eta^3 + Ma_{012}^3) + 48(\eta a_{012})^2 - 48(G\eta^2 + G^*a_{012}^2) \\ + 24B\eta a_{012} + 24(C\eta + Ea_{012}) + D = 0. \end{aligned} \tag{3.36}$$

Using this, we see that applying formulas (3.33) defining Σ_0 (see especially the last two formulas of (3.33) and (3.35)) to the left hand side of (3.32) or (3.36) multiplies it by M^2. Q.E.D.

3.2.4 A birational map of the spherical space of symmetric 3×3 matrices onto the space V of marked cubics

We write $\widehat{m}$ for the adjoint matrix of a symmetric 3×3-matrix m and $D(\cdot,\cdot,\cdot)$ for the mixed discriminant of three symmetric 3×3 matrices. For a triple $\mathbf{m} = (m_0, m_1, m_2)$, we define a ternary cubic by

$$F_{\mathbf{m}} = 6\det(x_0m_0 + x_1m_1 + x_2m_2),$$

and a number $\theta(\mathbf{m})$ by

$$\begin{aligned} \theta(\mathbf{m}) = 2(&2D(\widehat{m_0}, \widehat{m_1}, \widehat{m_2}) - (D(m_0, m_1, m_2))^2 \\ &+ D(m_0, m_1, m_1)D(m_0, m_2, m_2) \\ &+ D(m_1, m_0, m_0)D(m_1, m_2, m_2) + D(m_2, m_0, m_0)D(m_2, m_1, m_1)). \end{aligned} \tag{3.37}$$

Theorem 3.3 *For a triple* $\mathbf{m} = (m_0 : m_1 : m_2) \in \mathbb{S}_2^+(\mathrm{Mat}_3(K))$, *the point* $(F_{\mathbf{m}}; \theta(\mathbf{m}))$ *belongs to* V, *and*

$$\alpha \colon \mathbb{S}_2^+(\mathrm{Mat}_3(K)) \to V \subset \mathbb{P}(1^{10}; 2) \quad \textit{given by } \alpha(\mathbf{m}) = (F_{\mathbf{m}}; \theta(\mathbf{m})) \qquad (3.38)$$

is a well-defined birational map, having the inverse

$$\beta \colon V \to \mathbb{S}_2^+(\mathrm{Mat}_3(K)), \quad \textit{given by } \beta(F;\theta) = (m_0 : m_1 : m_2),$$

where $x_0m_0+x_1m_1+x_2m_2 = \theta\,\mathrm{HE}(F)+\mathrm{HE}(\mathrm{He}(F))$; *see (3.4) for the Hessian matrix* $\mathrm{HE}(F)$.

The map α *is* $\mathcal{P}$*-anti-equivariant, and has the following compatibility with the action of the standard quadratic transformation*

$$\Sigma_0(\alpha(\mathbf{m})) = \alpha(S_0(\mathbf{m})).$$

Remark 3.3 Formula (3.37) is borrowed from the end of Salmon, Conic sections [15]. Salmon gives a different formula for $\theta(\mathbf{m})$, which he attributes to Burnside. Namely, write $[\cdot,\cdot,\cdot]$ for the determinant made up of three ternary linear forms, and

$$\det(x_0m_0 + x_1m_1 + x_2m_2) = \begin{vmatrix} A_{00} & A_{01} & A_{02} \\ A_{10} & A_{11} & A_{12} \\ A_{20} & A_{21} & A_{22} \end{vmatrix}$$

where $A_{ij} = A_{ij}(x_0, x_1, x_2)$ are linear forms and $A_{ij} = A_{ji}$ for $i, j = 0, 1, 2$. Then

$$\begin{aligned}
\theta(\mathbf{m}) = 2 \times \big(&-8[A_{01}, A_{12}, A_{20}]^2 + [A_{00}, A_{11}, A_{22}]^2 \\
&+ 4[A_{21}, A_{10}, A_{02}][A_{00}, A_{11}, A_{22}] + 4[A_{00}, A_{11}, A_{12}][A_{00}, A_{22}, A_{12}] \\
&+ 4[A_{11}, A_{22}, A_{02}][A_{11}, A_{00}, A_{02}] + 4[A_{22}, A_{00}, A_{01}][A_{22}, A_{11}, A_{01}] \\
&+ 8[A_{11}, A_{02}, A_{01}][A_{22}, A_{02}, A_{01}] + 8[A_{00}, A_{12}, A_{10}][A_{22}, A_{12}, A_{10}] \\
&+ 8[A_{00}, A_{21}, A_{20}][A_{11}, A_{21}, A_{20}] - 8[A_{00}, A_{02}, A_{01}][A_{11}, A_{22}, A_{12}] \\
&- 8[A_{11}, A_{10}, A_{12}][A_{22}, A_{00}, A_{20}] - 8[A_{22}, A_{10}, A_{12}][A_{00}, A_{11}, A_{01}]\big).
\end{aligned}$$

Salmon [15] also sketches a proof that α is well defined.

Proof of Theorem 3.3 We introduce some notation. Let m and m' be two symmetric 3×3-matrices and $\widehat{[m, m']}$ their mixed adjoint matrix, that is,

$$(\widehat{um + vm'}) = u^2\widehat{m} + uv\widehat{[m, m']} + v^2\widehat{m'}, \quad \text{in particular} \quad \widehat{[m, m]} = 2\widehat{m}.$$

For six ternary quadratic forms A, B, C, D, E, F (or the corresponding symmetric matrices), we write $[A, B, C, D, E, F]$ for the 6×6 determinant whose columns are these forms, written out as normalized coefficients in the order $00, 11, 22, 12, 02, 01$. Salmon's (or Burnside's) second invariant $M = M(\mathbf{m})$ is

$$M = \left[\widehat{[m_0, m_0]}, \widehat{[m_1, m_1]}, \widehat{[m_2, m_2]}, \widehat{[m_1, m_2]}, \widehat{[m_0, m_2]}, \widehat{[m_0, m_1]}\right].$$

The Aronhold invariants of the above symmetric determinant $F = F_{\mathbf{m}}$ are the following expressions (see Conic sections, [15], loc. cit.)

$$S(F) = \theta^2 - 24M, \quad T(F) = \theta^3 - 36\theta M, \quad R(F) = 432M^2(32M - \theta^2),$$

where $\theta = \theta(\mathbf{m})$ and $M = M(\mathbf{m})$. Hence $(F; \theta)$ satisfies (3.21) and belongs to $V \subset \mathbb{P}(1^{10}, 2)$. Thus the map α is well defined. Further, if $(F; \theta) \in V$, and if we identify $\mathbf{m}$ and the corresponding linear form with their matrix coefficients, then

$$\begin{aligned}\alpha(\beta(F; \theta)) &= \alpha(\theta \mathrm{HE}(F) + \mathrm{HE}(\mathrm{He}(F))) \\ &= \big(\mathrm{He}(\theta F + \mathrm{He}(F));\ 2T(\theta F + \mathrm{He}(F)\big) = (F; \theta)\end{aligned}$$

by the proof of Theorem 3.1, (A). Because they map between varieties of the same dimension, it is now obvious that α and β are birational. That α is compatible with the standard quadratic transformation follows from the observation that the mixed determinant of adjoint matrices in formula (3.37) corresponds to the η of Theorem 3.1 and from the behaviour of mixed determinants of the third order when the matrices involved are replaced by their inverses (or adjoints). Q.E.D.

3.2.5 An action of the Cremona group on the space of cubics

Consider the following two composite maps from the space of plane cubics to the spherical 2-space over 3×3-matrices:

$$\mathbb{P}(S^3(W^*)) \xrightarrow{\mathrm{he}} V \xrightarrow{\beta} \mathbb{S}_2^+(\mathrm{Mat}_3(K)),$$

$$\mathbb{P}(S^3(W^*)) \xrightarrow{\mathrm{ca}} V \xrightarrow{\beta} \mathbb{S}_2^+(\mathrm{Mat}_3(K)).$$

Each of these maps leads to an action of the Cremona group on the space of plane cubics, the first on the right, the second on the left. If C is a plane cubic defined by a form F, and $g \in \mathrm{Cr}(2, K)$ (or $g \in \mathrm{UCr}(2, K)$), then we may define

$$g(C) = (\beta \circ \mathrm{he})^{-1}(g((\beta \circ \mathrm{he})(F))) \quad \text{or} \quad g(C) = (\beta \circ \mathrm{ca})^{-1}(g((\beta \circ \mathrm{ca})(F))).$$

3.3 An action of the Cremona group of the plane on the space of quartics

Let K be an algebraically closed field of characteristic zero. Every ordered triple of symmetric 4×4-matrices $m_0, m_1, m_2 \in \mathrm{Mat}_4^+(K)$ defines a net of quadrics $x_0M_0 + x_1M_1 + x_2M_2 = 0$ in $\mathbb{P}^3$; here M_i is a quaternary quadratic form with matrix m_i, and the x_i are parameters in the net. $\mathrm{GL}(4,K)$-equivalence classes of stable nets correspond one-to-one to points $\mathbf{m} = (m_0 : m_1 : m_2) \in \mathbb{S}_2^+(\mathrm{Mat}_4(K))$. The discriminant curve $C(\mathbf{m})$ of such a point is well defined and also has degree 4. This curve has a marked even theta characteristic $\theta(\mathbf{m})$ (at least, provided that it is nonsingular, see [3]). Thus a point of spherical 2-space defines a point of the variety M_4^{ev} of plane quartics with a marked even theta characteristic. By results of Barth [3], the map

$$\gamma\colon \mathbb{S}_2^+(\mathrm{Mat}_4(K)) \to M_4^{\mathrm{ev}} \quad \text{given by } \mathbf{m} \mapsto (C(\mathbf{m}); \theta(\mathbf{m})) \tag{3.39}$$

is one-to-one on some open subset, and hence birational.

Moreover, every ternary quartic $F \in S^4(W^*)$ defines a pair $(S(F); \theta(F))$, where $S(F)$ is the Clebsch covariant of degree 4 for F, and $\theta(F)$ is an even theta characteristic of the plane quartic S whose equation is $S(F) = 0$ (at least, provided that F is weakly nondegenerate, see [8] for details). This map

$$\mathrm{Sc}\colon \mathbb{P}(S^4(W^*)) \to M_4^{\mathrm{ev}} \quad \text{given by } F \mapsto (S(F); \theta(F)) \tag{3.40}$$

is the Scorza map. By a theorem of Scorza (see [8], 7.8), Sc is a $\mathcal{P}$-equivariant birational isomorphism. Thus, we get the following possibility to define a (right) action of the Cremona group on the space of plane quartics: if $F \in \mathbb{P}(S^4(W^*))$, and $g \in \mathrm{Cr}(2,K)$ (or $g \in \mathrm{UCr}(2,K)$), then we may define

$$g(F) = \mathrm{Sc}^{-1}(\gamma(g(\gamma^{-1}(\mathrm{Sc}(F))))).$$

Remark 3.4 Let $X \subset \mathbb{S}_2^+(A)$ be the subset defined by the equations

$$\det(m_i\widehat{m_j}m_k - m_k\widehat{m_j}m_i) = 0,$$

where (i,j,k) is an arbitrary permutation of (0, 1, 2). Equivalently, X is the subvariety whose generic point $\mathbf{x} = (x_0 : x_1 : x_2)$ satisfies the equations

$$\det(x_i x_j^{-1} x_k - x_k x_j^{-1} x_i) = 0.$$

In other words, Barth's commutators (see [3]) for $\mathbf{x}$ have rank ≤ 2.

The variety X is preserved by collineations and the standard quadratic transformation; this is clear for collineations. As for the standard quadratic transformation, S_0 substitutes $x_i \mapsto x_i^{-1}$, and

$$\begin{aligned}\det(x_i^{-1}x_jx_k^{-1} - x_k^{-1}x_jx_i^{-1}) &= \det(x_i^{-1}(x_jx_k^{-1}x_i - x_ix_k^{-1}x_j)x_i^{-1}) \\ &= (\det(x_i))^{-2}\det(x_jx_k^{-1}x_i - x_ix_k^{-1}x_j).\end{aligned}$$

Therefore the action of the Cremona group we have just constructed on the space of quartics with an even theta characteristic extends Artamkin's action (see the Introduction) on the space of special marked quartics corresponding to certain vector bundles.

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