

BIRATIONAL MAPS WITH ARITHMETICALLY COHEN-MACAULAY GRAPHS

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ABSTRACT. A rational map whose source and image are projectively embedded varieties is said to have an *arithmetically Cohen–Macaulay graph* if the Rees algebra of one (hence any) of its base ideals is a Cohen–Macaulay ring. If the map is birational onto the image one considers how this property forces an upper bound on the degree of a representative of the map. In the plane case a complete description is given of the Cremona maps with Cohen–Macaulay graph, while in arbitrary dimension n it is shown that a Cremona map with Cohen–Macaulay graph has degree at most n^2 .

INTRODUCTION

The overall goal of this work is to establish upper bounds for the degrees of representatives of a birational map. The guiding idea is to obtain such bounds that are simply expressed in terms of known numerical invariants, such as dimension, relation type, regularity and so forth.

A more specific goal of this paper is to study how the Cohen–Macaulay property of the graph of the map has any impact on these bounds. Here we think of the graph of the map in terms of the Rees algebra $\mathcal{R}_A(I)$ of a base ideal $I \subset A$ of the map, where A denotes the homogeneous coordinate ring of the source projective variety. The algebra $\mathcal{R}_A(I)$ is sufficiently stable to warrant a bona fide replacement of the graph - this and other aspects of the theory will be explained in the later sections. The property that $\mathcal{R}_A(I)$ be a Cohen–Macaulay ring - i.e., that bi-projective structure of the graph of the graph be arithmetically Cohen–Macaulay - is stronger than the property that the graph be a Cohen–Macaulay variety since the latter is a local property at geometric points off the bi-irrelevant ideal. However, for the sake of brevity we will often refer to a Cohen–Macaulay graph when the stronger property holds.

We now describe the contents of the sections.

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The first section is a recap of the main terminology and tools, with a pointer to the aspects that will play a central role in the statements and proofs of subsequent sections. As such, Section 1.2 brings up how one deals appropriately with the notion of a base ideal and its associated algebraic constructions.

The second section contains the main results. These are bounds for the degrees of the forms defining a rational map in case the map is birational and/or the associated graph is Cohen–Macaulay.

The first of these results gives a full description of the plane Cremona maps having a saturated base ideal $I \subset A$ such that $\mathcal{R}_A(I)$ is Cohen–Macaulay. The proof is mostly a sharp elementary calculation culminating with the upfront use of the main criterion of [DHS]. Later in the section we give a homological argument showing that in the plane case the assumption that $\mathcal{R}_A(I)$ is Cohen–Macaulay actually implies that I be saturated.

The second result states two bounds for a birational map onto the image, the second of these holding when the graph is in addition Cohen–Macaulay. Both bounds seem to be new in the literature.

We then state a crucial lemma which encapsulates a couple of estimates around numerical homological invariants of an ideal of a local ring. This lemma is applied to derive some stronger bounds in the case of a Cremona map in arbitrary dimension whose graph is Cohen–Macaulay. In addition one proves that a plane Cremona map having Cohen–Macaulay graph has a saturated base ideal. Again such results seem to be new.

The third section contains a few ramblings on the subject. We first focus on bounding the defining degrees of a rational map having a base ideal I of grade at least 2 – a hypothesis saying that the map has essentially a unique base ideal. The statement gives an upper bound for the degree of the generators of I by invoking its syzygies.

Next one proceeds to a comparison between some of the bounds and to some questions on the asymptotic behavior of the regularity of the powers of the base ideal of a birational map.

The section is wrapped up with a weak converse to a result of [SUV] concerning an upper bound for the a -invariant. The result implies that if $\mathbb{P}_k^{d-1} \dashrightarrow Y \subseteq \mathbb{P}_k^m$ is a rational representation of \mathbb{P}_k^{d-1} whose base ideal $I \subset A = k[x_0, \dots, x_{d-1}]$ has codimension 2 and is not generically a complete intersection then $\mathcal{G} = \mathcal{G}_A(I)$ is not Gorenstein.

1. SETUP AND TERMINOLOGY

1.1. Homological recap. Let (A, \mathfrak{m}) denote a Noetherian local ring of dimension d and let I stand for a proper ideal of A . Set $\mathcal{R}_A(I) := A[It] \subset A[t]$, the Rees algebra of I over A and $\mathcal{G}_I(A) := A/I \otimes_A \mathcal{R}_A(I)$, the associated graded algebra. Let $S = A[y_1, \dots, y_r]$ denote a standard graded *local polynomial ring over A and let \mathcal{J} stand for a presentation ideal of $\mathcal{R}_A(I)$ over S . Clearly, then $\mathcal{R}_A(I) \simeq S/\mathcal{J}$ is standard graded *local, with

homogeneous maximal ideal $\mathfrak{N} := (\mathfrak{m}, (\mathcal{R}_A(I))_+)$. The ideal \mathcal{J} is a homogeneous ideal of S . The maximum degree of the elements of a minimal generating set of \mathcal{J} is called the relation type of I . The relation type depends only on the ideal I and is independent of its generators. Similarly, residue algebra $\mathcal{G}_I(A)$ is a *local ring with *maximal ideal $\mathfrak{M} := (\mathfrak{m}, \mathcal{G}_I(A)_+)$.

We will be interested in the case where $\mathcal{R}_A(I)$ is a Cohen–Macaulay ring. For this, recall more generally that if S is a finitely generated \mathbb{N} -graded algebra over a local ring (A, \mathfrak{m}) and its maximal ideal or a standard graded algebra over a field and its irrelevant ideal, with $S_0 = A$, then S is a Cohen–Macaulay ring if and only if the local ring $T := S_{(\mathfrak{m}, S_+)}$ is Cohen–Macaulay ([HR] or [BH, Exercise 2.1.17]).

An ideal $J \subset I$ is called a reduction of I if $JI^n = I^{n+1}$ for some integer n . The reduction J is said to be a *minimal reduction* if it does not contain properly another reduction of I . The *reduction exponent* of I with respect to the minimal reduction J , denoted by $r_J(I)$, is the least integer $n \geq 0$ such that $JI^n = I^{n+1}$. The (absolute) reduction exponent of I is $\min\{r_J(I) : J \text{ is a minimal reduction of } I\}$. The *analytic spread* of I is $\ell(I) := \dim \mathcal{R}_A(I)/\mathfrak{m}\mathcal{R}_A(I)$. If the residue field A/\mathfrak{m} is infinite then the analytic spread coincides with the minimum number of the generators of a minimal reduction of I .

Let $R = \bigoplus_{n \geq 0} R_n$ be a positively graded *local Noetherian ring of dimension d with graded maximal ideal \mathfrak{n} . For a graded R -module M , $\text{indeg}(M) := \inf\{i : M_i \neq 0\}$ and $\text{end}(M) := \sup\{i : M_i \neq 0\}$.

If M is a finitely generated graded R -module, the Castelnuovo–Mumford regularity of M is defined as

$$\text{reg}(M) := \max\{\text{end}(H_{R_+}^i(M)) + i\}.$$

One can also introduce the regularity with respect to the maximal ideal \mathfrak{M} , namely, $\text{reg}_{\mathfrak{M}}(M) := \max\{\text{end}(H_{\mathfrak{M}}^i(M)) + i\}$. Likewise, set $a_i^*(M) := \text{end}(H_{\mathfrak{M}}^i(M))$.

In particular, if $\dim A = d$ then $a_d^*(\mathcal{G}_I(A)) = \text{end}(H_{\mathfrak{M}}^d(\mathcal{G}_I(A)))$, an invariant of relevance in this work.

Finally, if A is standard graded and $I \subset A$ is an ideal generated by forms of the same degree, then one can also consider (\mathbf{x})-regularity by defining it to be $\text{reg}_{\mathbf{x}}(\mathcal{R}_A(I)) := \max\{\text{end}(H_{(\mathbf{x})}^i(\mathcal{R}_A(I))) + i\}$, where the grading has been reseted to $\deg(\mathbf{x}) = 1$ and $\deg(\mathbf{y}) = 0$. This notion will be frequently used throughout.

The above four notions – Cohen–Macaulayness, reduction number, regularity and a -invariant – are inextricably tied together and there is quite a large publication list on the subject by several authors going back at least to the early nineties. Our purpose is to draw on parts of this knowledge in order to prove one of the main theorems in the paper.

1.2. Elements of birational maps. Let $\mathfrak{F} : X \dashrightarrow Y$ denote a birational map of (reduced and irreducible) projective varieties over an algebraically closed field k . For simplicity, assume that $X \subset \mathbb{P}_k^n$ and $Y \subset \mathbb{P}_k^m$ are non-degenerate embeddings. Let $A := S(X)$ and $B := S(Y)$ stand for the corresponding homogeneous coordinate rings. Set $d = \dim A = \dim B$.

Now, quite generally, a rational map $\mathfrak{F} : X \dashrightarrow \mathbb{P}_k^m$ is defined by forms $f_0, \dots, f_m \in A$ of the same degree – the tuple (f_0, \dots, f_m) is called a *representative* of the map and the f_j 's are its coordinates. If no confusion arises, we call the common degree of the f_j 's the *degree of the representative*. Set $I := (f_0, \dots, f_m) \subset A$ for the ideal generated by the coordinates of the representative.

It is a fact that the set of representatives of \mathfrak{F} correspond bijectively to the homogeneous vectors in the rank one graded A -module $\text{Hom}(I, A)$. In particular, \mathfrak{F} is uniquely represented up to proportionality if and only if I has grade at least two (see [S, Proposition 1.1 and Definition 1.2]). We emphasize that two representatives of \mathfrak{F} share the same module of syzygies, generate isomorphic k -subalgebras of A and the corresponding base ideals of A have naturally isomorphic Rees algebras (see [S] and [DHS] for the details).

Choosing a (finite) minimal set of homogeneous generators of $\text{Hom}(I, A)$ gives a well defined finite set of degrees of representatives of the map \mathfrak{F} .

For any given representative (f_0, \dots, f_m) of the rational map, one calls *base ideal* the corresponding ideal $I = (f_0, \dots, f_m) \subset A$. The base locus, in the sense of the exact subscheme where the map is not defined is given by another ideal containing every such I (see [S, Proposition 1.1 (2)]).

An effective algebraic criterion for the birationality of \mathfrak{F} onto Y was established in [DHS, Theorem 2.18]. The appropriate tool to deal with this question is the graph of \mathfrak{F} which, algebraically, is thought of as the Rees algebra of a given base ideal $(\mathbf{f}) \subset A(X)$ – this idea goes back at least to [RuS].

2. UPPER BOUNDS WHEN THE GRAPH IS COHEN–MACAULAY

2.1. Plane Cremona case. Although our main concern is the use of homological tools to understand the impact of the Cohen–Macaulay property, in a special case one can get around with more elementary methods – this is the purpose of this short part.

Namely, we have the following result:

Proposition 2.1. *Let $I \subset A := k[x, y, z]$ denote the base ideal of a plane Cremona map of degree $\delta \geq 1$. The following two conditions are equivalent:*

- (a) *I is saturated and the Rees algebra $\mathcal{R} := \mathcal{R}_A(I)$ of I is Cohen–Macaulay.*
- (b) *$\delta \leq 3$ or else $\delta = 4$ and the map is not a de Jonquières map.*

Now, any other minimal generator of \mathcal{J} being still a maximal minor thereof has to include the first and last rows of Ψ . This leaves no choice for the bidegrees of the remaining minimal generators of \mathcal{J} except to have the form $(t_j, 2)$, for some $t_j \geq 2$, $2 \leq j \leq m$. Moreover, necessarily, $t_j = r_1 + r_2 + \cdots + \widehat{r_j} + \cdots + r_m + r'_m$.

On the other hand, since I is the base ideal of a Cremona map, then \mathcal{J} admits at least two minimal generators of bidegree $(1, s)$, for some $s \geq 1$ (see [DHS, Theorem 2.18]).

Say, $r' \leq r$. If $r' = 1$, and since one is assuming that I is saturated, then one can show that the map is a de Jonquières map – a full proof is given in [RaS, Proposition 3.4]), but the main line of argument is as follows: if $\text{cod}(I_1(\phi)) = 3$ then I is a generically a complete intersection and, hence of linear type since it is an almost complete intersection; but this not possible by [DHS, Proposition 3.4]. Thus, $I_1(\phi)$ has codimension ≤ 2 and up to a change of variables one may assume that the first column of ϕ has coordinates $x, y, 0$. Since the map is assumed to a Cremona map, the coordinates of the second column can be taken to be z -monoids (see, e.g., [Pan, Corollaire 2.3]).

Now, by the “only if” part of [HS, Theorem 2.7 (iii)], one has $d \leq 3$.

Thus, assume that $r' \geq 2$. Then there are two distinct minimal generators of bidegree $(1, 2)$. Letting Δ_1 be one of them, we must have $1 = t_j = r_1 + r_2 + \cdots + \widehat{r_j} + \cdots + r_m + r'_m$, for for some $2 \leq j \leq m$. It follows that $r = (r_1 + r_2 + \cdots + \widehat{r_j} + \cdots + r_m) + r_j \leq r_j + 1$. Since the row where the entry a_j appears is omitted by the minor Δ_1 , it must appear in the other minor Δ_2 of bidegree $(1, 2)$. By a similar token, $1 = t_{j'}$, with $j' \neq j$, and hence we must have $r_j \leq 1$. We conclude that $r \leq r_j + 1 \leq 2$. Since we are assuming that $2 \leq r' \leq r$, it follows finally that $r = r' = 2$, in which case the map has degree 4 and is necessarily non de Jonquières.

(b) \Rightarrow (a) This implication is mostly an educated exercise.

Assume first that $\delta = 2$, i.e., the map is a quadratic Cremona map. This case is sufficiently well-known: the ideal is generated by the 2-minors of a rank two 2×3 matrix with linear entries. Therefore, A/I is a Cohen–Macaulay ring – equivalently here I is saturated – and, moreover, I is an ideal of linear type, i.e., \mathcal{R} coincides with the symmetric algebra of I , the latter being a complete intersection.

If $\delta = 3$ the map is a de Jonquières map as stems easily from the equations of condition, hence the result follows, e.g., from [HS, Corollary 2.5] for the saturation and from the “if” part of [HS, Theorem 2.7 (iii)], for the Cohen–Macaulayness.

Finally, assume that $\delta = 4$ and that the map is not a de Jonquières map. It is classically known – or easy to verify – that there is only one additional proper homaloidal type for this degree, namely, $(4; 2^3, 1^3)$. Still there is the question here of the base points, since if they are not general it may happen that such a Cremona map or its inverse, though of the same homaloidal type, may have infinitely near base points. Alas, it is not clear a

priori whether these are obstacles for showing the stated result. Fortunately, somehow mysteriously, the algebra gets around it in all cases! For the proof we refer to the recent [CRS, Proposition A.1 and Remark A.3], where all is needed is that we may assume that three of the base points be proper and non-aligned. \square

Remark 2.2. If I is not saturated then to proceed elementarily as in the above argument, would require knowing the number and degrees of a set of minimal syzygies of I . There is a strong suspicion about how these ought to be, while there is at least one large class matching this expectation ([RaS, Proposition 4.12]). Note that, for an arbitrary ideal I the hypothesis that \mathcal{R} is Cohen–Macaulay does not imply that I is saturated. For example, this is the case of an almost complete intersection $I \subset R$ in a Cohen–Macaulay ring, such that I is a non-saturated generically complete intersection. It is known – see, e.g., [HSV] –, that I is of linear type (in particular, has maximal analytic spread) and its symmetric algebra is Cohen–Macaulay. (Perhaps the simplest explicit example has $R = k[x, y, z]$ and I the ideal generated by the partial derivatives of the binomial $y^2z - x^3$.)

Note, however, that the base ideal of a plane Cremona map, although being an almost complete intersection of maximal analytic spread, is very rarely generically a complete intersection ([DHS, Corollary 3.6]). And in fact, we will prove in Proposition 2.6 that, for the base ideal of a plane Cremona map, the hypothesis that \mathcal{R} is Cohen–Macaulay *does* imply that I is saturated. This will allow us to remove the extra assumption in item (a) of the above proposition.

2.2. General case. From previous parts, given a rational map $\mathfrak{F} : X \dashrightarrow \mathbb{P}^m$, the Rees algebra $\mathcal{R}_A(I)$ of one of its base ideals $I \subset A$ depends only on the map and not on this particular base ideal. Since the biprojective variety defined by $\mathcal{R}_A(I)$ gives a version of the graph of \mathfrak{F} , if no confusion arises any property of the Rees algebra will be said to be a property of the latter.

Recall that $\delta(\mathfrak{F})$ stands for the least possible degree of the base ideals of the map. For the next proposition we introduce a new numerical invariant of a rational map $\mathfrak{F} : X \subset \mathbb{P}_k^n \dashrightarrow \mathbb{P}_k^m$. It will be a consequence of the following lemma.

Lemma 2.3. *Let $I, J \subset A$ denote two base ideals of a rational map $\mathfrak{F} : X \subset \mathbb{P}_k^n \dashrightarrow \mathbb{P}_k^m$, where A stands for the homogeneous coordinate ring of X . Let δ_I, δ_J denote the respective degrees of the generators. Then*

$$\operatorname{reg}(I^r) - r\delta_I = \operatorname{reg}(J^r) - r\delta_J,$$

for every $r \geq 1$.

Proof. As observed in Section 1.2, I and J have the same syzygies. This is because being base ideals of the same rational map they come from equivalent representatives in the

sense of [DHS, Section 2.2] – that is, two sets of homogeneous coordinates of the same point of $\mathbb{P}_{K(A)}^n$, where $K(R)$ is the total ring of fractions of A . It follows that, for any $r \geq 1$, the powers I^r and J^r are base ideals of another rational map, and hence by the same token they too have the same syzygies. Writing A as a residue of the polynomial ring $R = k[x_0, \dots, x_n]$, let

$$\cdots \longrightarrow \bigoplus_i R(-r \delta_I - i) \longrightarrow R(-r \delta_I)^{N_r} \longrightarrow I^r \longrightarrow 0 \quad (3)$$

denote the minimal free resolution of I^r over R , for suitable N_r . Then the above discussion implies that the minimal free resolution of J^r over R has the form

$$\cdots \longrightarrow \bigoplus_i R(-r \delta_J - i) \longrightarrow R(-r \delta_J)^{N_r} \longrightarrow J^r \longrightarrow 0$$

where the maps and the standard degrees i 's are the same as those of (3).

The stated equality follows immediately from this. \square

Clearly, $\text{reg}(I^r) \geq r\delta_I$ for every $r \geq 1$. Thus, one gets a numerical function $\mathfrak{f} : \mathbb{N} \rightarrow \mathbb{N}$ which depends solely on the map \mathfrak{F} , defined by $\mathfrak{f}(r) = \text{reg}(I^r) - r\delta_I$, for any choice of a base ideal I . This function attains a maximum which, according to Chardin ([Ch, Theorem 3.5]), is the \mathbf{x} -regularity $\text{reg}_{\mathbf{x}}(\mathcal{R}_A(I))$ introduced in Section 1.1, thus retrieving another numerical invariant of \mathfrak{F} (since as remarked in Section 1.2, the Rees algebra is independent of the chosen base ideal).

In the case where \mathfrak{F} is birational onto its image, one can state a bound for the minimum degree $\delta(\mathfrak{F})$ of a representative:

Theorem 2.4. *Let $\mathfrak{F} : X \subset \mathbb{P}_k^n \dashrightarrow Y \subset \mathbb{P}_k^m$ denote a birational map of reduced and irreducible nondegenerate projective varieties over an algebraically closed field k and let I stand for a base ideal of \mathfrak{F} generated in degree δ . Then*

$$\delta(\mathfrak{F}) \leq m \sup_{r \geq 1} \{\text{reg}(I^r) - r\delta + 1\}, \quad (4)$$

If moreover the graph of \mathfrak{F} is Cohen-Macaulay then

$$\delta(\mathfrak{F}) \leq m \sup_{r \geq 1} \{\text{reg}(I^r) - r\delta + 1\} \leq m d \quad (5)$$

where $d = \dim(X) + 1$.

Proof. Let A and B stand, respectively, for the homogeneous coordinate rings of X and Y . Let $I' \subset B$ denote the ideal generated by the forms of a representative of the inverse map to \mathfrak{F} . Consider a presentation $\mathcal{R}_B(I') \simeq B[x_0, \dots, x_n]/\mathcal{J}_{I'}$, where B is concentrated in degree zero. According to [DHS, Theorem 2.18 Supplement (ii)], the map \mathfrak{F} , considered as the inverse map to its own inverse, has a representative whose coordinates are suitable $m \times m$ minors of the Jacobian matrix $\text{Jac}_{\mathbf{y}}((\mathcal{J}_{I'})_{(*,1)})$ and, clearly, the degree of these coordinates is an upper bound for the minimum degree $\delta(\mathfrak{F})$

of a representative. Therefore, to bound $\delta(\mathfrak{F})$ one is led to bounding from above the \mathbf{x} -degree of the minimal generators of $\mathcal{J}_{I'}$, i.e. the relation type of the ideal $I' \subset B$.

Now, quite generally, the relation type of the ideal $I' \subset B$ is bounded by $\text{reg}(\mathcal{R}_B(I')) + 1$, where the regularity is computed with respect to the the irrelevant ideal $\mathcal{R}_B(I')_+$ ([BS, 15.3.1]).

We claim that $\text{reg}_{\mathbf{x}}(\mathcal{R}_A(I)) = \text{reg}(\mathcal{R}_B(I'))$.

For this, we note that birationality implies an isomorphism $\mathcal{R}_B(I') \simeq \mathcal{R}_A(I)$ of bigraded k -algebras ([S, Proposition 2.1]) naturally based on the standard bigraded structure of the polynomial ring $R[\mathbf{y}] = k[\mathbf{x}, \mathbf{y}]$. If one changes this grading by setting $\text{deg}(\mathbf{y}) = 0$ and $\text{deg}(\mathbf{x}) = 1$, it obtains an isomorphism of standard graded k -algebras. It follows that $H_{(\mathbf{x})}^i(\mathcal{R}_B(I')) \simeq H_{(\mathbf{x})}^i(\mathcal{R}_A(I))$ for every i . Since $\text{reg}(\mathcal{R}_B(I')) + 1$ is computed with respect to the the irrelevant ideal $\mathcal{R}_B(I')_+$, it is defined in terms of local cohomology based on the ideal $B[\mathbf{x}]_+ = (\mathbf{x})$. Then the above isomorphisms of local cohomology modules imply that $\text{reg}_{\mathbf{x}}(\mathcal{R}_A(I)) = \text{reg}(\mathcal{R}_B(I'))$.

As a consequence, $\delta(\mathfrak{F}) \leq m(\text{reg}_{\mathbf{x}}(\mathcal{R}_A(I)) + 1)$. Applying Chardin's equality mentioned earlier, we obtain the first inequality.

If, in addition, the graph of \mathfrak{F} is Cohen-Macaulay, then the isomorphism $\mathcal{R}_B(I') \simeq \mathcal{R}_A(I)$ of bigraded structures implies that $\mathcal{R}_B(I')$ is Cohen-Macaulay. By [AHT, Proposition 6.2], one has $\text{reg}(\mathcal{R}_B(I')) \leq \ell(I') - 1$. Since $\ell(I') = \dim(X) + 1$ we are through for the second bound. \square

We note that the bounds just obtained are expressed solely in terms of the ambient dimension, whereas the Cohen-Macaulayness of the graph is a property of the map.

For the subsequent result we will use the following lemma – essentially a collection of known results in a form that suits our purpose in this part.

Lemma 2.5. *Let (A, \mathfrak{m}) be a Cohen–Macaulay local ring of dimension d and let I be an ideal of A with analytic spread ℓ and reduction number $r(I)$. If $\mathcal{R}_A(I)$ is Cohen–Macaulay, one has*

$$\text{reg}(\mathcal{R}_A(I)) \leq \begin{cases} r(I) & \text{if } \ell = \text{ht}(I) \\ \max\{r(I), \ell - 2\} & \text{otherwise.} \end{cases}$$

Proof. Set $\mathcal{R} := \mathcal{R}_A(I)$ and $\mathcal{G} := \mathcal{G}_I(A)$. Since both A and \mathcal{R} are Cohen-Macaulay then \mathcal{G} is Cohen-Macaulay and moreover $a_d^*(\mathcal{G}) \leq -1$ [TI, Theorem 1.1]. According to [HHK, Corollary 4.2], one has

$$a_d^*(\mathcal{G}) = \max\{r(I) - \ell(I), a(\mathcal{G}_{\mathfrak{p}}) \mid \mathfrak{p} \in V(I) \setminus \mathfrak{m}\}$$

where $a(\mathcal{G}_{\mathfrak{p}})$ is the a -invariant of $\mathcal{G}_{\mathfrak{p}}$ over $A_{\mathfrak{p}}$. A recursion then yields

$$a_d^*(\mathcal{G}) = \max\{r(I_{\mathfrak{p}}) - \ell(I_{\mathfrak{p}}) \mid \mathfrak{p} \in V(I)\}. \quad (6)$$

On the other hand, by [AHT, Corollary 4.11], $\text{reg}(\mathcal{G}) \leq \max\{r_{\ell-1}(I), r(I)\}$, where

$$r_{\ell-1}(I) = \begin{cases} -1, & \text{if } \ell = \text{ht}(I) \\ \max\{r(I_{\mathfrak{p}}) - \ell(I_{\mathfrak{p}}) : \text{ht}(\mathfrak{p}) \leq \ell - 1 \text{ and } \ell(I_{\mathfrak{p}}) = \text{ht}(\mathfrak{p})\} + \ell - 1, & \text{otherwise.} \end{cases}$$

Since $a_d^*(\mathcal{G}) \leq -1$, (6) implies that $r(I_{\mathfrak{p}}) - \ell(I_{\mathfrak{p}}) \leq -1$ for all $\mathfrak{p} \in V(I)$. Therefore $r_{\ell-1}(I) \leq \ell - 2$ if $\ell \neq \text{ht}(I)$. The assertion now follows because $\text{reg}(\mathcal{R}) = \text{reg}(\mathcal{G})$ by [T, Corollary 3.3]. \square

Theorem 2.6. (*k infinite*) *Let $\mathfrak{F} : \mathbb{P}_k^n \dashrightarrow \mathbb{P}_k^n$ be a Cremona map having Cohen–Macaulay graph. Then the respective base ideals of \mathfrak{F} and its inverse are generated in degree $\leq n^2$.*

Furthermore, if $I \subset A = k[x_0, \dots, x_n]$ stands for the base ideal of \mathfrak{F} , then the saturation of I is the ideal $(I :_A A_+^{n-2})$. In particular, the base ideal of a plane Cremona map with Cohen–Macaulay graph is saturated.

Proof. Set $I \subset A$ for the base ideal of the map. Since the generators of I are analytically independent and k is infinite, the reduction number $r(I) = 0$. From Lemma 2.5 one has $\text{reg}(\mathcal{R}_A(I)) \leq n - 1$, hence the relation type of I is $\leq n$. Therefore the same reasoning as in the proof of Theorem 2.4 shows that $\delta(\mathfrak{F})$ and $\delta(\mathfrak{F}^{-1})$ are at most n^2 .

As earlier, twisting around by considering the graph of the inverse map, we derive the inequality $\text{reg}_{\mathbf{x}}(\mathcal{R}_A(I)) \leq n - 1$. It then follows from Chardin’s equality that $\text{reg}(I) - \delta \leq n - 1$ where δ is the degree of the generators of I . Therefore $\text{reg}(A/I) \leq \delta + n - 2$. In particular $\text{end}(H_{(\mathbf{x})}^0(A/I)) = \text{end}(I^{\text{sat}}/I) \leq \delta + n - 2$. On the other hand, by [PR, Proposition 1.2 (i)] (also [D, Proposition 1.3.6]) one always has $\text{indeg}(I^{\text{sat}}/I) \geq \delta + 1$ whenever I is the base ideal of a Cremona map. Therefore $(A_+)^{n-2} \frac{I^{\text{sat}}}{I} = 0$ which proves the assertion. \square

3. UPPER BOUNDS FOR THE DEGREE OF A LINEAR SYSTEM

3.1. A naïve upper bound. In this short section we give an upper bound for the degree of a linear system defining a rational map, under the assumption that it generates an ideal of grade at least 2 – an assumption meaning that the map has essentially a unique base ideal.

Proposition 3.1. *Let $\mathfrak{F} : X \subset \mathbb{P}_k^n \dashrightarrow \mathbb{P}_k^m$ denote a rational map. Suppose that \mathfrak{F} admits a base ideal $I \subset A$ of grade at least 2. If δ denote the degree of the forms generating I , one has*

$$\delta \leq \frac{m}{m+1} e,$$

where e is the largest twist in the graded presentation of I over A .

The proposition is an immediate consequence of the following general lemma which seems to be folklore.

Lemma 3.2. *Let R be a commutative Noetherian ring and $I \subset R$ be an ideal of grade at least two. Fixing a set $\{f_0, \dots, f_m\}$ of generators of I , let M denote a corresponding presentation matrix. Then there exists an $(m+1) \times m$ submatrix of M of rank m , and a non-zero element $h \in R$ such that $hf_i = \Delta_i$, where Δ_i denotes the i -th minor of this submatrix.*

In particular, if R is standard graded and if I is homogenous and generated in a single degree δ and related in degree at most $b_1(I)$ then $b_1(I) \geq \frac{m+1}{m} \delta$.

Proof. Consider the free presentation $R^p \xrightarrow{M} R^{m+1} \xrightarrow{\mathbf{f}} R \rightarrow R/I \rightarrow 0$, where \mathbf{f} denotes the vector $(f_0 \cdots f_m)$. Since $\text{grade}(I) \geq 2$, dualizing into R yields the exact sequence $0 \rightarrow R^* \xrightarrow{\mathbf{f}^t} R^{m+1*} \xrightarrow{M^t} R^{p*}$ and moreover $\text{Im}(M)$ has rank m . Let N be an $m \times (m+1)$ submatrix of M^t of rank m and denote the (ordered) $m \times m$ minors of N by $\Delta = \Delta_0, \dots, \Delta_m$. For an arbitrary row of M^t say $(r_{0i} \cdots r_{mi})$, $\sum_{j=0}^m r_{ji} \Delta_j$ is the determinant of an $(m+1) \times (m+1)$ matrix which is either a matrix with two equal rows or it is a $(m+1) \times (m+1)$ submatrix of M^t . In any case the associated determinant is zero. Therefore in the following diagram the compositions of any two compatible maps are zero

$$\begin{array}{ccccc} R^* & \xrightarrow{\mathbf{f}^t} & R^{m+1*} & \xrightarrow{M^t} & R^{p*} \\ & & \uparrow (\Delta) & & \\ & & R & & \end{array}$$

Since $\ker(M^t) = \text{Im } \mathbf{f}^t$, one has $\Delta = (\Delta_0 \cdots \Delta_m) \in \text{Im}(\mathbf{f}^t)$; which proves the first assertion. To see the degree assertion in the equihomogeneous case, note that in the graded case h is homogeneous, hence the equality $hf_i = \Delta_i$ implies that $\delta \leq \deg(\Delta_i) \leq m(b_1(I) - \delta)$. \square

3.2. Confronting bounds in the birational case. Since the bound in Proposition 3.1 is quite natural, it makes sense confronting it with the birational bound obtained in Theorem 2.4. For this we assume that the map \mathfrak{F} is birational, as in Theorem 2.4, and $I \subset A$ is a base ideal of grade ≥ 2 , as in Proposition 3.1. In this situation, the representative of \mathfrak{F} is uniquely defined up to scalars (see [S, Proposition 1.1]), hence $\delta(\mathfrak{F}) = \delta$. Let us suppose in addition that the function \mathfrak{f} attains its maximum at $r = 1$ – a hypothesis we will digress on in a minute.

Then the second bound gives $\delta \leq m(\text{reg}(I) - \delta + 1)$, i.e., $\delta \leq \frac{m}{m+1}(\text{reg}(I) + 1)$. Since $\text{reg}(I) \geq e - 1$, we see that the first bound is better.

On the other hand, the second bound is more comprehensive and besides it has the advantage of providing a lower bound for \mathfrak{f} in terms of an invariant of the map \mathfrak{F} and not of a particular base ideal I .

A question arises naturally in the birational case, as to what values of r give the maximum of \mathfrak{f} defined by $\mathfrak{f}(r) = \text{reg}(I^r) - r\delta_I$, for any choice of a base ideal I . It is conceivable that, for arbitrary homogeneous ideals generated in a fixed degree, the behavior is quite erratic. However, for a base ideal I of a birational map, one knows, for example, that far out powers of I will have quite a bit of linear syzygies, while the regularity is quite sensitive to those.

And, in fact, the maximum may fail to be attained at $r = 1$, as is shown by the following example of Terai (see also [St]):

Example 3.3. Let

$$I = (\mathbf{f}) := (abc, abf, ace, ade, adf, bcd, bde, bef, cdf, cef) \subset A := \mathbb{Q}[a, b, c, d, e, f].$$

The main highlights of this example are:

- (1) I has linear resolution – in particular, $\text{reg}(I) = 3$.
- (2) I^2 has codepth zero and the last syzygies have standard degree 2, hence $\text{reg}(I^2) \geq 12 - 5 = 7$ (in fact, one has $= 7$).
- (3) I defines a birational map onto the image.

The first two features are directly checked in a computer program. To see (3), it suffices to know that I is linearly presented (or even still that the linear part of the syzygies has maximal rank, i.e., 9) and that $\dim[\mathbf{f}] = 6$ (maximum). These two properties imply birationality by [DHS, Theorem 3.2]. For the dimension, since we are in characteristic zero, it suffices to check that the Jacobian matrix of \mathbf{f} has rank 6.

The above question gets streamlined in the particular case of Cremona maps. Here it might be tempting to guess that the maximum is attained at $r = 1$, but at the moment one lacks sufficient preliminary evidence.

3.3. Cohen–Macaulay graphs and the a -invariant. As a final piece, we comment on the a -invariant vis-à-vis the Cohen–Macaulay property. We have seen the relevance of the inequality $a_d^*(\mathcal{G}) \leq -1$ in the proof of Lemma 2.5. Bounding the a -invariant of \mathcal{G} below -1 may require strong additional assumptions. In [SUV, Theorem 2.4], e.g., it is proved that, for an ideal I of codimension g such that \mathcal{G} is Cohen–Macaulay and I satisfies the condition (\mathcal{F}_1) , one has $a_d^*(\mathcal{G}) \leq -g$. The latter condition means that $\mu(I_{\mathfrak{p}}) \leq \text{ht}(\mathfrak{p})$ for any prime $\mathfrak{p} \supseteq I$, thus imposing a severe restriction on the nature of I . In particular, it implies that I is generically a complete intersection which, as remarked earlier, is a rare event among base ideals of Cremona maps ([DHS, Corollary 3.6]). Curiously, we can file the following weak converse to the above result.

Proposition 3.4. *Let (A, \mathfrak{m}) denote a Cohen–Macaulay local ring of dimension d and with infinite residue field. Let $I \subset A$ stand for an equidimensional ideal of codimension*

g such that $\mathcal{G} := \mathcal{G}_A(I)$ is Cohen-Macaulay. If $a_d^*(\mathcal{G}) \leq -g$ then I is generically complete intersection.

Proof. [HHK, 4.2] entails the equality $a_d^*(\mathcal{G}) = \max\{r(I_{\mathfrak{p}}) - \ell(I_{\mathfrak{p}}) \mid \mathfrak{p} \in V(I)\}$. Let $\mathfrak{p} \in V(I)$ have codimension g . Since $a_d^*(\mathcal{G}) \leq -g$, one has $r(I_{\mathfrak{p}}) - \ell(I_{\mathfrak{p}}) \leq -g$, and since I is assumed to be equidimensional, then $\ell(I_{\mathfrak{p}}) = \text{ht}(\mathfrak{p}) = g$. Thus, $r(I_{\mathfrak{p}}) \leq 0$ which means that $I_{\mathfrak{p}}$ has no proper reduction. On the other hand, since the residue field is infinite, $I_{\mathfrak{p}}$ possesses a minimal reduction with $\ell(I_{\mathfrak{p}})$ generators. Of course, $\ell(I_{\mathfrak{p}}) \leq \mu(I_{\mathfrak{p}})$. Necessarily then $\ell(I_{\mathfrak{p}}) = \mu(I_{\mathfrak{p}})$, that is $\mu(I_{\mathfrak{p}}) = g$ which proves the assertion. \square

In particular, if $\mathbb{P}_k^{d-1} \dashrightarrow Y \subseteq \mathbb{P}_k^m$ is a rational representation of \mathbb{P}_k^{d-1} whose base ideal $I \subset A = k[x_0, \dots, x_{d-1}]$ has codimension 2 and is not generically a complete intersection then $\mathcal{G} = \mathcal{G}_A(I)$ is not Gorenstein, as otherwise it would imply the inequality $a_d^*(\mathcal{G}) \leq -2$ by [HHK, Theorem 5.2 and Remark 5.3].

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BOUNDS ON DEGREES OF BIRATIONAL MAPS WITH ARITHMETICALLY COHEN-MACAULAY GRAPHS

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ABSTRACT. A rational map whose source and image are projectively embedded varieties is said to have an *arithmetically Cohen–Macaulay graph* if the Rees algebra of one (hence any) of its base ideals is a Cohen–Macaulay ring. The main objective of this paper is to obtain an upper bound on the degree of a representative of the map in case it is birational onto the image with an arithmetically Cohen–Macaulay graph. In the plane case a complete description is given of the Cremona maps with arithmetically Cohen–Macaulay graph, while in arbitrary dimension n it is shown that a Cremona map with arithmetically Cohen–Macaulay graph has degree at most n^2 .

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INTRODUCTION

The overall goal of this work is to establish upper bounds for the degrees of representatives of a birational map. The guiding idea is to obtain such bounds that are simply expressed in terms of known numerical invariants, such as dimension, relation type, regularity and so forth.

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An intriguing related matter considered by some authors is bounding the degrees of representatives of the inverse map in terms of invariants of the originally given birational map. It is a classical fact that if $\mathfrak{F} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is a Cremona map presented by polynomials of degree d then its inverse is also represented in degree d which is essentially an application of Bezout's theorem (see, e.g., [A]). The problem of determining a universal upper bound for the degree of the inverse of Cremona maps of \mathbb{P}^n was settled by O. Gabber and published in [BCW]: if $\mathfrak{F} : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is a Cremona map represented by polynomials of degree d then its inverse is represented in degree at most d^{n-1} . This upper bound is sharp as the map \mathfrak{F} represented by

$$(x_0^d : x_1x_0^{d-1} : x_2x_0^{d-1} + x_1^d : x_3x_0^{d-1} + x_2^d : \cdots : x_nx_0^{d-1} + x_{n-1}^d)$$

has inverse of degree exactly d^{n-1} .

Enlarging the picture from Cremona maps to arbitrary birational maps $\mathfrak{F} : X \dashrightarrow Y$ where $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$, one faces preliminarily the fact that both the map and its inverse can be defined (represented) by different sets of forms of fixed degree, where this degree may vary from set to set. One way to circumvent this technicality is to think about the smallest such degree – in fact, it follows from [S, Proposition 1.1] that there are natural minimum and *maximum* such degrees for any rational map $\mathfrak{F} : X \dashrightarrow Y$, while from Theorem 1.1 these degrees in the case of the inverse of a given birational map $\mathfrak{F} : X \dashrightarrow Y$ are in principle bounded in terms of the invariants of \mathfrak{F} .

Anyway one looks, it becomes evident that the problem of finding an upper bound for the possible degrees of the inverse map becomes very subtle. As far as the authors know there is no known general upper bound for arbitrary X and Y . However, based on a recent bound for the degrees of reduced Groebner bases established in [MR], we will state one such bound. Unfortunately, as expected from Groebner basis theory, the resulting bound is doubly exponential.

Although our main goal does not focus on this question, it remains interesting not only in its own, but also in a variety of applications from the Jacobian conjecture to recent work such as [BBV].

The more specific goal of this paper is to study how the Cohen–Macaulay property of the graph of the map has any impact on these bounds. Under this requirement a much better bound for the minimum defining degree of the inverse map to a birational map is obtained in terms of the given data (Theorem 2.5). Here we think of the graph of the map in terms of the Rees algebra $\mathcal{R}_A(I)$ of a base ideal $I \subset A$ of the map, where A denotes the homogeneous coordinate ring of the source projective variety. The algebra $\mathcal{R}_A(I)$ is sufficiently stable to warrant a bona fide replacement of the graph – this and other aspects of the theory will be explained in the later sections. The property that $\mathcal{R}_A(I)$ be a Cohen–Macaulay ring – i.e., that bi-projective structure of the graph be arithmetically

Cohen–Macaulay – is stronger than the property that the graph be a Cohen–Macaulay variety since the latter is a local property at geometric points off the bi-irrelevant ideal. However, for the sake of brevity we will often refer to a Cohen–Macaulay graph when the stronger property holds.

We now describe the contents of the sections.

The first section is a recap of the main terminology and tools, with a pointer to the aspects that will play a central role in the statements and proofs of subsequent sections. The first subsection is a recap of homological gadgets, while the second discusses shortly the notion of a base ideal of a rational map and its associated algebraic constructions. We restate the main effective birationality criterion of [DHS] and as an application we give a general bound for the possible degrees of the inverse map to an arbitrary birational map.

The second section contains the main results. It is further subdivided in four subsections.

The first subsection is a direct attack on the question of obtaining an exact bound for the defining degree of a Cremona map for which the corresponding graph is arithmetically Cohen–Macaulay. The result only covers the plane case and shows that this bound is 4 – a particular case of a more general result to be subsequently proved. In addition, one obtains a precise classification of the plane Cremona maps for which the corresponding graph is arithmetically Cohen–Macaulay.

The second and third subsections are concerned with the main results of the paper, stating general bounds, first for the minimum defining degree of a birational map onto the image, and second for the minimum defining degree of a birational map onto the image when the graph is in addition Cohen–Macaulay. The latter bound has a “symmetric” nature, thus also providing a bound for the minimum defining degree of the inverse map in terms of the given data. Both bounds seem to be new in the literature.

We also discuss a related question on the asymptotic behavior of the regularity of the powers of the base ideal of a birational map.

We then state a crucial lemma which encapsulates a couple of estimates around numerical homological invariants of an ideal of a local ring. This lemma is applied to derive some stronger bounds in the case of a Cremona map in arbitrary dimension whose graph is Cohen–Macaulay. In addition one proves that a plane Cremona map having Cohen–Macaulay graph has a saturated base ideal. Again such results seem to be new.

The last subsection contains some ramblings on bounding the defining degrees of a rational map having a base ideal I of grade at least 2 – a hypothesis saying that the map has essentially a unique base ideal. The statement gives an upper bound for the degree of the generators of I by invoking its syzygies. One proceeds to a comparison between this and the earlier bounds.

1. SETUP AND TERMINOLOGY

1.1. Homological recap. Let (A, \mathfrak{m}) denote a Noetherian local ring of dimension d and let I stand for a proper ideal of A . Set $\mathcal{R}_A(I) := A[It] \subset A[t]$, the Rees algebra of I over A and $\mathcal{G}_I(A) := A/I \otimes_A \mathcal{R}_A(I)$, the associated graded algebra. Let $S = A[y_1, \dots, y_r]$ denote a standard graded *local polynomial ring over A and let \mathcal{J} stand for a presentation ideal of $\mathcal{R}_A(I)$ over S . Clearly, then $\mathcal{R}_A(I) \simeq S/\mathcal{J}$ is standard graded *local, with homogeneous maximal ideal $\mathfrak{N} := (\mathfrak{m}, (\mathcal{R}_A(I))_+)$. The ideal \mathcal{J} is a homogeneous ideal of S . The maximum degree of the elements of a minimal generating set of \mathcal{J} is called the relation type of I . The relation type depends only on the ideal I and is independent of its generators. Similarly, the residue algebra $\mathcal{G}_I(A)$ is a *local ring with *maximal ideal $\mathfrak{M} := (\mathfrak{m}, \mathcal{G}_I(A)_+)$.

We will be interested in the case where $\mathcal{R}_A(I)$ is a Cohen–Macaulay ring. For this, recall more generally that if S is a finitely generated \mathbb{N} -graded algebra over a local ring (A, \mathfrak{m}) and its maximal ideal or a standard graded algebra over a field and its irrelevant ideal, with $S_0 = A$, then S is a Cohen–Macaulay ring if and only if the local ring $T := S_{(\mathfrak{m}, S_+)}$ is Cohen–Macaulay ([HR] or [BH, Exercise 2.1.17]).

An ideal $J \subset I$ is called a reduction of I if $JI^n = I^{n+1}$ for some integer n . The reduction J is said to be a *minimal reduction* if it does not contain properly another reduction of I . The *reduction exponent* of I with respect to the minimal reduction J , denoted by $r_J(I)$, is the least integer $n \geq 0$ such that $JI^n = I^{n+1}$. The (absolute) reduction exponent of I is $\min\{r_J(I) : J \text{ is a minimal reduction of } I\}$. The *analytic spread* of I is $\ell(I) := \dim \mathcal{R}_A(I)/\mathfrak{m}\mathcal{R}_A(I)$. If the residue field A/\mathfrak{m} is infinite then the analytic spread coincides with the minimum number of the generators of a minimal reduction of I .

Let $R = \bigoplus_{n \geq 0} R_n$ be a positively graded *local Noetherian ring of dimension d with graded maximal ideal \mathfrak{n} . For a graded R -module M , $\text{indeg}(M) := \inf\{i : M_i \neq 0\}$ and $\text{end}(M) := \sup\{i : M_i \neq 0\}$.

If M is a finitely generated graded R -module, the Castelnuovo–Mumford regularity of M is defined as

$$\text{reg}(M) := \max\{\text{end}(H_{R_+}^i(M)) + i\}$$

and the a -invariants of M are defined as $a_i(M) = \text{end}(H_{R_+}^i(M))$ as well as $a(M) := a_{\dim(M)}$.

One can also introduce the regularity with respect to the maximal ideal \mathfrak{M} , namely, $\text{reg}_{\mathfrak{M}}(M) := \max\{\text{end}(H_{\mathfrak{M}}^i(M)) + i\}$. Likewise, set $a_i^*(M) := \text{end}(H_{\mathfrak{M}}^i(M))$.

In particular, if $\dim A = d$ then $a_d^*(\mathcal{G}_I(A)) = \text{end}(H_{\mathfrak{M}}^d(\mathcal{G}_I(A)))$, an invariant of relevance in this work.

Finally, if A is standard graded and $I \subset A$ is an ideal generated by forms of the same degree, then one can also consider the (\mathbf{x}) -regularity by defining it to be $\text{reg}_{\mathbf{x}}(\mathcal{R}_A(I)) := \max\{\text{end}(H_{(\mathbf{x})}^i(\mathcal{R}_A(I))) + i\}$, where the grading has been reseted to $\deg(\mathbf{x}) = 1$ and $\deg(\mathbf{y}) = 0$. This notion will be frequently used throughout.

The above four notions – Cohen–Macaulayness, reduction number, regularity and a -invariant – are inextricably tied together and there is quite a large publication list on the subject by several authors going back at least to the early nineties. Our purpose is to draw on parts of this knowledge in order to prove one of the main theorems in the paper.

1.2. Elements of birational maps. Let $\mathfrak{F} : X \dashrightarrow Y$ denote a birational map of (reduced and irreducible) projective varieties over an algebraically closed field k . For simplicity, assume that $X \subset \mathbb{P}_k^n$ and $Y \subset \mathbb{P}_k^m$ are non-degenerate embeddings. Let $A := S(X)$ and $B := S(Y)$ stand for the corresponding homogeneous coordinate rings. Set $d = \dim A = \dim B$.

Now, quite generally, a rational map $\mathfrak{F} : X \dashrightarrow \mathbb{P}_k^m$ is defined by forms $f_0, \dots, f_m \in A$ of the same degree – the tuple (f_0, \dots, f_m) is called a *representative* of the map and the f_j 's are its coordinates. If no confusion arises, we call the common degree of the f_j 's the *degree of the representative*. Set $I := (f_0, \dots, f_m) \subset A$ for the ideal generated by the coordinates of the representative.

It is a fact that the set of representatives of \mathfrak{F} correspond bijectively to the homogeneous vectors in the rank one graded A -module $\text{Hom}(I, A)$. In particular, \mathfrak{F} is uniquely represented up to proportionality if and only if I has grade at least two (see [S, Proposition 1.1 and Definition 1.2]). We emphasize that two representatives of \mathfrak{F} share the same module of syzygies, generate isomorphic k -subalgebras of A and the corresponding base ideals of A have naturally isomorphic Rees algebras (see [S] and [DHS] for the details).

Choosing a (finite) minimal set of homogeneous generators of $\text{Hom}(I, A)$ gives a well defined finite set of degrees of representatives of the map \mathfrak{F} .

For any given representative (f_0, \dots, f_m) of the rational map, one calls *base ideal* the corresponding ideal $I = (f_0, \dots, f_m) \subset A$. The base locus, in the sense of the exact subscheme where the map is not defined is given by another ideal containing every such I (see [S, Proposition 1.1 (2)]).

An effective algebraic criterion for the birationality of \mathfrak{F} onto Y was established in [DHS, Theorem 2.18]. The appropriate tool to deal with this question is the graph of \mathfrak{F} which, algebraically, is thought of as the Rees algebra of a given base ideal $(\mathbf{f}) \subset A = S(X)$ – this idea goes back at least to [RuS].

For the sake of the reader's convenience and handy reference, we state the main features of this criterion.

Let $\mathfrak{F} : X \dashrightarrow \mathbb{P}_k^m$ denote a rational map with image $Y \subset \mathbb{P}_k^m$. Denote $A = k[\mathbf{X}]/\mathfrak{a}$ and $B = k[\mathbf{Y}]/\mathfrak{b}$ the respective homogeneous coordinate rings of X and Y . For simplicity we assume that X and Y are non-degenerated, i.e., neither \mathfrak{a} nor \mathfrak{b} contains nonzero linear forms. Suppose that \mathfrak{F} is represented by a set $\mathbf{f} \subset A$ of forms of fixed degree. Let $A[\mathbf{Y}] = A[Y_0, \dots, Y_m]$ be a polynomial ring over A with the standard bigrading, i.e., where $\deg(x_i) = (1, 0)$ and $\deg(Y_j) = (0, 1)$. Since \mathbf{f} is generated in a fixed degree, the Rees algebra

$$\mathcal{R}_A((\mathbf{f})) := A \oplus I \oplus I^2 \oplus \dots \simeq A[It] \subset A[t]$$

is a standard bigraded k -algebra. Mapping $Y_j \mapsto f_j t$ yields a presentation $A[\mathbf{Y}]/\mathcal{J} \simeq \mathcal{R}_A((\mathbf{f}))$, with \mathcal{J} a bihomogeneous ideal.

Next, one can pick a minimal set of generators of the ideal $(\mathcal{J}_{1,*})$ consisting of a finite number of forms of bidegree $(1, q)$, for various q 's. Let $\{P_1, \dots, P_s\} \subset k[\mathbf{X}, \mathbf{Y}]$ denote liftings of these bifurms. Consider the Jacobian matrix of the polynomials $\{P_1, \dots, P_s\}$ with respect to \mathbf{X} , a matrix with entries in $k[\mathbf{Y}]$. Write ψ for the corresponding matrix over $B = k[\mathbf{Y}]/\mathfrak{b}$.

Theorem 1.1. ([DHS, Theorem 2.18]) *With the above notations, \mathfrak{F} is birational if and only if $\text{rank}(\psi) = n$. If this is the case then*

- (i) *A representative of the inverse map over B is given by the coordinates of any homogeneous vector of positive degree in the (rank one) null space of ψ over B .*
- (ii) *A representative as in (i) can be taken to be the ordered set of the signed n -minors of an arbitrary $n \times (n+1)$ submatrix of ψ of rank n .*

Now, drawing upon this criterion and a recent bound established in [MR], one obtains the following general bound for the possible degrees of the inverse map to a birational map:

Proposition 1.2. *Let $X \subset \mathbb{P}_k^n$ denote a closed reduced and irreducible subvariety, and let $\mathfrak{F} : X \dashrightarrow \mathbb{P}_k^m$ denote a birational map with image $Y \subset \mathbb{P}_k^m$, admitting a representative of degree $d \geq 1$. Assume that X and Y are non-degenerated. Then the degree of one of representative of the inverse map \mathfrak{F}^{-1} is bounded above by the number*

$$2n \left[\frac{1}{2} \delta^{2(n+m+1-\dim(X))^2} + \delta \right]^{2(\dim(X)+2)}$$

where d_0 is the maximum degree of a set of minimal generators of the homogeneous defining ideal of X in the given embedding (with the convention that $d_0 = 0$ if $X = \mathbb{P}_k^n$) and $\delta = \max\{d+1, d_0\}$.

Proof. Keeping the previously established notation, let $A = k[\mathbf{X}]/\mathfrak{a}$ be the coordinate ring of X , let $I := (\mathbf{f}) \subset A$ denote the ideal generated by the forms of the given representative of \mathfrak{F} , and let $\mathcal{R}_A(I) \simeq A[\mathbf{Y}]/\mathcal{J}$ be the corresponding Rees algebra. According

to Theorem 1.1 a representative for \mathfrak{F}^{-1} is given by certain $n \times n$ minors of ψ . Since the entries of ψ are of degree at most $\deg_y(\mathcal{J}_{1,*})$, it follows that the degree of any representative of \mathfrak{F}^{-1} is bounded by $n \cdot \text{reltype}_R(I)$, where $\text{reltype}_R(I)$ denotes the relation type of I in R (this is independent of \mathcal{J}).

To compute $\mathcal{J} \subset k[\mathbf{X}, \mathbf{Y}]$, one takes a new variable t over $k[\mathbf{X}, \mathbf{Y}]$. Then $\mathcal{J} = (Y_0 - f_0t, \dots, Y_m - f_mt, \mathfrak{a}) \cap k[\mathbf{X}, \mathbf{Y}]$. To obtain explicit generators of this elimination ideal one considers an elimination term order such as $t > \mathbf{X} > \mathbf{Y}$ and computes the Grobner basis of the ideal $J := (Y_0 - f_0t, \dots, Y_m - f_mt, \mathfrak{a})$ with respect to this order.

Notice that t is a non-zero divisor on J and that $k[\mathbf{X}, \mathbf{Y}, t]/(J, t) \simeq A$, hence

$$\dim(k[\mathbf{X}, \mathbf{Y}, t]/J) = \dim(A) + 1 = \dim(X) + 2.$$

We now apply a theorem of Mayer and Ritscher [MR] saying that for an arbitrary ideal of dimension r in a polynomial ring with l variables over an infinite field, generated in total degrees $d_1 \geq d_2 \geq \dots \geq d_s$, the degrees of the elements of a reduced Grobner basis are bounded by

$$2^{\lceil \frac{1}{2}((d_1 \dots d_{l-r})^{2(l-r)} + d_1) \rceil}.$$

We apply this result to the above ideal $J \subset k[\mathbf{X}, \mathbf{Y}]$, noting that $l = n + m + 3$ and $r = \dim(X) + 2$ in the present situation. \square

Remark 1.3. A much better bound is obtained under the requirement that the graph of the given birational map is arithmetically Cohen–Macaulay (Theorem 2.5). The above bound is unfortunately very coarse and far from even recovering the optimal bound in the case of Cremona maps. In any case it motivates the question as to whether, given closed subvarieties $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$, there exists a function $\Phi(d, d_0, e_0, \dim(X), n, m)$ bounding the degree of any representative of the inverse map to any birational map from X onto Y having a representative of degree d . Here, d_0 (respectively, e_0) denotes the maximum degree of a minimal generator of the homogeneous defining ideal of X (respectively, Y).

Since very little is known about this question, one may as well ask whether Φ is polynomial as a function of d .

An affirmative answer to this question is not only interesting in its own, but also it has a variety of applications from the Jacobian conjecture to more recent speculation such as [BBV].

2. UPPER BOUNDS WHEN THE GRAPH IS COHEN–MACAULAY

2.1. Plane Cremona case. Although our main concern is the use of homological tools to understand the impact of the Cohen–Macaulay property, in a special case one can get

Since the row where the entry a_j appears is omitted by the minor Δ_1 , it must appear in the other minor Δ_2 of bidegree $(1, 2)$. By a similar token, $1 = t_{j'}$, with $j' \neq j$, and hence we must have $r_j \leq 1$. We conclude that $r \leq r_j + 1 \leq 2$. Since we are assuming that $2 \leq r' \leq r$, it follows finally that $r = r' = 2$, in which case the map has degree 4 and is necessarily non de Jonquières.

(b) \Rightarrow (a) This implication is mostly picking up from various results in the literature.

Assume first that $d = 2$, i.e., the map is a quadratic Cremona map. This case is sufficiently well-known: the ideal is generated by the 2-minors of a rank two 2×3 matrix with linear entries. Therefore, A/I is a Cohen–Macaulay ring – equivalently, in this situation, I is saturated – and, moreover, I is an ideal of linear type, i.e., \mathcal{R} coincides with the symmetric algebra of I , the latter being a complete intersection.

If $d = 3$ the map is a de Jonquières map as a consequence of the well-known equations of condition satisfied by a Cremona map. In this case, saturation follows, e.g., from [HS, Corollary 2.5] and Cohen–Macaulayness follows from the “if” part of [HS, Theorem 2.7 (iii)].

Finally, assume that $d = 4$. By [HS, Theorem 1.5 (i)], the base ideal is saturated. Since the map is not a de Jonquières map there is only one additional proper homaloidal type for this degree, namely, $(4; 2^3, 1^3)$ (this follows easily from the Hudson test as stated in [A]). Since we are assuming that three of the base points are proper and non-aligned then the result is proved in [CRS, Proposition A.1 and Remark A.3]. \square

Remark 2.2. If I is not saturated then to proceed elementarily as in the above argument, would require knowing the number and degrees of a set of minimal syzygies of I . There is a strong suspicion about how these ought to be, while there is at least one large class matching this expectation ([RaS, Proposition 4.12]). Note that, for an arbitrary ideal I the hypothesis that \mathcal{R} is Cohen–Macaulay does not imply that I is saturated. For example, this is the case of an almost complete intersection $I \subset R$ in a Cohen–Macaulay ring, such that I is a non-saturated generically complete intersection. It is known – see, e.g., [HSV] –, that I is of linear type (in particular, has maximal analytic spread) and its symmetric algebra is Cohen–Macaulay. (Perhaps the simplest explicit example has $R = k[x, y, z]$ and I the ideal generated by the partial derivatives of the binomial $y^2z - x^3$.)

One should observe, however, that the base ideal of a plane Cremona map, although being an almost complete intersection of maximal analytic spread, is very rarely generically a complete intersection ([DHS, Corollary 3.6]). And in fact, we will prove in Proposition 2.7 that, for the base ideal of a plane Cremona map, the hypothesis that \mathcal{R} is Cohen–Macaulay *does* imply that I is saturated. This will allow us to remove the extra assumption in item (a) of the above proposition.

2.2. General case: preliminaries. From previous parts, given a rational map $\mathfrak{F} : X \dashrightarrow \mathbb{P}^m$, the Rees algebra $\mathcal{R}_A(I)$ of one of its base ideals $I \subset A$ depends only on the map and not on this particular base ideal. Since the biprojective variety defined by $\mathcal{R}_A(I)$ is the graph of \mathfrak{F} , if no confusion arises any property of the Rees algebra will be said to be a property of the latter.

Let us denote by $\delta(\mathfrak{F})$ the least possible degree of the forms of a base ideal of the map. For the next proposition we introduce a new numerical invariant of a rational map $\mathfrak{F} : X \subset \mathbb{P}_k^n \dashrightarrow \mathbb{P}_k^m$. It will be a consequence of the following lemma.

Lemma 2.3. *Let $I, J \subset A$ denote two base ideals of a rational map $\mathfrak{F} : X \subset \mathbb{P}_k^n \dashrightarrow \mathbb{P}_k^m$, where A stands for the homogeneous coordinate ring of X . Let δ_I, δ_J denote the respective degrees of the generators. Then*

$$\operatorname{reg}(I^r) - r\delta_I = \operatorname{reg}(J^r) - r\delta_J,$$

for every $r \geq 1$.

Proof. As observed in Section 1.2, I and J have the same syzygies. This is because being base ideals of the same rational map they come from equivalent representatives in the sense of [DHS, Section 2.2] – that is, two sets of homogeneous coordinates of the same point of $\mathbb{P}_{K(A)}^n$, where $K(R)$ is the total ring of fractions of A . It follows that, for any $r \geq 1$, the powers I^r and J^r are base ideals of another rational map, and hence by the same token they too have the same syzygies. Writing A as a residue of the polynomial ring $R = k[x_0, \dots, x_n]$, let

$$\cdots \longrightarrow \bigoplus_i R(-r\delta_I - i) \longrightarrow R(-r\delta_I)^{N_r} \longrightarrow I^r \longrightarrow 0 \quad (3)$$

denote the minimal free resolution of I^r over R , for suitable N_r . Then the above discussion implies that the minimal free resolution of J^r over R has the form

$$\cdots \longrightarrow \bigoplus_i R(-r\delta_J - i) \longrightarrow R(-r\delta_J)^{N_r} \longrightarrow J^r \longrightarrow 0$$

where the maps and the standard degrees i 's are the same as those of (3).

The stated equality follows immediately from this. \square

Clearly, $\operatorname{reg}(I^r) \geq r\delta_I$ for every $r \geq 1$. Thus, one gets a numerical function $\mathfrak{f} : \mathbb{N} \rightarrow \mathbb{N}$ which depends solely on the map \mathfrak{F} , defined by $\mathfrak{f}(r) = \operatorname{reg}(I^r) - r\delta_I$, for any choice of a base ideal I . This function attains a maximum which, according to Chardin-Romer ([Ch, Theorem 3.5]), is the \mathbf{x} -regularity $\operatorname{reg}_{\mathbf{x}}(\mathcal{R}_A(I))$ introduced in Section 1.1, thus retrieving another numerical invariant of \mathfrak{F} (since as remarked in Section 1.2, the Rees algebra is independent of the chosen base ideal).

As a parenthesis, a question arises naturally in the birational case, as to what values of r give the maximum of \mathfrak{f} defined by $\mathfrak{f}(r) = \operatorname{reg}(I^r) - r\delta_I$, for any choice of a base ideal

I. It is conceivable that, for arbitrary homogeneous ideals generated in a fixed degree, the behavior is quite erratic. However, for a base ideal I of a birational map, one knows, for example, that far out powers of I will have quite a bit of linear syzygies, while the regularity is quite sensitive to those.

And, in fact, the maximum may fail to be attained at $r = 1$, as is shown by the following example of Terai (see also [St]):

Example 2.4. Let

$$I = (\mathbf{f}) := (abc, abf, ace, ade, adf, bcd, bde, bef, cdf, cef) \subset A := \mathbb{Q}[a, b, c, d, e, f].$$

The main highlights of this example are:

- (1) I has linear resolution – in particular, $\text{reg}(I) = 3$.
- (2) I^2 has codepth zero and the last syzygies have standard degree 2, hence $\text{reg}(I^2) \geq 12 - 5 = 7$ (in fact, one has $= 7$).
- (3) I defines a birational map onto the image.

The first two features are directly checked in a computer program. To see (3), it suffices to know that I is linearly presented (or even still that the linear part of the syzygies has maximal rank, i.e., 9) and that $\dim[\mathbf{f}] = 6$ (maximum). These two properties imply birationality by [DHS, Theorem 3.2]. For the dimension, since we are in characteristic zero, it suffices to check that the Jacobian matrix of \mathbf{f} has rank 6.

The above question gets streamlined in the particular case of Cremona maps. Here it might be tempting to guess that the maximum is attained at $r = 1$, but at the moment we lack sufficient preliminary evidence.

2.3. General case: main theorems. In the case where \mathfrak{F} is birational onto its image, one can state a bound for the minimum degree $\delta(\mathfrak{F})$ of a representative:

Theorem 2.5. *Let $X \subset \mathbb{P}_k^n$ denote a closed reduced and irreducible nondegenerate subvariety and let $\mathfrak{F} : \dashrightarrow \mathbb{P}_k^m$ denote a birational map with nondegenerate image Y . Let there be given an arbitrary representative of \mathfrak{F} , δ its degree, and I the base ideal generated by the forms of this representative. Then*

$$\delta(\mathfrak{F}) \leq m \sup_{r \geq 1} \{\text{reg}(I^r) - r\delta + 1\}. \quad (4)$$

If moreover the graph of \mathfrak{F} is Cohen-Macaulay then

$$\delta(\mathfrak{F}) \leq m \sup_{r \geq 1} \{\text{reg}(I^r) - r\delta + 1\} \leq m d \quad (5)$$

and, similarly,

$$\delta(\mathfrak{F}^{-1}) \leq n d, \quad (6)$$

where $d = \dim(X) + 1$.

Proof. Let A and B stand, respectively, for the homogeneous coordinate rings of X and Y . Let $I' \subset B$ denote the ideal generated by the forms of a representative of the inverse map to \mathfrak{F} . Consider a presentation $\mathcal{R}_B(I') \simeq B[x_0, \dots, x_n]/\mathcal{J}_{I'}$, where B is concentrated in degree zero. According to Theorem 1.1, the map \mathfrak{F} , considered as the inverse map to its own inverse, has a representative whose coordinates are suitable $m \times m$ minors of the Jacobian matrix $\text{Jac}_{\mathbf{y}}((\mathcal{J}_{I'})_{(*,1)})$ and, clearly, the degree of these coordinates is an upper bound for the minimum degree $\delta(\mathfrak{F})$ of a representative. Therefore, to bound $\delta(\mathfrak{F})$ one is led to bounding from above the \mathbf{x} -degree of the minimal generators of $\mathcal{J}_{I'}$, i.e. the relation type of the ideal $I' \subset B$.

Now, quite generally, the relation type of the ideal $I' \subset B$ is bounded by $\text{reg}(\mathcal{R}_B(I')) + 1$, where the regularity is computed with respect to the the irrelevant ideal $\mathcal{R}_B(I')_+$ ([BS, 15.3.1]).

We claim that $\text{reg}_{\mathbf{x}}(\mathcal{R}_A(I)) = \text{reg}(\mathcal{R}_B(I'))$.

For this, we note that birationality implies an isomorphism $\mathcal{R}_B(I') \simeq \mathcal{R}_A(I)$ of bigraded k -algebras ([S, Proposition 2.1]) naturally based on the standard bigraded structure of the polynomial ring $R[\mathbf{y}] = k[\mathbf{x}, \mathbf{y}]$. If one changes this grading by setting $\deg(\mathbf{y}) = 0$ and $\deg(\mathbf{x}) = 1$, it obtains an isomorphism of standard graded k -algebras. It follows that $H_{(\mathbf{x})}^i(\mathcal{R}_B(I')) \simeq H_{(\mathbf{x})}^i(\mathcal{R}_A(I))$ for every i . Since $\text{reg}(\mathcal{R}_B(I')) + 1$ is computed with respect to the the irrelevant ideal $\mathcal{R}_B(I')_+$, it is defined in terms of local cohomology based on the ideal $B[\mathbf{x}]_+ = (\mathbf{x})$. Then the above isomorphisms of local cohomology modules imply that $\text{reg}_{\mathbf{x}}(\mathcal{R}_A(I)) = \text{reg}(\mathcal{R}_B(I'))$.

As a consequence, $\delta(\mathfrak{F}) \leq m(\text{reg}_{\mathbf{x}}(\mathcal{R}_A(I)) + 1)$. Applying Chardin-Romer 's equality mentioned earlier, we obtain the first inequality.

If, in addition, the graph of \mathfrak{F} is Cohen-Macaulay, then the isomorphism $\mathcal{R}_B(I') \simeq \mathcal{R}_A(I)$ of bigraded structures implies that $\mathcal{R}_B(I')$ is Cohen-Macaulay. By [AHT, Proposition 6.2], one has $\text{reg}(\mathcal{R}_B(I')) \leq \ell(I') - 1$. Since $\ell(I') = \dim(X) + 1$ we are through for the second bound.

The third bound is a consequence of the symmetry of the argument and the fact that both \mathfrak{F} and its inverse share the same graph. \square

We note that the bounds just obtained are expressed solely in terms of the ambient dimension, whereas the Cohen-Macaulayness of the graph is a property of the map.

For the subsequent result we will use the following lemma – essentially a collection of known results in a form that suits our purpose in this part.

Lemma 2.6. *Let (A, \mathfrak{m}) be a Cohen–Macaulay local ring of dimension d and let I be an ideal of A with analytic spread ℓ and reduction number $r(I)$. If $\mathcal{R}_A(I)$ is Cohen–Macaulay, one has*

$$\operatorname{reg}(\mathcal{R}_A(I)) \leq \begin{cases} r(I) & \text{if } \ell = \operatorname{ht}(I) \\ \max\{r(I), \ell - 2\} & \text{otherwise.} \end{cases}$$

Proof. Set $\mathcal{R} := \mathcal{R}_A(I)$ and $\mathcal{G} := \mathcal{G}_I(A)$. Since both A and \mathcal{R} are Cohen–Macaulay then \mathcal{G} is Cohen–Macaulay and moreover $a_d^*(\mathcal{G}) \leq -1$ [TI, Theorem 1.1]. According to [HHK, Corollary 4.2], one has

$$a_d^*(\mathcal{G}) = \max\{r(I) - \ell(I), a(\mathcal{G}_{\mathfrak{p}}) \mid \mathfrak{p} \in V(I) \setminus \mathfrak{m}\}$$

where $a(\mathcal{G}_{\mathfrak{p}})$ is the a -invariant of $\mathcal{G}_{\mathfrak{p}}$ over $A_{\mathfrak{p}}$. A recursion then yields

$$a_d^*(\mathcal{G}) = \max\{r(I_{\mathfrak{p}}) - \ell(I_{\mathfrak{p}}) \mid \mathfrak{p} \in V(I)\}. \quad (7)$$

On the other hand, by [AHT, Corollary 4.11], $\operatorname{reg}(\mathcal{G}) \leq \max\{r_{\ell-1}(I), r(I)\}$, where

$$r_{\ell-1}(I) = \begin{cases} -1, & \text{if } \ell = \operatorname{ht}(I) \\ \max\{r(I_{\mathfrak{p}}) - \ell(I_{\mathfrak{p}}) : \operatorname{ht}(\mathfrak{p}) \leq \ell - 1 \text{ and } \ell(I_{\mathfrak{p}}) = \operatorname{ht}(\mathfrak{p})\} + \ell - 1, & \text{otherwise.} \end{cases}$$

Since $a_d^*(\mathcal{G}) \leq -1$, (7) implies that $r(I_{\mathfrak{p}}) - \ell(I_{\mathfrak{p}}) \leq -1$ for all $\mathfrak{p} \in V(I)$. Therefore $r_{\ell-1}(I) \leq \ell - 2$ if $\ell \neq \operatorname{ht}(I)$. The assertion now follows because $\operatorname{reg}(\mathcal{R}) = \operatorname{reg}(\mathcal{G})$ by [T, Corollary 3.3]. \square

Theorem 2.7. (*k infinite*) *Let $\mathfrak{F} : \mathbb{P}_k^n \dashrightarrow \mathbb{P}_k^n$ be a Cremona map having Cohen–Macaulay graph. Then the respective base ideals of \mathfrak{F} and its inverse are generated in degree $\leq n^2$.*

Furthermore, if $I \subset A = k[x_0, \dots, x_n]$ stands for the base ideal of \mathfrak{F} , then the saturation of I is the ideal $(I :_A A_+^{n-2})$. In particular, the base ideal of a plane Cremona map with Cohen–Macaulay graph is saturated.

Proof. Set $I \subset A$ for the base ideal of the map. Since the generators of I are analytically independent and k is infinite, the reduction number $r(I) = 0$. From Lemma 2.6 one has $\operatorname{reg}(\mathcal{R}_A(I)) \leq n - 1$, hence the relation type of I is $\leq n$. Therefore the same reasoning as in the proof of Theorem 2.5 shows that $\delta(\mathfrak{F})$ and $\delta(\mathfrak{F}^{-1})$ are at most n^2 .

As earlier, twisting around by considering the graph of the inverse map, we derive the inequality $\operatorname{reg}_{\mathbf{x}}(\mathcal{R}_A(I)) \leq n - 1$. It then follows from Chardin–Romer’s equality that $\operatorname{reg}(I) - \delta \leq n - 1$ where δ is the degree of the generators of I . Therefore $\operatorname{reg}(A/I) \leq \delta + n - 2$. In particular $\operatorname{end}(H_{(\mathbf{x})}^0(A/I)) = \operatorname{end}(I^{\operatorname{sat}}/I) \leq \delta + n - 2$. On the other hand, by [PR, Proposition 1.2 (i)] (also [D, Proposition 1.3.6]) one always has $\operatorname{indeg}(I^{\operatorname{sat}}/I) \geq \delta + 1$ whenever I is the base ideal of a Cremona map. Therefore $(A_+)^{n-2} \frac{I^{\operatorname{sat}}}{I} = 0$ which proves the assertion. \square

2.4. Confronting a naïve upper bound. In this short piece we deviate somewhat from our main objectives by we giving an upper bound for the degree of a linear system defining a rational map, under the assumption that it generates an ideal of grade at least 2 – an assumption meaning that the map has essentially a unique base ideal.

Proposition 2.8. *Let $\mathfrak{F} : X \subset \mathbb{P}_k^n \dashrightarrow \mathbb{P}_k^m$ denote a rational map. Suppose that \mathfrak{F} admits a base ideal $I \subset A$ of grade at least 2. If δ denote the degree of the forms generating I , one has*

$$\delta \leq \frac{m}{m+1} e,$$

where e is the largest twist in the graded presentation of I over A .

The proposition is an immediate consequence of the following general lemma which seems to be folklore.

Lemma 2.9. *Let R be a commutative Noetherian ring and $I \subset R$ be an ideal of grade at least two. Fixing a set $\{f_0, \dots, f_m\}$ of generators of I , let M denote a corresponding presentation matrix. Then there exists an $(m+1) \times m$ submatrix of M of rank m , and a non-zero element $h \in R$ such that $hf_i = \Delta_i$, where Δ_i denotes the i -th minor of this submatrix.*

In particular, if R is standard graded and if I is homogenous and generated in a single degree δ and related in degree at most $b_1(I)$ then $b_1(I) \geq \frac{m+1}{m} \delta$.

Proof. Consider the free presentation $R^p \xrightarrow{M} R^{m+1} \xrightarrow{\mathbf{f}} R \rightarrow R/I \rightarrow 0$, where \mathbf{f} denotes the vector $(f_0 \cdots f_m)$. Since $\text{grade}(I) \geq 2$, dualizing into R yields the exact sequence $0 \rightarrow R^* \xrightarrow{\mathbf{f}^t} R^{m+1*} \xrightarrow{M^t} R^{p*}$ (as is well-known, the natural inclusion $R \hookrightarrow \text{Hom}(I, R)$ is an equality if and only if I has grade at least 2) and moreover $\text{Im}(M)$ has rank m . Let N be an $m \times (m+1)$ submatrix of M^t of rank m and denote the (ordered) $m \times m$ minors of N by $\Delta = \Delta_0, \dots, \Delta_m$. For an arbitrary row of M^t , say, $(r_{0i} \cdots r_{mi})$, $\sum_{j=0}^m r_{ji} \Delta_j$ is the determinant of an $(m+1) \times (m+1)$ matrix which is either a matrix having two equal rows or an $(m+1) \times (m+1)$ submatrix of M^t . In any case the corresponding determinant vanishes. Therefore in the following diagram the compositions of any two compatible maps are zero

$$\begin{array}{ccccc} R^* & \xrightarrow{\mathbf{f}^t} & R^{m+1*} & \xrightarrow{M^t} & R^{p*} \\ & & \uparrow (\Delta) & & \\ & & R & & \end{array} .$$

Since $\ker(M^t) = \text{Im } \mathbf{f}^t$, one has $\Delta = (\Delta_0 \cdots \Delta_m) \in \text{Im}(\mathbf{f}^t)$; which proves the first assertion. To see the degree assertion in the equihomogeneous case, note that in the graded case h is homogeneous, hence the equality $hf_i = \Delta_i$ implies that $\delta \leq \deg(\Delta_i) \leq m(b_1(I) - \delta)$. \square

Since the bound in Proposition 2.8 is quite natural, it makes sense confronting it with the birational bound obtained in Theorem 2.5. For this we assume that the map \mathfrak{F} is birational, as in Theorem 2.5, and $I \subset A$ is a base ideal of grade ≥ 2 , as in Proposition 2.8. In this situation, the representative of \mathfrak{F} is uniquely defined up to scalars (see [S, Proposition 1.1]), hence $\delta(\mathfrak{F}) = \delta$. Let us suppose in addition that the function \mathfrak{f} attains its maximum at $r = 1$ – a hypothesis we will digress on in a minute.

Then the second bound gives $\delta \leq m(\text{reg}(I) - \delta + 1)$, i.e., $\delta \leq \frac{m}{m+1}(\text{reg}(I) + 1)$. Since $\text{reg}(I) \geq e - 1$, we see that the first bound is better.

On the other hand, the second bound is more comprehensive and besides it has the advantage of providing a lower bound for \mathfrak{f} in terms of an invariant of the map \mathfrak{F} and not of a particular base ideal I .

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