THE DECOMPOSITION GROUP OF A LINE IN THE PLANE

ISAC HEDÉN AND SUSANNA ZIMMERMANN

Abstract. We show that the decomposition group of a line L in the plane, i.e. the subgroup of plane birational transformations that send L to itself birationally, is generated by its elements of degree 1 and one element of degree 2, and that it does not decompose as a non-trivial amalgamated product.

1. Introduction

We denote by $\mathrm{Bir}(\mathbb{P}^2)$ the group of birational transformations of the projective plane $\mathbb{P}^2 = \mathrm{Proj}(k[x,y,z])$, where k is an algebraically closed field. Let $C \subset \mathbb{P}^2$ be a curve, and let

$$\mathrm{Dec}(C) = \{\varphi \in \mathrm{Bir}(\mathbb{P}^2),\ \varphi(C) \subset C \text{ and } \varphi|_C : C \dashrightarrow C \text{ is birational}\}.$$

This group has been studied for curves of genus ≥ 1 in [BPV2009], where it is linked to the classification of finite subgroups of $\mathrm{Bir}(\mathbb{P}^2)$.

For the decomposition group of a line, they give the following result: for any line $L \subset \mathbb{P}^2$, the action of $\mathrm{Dec}(L)$ on L induces a split exact sequence

$$0 \longrightarrow \mathrm{Ine}(L) \longrightarrow \mathrm{Dec}(L) \longrightarrow \mathrm{PGL}_2 = \mathrm{Aut}(L) \longrightarrow 0$$

and $\mathrm{Ine}(L)$ is neither finite nor abelian and also it doesn't leave any pencil of rational curves invariant [BPV2009, Proposition 4.1]. Further they ask the question whether $\mathrm{Dec}(L)$ is generated by its elements of degree 1 and 2 [BPV2009, Question 4.1.2].

We give a affirmative answer to their question in the form of the following result, similar to the Noether-Castelnuovo theorem [Cas1901] which states that $\mathrm{Bir}(\mathbb{P}^2)$ is generated by $\sigma\colon [x:y:z] \mapstochar\dashrightarrow [yz:xz:xy]$ and $\mathrm{Aut}(\mathbb{P}^2) = \mathrm{PGL}_3$.

Theorem 1. *For any line $L \subset \mathbb{P}^2$, the group $\mathrm{Dec}(L)$ is generated by $\mathrm{Dec}(L) \cap \mathrm{PGL}_3$ and any of its quadratic elements having three proper base-points in $\mathbb{P}^2$.*

The similarities between $\mathrm{Dec}(L)$ and $\mathrm{Bir}(\mathbb{P}^2)$ go further than this. Cornulier shows in [Cor2013] that $\mathrm{Bir}(\mathbb{P}^2)$ cannot be written as an amalgamated product in any nontrivial way, and we modify his proof to obtain an analogous result for $\mathrm{Dec}(L)$.

Theorem 2. *The decomposition group $\mathrm{Dec}(L)$ of a line $L \subset \mathbb{P}^2$ does not decompose as a non-trivial amalgam.*

The article is organised as follows: in Section 2 we show that for any element of $\mathrm{Dec}(L)$ we can find a decomposition in $\mathrm{Bir}(\mathbb{P}^2)$ into quadratic maps such that the successive images of L are curves (Proposition 2.4), i.e. the line is not contracted to a point at any time. We then show in Section 3 that we can modify this decomposition, still in $\mathrm{Bir}(\mathbb{P}^2)$, into de Jonquières maps where all of the sucessive images of L have degree 1, i.e. they are lines. Finally we prove Theorem 1. Our main sources of inspiration for technique and ideas here have been [AC2002, §8.4, §8.5] and [Bla2012]. In Section 4 we

Date: January 13, 2016.

2010 *Mathematics Subject Classification.* 14E07.

The first named author is an International Research Fellow of the Japanese Society for the Promotion of Sciences (JSPS). Both authors acknowledge support by the Swiss National Science Foundation Grant "Birational Geometry" PP00P2_153026.

prove Theorem 2 using ideas that are strongly inspired by [Cor2013].

Acknowledgement: The authors would like to thank Jérémy Blanc for helpful discussions, and Yves de Cornulier for kindly answering their questions.

2. Avoiding to contract L

Given a birational map $\rho\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, the Noether-Castelnuovo theorem states that there is a decomposition $\rho = \rho_m\rho_{m-1}\dots\rho_1$ of ρ where each ρ_i is a quadratic map with three proper base points. This decomposition is far from unique, and the aim of this section is to show that if $\rho \in \mathrm{Dec}(L)$, we can choose the ρ_i so that none of the successive birational maps $(\rho_i \dots \rho_1\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2)_{i=1}^m$ contracts L to a point. This is Proposition 2.4.

Given a birational map $\varphi : X \dashrightarrow Y$ between smooth projective surfaces, and a curve $C \subset X$ which is contracted by φ, we denote by $\pi_1\colon Z_1 \to Y$ the blowup of the point $\varphi(C) \in Y$. If C is contracted also by the birational map $\pi_1^{-1}\varphi\colon X \dashrightarrow Z_1$, we denote by $\pi_2\colon Z_2 \to Z_1$ the blowup of $(\pi_1^{-1}\varphi)(C) \in Z_1$ and consider the birational map $(\pi_1\pi_2)^{-1}\varphi\colon X \dashrightarrow Z_2$. If this map too contracts C, we denote by $\pi_3\colon Z_3 \to Z_2$ the blowup of the point onto which C is contracted. Repeating this procedure a finite number of times $D \in \mathbb{N}$, we finally arrive at a variety $Z := Z_D$ and a birational morphism $\pi := \pi_1\pi_2\cdots\pi_D\colon Z \to Y$ such that $(\pi^{-1}\varphi)$ does not contract C. Then $(\pi^{-1}\varphi)|_C\colon C \dashrightarrow (\pi^{-1}\varphi)(C)$ is a birational map.

Definition 2.1. In the above situation, we denote by $D(C,\varphi) \in \mathbb{N}$ the minimal number of blowups which are needed in order to not contract the curve C and we say that C is contracted $D(C,\varphi)$ times by φ. In particular, a curve C is sent to a curve by φ if and only if $D(C,\varphi) = 0$.

Let $f\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a birational map, and let p belong to $\mathbb{P}^2$ as a proper or infinitely near point. If p is not a base point of f, we define $f_\bullet(p)$ via a minimal resolution

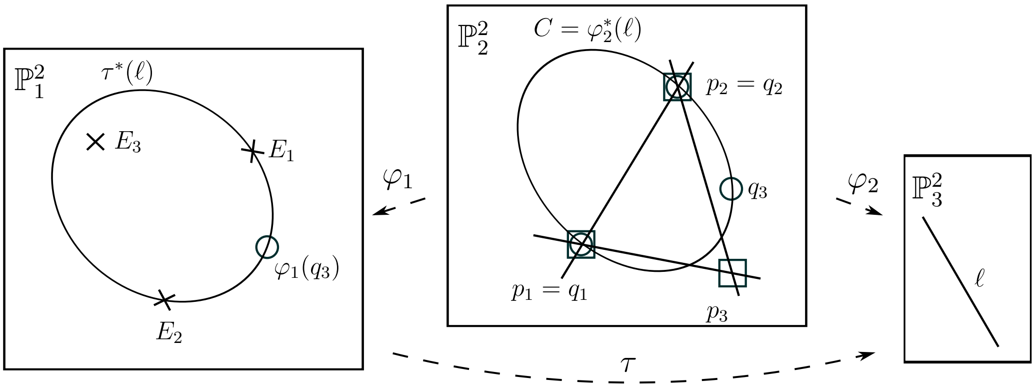
$$\begin{array}{ccc} & S & \\ {}^{\nu_1}\swarrow & & \searrow^{\nu_2} \\ \mathbb{P}^2 & \overset{f}{\dashrightarrow} & \mathbb{P}^2 \end{array}$$

where ν_1, ν_2 are sequences of blow-ups, as follows. Not being a base point of f, p corresponds via ν_1 to a point of S or infinitely near. Applying ν_2 to this point, we obtain $f_\bullet(p)$; this point belongs to $\mathbb{P}^2$ as a proper or infinitely near point.

We recall the following well known result, which will be used explicitly or implicitly a number of times in the sequel.

Lemma 2.2. *Let $\varphi_1, \varphi_2 \in \mathrm{Bir}(\mathbb{P}^2)$ be birational maps of degree 2 with proper base points p_1, p_2, p_3 and q_1, q_2, q_3 respectively. If φ_1 and φ_2 have (exactly) two common base points, say $p_1 = q_1$ and $p_2 = q_2$, then the composition $\tau = \varphi_2\varphi_1^{-1}$ is quadratic. Furthermore the three base points of τ are proper points of $\mathbb{P}^2$ if and only if q_3 is not on any of the lines joining two of the p_i.*

Proof. The lemma is proved by the below figure, where squares and circles in $\mathbb{P}^2_2$ denote the base points of φ_1 and φ_2 respectively. The crosses in $\mathbb{P}^2_1$ denote the base points of φ_1^{-1} (corresponding to the lines in $\mathbb{P}^2_2$), and the conics in $\mathbb{P}^2_1$ and $\mathbb{P}^2_2$ denote the pullback of a general line $\ell \in \mathbb{P}^2_3$.



It follows that the base points of τ are $E_1, E_2, \varphi_1(q_3)$ if q_3 is not on any of the three lines. If q_3 is on one of the three lines, $(\varphi_1)_\bullet(q_3)$ will be a point infinitely near to the corresponding $E_i \in \mathbb{P}^2_1$. □

The following lemma describes how the number of times that a line is contracted changes when composing with a quadratic transformation of $\mathbb{P}^2$ with three proper base points.

Lemma 2.3. *Let $\rho\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a birational map and let $\varphi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a quadratic birational map with base points $q_1, q_2, q_3 \in \mathbb{P}^2$. For $1 \leq i < j \leq 3$ we denote by $\ell_{ij} \subset \mathbb{P}^2$ the line which joins the base points q_i and q_j. If $D(L,\rho) = k \geq 1$, we have*

$$D(L,\varphi\rho) = \begin{cases} k+1 & \text{if } \rho(L) \in (\ell_{12} \cup \ell_{13} \cup \ell_{23}) \setminus \mathrm{Bp}(\varphi), \\ k & \text{if } \rho(L) \notin \ell_{12} \cup \ell_{13} \cup \ell_{23}, \\ k & \text{if } \rho(L) = q_i \text{ for some } i, \text{ and } \varphi_\bullet(\rho(L)) \in \mathrm{Bp}(\varphi^{-1}), \\ k-1 & \text{if } \rho(L) = q_i \text{ for some } i, \text{ and } \varphi_\bullet(\rho(L)) \notin \mathrm{Bp}(\varphi^{-1}). \end{cases}$$

Proof. We consider the minimal resolutions of φ; in figures 1-4, the filled black dots denote the successive images of L, i.e. $\rho(L)$, $(\pi^{-1}\rho)(L)$ and $(\eta\pi^{-1}\rho)(L)$ respectively.

We argue by figure 1 and 2 in the case where $\rho(L)$ does not coincide with any of the base points of φ. If $\rho(L) \in \ell_{ij}$ for some i,j, then $D(L,\varphi\rho) = D(L,\rho)+1$, since ℓ_{ij} is contracted by φ. Otherwise, the number of times L is contracted does not change. Suppose that $\rho(L) = q_i$ for some i. If $D(L,\rho) = 1$, we have $(\pi^{-1}\rho)(L) = E_i$, and then

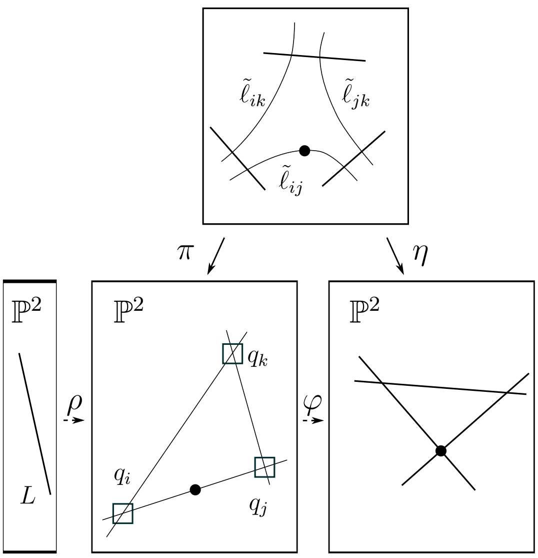


Figure 1: $D(L,\varphi\rho) = k+1$;
$\rho(L) \in (\ell_{12} \cup \ell_{13} \cup \ell_{23}) \setminus \mathrm{Bp}(\varphi)$.

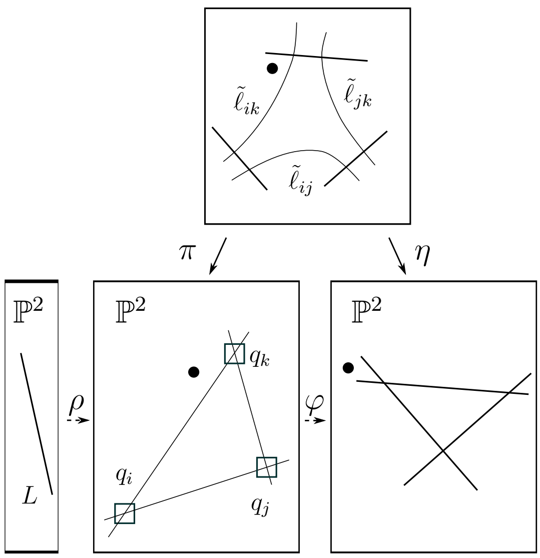


Figure 2: $D(L,\varphi\rho) = k$;
$\rho(L) \notin \ell_{12} \cup \ell_{13} \cup \ell_{23}$.

clearly $D(L, \varphi\rho) = 0$ since E_i is not contracted by η. If $D(L, \rho) \geq 2$ we argue by the figures 3 and 4.

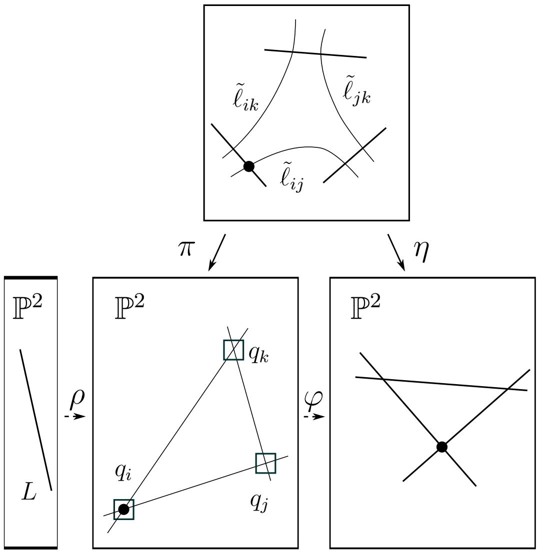


Figure 3: $D(L, \varphi\rho) = k$;
$\varphi_\bullet(\rho(L)) \in \mathrm{Bp}(\varphi^{-1})$.

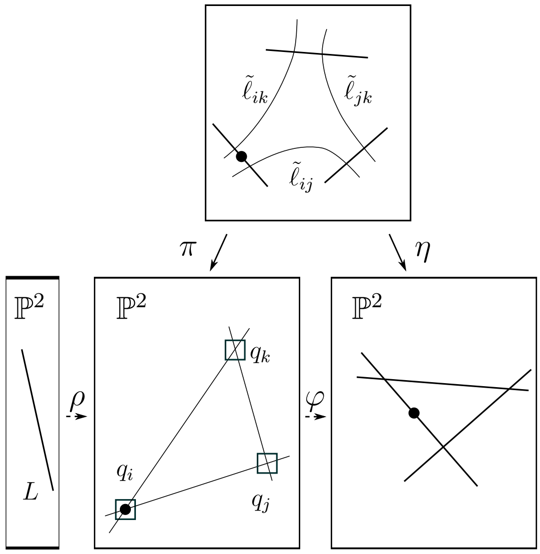


Figure 4: $D(L, \varphi\rho) = k - 1$;
$\varphi_\bullet(\rho(L)) \notin \mathrm{Bp}(\varphi^{-1})$.

□

Proposition 2.4. *For any given element $\rho \in \mathrm{Dec}(L)$, there is a decomposition of ρ into quadratic maps $\rho = \rho_m \ldots \rho_1$ with three proper base points such that none of the successive compositions $(\rho_i \ldots \rho_1)_{i=1}^m$ contract L to a point.*

Proof. Let $\rho = \rho_m \ldots \rho_1$ be a decomposition of ρ into quadratic maps with only proper base points. We can assume that $d := \max\{D(L, \rho_j \ldots \rho_1) \mid 1 \leq j \leq m\} > 0$, otherwise we are done. Let $n := \max\{j \mid D(L, \rho_j \ldots \rho_1) = d\}$. We denote the base points of ρ_n^{-1} and ρ_{n+1} by p_1, p_2, p_3 and q_1, q_2, q_3 respectively.

We first look at the case where $D(L, \rho_{n-1} \ldots \rho_1) = D(L, \rho_{n+1} \ldots \rho_1) = d - 1$. Here both ρ_n^{-1} and ρ_{n+1} have a base point at $(\rho_n \ldots \rho_1)(L) \in \mathbb{P}^2$, and we may assume that this point is $p_1 = q_1$ – see Figure 5. Interchanging the roles of q_2 and q_3 if necessary, we may assume that p_1, p_2, q_2 are not collinear. Let $r \in \mathbb{P}^2$ be a general point, and let c_1 and c_2 denote quadratic maps with base points $[p_1, p_2, r]$ and $[p_1, q_2, r]$ respectively; then the maps τ_1, τ_2, τ_3 (defined by the commutative diagram in Figure 5) are quadratic with three proper base-points in $\mathbb{P}^2$. Note that $D(L, \tau_i \ldots \tau_1 \rho_{n-1} \ldots \rho_1) = d-1$ for $i = 1, 2, 3$. Thus we obtained a new decomposition of ρ into quadratic maps with three proper base points

$$\rho = \rho_m \ldots \rho_{n+2} \tau_3 \tau_2 \tau_1 \rho_{n-1} \ldots \rho_1,$$

where the number of instances where L is contracted d times has decreased by 1.

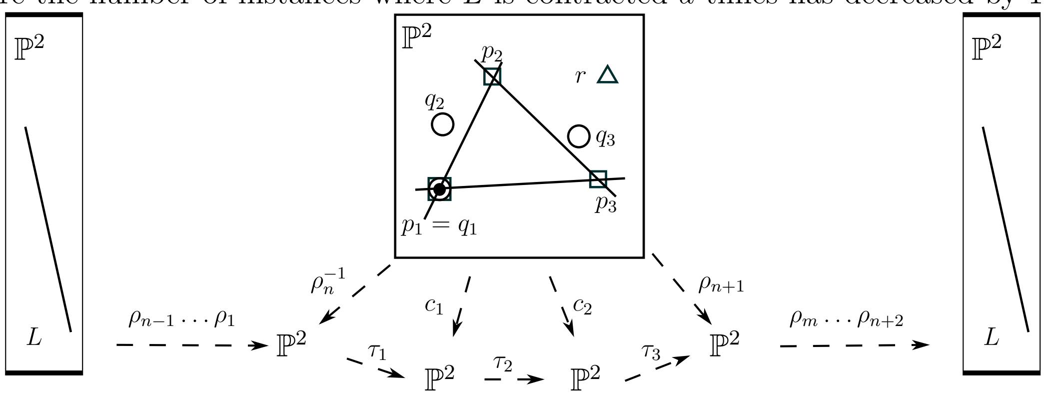


Figure 5

Now assume instead that $D(L, \rho_{n-1} \dots \rho_1) = d$ and $D(L, \rho_{n+1} \dots \rho_1) = d - 1$. Then $(\rho_n \dots \rho_1)(L)$ is a base point of ρ_{n+1}, which we may assume to be q_1. Furthermore $(\rho_n \dots \rho_1)(L)$ either does not lie on a line joining two base points of ρ_n^{-1}, or $D(L, \rho_n \dots \rho_1) \geq 2$ and $(\rho_n \dots \rho_1)(L)$ is a base point of ρ_n^{-1} (which we may assume to be p_1, and equal to q_1), at the same time as $(\rho_{n-1} \dots \rho_1)(L)$ is a base point of ρ_n.

We consider the first case. The lines through q_1 and the p_i, $i = 1, 2, 3$ define three tangent directions at q_1. At least two of them have to be different from the tangent direction given by $(\rho_n \dots \rho_1)(L)$, and we may assume that these are given by p_2, p_3. Then with a quadratic map $c_1 := [q_1, p_2, p_3]$ with base points q_1, p_2, p_3, we have $D(L, c_1\rho_n \dots \rho_1) = D(L, \rho_n \dots \rho_1) - 1$. Let $r, s \in \mathbb{P}^2$ be two general points and define c_2, c_3, c_4 with three proper base points respectively as $[q_1, r, p_3]$, $[q_1, r, s]$, $[q_1, q_2, s]$. Note that the corresponding maps $\tau_1, \dots, \tau_5$, defined in an analogous way as in Figure 5, are quadratic with three proper base points. Note also that $D(L, c_i\rho_n \dots \rho_1) = D(L, \rho_n \dots \rho_1) - 1$ for $i = 2, 3, 4$. Only for $i = 4$ this is not immediately clear, so suppose that this is not the case, i.e. $D(L, c_4\rho_n \dots \rho_1) = D(L, \rho_n \dots \rho_1)$. It follows that the tangent direction corresponding to $(\rho_n \dots \rho_1)(L)$ is given by the line through q_1 and q_2, but this is not possible by the assumption that $D(L, \rho_{n+1} \dots \rho_1) = d - 1$.

In the second case we have $p_1 = q_1$ and the tangent direction at $p_1 = q_1$ corresponding to $(\rho_n \dots \rho_1)(L)$ is the direction either of the line through p_1 and p_2 or the line through p_1 and p_3 (see Figure 3). By interchanging the roles of p_2 and p_3 if necessary, we may assume that it corresponds to the direction of the line through p_1 and p_3. Interchanging the roles of q_2 and q_3 if necessary, we may assume that p_1, q_2, p_3 are not collinear. Let $r, s \in \mathbb{P}^2$ be general points and define quadratic maps c_1, c_2, c_3 with three proper base points respectively by $[p_1, p_2, s]$, $[p_1, r, s]$, $[p_1, r, q_2]$. Then the corresponding maps $\tau_1, \tau_2, \tau_3, \tau_4$ are quadratic with three proper base points and $D(L, c_i\rho_n \dots \rho_1) = D(L, \rho_n \dots \rho_1) - 1$ for $i = 1, 2, 3$. The latter holds for c_1 since the direction given by p_1 and p_2 is different from the tangent direction corresponding to $(\rho_n \dots \rho_1)(L)$, and for c_3 it follows from the assumption that the image of L is contracted $d - 1$ times by $(\rho_{n+1} \dots \rho_1)$ and that p_1, q_2, p_3 are not collinear.

Both in the first and second case, we again arrive at a new decomposition into quadratic maps with three proper base points

$$\rho = \rho_m \dots \rho_{n+2}\tau_j \dots \tau_1\rho_{n-1} \dots \rho_1 \qquad (j \in \{4, 5\}),$$

where the number of instances where L is contracted d times has decreased by 1, and we conclude by induction. □

3. Avoiding to send L to a curve of degree higher than 1.

By proposition 2.4, any element $\rho \in \mathrm{Dec}(L)$ can be decomposed as

$$\rho = \rho_m \dots \rho_1$$

where each ρ_j is quadratic with three proper base points, and all of the successive images $((\rho_i \dots \rho_1)(L))_{i=1}^m$ of L are curves. The aim of this section is to show that the ρ_j even can be chosen so that all of these curves have degree 1. That is, we find a decomposition of ρ into quadratic maps such that all the successive images of L are lines. This means in particular that $\mathrm{Dec}(L)$ is generated by its elements of degree 1 and 2.

Definition 3.1. A birational transformation of $\mathbb{P}^2$ is called de Jonquières if it preserves the pencil of lines passing through $[1 : 0 : 0] \in \mathbb{P}^2$. These transformations form a subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ which we denote by $\mathcal{J}$.

Remark 3.2. In [AC2002], a de Jonquières map is defined by the slightly less restrictive property that it sends a pencil of lines to a pencil of lines. Given a map with this property, we can always obtain an element in $\mathcal{J}$ by composing from left and right with elements of PGL_3.

For a curve $C \subset \mathbb{P}^2$ and a point p in $\mathbb{P}^2$ or infinitely near, we denote by $m_C(p)$ the multiplicity of C in p. If it is clear from context which curve we are referring to, we will use the notation $m(p)$.

Lemma 3.3. *Let $\varphi \in \mathcal{J}$ be of degree $e \geq 2$, and $C \subset \mathbb{P}^2$ a curve of degree d. Suppose that*

$$\deg(\varphi(C)) \leq d.$$

Then there exist two base-points q_1, q_2 of φ different from $[1:0:0]$ such that

$$m_C([1:0:0]) + m_C(q_1) + m_C(q_2) \geq d.$$

This inequality can be made strict in case $\deg(\varphi(C)) < d$, with a completely analogous proof.

Proof. Since $\varphi \in \mathcal{J}$ is of degree e, it has exactly $2e-1$ base-points $r_0 := [1:0:0], r_1, \dots, r_{2e-2}$ of multiplicity $e-1, 1, \dots, 1$ respectively. Then

$$\begin{aligned} d \geq \deg(\varphi(C)) =& ed - (e-1)m_C(r_0) - \sum_{i=1}^{e-1}(m_C(r_{2i-1}) + m_C(r_{2i})) \\ =& d + \sum_{i=1}^{e-1}(d - m_C(r_0) - m_C(r_{2i-1}) - m_C(r_{2i})) \end{aligned}$$

Hence there exist i_0 such that $d \leq m_C(r_0) + m_C(r_{2i_0-1}) + m_C(r_{2i_0})$. □

Remark 3.4. Note also that we can choose the points q_1, q_2 such that q_1 either is a proper point in $\mathbb{P}^2$ or in the first neighbourhood of $[1:0:0]$, and that q_2 either is proper point of $\mathbb{P}^2$ or is in the first neighbourhood of $[1:0:0]$ or q_1.

Remark 3.5. A quadratic map sends a pencil of lines through one of its base points to a pencil of lines, and we conclude from Proposition 2.4 and Remark 3.2 that there exists maps $\alpha_1, \dots, \alpha_{m+1} \in \mathrm{PGL}_3$ and $\rho_i \in \mathcal{J}$ such that

$$\rho = \alpha_{m+1}\rho_m\alpha_m\rho_{m-1}\alpha_{m-1}\dots\alpha_2\rho_1\alpha_1$$

and such that all of the successive images of L with respect to this decomposition are curves.

Proposition 3.6. *Let $\rho \in \mathrm{Dec}(L)$. Then there exists $\rho_i \in \mathcal{J}$ and $\alpha_i \in \mathrm{PGL}_3$ such that $\rho = \alpha_{m+1}\rho_m\alpha_m\rho_{m-1}\alpha_{m-1}\dots\alpha_2\rho_1\alpha_1$ and all of the successive images of L are lines.*

Proof. Start with a decomposition $\rho = \alpha_{m+1}\rho_m\alpha_m\rho_{m-1}\alpha_{m-1}\dots\alpha_2\rho_1\alpha_1$ as in Remark 3.5.

Denote $C_i := (\rho_i\alpha_i\cdots\rho_1\alpha_1)(L) \subset \mathbb{P}^2$, $d_i := \deg(C_i)$ and let

$$D := \max\{d_i \mid i = 1, \dots, m\}, \quad n := \max\{i \mid D = d_i\}, \quad k := \sum_{i=1}^{n}(\deg\rho_i - 1).$$

We use induction on the lexicographically ordered pair (D, k).

We may assume that $D > 1$, otherwise our goal is already achieved. We may also assume that $\alpha_{n+1} \notin \mathcal{J}$, otherwise the pair (D, k) decreases as we replace the three maps $\rho_{n+1}, \alpha_{n+1}, \rho_n$ by their composition $\rho_{n+1}\alpha_{n+1}\rho_n \in \mathcal{J}$. Indeed, either D decreases, or D stays the same while k decreases by $\deg\rho_n - 1$. Using Lemma 3.3, we find simple base points p_1, p_2 of ρ_n^{-1} and simple base points $\tilde{q}_1, \tilde{q}_2$ of ρ_{n+1}, all different from $p_0 := [1:0:0]$, such that

$$m_{C_n}(p_0) + m_{C_n}(p_1) + m_{C_n}(p_2) \geq D$$

and

$$m_{\alpha_{n+1}(C_n)}(p_0) + m_{\alpha_{n+1}(C_n)}(\tilde{q}_1) + m_{\alpha_{n+1}(C_n)}(\tilde{q}_2) > D.$$

We choose $p_1, p_2, \tilde{q}_1, \tilde{q}_2$ as in Remark 3.4. Let $q_0 := (\alpha_{n+1}^{-1})_\bullet(p_0)$, $q_1 := (\alpha_{n+1}^{-1})_\bullet(\tilde{q}_1)$, and $q_2 := (\alpha_{n+1}^{-1})_\bullet(\tilde{q}_2)$. Note that p_0 and q_0 are two distinct points of $\mathbb{P}^2$ since $\alpha_{n+1} \notin \mathcal{J}$. We number the points so that $m(p_1) \geq m(p_2)$, $m(\tilde{q}_1) \geq m(\tilde{q}_2)$ and so that if p_i (resp. $\tilde{q}_i$) is infinitely near p_j (resp. $\tilde{q}_j$), then $j < i$.

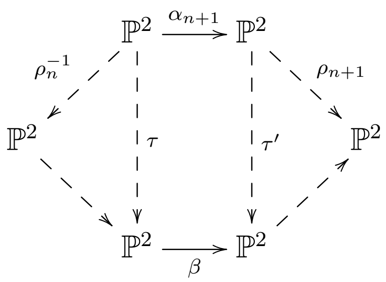
We study two cases separately which depending on the multiplicities of the base-points.

Case (a): $m(q_0) \geq m(q_1)$ and $m(p_0) \geq m(p_1)$. Then we find two quadratic maps $\tau', \tau \in \mathcal{J}$ and $\beta \in \mathrm{PGL}_3$ so that $\rho_{n+1}\alpha_{n+1}\rho_n = (\rho_{n+1}\tau^{-1})\beta(\tau\rho_n)$ and so that the pair (D, k) is reduced as we replace the sequence $(\rho_{n+1}, \alpha_{n+1}, \rho_n)$ by $(\rho_{n+1}\tau^{-1}, \beta, \tau\rho_n)$. The procedure goes as follows.

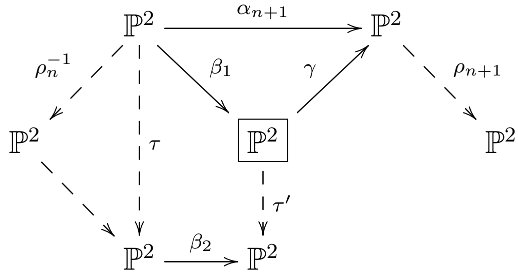
If possible we choose a point $r \in \{p_1, q_1\} \setminus \{p_0, q_0\}$. Should this set be empty, i.e. $p_0 = q_1$ and $p_1 = q_0$, we choose $r = q_2$ instead. The ordering of the points implies that the point r is either a proper point in $\mathbb{P}^2$ or in the first neighbourhood of p_0 or q_0. Furthermore, the assumption implies that $m(p_0) + m(q_0) + m(r) > D$, so r is not on the line passing through p_0 and q_0. In particular, there exists a quadratic map $\tau \in \mathcal{J}$ with base points p_0, q_0, r; then

$$\deg(\tau(C_n)) = 2D - m(p_0) - m(q_0) - m(r) < D.$$

Choose $\beta \in \mathrm{PGL}_3$ so that the quadratic map $\tau' := \beta\tau(\alpha_{n+1})^{-1}$ in the below commutative diagram is de Jonquières – this is possible since τ has q_0 as a base point. This decreases the pair (D, k).



Case (b): $m(p_0) < m(p_1)$. Let τ be a quadratic de Jonquières map with base points p_0, p_1, p_2. This is possible since our assumption implies that p_1 is a proper base point and because p_0, p_1, p_2 are base-points of ρ_n^{-1} of multiplicity $\deg \rho_n - 1, 1, 1$ respectively and hence not collinear. Choose $\beta_1 \in \mathrm{PGL}_3$ which exchanges p_0 and p_1, let $\gamma = \alpha_{n+1}\beta_1^{-1}$ and choose $\beta_2 \in \mathrm{PGL}_3$ so that $\tau' := \beta_2\tau\beta_1^{-1} \in \mathcal{J}$. The latter is possible since $\beta_1^{-1}(p_0) = p_1$ is a base point of τ, and we have the following diagram.



Since $\deg(\tau\rho_n) = \deg \rho_n - 1$, the pair (D, k) stays unchanged as we replace the sequence (α_{n+1}, ρ_n) in the decomposition of ρ by the sequence $(\gamma, (\tau')^{-1}, \beta_2, \tau\rho_n)$. In the new decomposition of ρ the maps $(\tau')^{-1}$ and γ play the roles that ρ_n and α_{n+1}

respectively played in the previous decomposition. In the squared $\mathbb{P}^2$, we have

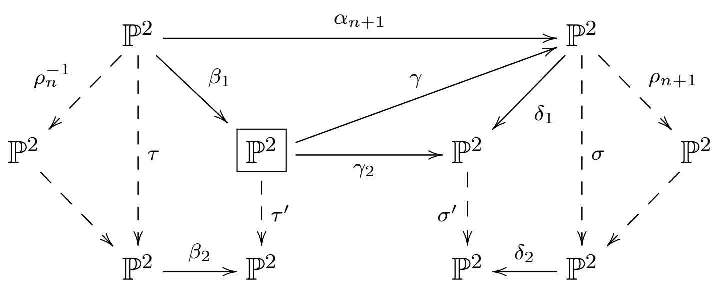
$$m(p_0) = m(\beta_1(p_1)) > m(\beta_1(p_0)) = m(p_1).$$

Define $q_0' := \gamma^{-1}(p_0)$, $q_1' := \gamma^{-1}(\tilde{q}_1)$, $q_2' := \gamma^{-1}(\tilde{q}_2)$, and note that $q_0' = \beta_1(q_0)$, $q_1' = \beta_1(q_1)$ and $q_2' = \beta_1(q_2)$. In the new decomposition these points play the roles that q_0, q_1, q_2 played in the previous decomposition.

If $m(q_0') \geq m(q_1')$, we continue as in case (a) with the points $p_0, p_1, \beta_1(p_2)$ and q_0', q_1', q_2'.

If $m(q_0') < m(q_1')$, we replace the sequence (ρ_{n+1}, γ) by a new sequence such that, similar to case (a), the roles of q_0' and q_1' are exchanged, and we will do this without touching $p_0, p_1, \beta(b_2)$. The replacement will not change (D, k) and we can apply case (a) to the new sequence.

As $m(q_0') < m(q_1')$, the point q_1' is a proper point of $\mathbb{P}^2$. Analogously to the previous case, there exists $\sigma \in \mathcal{J}$ with base-points $\gamma(q_0') = p_0, \gamma(q_1') = \tilde{q}_1, \gamma(q_2') = \tilde{q}_2$, and there exists $\delta_1 \in \mathrm{PGL}_3$ which exchanges p_0 and $\tilde{q}_1$. Since $\delta_1^{-1}(p_0) = \tilde{q}_1$ is a base-point of σ, there furthermore exists $\delta_2 \in \mathrm{PGL}_3$ such that $\sigma' := \delta_2\sigma\delta_1^{-1} \in \mathcal{J}$. Let $\gamma_2 := \delta_1\gamma$.



Replacing the sequence (ρ_{n+1}, γ) with $(\rho_{n+1}\sigma^{-1}, \delta_2^{-1}, \sigma', \delta_1\gamma)$ does not change the pair (D, k). The latest position with the highest degree is still the squared $\mathbb{P}^2$ but in the new sequence we have

$$m(\gamma_2^{-1}(p_0)) = m(\beta_1(q_1)) > m(\beta_1(q_0)) = m(\gamma_2^{-1}(\delta_1(\tilde{q}_1)))$$

Since $p_0, p_1, \beta_1(p_2)$ were undisturbed, the inequality $m(p_0) > m(p_1)$ still holds, and we proceed as in case (a).

In this proof, we have used several different quadratic maps $\tau, \tau', \sigma, \sigma'$. Note that none of these can contract C (or an image of C), since quadratic maps only can contract curves of degree 1. $\square$

Remark 3.7. Suppose that $\rho \in \mathcal{J}$ preserves a line L. Then the Noether-equalities imply that L passes either through $[1:0:0]$ and no other base-points of ρ, or that it passes through exactly $\deg\rho - 1$ simple base-points of ρ and not through $[1:0:0]$.

Lemma 3.8. *Let $\rho \in \mathcal{J}$ be of degree ≥ 2 and let L be a line passing through exactly $\deg\rho - 1$ simple base-points of ρ and not through $[1:0:0]$. Then there exist $\rho_1, \ldots, \rho_i \in \mathcal{J}$ of degree 2 such that $\rho = \rho_m \cdots \rho_1$ and the successive images of L are lines.*

Proof. Note that the curve $\rho(L)$ is a line. Call $p_0 := [1:0:0], p_1, \ldots, p_{2d-2}$ the base-point of ρ. Without loss of generality, we can assume that $p_1, \ldots, p_{d-1}$ are the simple base-points of ρ that are contained in L and that p_1 is a proper base-point in $\mathbb{P}^2$. We do induction on the degree of ρ.

If there is no simple proper base-point p_i, $i \geq d$, of ρ in $\mathbb{P}^2$ that is not on L, choose a general point $r \in \mathbb{P}^2$. There exists a quadratic transformation $\tau \in \mathcal{J}$ with base-points p_0, p_1, r. The transformation $\rho\tau^{-1} \in \mathcal{J}$ is of degree $\deg\rho$ and sends the line $\tau(L)$ (which does not contain $[1:0:0]$) onto the line $\rho(L)$. The point $\rho_\bullet(r) = \rho(r)$ is a proper base-point of $(\rho\tau^{-1})^{-1}$ in $\mathbb{P}^2$ not on the line $\rho(L)$.

So, we can assume that there exists a proper base-point of ρ in $\mathbb{P}^2$ that is not on L, lets call it p_d. The points p_0, p_1, p_d are not collinear (because of their multiplicities), hence there exists $\tau \in \mathcal{J}$ of degree 2 with base-points p_0, p_1, p_d. The map $\rho\tau^{-1} \in \mathcal{J}$ is of degree $\deg \rho - 1$ and $\tau(L)$ is a line passing through exactly $\deg \rho - 2$ simple base-points of $\rho\tau^{-1}$ and not through $[1:0:0]$. □

Lemma 3.9. *Let $\rho \in \mathcal{J}$ be of degree ≥ 2 and let L be a line passing through $[1:0:0]$ and no other base-points of ρ. Then there exist $\rho_1, \ldots, \rho_m \in \mathcal{J}$ of degree 2 such that $\rho = \rho_m \cdots \rho_1$ and the successive images of L are lines.*

Proof. Note that the curve $\rho(L)$ is a line. We use induction on the degree of ρ.

Assume that ρ has no simple proper base-points, i.e. all simple base-points are infinitely near $p_0 := [1:0:0]$. There exists a base-point p_1 of ρ in the first neighbourhood of p_0. Choose a general point $q \in \mathbb{P}^2$. There exists $\tau \in \mathcal{J}$ quadratic with base-points p_0, p_1, q. The map $\rho\tau^{-1} \in \mathcal{J}$ is of degree $\deg \rho$ and $\tau(L)$ is a line passing through the base-point p_0 of $\rho\tau^{-1}$ of multiplicity $\deg \rho - 1$ and through no other base-points of $\rho\tau^{-1}$. Moreover, the point $\rho(q)$ is a (simple proper) base-point of $\tau\rho^{-1}$. Therefore, $\tau\rho^{-1}$ has a simple proper base-point in $\mathbb{P}^2$ and sends the line $\rho(L)$ onto the line $\tau(L)$, both of which pass through p_0 and no other base-points.

So, we can assume that ρ has at least one simple proper base-point p_1. Let p_2 be a base-point of ρ that is a proper point of $\mathbb{P}^2$ or in the first neighbourhood of p_0 or p_1. Because of their multiplicities, the points p_0, p_1, p_2 are not collinear. Hence there exists $\tau \in \mathcal{J}$ quadratic with base-points p_0, p_1, p_2. The map $\rho\tau^{-1}$ is a map of degree $\deg \rho - 1$ and $\tau(L)$ is a line passing through p_0 and no other base-points. □

Lemma 3.10. *Let $\rho \in \mathcal{J}$ of degree 2 that sends a line L onto a line. There exist $\rho_1, \ldots, \rho_n \in \mathcal{J}$ of degree 2 with only proper base-points of $\mathbb{P}^2$ such that*

$$\rho = \rho_n \cdots \rho_1$$

and the successive images of L are lines.

Proof. Since ρ sends L onto a line, the line L must pass through exactly one of the base-points of ρ. In particular, this base-point is a proper point of $\mathbb{P}^2$.

Suppose that ρ has apart from $[1:0:0]$ exactly one other proper base-point q. Pick a general point $r \in \mathbb{P}^2$ not contained in L. There exists $\rho_1 \in \mathcal{J}$ of degree 2 with base-points $[1:0:0], r, q$. The transformation $\rho_2 := \rho\rho_1^{-1} \in \mathcal{J}$ is of degree 2, sends L onto a line, and has three proper base-points. Since ρ_1 also sends L onto a line, we conclude with $\rho = \rho_2\rho_1$.

Suppose that $[1:0:0]$ is the only proper base-point of ρ. Then ρ has a base-point q which is in the first neighbourhood of $[1:0:0]$. Pick a general point $r \in \mathbb{P}^2$ that is not on L. There exists $\rho_1 \in \mathcal{J}$ of degree 2 with base-points $[1:0:0], q, r$. Since $q \notin L$, the map $\rho_2 := \rho\rho_1^{-1} \in \mathcal{J}$ is of degree 2, sends L onto a line, and has at exactly one proper base-point other than $[1:0:0]$. Since ρ_1 sends L onto a line, we can apply the first case to ρ_1, ρ_2. □

Theorem 1. *For any line L, the group $\mathrm{Dec}(L)$ is generated by $\mathrm{Dec}(L) \cap \mathrm{PGL}_3$ and any of its quadratic elements having three proper base-points in $\mathbb{P}^2$.*

Proof. By conjugating with an appropriate automorphism of $\mathbb{P}^2$, we can assume that L is given by $x = y$. Note that the standard quadratic involution $\sigma\colon [x:y:z] \dashrightarrow [yz:xz:xy]$ is contaid in $\mathrm{Dec}(L)$. It follows from Proposition 3.6, Remark 3.7, and Lemmata 3.8, 3.9 and 3.10 that every element $\rho \in \mathrm{Dec}(L)$ has a composition $\rho = \alpha_{m+1}\rho_m\alpha_m\rho_m\alpha_{m-1}\cdots\alpha_2\rho_1\alpha_1$, where $\alpha_i \in \mathrm{PGL}_3$ and $\rho_i \in \mathcal{J}$ are quadratic with only proper base-points in $\mathbb{P}^2$ such that the successive images of L are lines. By composing the ρ_i from the left and the right with linear maps, we obtain a decomposition

$$\rho = \alpha_{m+1}\rho_m\alpha_m\rho_m\alpha_{m-1}\cdots\alpha_2\rho_1\alpha_1$$

where $\alpha_i \in \mathrm{PGL}_3 \cap \mathrm{Dec}(L)$ and $\rho_i \in \mathrm{Dec}(L)$ are of degree 2 with only proper base-points in $\mathbb{P}^2$. It therefore suffices to show that for any quadratic element $\rho \in \mathrm{Dec}(L)$ having three proper base-points in $\mathbb{P}^2$ there exist $\alpha, \beta \in \mathrm{Dec}(L) \cap \mathrm{PGL}_3$ such that $\sigma = \beta\rho\alpha$.

By Remark 3.7, for any quadratic element of $\mathrm{Dec}(L)$ the line L passes through exactly one of its base-points in $\mathbb{P}^2$.

Let $q_1 = [1:0:0]$, $q_2 = [0:1:0]$, $q_3 = [0:0:1]$. They are the base-points of σ, and σ sends the pencil of lines through q_i onto itself. Furthermore, $q_3 \in L$ but $q_1, q_2 \notin L$. Let $s := [1:1:1] \in L$. Remark that $\sigma(s) = s$ and that no three of q_1, q_2, q_3, s are collinear.

Let $\rho \in \mathrm{Dec}(L)$ be another quadratic map having three proper base-points in $\mathbb{P}^2$. Let p_1, p_2, p_3 (resp. p_1', p_2', p_3') be its base-points (resp. the ones of ρ^{-1}). Say L passes through p_1 and ρ sends the pencil of lines through p_i onto the pencil of lines through p_i', $i = 1, 2, 3$. Pick a point $r \in L \setminus \{p_1\}$, not collinear with p_2, p_3. Then no three of p_1, p_2, p_3, r (resp. $p_1', p_2', p_3', \rho(r)$) are collinear. In particular, there exist $\alpha, \beta \in \mathrm{PGL}_3$ such that

$$\alpha\colon \begin{cases} q_i \mapsto p_i \\ s \mapsto r \end{cases}, \quad \beta\colon \begin{cases} p_i' \mapsto q_i \\ \rho(r) \mapsto s \end{cases}$$

Note that $\alpha, \beta \in \mathrm{Dec}(L) \cap \mathrm{PGL}_3$. Furthermore, the quadratic maps $\sigma, \rho' := \beta\rho\alpha \in \mathrm{Dec}(L)$ and their inverse all have the same base-points (namely q_1, q_2, q_3) and both σ, ρ' send the pencil through q_i onto itself. Since moreover $\rho'(s) = \sigma(s) = s$, we have $\sigma = \rho'$. □

4. Dec(L) is not an amalgam

Just like $\mathrm{Bir}(\mathbb{P}^2)$, its subgroup $\mathrm{Dec}(L)$ is generated by its linear elements and one quadratic element (Theorem 1). In [Cor2013, Corollary A.2], it is shown that $\mathrm{Bir}(\mathbb{P}^2)$ is not an amalgamated product. In this section we adjust the proof to our situation and prove that the same statement holds for $\mathrm{Dec}(L)$.

The notion of being an amalgamated product is closely related to actions on trees, or, in this case, $\mathbb{R}$-trees.

Definition and Lemma 4.1. A *real tree*, or $\mathbb{R}$-tree, can be defined in the following three equivalent ways [Cis2001]:

1. A geodesic space which is 0-hyperbolic in the sense of Gromov.
2. A uniquely geodesic metric space for which $[a, c] \subset [a, b] \cup [b, c]$ for all a, b, c.
3. A geodesic metric space with no subspace homeomorphic to the circle.

We say that a real tree is a *complete real tree* if it is complete as a metric space.

Every ordinary tree can be seen as real tree by endowing it with the usual metric but not every real tree is isometric to an ordinary tree (endowed with the usual metric) [Cis2001, §II.2, Proposition 2.5, Example].

Definition 4.2. A group G has the *property* $(\mathrm{F}\mathbb{R})_\infty$ if for every isometric action of G on a complete real tree, every element has a fixed point.

We summarize the discussion in [Cor2013, before Remark A.3] in the following result.

Lemma 4.3. *If a group G has property $(\mathrm{F}\mathbb{R})_\infty$, it does not decompose as non-trivial amalgam.*

We will devote the rest of this section to proving Proposition 4.4 and thereby showing that $\mathrm{Dec}(L)$ is not an amalgam.

Proposition 4.4. *The decomposition group $\mathrm{Dec}(L)$ has property $(\mathrm{F}\mathbb{R})_\infty$.*

By convention, from now on, $\mathcal{T}$ will denote a complete real tree and all actions on $\mathcal{T}$ are assumed to be isometric.

Definition 4.5. Let $\mathcal{T}$ be a complete real tree.

(1) A *ray* in $\mathcal{T}$ is a geodesic embedding $(x_t)_{t\geq 0}$ of the half-line.
(2) An *end* in $\mathcal{T}$ is an equivalence class of rays, where we say that two rays x and y are equivalent if there exists $t, t' \in \mathbb{R}$ such that $\{x_s;\ s \geq t\} = \{y'_s;\ s' \geq t'\}$.
(3) Let G be a group of isometries of $\mathcal{T}$ and ω an end in $\mathcal{T}$ represented by a ray $(x_t)_{t\geq 0}$. The group G *stably fixes the end* ω if for every $g \in G$ there exists $t_0 := t_0(g)$ such that g fixes x_t for all $t \geq t_0$.

Remark 4.6. [Cor2013, Lemma A.9] For a group G, property $(\mathrm{F}\mathbb{R})_\infty$ is equivalent to each of the following statements:

(1) For every isometric action of G on a complete real tree, every finitely generated subgroup has a fixed point.
(2) Every isometric action of G on a complete real tree has a fixed point or stably fixes an end.

Definition 4.7. For a line $L \subset \mathbb{P}^2$, define $\mathcal{A}_L := \mathrm{PGL}_3 \cap \mathrm{Dec}(L)$. If L is given by the equation $f = 0$, we also use the notation $\mathcal{A}_{\{f=0\}}$.

Lemma 4.8. *For any line $L \subset \mathbb{P}^2$ the group $\mathcal{A}_L$ has property* $(\mathrm{F}\mathbb{R})_\infty$.

Proof. Since for two lines L and L' the groups $\mathrm{Dec}(L)$ and $\mathrm{Dec}(L')$ are conjugate, it is enough to prove the lemma for one line, say the line given by $x = 0$. Note that $A = (a_{ij})_{1\leq i,j\leq 3} \in \mathrm{PGL}_3$ is in $\mathcal{A}_{\{x=0\}}$ if and only if $a_{12} = a_{13} = 0$.

Let $\mathcal{A}_{\{x=0\}}$ act on $\mathcal{T}$ and let $F \subset \mathcal{A}_{\{x=0\}}$ be a finite subset. The elements of F can be written as a product of elementary matrices contained in $\mathcal{A}_{\{x=0\}}$; let A be the (finitely generated) subring of k generated by all entries of the elementary matrices contained in $\mathcal{A}_{\{x=0\}}$ that are needed to obtain the elements in F. Then F is contained in $\mathrm{EL}_3(A)$, the subgroup of $\mathrm{SL}_3(A)$ generated by elementary matrices. By the Shalom-Vaserstein theorem (see [EJZ010, Theorem 1.1]), $\mathrm{EL}_3(A)$ has Kazhdan's property (T) and in particular (as $\mathrm{EL}_3(A)$ is countable) has a fixed point in $\mathcal{T}$ [Wat1982, Theorem 2], so F has a fixed point in $\mathcal{T}$. It follows that the subgroup of $\mathcal{A}_{\{x=0\}}$ generated by F has a fixed point [Ser1977, §I.6.5, Corollary 3]. In particular, by Remark 4.6 (1), $\mathcal{A}_{\{x=0\}}$ has property $(\mathrm{F}\mathbb{R})_\infty$. □

From now on, we fix L to be the line given by $x = y$. It is enough to prove Proposition 4.4 for this line since $\mathrm{Dec}(L)$ and $\mathrm{Dec}(L')$ are conjugate groups (by linear elements) for all lines L and L'. As before, we denote the standard quadratic involution by $\sigma \in \mathrm{Bir}(\mathbb{P}^2)$; with our choice of L, it is contained in $\mathrm{Dec}(L)$.

Let $\mathcal{D}_L \subset \mathrm{PGL}_3$ be the subgroup of diagonal matrices that send L onto L, i.e.

$$\mathcal{D}_L := \{\mathrm{diag}(s, s, t)\, s, t \in \mathbb{C}^*\} \subset \mathrm{PGL}_3.$$

Lemma 4.9. *We have $\langle \mathcal{D}_L, \mu_1, \mu_2, P\rangle = \mathcal{A}_L$, with the three involutions*

$$\mu_1 := \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \in \mathcal{A}_L,\ \mu_2 := \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \in \mathcal{A}_L, \quad \text{and } P := \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathcal{A}_L.$$

Proof. Given any $\lambda \in \mathbb{C}^*$, the matrices

$$A_\lambda := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lambda & 0 & 1 \end{bmatrix},\ B_\lambda := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \lambda & 1 \end{bmatrix},\ \text{ and } C_\lambda := \begin{bmatrix} 1 & 0 & \lambda \\ 0 & 1 & \lambda \\ 0 & 0 & 1 \end{bmatrix}$$

belong to $\langle \mathcal{D}_L, \mu_1, \mu_2, P\rangle$. Indeed, we have $A_\lambda = \mathrm{diag}(-\lambda^{-1}, -\lambda^{-1}, 1)\cdot\mu_2\cdot\mathrm{diag}(\lambda, \lambda, 1)$, $B_\lambda = PA_\lambda P$ and $C_\lambda = \mathrm{diag}(1, 1, \lambda^{-1}) \cdot \mu_1 \cdot \mathrm{diag}(-1, -1, \lambda)$.

Left multiplication by these corresponds to three types of row operations on matrices in PGL_3 and right multiplication corresponds in the same way to three types of column operations. We denote them respectively by $r_1, r_2, r_3, c_1, c_2, c_3$, and we write d for multiplication by an element in $\mathcal{D}_L$.

Let $A = (a_{ij})_{1\leq i,j\leq 3} \in \mathrm{PGL}_3$ be a matrix which is in $\mathcal{A}_L$, i.e. such that $a_{13} = a_{23}$ and $a_{11}+a_{12} = a_{21}+a_{22}$. We proceed as follows, using only the above mentioned operations.

$$
\begin{aligned}
A \quad = \quad & \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \xrightarrow{d} \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & 1 \end{bmatrix} \xrightarrow{r_3} \begin{bmatrix} * & * & 0 \\ y & z & 0 \\ * & * & 1 \end{bmatrix} \xrightarrow{c_1 \text{ and } c_2} \begin{bmatrix} * & * & 0 \\ y & z & 0 \\ -y & -z & 1 \end{bmatrix} \\
\xrightarrow{r_3} & \begin{bmatrix} * & * & 1 \\ 0 & 0 & 1 \\ -y & -z & 1 \end{bmatrix} \xrightarrow{d} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ * & * & 1 \end{bmatrix} \xrightarrow{r_1} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & * & * \end{bmatrix} \xrightarrow{r_2} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & * & 0 \end{bmatrix} \\
\xrightarrow{d} & \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{r_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{c_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{r_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \xrightarrow{d} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

In the first step (d) we assume that $a_{33} \neq 0$ – this can always be achieved by performing a row operation of type r_1 on A if necessary. In the second step (r_3), we use that $a_{13} = a_{23}$. The entries on place $(2,1)$ and $(2,2)$ after the second step are denoted by y and z respectively. In the fifth step (d), we use that the entry on place $(1,1)$ is nonzero; this follows from the assumption $a_{11} + a_{12} = a_{21} + a_{22}$ and that A is invertible. ☐

Lemma 4.10. *Suppose that* $\mathrm{Dec}(L)$ *acts on* $\mathcal{T}$ *so that* $\mathcal{A}_L$ *has no fixed points. Then* $\mathrm{Dec}(L)$ *stably fixes an end.*

Proof. Since $\mathcal{A}_L$ has property $(\mathrm{F}\mathbb{R})_\infty$ and has no fixed points, it stably fixes an end (Remark 4.6 (2)). Observe that this fixed end is unique: if $\mathcal{A}_L$ stably fixes two different ends ω_1, ω_2, then $\mathcal{A}_L$ pointwise fixes the line joining the two ends and has therefore fixed points (this uses that the only isometries on $\mathbb{R}$ are translations and reflections [Cis2001, §I.2, Lemma 2.1]).

Let ω, respresented by the ray $(x_t)_{t\geq 0}$, be the unique end which is stably fixed by $\mathcal{A}_L$ and define $C := \langle \mathcal{D}_L, P\rangle$. Being a subgroup of $\mathcal{A}_L$, C obviously also stably fixes ω. Note that the end $\sigma\omega$ is stably fixed by $\sigma\mathcal{A}_L\sigma^{-1}$. In particular, since $\sigma C\sigma^{-1} = C$, the end $\sigma\omega$ is also stably fixed by C. If $\sigma\omega = \omega$, then ω is stably fixed by σ and by Theorem 1, ω is stably fixed by $\mathrm{Dec}(L)$. Otherwise, let l be the line joining ω and $\sigma\omega \neq \omega$. Since C stably fixes ω and $\sigma\omega$, it stably fixes both ends of l. In particular, the line l is pointwise fixed by C. Since $\mu_1, \mu_2 \in \mathcal{A}_L$, μ_1, μ_2 stably fix the end ω and therefore, x_t is fixed by μ_1, μ_2 for $t \geq t_0$ for some t_0, and hence, by Lemma 4.9, x_t is fixed by all of $\mathcal{A}_L$ for $t \geq t_0$, contradicting the assumption. ☐

Proof of Proposition 4.4. Recall that $\mu_1, \mu_2 \in \mathcal{A}_L$ and note that $\sigma\mu_1$ has order 3 and that $\sigma\mu_2$ has order 6. It follows that

$$\sigma = (\mu_1\sigma)\mu_1(\mu_1\sigma)^{-1}$$

By Theorem 1, $\mathrm{Dec}(L)$ is generated by σ and $\mathcal{A}_L$. It follows that $\mathcal{A}_1 := \mathcal{A}_L$ and $\mathcal{A}_2 := \sigma\mathcal{A}_L\sigma$ generate $\mathrm{Dec}(L)$.

Consider an action of $\mathrm{Dec}(L)$ on $\mathcal{T}$. It induces an action of $\mathcal{A}_L$, which has property $(\mathrm{F}\mathbb{R})_\infty$ by Lemma 4.8 (i.e. $\mathcal{A}_L$ has a fixed point or stably fixes an end by Remark 4.6 (2)). If $\mathcal{A}_L$ has no fixed point, Lemma 4.10 implies that $\mathrm{Dec}(L)$ stably fixes an end, and then we are done.

Assume that $\mathcal{A}_L$ has a fixed point. We conclude the proof by showing that in this case, even $\mathrm{Dec}(L)$ has a fixed point.

For $i = 1, 2$, let $\mathcal{T}_i$ be the set of fixed points of $\mathcal{A}_i$. The two trees are exchanged by σ. If $\mathcal{T}_1 \cap \mathcal{T}_2 \neq \emptyset$, $\mathrm{Dec}(L)$ has a fixed point since $\langle \mathcal{A}_1, \mathcal{A}_2 \rangle = \mathrm{Dec}(L)$. Let us consider the case where $\mathcal{T}_1$ and $\mathcal{T}_2$ are disjoint.

Let $\mathcal{S} := [x_1, x_2]$, $x_i \in \mathcal{T}_i$, be the minimal segment joining the two trees and $s > 0$ its length. Let $C := \langle \mathcal{D}_L, P \rangle$. Then $\mathcal{S}$ is pointwise fixed by $C \subset \mathcal{A}_1 \cap \mathcal{A}_2$ and reversed by σ. For $i = 1, 2$, the image of $\mathcal{S}$ by μ_i is a segment $\mu_i(\mathcal{S}) = [x_1, \mu_i x_2]$. By Lemma 4.9, $\langle C, \mu_1, \mu_2 \rangle = \mathcal{A}_1$, so it follows that for $i = 1$ or $i = 2$, we have $\mu_i(\mathcal{S}) \cap \mathcal{S} = \{x_1\}$. Otherwise, because $\mathcal{T}$ is a tree and $\mathcal{A}_1$ acts by isometries, both μ_1, μ_2 fix $\mathcal{S}$ pointwise and so $\mathcal{A}_1$ fixes $\mathcal{S}$ pointwise and in particular it fixes x_2 – this would contradict $\mathcal{T}_1 \cap \mathcal{T}_2 = \emptyset$. Choose an element $I \in \{1, 2\}$ such that $\mu_I(\mathcal{S}) \cap \mathcal{S} = \{x_1\}$.

Finally we arrive at a contradiction by computing $d(x_1, (\sigma\mu_I)^k x_1)$ in two different ways. On the one hand we see that this distance is sk, on the other hand we have $(\sigma\mu_I)^6 = 1$. More generally, we show that

$$d(\ (\sigma\mu_I)^k x_1,\ (\sigma\mu_I)^l x_1) = |k - l| s$$

for all k, l. Since we are on a real tree, it suffices to show this for k, l with $|k - l| \leq 2$ (cf. [Cor2013, Lemma A.4]). By translation, we only have to check it for $l = 0, k = 1, 2$. For $k = 1$, we have $d(\sigma\mu_I x_1, x_1) = d(\sigma x_1, x_1) = d(x_2, x_1) = s$. For $k = 2$, the segment $\mu_I(\mathcal{S}) = [x_1, \mu_I x_2]$ intersects $\mathcal{S}$ only at x_1. In particular, $d(\mu_I x_2, x_2) = 2s$ and hence

$$d(\sigma\mu_I\sigma\mu_I x_1, x_1) = d(\sigma\mu_I\sigma x_1, x_1) = d(\mu_I\sigma x_1, \sigma x_1) = d(\mu_I x_2, x_2) = 2s.$$

It follows that $\mathcal{T}_1$ and $\mathcal{T}_2$ cannot be disjoint, and we are done. □

References

[AC2002] Maria Alberich-Carramiñana, *Geometry of the plane Cremona maps.* Lecture Notes in Mathematics **1769**, Springer-Verlag, Berlin, 2002.

[Bla2012] Jérémy Blanc: *Simple Relations in the Cremona Group*, Proc. Amer. Math. Soc. **140** (2012), 1495–1500.

[BPV2009] Jérémy Blanc, Ivan Pan, Thierry Vust *On birational transformations of pairs in the complex plane* Geom. Dedicata **139** (2009), 57–73.

[Cas1901] Guido Castelnuovo: *Le trasformazioni generatrici del gruppo cremoniano nel piano*, Atti della R. Accad. delle Scienze di Torino **36** (1901), 861–874.

[Cis2001] Ian Chiswell: *Introduction to Λ-trees*, World Scientific, River Edge, NJ, 2001.

[Cor2013] Yves de Cornulier: *The Cremona Group is not an amalgam*, appendix of the article Serge Cantat, Stéphane Lamy: *Normal subgroups in the Cremona group*, Acta Math., **210** (2013), 31–94.

[EJZ010] Mikhail Ershov, Andrei Jaikin-Zapirain: *Property (T) for noncommutative universal lattices*, Invent. Math., **179** (2010), 303–347.

[Ser1977] Jean-Pierre Serre: *Arbres, Amalgames,* SL_2. Asterisque **46**, Soc. Math. France, 1977. English translation: *Trees*, Springer 1980.

[Wat1982] Yasuo Watanati: *Property (T) of Kazhdan implies property (FA) of Serre.* Math. Japonica **27** (1982), 97–103.

Isac Hedén, Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502 Japan

E-mail address: Isac.Heden@kurims.kyoto-u.ac.jp

Susanna Zimmermann, Departement Mathematik und Informatik, Universität Basel, Spiegelgasse 1, 4051 Basel, Switzerland

E-mail address: Susanna.Zimmermann@unibas.ch