CURVES OF CONTACT OF ANY ORDER ON ALGEBRAIC SURFACES

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I. *Contact along a Simple Curve.*

A recent paper* referred incidentally to contact of cubic surfaces along a straight line; the present investigation generalizes these ideas and determines the postulation and equivalence† of contact of any order of a family of algebraic surfaces along a curve, subject only to certain restrictions as to the nature of the singularities.

Let there be given a curve C_m of order m and also a surface f_κ of degree κ passing through C, and let us find the postulation, or number of independent conditions imposed upon a surface f_λ which has i-contact (contact of order $i-1$) with f_κ along C; and the equivalence, or number absorbed of the $\lambda\mu\nu$ intersections of three general surfaces f_λ, f_μ, f_ν, each of which has i-contact with f_κ along C.

Let C have no singular points except d double points with distinct tangents; let r be its rank, and† $\rho = r+2d+2m$.

Let‡ $(P_i)_\lambda = \frac{1}{2}i(i+1)(\lambda+2)m-\frac{1}{12}i(i+1)(2i+1)\rho$

$= \text{postulation of } C \text{ as an } i\text{-fold curve on } f_\lambda,$

$E_i = i^2(\lambda+\mu+\nu)m-i^3\rho$

$= \text{equivalence of } C \text{ as a common } i\text{-fold curve on } f_\lambda, f_\mu, f_\nu.$

It will appear that, even if f_κ had no singular points, f_λ would have at points of C a certain number of nodes of the type B_i (each of which reduces the class of the surface by i). It is therefore natural to allow f_κ to have D_l nodes of the type B_l ($l = 2, 3, \ldots, i$; B_2's are conic nodes, the

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* *Proc. London Math. Soc.*, Ser. 2, Vol. 10, p. 18, 1911. See also Loria, *Atti R. Acc. Sci., Torino*, tom. XXVI, p. 294, footnote, 1891.

† Cayley, *Coll. Math. Papers*, Vol. VII, p. 189, 1869 (where ρ is used for $r+2d$).

‡ Noether, *An. di Mat.*, Ser. 2, tom. V, p. 163, 1871.

other B's are binodes). Then f_λ has the same nodes except the B_i's, which are in general different both in number and position from those of f_κ. We assume that no surface has on C any more complicated singular point, thus excluding unodes and triple points, and also that at none of the nodes does C have a double point, or touch the edge of a binode.

In the case of simple contact, the tangent planes to f_λ, f_κ must coincide at all points of C at which both are determinate. If the point is a node on f_κ and not on f_λ, then the tangent plane to f_λ coincides with the limit to the tangent planes to f_κ at adjacent points of C; there is no necessity for f_λ to have a node, and all singularities are variable. But in the case of osculation, not only do the tangent planes coincide, but also the sections of f_λ, f_κ by any plane have the same curvature at points of C. Now as we approach a conic node of f_κ, this curvature becomes infinite for f_κ and therefore for f_λ, which must also have a conic node; the singularity is fixed by the condition of osculation. But for a binode this does not hold, since the curvature does not become infinite, provided the edge does not touch C. In the general case of i-contact, all variable singularities are B_i's, and all B_i's are variable.

For, let the origin O be a simple point on f_λ and a node on f_κ; if it were a conic node, we have seen that there could not be more than simple contact; it is therefore a binode, and one sheet of f_κ has $(i-1)$-contact with f_λ.

Let $$f_\lambda = \phi_1+\phi_2+\ldots+\phi_l+\ldots+\phi_\lambda,$$

where ϕ_l is a homogeneous function of xyz of degree l, and let

$$F_l = \phi_1+\phi_2+\ldots+\phi_l.$$

Then f_κ must have the form

$$f_\kappa = F_{i-2}M+\psi_i+\ldots+\psi_\kappa,$$

where M is a surface having $(i-2)$-contact at O with the second sheet of f_κ. From this form it follows that O is a B_i on f_κ.

Conversely, if O is a B_i, then f_κ has this form, and we can prove that there exists a surface f', having i-contact with f along C, but not singular at O, and independent of f in the sense of not being merely the sum of a multiple of f and terms which, equated to zero, give a surface having C as an i-fold curve, and which will be called i-fold terms. Stated algebraically, there exist multipliers M, M', such that

$$Mf'-M'f \equiv i\text{-fold terms.} \tag{I}$$

Now, if M is arbitrary, there are no solutions of (I) other than

$$\left.\begin{aligned} f' &= af + i\text{-fold terms} \\ M' &= aM + (i-1)\text{-fold terms} \end{aligned}\right\}, \tag{II}$$

which are not useful for our purpose, as f' is not independent of f.

Now (I) imposes a certain number of independent conditions, which, if f, M are general, fall upon the coefficients of f', M' and reduce them to the form (II). But, if f, M are specially related, a certain number of the conditions are satisfied by reason of these relations, and do not fall upon f', M'. In this case, certain arbitrary constants are left in f', besides those of (II), and there exist useful solutions of (I), whose number is that of the conditions transferred from f', M' to f, M.

Now the conditions imposed by (I) are concerned only with the behaviour of $Mf'-M'f$ at points of C; therefore this transference can only take place by reason of the behaviour of M at its points of intersection with C. Take one of these as origin; we have

$$M'f = Mf' + (x, y, z)^i + \dots .$$

Since M passes through O, and f', which contains C, also passes through O, the right hand side has a B_i at O, or a more complicated singularity if f' is singular. Therefore $M'f$ has a sheet through O which has $(i-1)$-contact with M, and if f is general, this sheet is M', and therefore $\frac{1}{2}i(i-1)$ of the conditions imposed by (I) express that M' has $(i-1)$-contact with M.

If f is singular, even if it has a sheet having l-contact with M at O, where $l < i-1$, these conditions still fall on M', and it also follows that f' has a singular point at O of the same nature as that of f. But if f has a B_i, one sheet having $(i-1)$-contact with M, then all these conditions fall away; M' does not even pass through O, and f' has a simple point there; and there exist exactly $\frac{1}{2}i(i-1)$ independent useful terms in f'.

Any multiple of f' has the same properties as f', so only $\frac{1}{2}i(i-1)$ are independent, *i.e.*, Mf' ceases to be independent when M satisfies a certain set of $\frac{1}{2}i(i-1)$ conditions, which are those of $(i-1)$-contact at O with the second sheet of f.

In the same way, corresponding to each B_l of f_κ there exists a surface f' having l-contact with f along C, and having a simple point instead of the B_l, and such that, if M is any surface having $(l-1)$-contact with the second sheet of f, then Mf' is the sum of a multiple of f and l-fold terms. With slight modifications all this applies to the case $l=2$, when the point is a conic node. Then in the intersection of f_κ with $f'^a\phi$, where ϕ is a

β-fold term, C counts $al+\beta$ times, and if this $\geqslant i$, the term when added to f_λ does not disturb its i-contact with f_κ.

Therefore, in the equation of the general surface f_λ we admit the following terms:—

(i) All i-fold and higher terms, in number $\binom{\lambda+3}{3}-(P_i)_\lambda$.

(ii) Multiples of f_κ; in order not to repeat terms of (i) we exclude from the multiplier all $(i-1)$-fold and higher terms. The number remaining is $(P_{i-1})_{\lambda-\kappa}$.

(iii) Terms such as $f'^a\phi$, where f' is the surface having l-contact with f_κ corresponding to any B_l of f_κ, and ϕ is a β-fold term where

$$al+\beta \geqslant i \quad (l = 2, 3, \ldots, i).$$

It remains to consider the number of independent terms in (iii). Let O be a B_l on f_κ, and let xy be the biplanes; let $x = z = 0$ be the tangent to C at O, and let C be given as the common intersection of as many surfaces g, g', ... as may be necessary. We may assume

$$\begin{aligned} g &= x+(x, y, z)^2+\ldots, \\ g' &= z+(x, y, z)^2+\ldots, \\ g'', \ldots &= \qquad (x, y, z)^l+\ldots, \ldots, \\ f' &= g+(g, g', \ldots)(x, y, z)^1+\ldots, \\ M &= y+(x, y, z)^2+\ldots; \end{aligned}$$

and therefore

$$f'y = \text{multiple of } f_\kappa + l\text{-fold terms} + f' \{(x, y, z)^2+\ldots\},$$

also $f'(x, y, z)^l =$ multiple of $f_\kappa + l$-fold terms.

Now $f'^a\phi$ is the sum of terms such as $f'^a(g, g', \ldots\overset{\beta}{)}(x, y, z)^\gamma$. By means of the above equations we can eliminate x, z, g'', ..., g, y from all the terms with $\gamma = 1, 2, \ldots, l-1$, and omit all the terms with $\gamma \geqslant l$, or with $a+\beta \geqslant i$, replacing them by sums of terms counted under (i), (ii), or under other values of a, β.

Let $i = al+b$, $0 \leqslant b < l$. Then the only terms to admit are

$$f'^a g'^\beta; \qquad a = 1, 2, \ldots, a, \quad \beta = i-al, \ldots, i-a-1\, i;$$

and $\quad f'^{a+1}(g, g')^\beta; \qquad \beta = 0, 1, \ldots, i-a-2.$

Therefore the number of independent terms formed corresponding to each B_l of f_κ is

$$\sum_{a=1}^{a} a(l-1)+\sum_{\beta=0}^{i-a-2}(\beta+1)=\tfrac{1}{2}\{i(i-a)-b(a+1)\}.$$

We verify that this is 0 at a simple point, where $l=1$, $a=i$, $b=0$; and is $\frac{1}{2}i(i-1)$ at a variable singularity, where $l=i$, $a=1$, $b=0$.

The postulation on f_λ of i-contact with f_κ is therefore

$$P=\binom{\lambda+3}{3}-\left[\binom{\lambda+3}{3}-(P_i)_\lambda+(P_{i-1})_{\lambda-\kappa}+\tfrac{1}{2}\sum_{l=2}^{i}\{i(i-a)-b(a+1)\}D_l\right]$$

$$=\{i(\lambda+2)+\tfrac{1}{2}i(i-1)\kappa\}m-\tfrac{1}{2}i^2\rho-\tfrac{1}{2}\Sigma\{i(i-a)-b(a+1)\}D_l$$

$$=i(P_1)_\lambda-\tfrac{1}{2}i(i-1)(\rho-\kappa m)-\tfrac{1}{2}\Sigma\{i(i-a)-b(a+1)\}D_l,$$

the sum extending to all nodes of f_κ.

For example, in the paper referred to on p. 398, there are found six cases of osculation of cubic surfaces along a straight line, for which

$$\begin{array}{rcllllll} D_2 &=& 0 & 1 & 2 & 0 & 1 & 0, \\ D_3 &=& 0 & 0 & 0 & 1 & 1 & 2, \\ P &=& 15 & 13 & 11 & 12 & 10 & 9. \end{array}$$

Here $i=3$, $\kappa=\lambda=3$, $m=1$, $\rho=2$, and though the expression for $(P_{i-1})_{\lambda-\kappa}$ is not strictly applicable, yet it gives the right value 1; these results verify the formula, which becomes

$$P=15-2D_2-3D_3.$$

In order to find the equivalence, we need to know the behaviour, at the nodes, of the residual intersection $C'_{\lambda\mu-im}$ of f_λ, f_μ.

(1) At a B_i of f_λ, we suppose that f_μ is not singular, and assume

$$f_\mu=F_{i-2}+\psi_{i-1}+\ldots,$$

$$f_\lambda=F_{i-2}M+\phi_i+\ldots,$$

where $$M=y+(x,\,y,\,z)^2+\ldots.$$

The tangents to the total intersection $iC+C'$ are given by

$$x=0,\quad y\psi_{i-1}=\phi_i,$$

and there are i branches, viz., C counted i times, and no branch of C'.

Any other curve C'' lying on the second sheet of f_λ meets f_μ in one point only at O.

(2) At a common B_l $(i > l > 2)$ of f_λ, f_μ, we assume

$$f_\lambda = F_{l-2}M + \phi_l + \ldots,$$

and f_μ differs from a multiple of f_λ by a set of terms of which the important parts are

$$\phi_i,\ F\phi_{i-l},\ F^2\phi_{i-2l},\ \ldots,\ F^a\phi_b,\ F^{a+1},\ \ldots.$$

There are two sets of branches of the total intersection:

(i) On the first, F is small of order l, having one of a different values given by

$$\phi_i + F\phi_{i-l} + \ldots + F^a\phi_b = 0,$$

and the important term in f_λ is ϕ_l; this leads to C counted al times.

(ii) If $b > 0$, there is a second set of branches on which F is small of order b, given by

$$F^a\phi_b + F^{a+1} = 0,$$

and the important term in f_λ is $-\phi_b M$; this leads to C counted b times, and a single branch of C', which touches the edge of the binode (unless $b = l-1$), and meets the third surface f_ν in $b(a+1)$ points coinciding at O. Here C'' meets f_μ in $a+1$ points at O.

But if $b = 0$, this second set is absent; there is no branch of C', and C'' meets f_μ in a points only.

(3) At a B_2, the term FM is absent, and is replaced by ϕ_2. A similar argument leads to the same result, putting $l = 2$.

In every case C' meets f_ν in $b(a+1)$ points coinciding at a B_l. Let it also meet C in I simple points, at each of which it meets f_ν in i points. Then the number of free intersections (*i.e.*, not lying on C) of f_λ, f_μ, f_ν, or of C', f_ν is

$$\nu(\lambda\mu - im) - iI - \sum_{l=2}^{i} b(a+1)D_l.$$

The rest of the $\lambda\mu\nu$ intersections are absorbed by C, and the equivalence is

$$E = i\nu m + iI + \Sigma b(a+1)D_l.$$

To find I, consider* an auxiliary non-singular surface f_{ϖ}, passing through C, but not touching f_λ along it. This meets C' in $I+\Sigma_1 D_l$ points on C (where Σ_1 extends to all nodes of f_λ for which $b>0$, and Σ_2 to those for which $b=0$), and in

$$\varpi(\lambda\mu-im)-I-\Sigma_1 D_l$$

other points, which are the free intersections of $f_\lambda, f_\mu, f_\varpi$.

But the residual intersection $C''_{\lambda\varpi-m}$ of f_λ, f_ϖ passes once through each of the nodes of f_λ and meets f_μ in $\Sigma_1(a+1)D_l+\Sigma_2 aD_l$ points at these nodes; let it also meet C in I' simple points, at each of which it meets f_μ in i points; then C'' meets f_μ in

$$\mu(\lambda\varpi-m)-iI'-\Sigma aD_l-\Sigma_1 D_l$$

other points, which are again the free intersections of $f_\lambda, f_\mu, f_\varpi$. Equating these values

$$I=(\mu-i\varpi)m+iI'+\Sigma aD_l.$$

To find I', consider† the locus $f_{\lambda+\varpi-2}$ of points whose polar planes with regard to f_λ, f_ϖ meet in a straight line which meets an arbitrary straight line; this locus meets C in $(\lambda+\varpi-2)m$ points consisting of:

the r points of C at which the tangent line meets the arbitrary straight line;

$\left.\begin{array}{l}\text{the } d \text{ double points of } C\\ \text{the } I' \text{ intersections of } C, C''\end{array}\right\}$ which are points of contact of f_λ, f_ϖ;

and the ΣD_l nodes of f_λ.

$$\text{Therefore } I'=(\lambda+\varpi)m-\rho-\Sigma D_l$$

$$I=(i\lambda+\mu)m-i\rho-\Sigma(i-a)D_l,$$

$$E=i(i\lambda+\mu+\nu)m-i^2\rho-\Sigma\{i(i-a)-b(a+1)\}D_l$$

$$=iE_1-i(i-1)(\rho-\lambda m)-\Sigma,$$

the sum extending to all nodes of f_λ.

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* Cayley, *Coll. Math. Papers*, Vol. v, p. 350, 1869.

† Salmon, *Geometry of Three Dimensions*, 5th ed., § 355, p. 371, 1912.

Now E must be symmetrical in λ, μ, ν; but if $l < i$, D_l is a constant; let D_λ be the value of D_i for f_λ.

Therefore $\lambda m - D_\lambda$ is independent of λ, and equal to $\kappa m - D_\kappa$. Hence even if f_κ has no nodes, f_λ will have B_i's if $\lambda > \kappa$.

The postulation and equivalence can be expressed in the form

$$P = i(P_1)_\lambda - N, \qquad E = iE_1 - 2N,$$

where $$N = \tfrac{1}{2}i(i-1)(\rho - \lambda m + D_\lambda) + \tfrac{1}{2}\sum_{l=2}^{i-1}\{i(i-a) - b(a+1)\}\, D_l,$$

the sum extending to all the fixed singularities. Hence, if, and only if, $N \geqslant 0$, the curve of i-contact can be regarded as the limit of i distinct adjacent curves of intersection, and these meet one another in N points.

The foregoing also applies when f_κ has, in addition, a certain number of nodes at double points of C, provided the two branches do not lie on the same sheet at a binode. Neither C' nor C'' passes through such a point, which only affects the results by entering into ρ.

For example, let C_m be the twisted cubic given by

$$g = y - x^2, \qquad g' = z - xy, \qquad g'' = xz - y^2,$$

and let $$f_\kappa = xg + zg' + z^2 g'',$$

with a conic node at the origin. We have

$$f' = g + yg' + yzg'',$$

touching f_κ along C.

In the family f_λ osculating f_κ along C the admissible terms are:

$$f_\kappa f_{\lambda-4}, \quad f'(g, g', g'' \between 1, x, y, z)^{\lambda-6}, \quad (g, g', g'' \overset{3}{\between} 1, x, y, z)^{\lambda-6};$$

where, to avoid repetitions, we exclude 2-fold terms from the multiplier of f_κ, and all terms but (g, g') from the multiplier of f'.

The postulation is $\qquad 9\lambda + 7 = 3P_1 + 4,$

which agrees with the formula, putting

$$m = 3, \quad \rho = 10, \quad i = 3, \quad \kappa = 4, \quad D_\kappa = 0, \quad D_2 = 1.$$

Consider the intersection of the two members of the family f_λ, viz.,

$$f_\kappa f_{\lambda-4} \quad \text{and} \quad f_\kappa \phi_{\lambda-4} + f'(g, g').$$

We find that the residual C'_{λ^2-9} meets C in the node of f_κ and in $6\lambda - 8$ other points. It meets a third surface of the family in two points at the node, three points at each of its other intersections with C and in $\lambda^3-27\lambda+22$ free points.

The equivalence is $27\lambda-22 = 3E_1+8$, which agrees with the formula.

II. *Contact along a Multiple Curve.*

Now let C be a j-fold line on $f_\kappa, f_\lambda, \ldots$, and let each sheet of f_λ have i-contact with the corresponding sheet of f_κ along C, which counts jk times in the intersection, where $k = j+i-1$. Then k-fold terms can be added to f_λ without disturbing the contact, and terms must now be called independent which are not the sum of a multiple of f_κ and k-fold terms.

The only higher singularities of f_κ on C that we consider are $(j+1)$-fold points. As before, we can prove that the variable singularities are $(j+1)$-planar points, at which j of the tangent planes are continuous with the tangent planes at adjacent points of C, and the last sheet of f_κ has $(i-1)$-contact with a non-singular surface M; and there exists an independent surface f' having i-contact along C with all the sheets of f_κ, and such that $M'f'$ ceases to be independent if M' has $(i-1)$-contact with M. Let f_κ have D_κ such points, and also D_l points of a similar nature with l instead of i $(l = 2, 3, \ldots, i-1)$; the D_2 points are conical. Let all these singular points fall at simple points of C; then a double point of C is a j-fold point of f_κ at which all the j tangent planes along each branch coincide with the plane of the two tangents to C. The residual does not pass through such a point.

In the equation of the general surface f_λ we admit the following terms:—

(i) All k-fold and higher terms, in number $\binom{\lambda+3}{3}-(P_k)_\lambda$.

(ii) Multiples of f_κ; the number of independent terms is $(P_{i-1})_{\lambda-\kappa}$.

(iii) Terms such as $\quad f'^{\alpha}(g, g', \ldots \overset{\beta}{\rangle\!\langle} x, y, z)^{\gamma},$

where $\qquad a(j+l-1)+\beta \geqslant k > aj+\beta.$

As before, we need only consider $\gamma = 0$, and we exclude $g'', \ldots$; since $f' = (g, g', \ldots)^j$ we can also exclude g^j and higher powers of g. Now let

$$k = a(j+l-1)+b, \quad 0 \leqslant b < j+l-1.$$

(1) Then, if $b \geqslant j$, the terms to admit are

$$f'^{\alpha} g'^{\beta-j+1}(g, g')^{j-1}, \quad \alpha = 1, 2, \ldots, a, \quad \beta = k-\alpha(j+l-1) \ldots k-\alpha j-1,$$

$$\text{and } f'^{a+1}(g, g')^{\beta}, \qquad \beta = 0, 1, \ldots, k-(a+1)j-1,$$

the last value of β being positive when $b \geqslant j$, and the number of independent terms formed corresponding to each $(j+1)$-fold point of f_κ is

$$\sum_{\alpha=1}^{a} \alpha(l-1)j + \sum_{\beta=0}^{k-(a+1)j-1} (\beta+1).$$

(2) But, if $b < j$, and $k > (a+1)j$, then in the multiplier of f'^{a} we admit $(g, g')^{\beta}$ instead of $g'^{\beta-j+1}(g, g')^{j-1}$ $(\beta = b, b+1, \ldots, j-1)$, and the number must be diminished by $\frac{1}{2}(j-b)(j-b-1)$.

If $k \leqslant a(j+1)$, this series of values of β stops at $\beta = k-aj-1$ instead of $j-1$, but we must also omit f'^{a+1} entirely, giving the same reduction.

Let Σ extend to all the $(j+1)$-fold points of f_κ, and Σ_1 to those for which $b < j$; then the postulation is

$$\begin{aligned} P = {} & (P_k)_\lambda - (P_{i-1})_{\lambda-\kappa} \\ & - \Sigma\left[\tfrac{1}{2}j(a+1)(k-aj-b) + \tfrac{1}{2}\{k-(a+1)j\}\{k-(a+1)j+1\}\right] \\ & + \Sigma_1 \tfrac{1}{2}(j-b)(j-b-1). \end{aligned}$$

To determine the behaviour of the residual, let the origin be one of the D_l points; we assume

$$f_\lambda = FM + \phi_{j+l-1} + \ldots,$$

$$f_\mu = \text{multiple of } f_\lambda + \phi_k + F\phi_{k-(j+l-1)} + \ldots + F^{a}\phi_b + F^{a+1} + \ldots,$$

where the lowest terms in F are of degree j.

(1) If $b \geqslant j$, the term of lowest degree in f_μ is F^{a+1}.

(i) On one set of branches, F is small of order $j+l-1$, having one of a different values, and the important term in f_λ is ϕ_{j+l-1}; this leads to C counted $aj(j+l-1)$ times.

(ii) On the second set, F is small of order b, and the important term in f_λ is $-\phi_b M$; this leads to C counted bj times, and j branches of C', which touch $F = M = 0$ (unless $b = j+l-2$), and each meets f_ν in $b(a+1)$ points at O. Any other curve C'' on the last sheet of f_λ meets f_μ in $j(a+1)$ points at O.

(2) If $b < j$, the term of lowest degree in f_μ is $F^a \phi_b$; there are b branches of C', which touch $\phi_b = M = 0$, and each meets f_ν in $aj+b$ points at O. Here C'' meets f_μ in $aj+b$ points at O.

Let C' also meet C in I ordinary points, at each of which it meets f_ν in k points. Then the number of free intersections of f_λ, f_μ, f_ν is

$$\nu(\lambda\mu - jkm) - kI - \Sigma jb(a+1) + \Sigma_1 b(j-b).$$

To find I, consider the auxiliary surface f_ϖ, passing once only through C. The number of free intersections of C', f_ϖ is

$$\varpi(\lambda\mu - jkm) - I - \Sigma j + \Sigma_1 (j-b),$$

and this is equal to the number of free intersections of f_μ and the residual intersection C'' of f_λ, f_ϖ, viz.,

$$\mu(\lambda\varpi - jm) - kI' - \Sigma j(a+1) + \Sigma_1 (j-b),$$

where I' is the number of intersections of C'', C at ordinary points of f_λ. We find

$$I' = (\lambda + j\varpi)m - j\rho - \Sigma 1,$$

whence

$$I = (j\mu + k\lambda)m - jk\rho - \Sigma(k - aj),$$

and the equivalence is

$$E = jk(\lambda+\mu+\nu)m - k(j-k)\lambda m - jk^2\rho - \Sigma\{k(k-aj) - jb(a+1)\} - \Sigma_1 b(j-b),$$

the sums extending to the $(j+1)$-fold points of f_λ.

As before, $\lambda m - D_\lambda$ is independent of λ, and the postulation and equivalence can be expressed in the forms

$$P = (P_j)_\lambda + j(i-1)(P_1)_\lambda - jN_1 - N_2, \quad E = E_j + j(i-1)E_1 - (3j-1)N_1 - 2N_2,$$

where

$$N_1 = -(i-1)\lambda m + j(i-1)\rho + \Sigma\{k-(a+1)j\} + \Sigma_1 (j-b),$$

$$N_2 = -\tfrac{1}{2}(i-1)(i-2j)\lambda m + \tfrac{1}{2}j(i-1)(i-j-1)\rho + \Sigma[\tfrac{1}{2}(a+1)j(2j-i-b) + \tfrac{1}{2}k(i-j)] - \Sigma_1 \tfrac{1}{2}(j-b)(3j-b-1),$$

which would be the formulæ for a system consisting of a j-fold curve and $j(i-1)$ distinct simple curves having N_1 intersections with the j-fold curve and N_2 with one another.

For example, consider a quartic family containing a double straight line and four simple straight lines all passing through a common triple point of the surfaces at the origin:

$$f_\lambda = \phi_3+\phi_4, \quad f_\mu = \psi_3+\psi_4, \quad f_\nu = \chi_3+\chi_4,$$

where $\phi_3 \ldots \chi_4$ can be interpreted as plane curves having in common a double point and four simple points; therefore $P = 24$.

Now the residual C_8' projects into the plane septic $\phi_3\psi_4-\phi_4\psi_3$ (shewing that C' has one branch through the origin), which has a fixed quadruple point, four fixed double points, and also passes through the remaining four intersections of ϕ_3, ϕ_4. It therefore meets the corresponding curve formed from f_λ, f_ν, viz., $\phi_3\chi_4-\phi_4\chi_3$, in thirteen variable points which are the projections of the free intersections of f_λ, f_μ, f_ν; therefore $E = 51$.

Now let f_κ be the simplest surface containing the system, viz., a cubic cone, and let two of the simple lines approach the double line along each sheet of f_κ. In the limit, we have

$$m = 1, \quad \rho = 2, \quad j = 2, \quad i = 3, \quad \kappa = 3, \quad D_\kappa = 0, \quad D_2 = 1,$$

$$\lambda = \mu = \nu = 4,$$

and the formulæ are verified. Note that $N_1 = N_2 = 3$.

It is possible to carry the theory farther in many directions, *e.g.*, so as to take account of higher singularities of the curve and surfaces, or of the cases in which the curve is a cuspidal edge, curve of self-contact, &c., on f_κ. For a multiple curve we may also investigate the conditions that the sheets of f_λ fall into several sets, each set having contact of a given order with a given surface.

As a simple example, let C be a straight line of multiplicity j, a, β, γ on f_κ, f_λ, f_μ, f_ν respectively, and let j sheets of each surface touch the corresponding sheets of f_κ. If f_κ has no $(j+1)$-fold points, we find

$$P = (P_a)_\lambda+j\,(P_1)_\lambda-\{aN_1+N_2\},$$

$$E = E_{a\beta\gamma}+jE_1-\{(a+\beta+\gamma-1)N_1+2N_2\},$$

where $\qquad N_1 = 2j-\kappa, \quad N_2 = (\kappa-j)(j-1),$

as for a common straight line, of multiplicity a, β, γ on f_λ, f_μ, f_ν respectively, and j distinct simple lines having N_1 intersections with the multiple line and N_2 with one another. This system is only possible provided

$$0 \leqslant N_1 \leqslant j, \quad 0 \leqslant N_2 \leqslant \tfrac{1}{2}j\,(j-1),$$

which are equivalent to $\kappa \leqslant \tfrac{3}{2}j$, unless $j = 1$, when $\kappa \leqslant 2$.

The subject is full of curious paradoxes. For example, in the case of simple contact, let f_κ be a general cubic surface, and C a simple straight line; the correct formulæ are obtained by supposing f_λ to contain a pair of straight lines intersecting in -1 points. Again, let f_κ consist of a pair of quadrics with a common generator, and let C be this nodal straight line; then f_λ can be similarly regarded as containing a nodal line and two distinct simple lines, the latter not meeting the nodal line, but meeting one another in two points. The explanation in the first example is that f_κ does not contain any straight line adjacent to C, for its twenty-six other straight lines are all at a finite distance. In the second example, the two points of contact of the two quadrics which lie on C are points of self-contact (tacnodes) in the case which arises as a limit, and only uniplanar points (unodes) in the case to which the formulæ apply.