Curves of simple contact on algebraic surfaces.

By

Hilda P. Hudson of Cambridge.

I. Introduction.

The intersection of surfaces has been discussed chiefly with regard to postulation and equivalence[*]), i. e., the number of independent conditions imposed on a surface by making it contain a certain element, and the number of points of intersection of three surfaces absorbed by the common element. These numbers are important in connection with transformation-theory. In the plane, the common elements can only be points, and contact can always be regarded as the limit of intersection in distinct adjacent points; this is not true in three dimensions. Then there may also be common curves, and though a curve of contact does arise when two distinct curves of intersection move up and coincide, this is not the most general case of contact along the curve. If a surface contains a curve, it cannot be thought of as containing an adjacent curve, unless the surface is specialized in a way that involves a certain number of singular points of the surface lying on the curve.

This paper determines the postulation and equivalence of a curve of contact on a family of surfaces of sufficiently high degree. Noether's[**])

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[*]) Cayley, Proc. Lond. Math. Soc. 3, p. 127. = Coll. Math. Papers VII, p. 189, 1869.

[**]) Noether, Ann. di Mat. (2) 5 (1871), p. 163. See also Loria, Atti R. Acc. Sc. Torino 26 (1891), p. 294 footnote. Hudson, Proc. Lond. Math. Soc. (2) 10 (1911), p. 15.

lines are closely followed, but the results are also proved by other methods. The relation of the curve of contact to its residual is found; this information is useful in the theory of twisted curves. We first examine the cases which do arise from the coincidence of two adjacent curves; this leads to formulae which apply also in the general case, when the expression may be negative which was at first interpreted as the number of intersections of the two curves.

Approached in this way, the general case is that in which the family of surfaces, of degree λ say, are required to touch a given surface, of lower degree $\varkappa$, along the given curve of order m, where λ exceeds a certain limit depending on $\varkappa$, m. The formulae do not apply to the case, more general from another point of view, in which the surfaces do not touch a surface of lower degree along the curve. The other chief restriction is that the only singularities considered are multiple points of the simplest type, i. e., on surfaces, conic nodes, and on curves, double points with distinct tangents.

The theory of moduli*) is not directly applicable to the question, because the given conditions do not determine the modulus, without considerations such as those of § IV below. In certain cases, another method would be to use a birational transformation in which the given curve corresponds to a single point in the second space, and to carry out the enumerations with regard to the transformed figure; but this process only transfers the difficulties of the question.

The following symbols and technical terms will be used:

$$\binom{n}{3} = \frac{1}{6}\, n\, (n-1)\, (n-2),$$

$(P_i)_\lambda =$ postulation of C_m as an i-fold curve on a surface of degree λ

$$= \frac{1}{2}\, i(i+1)\, (\lambda+2)m - \frac{1}{12}\, i(i+1)\, (2i+1)\varrho,$$

$(E_i)_\lambda =$ equivalence of C_m as an i-fold curve on surfaces of degrees $\lambda, \mu, \nu,$

$$= i^2(\lambda+\mu+\nu)m - i^3\varrho,$$

where

$$\varrho = m(m+1) - 2h = r + 2m + 2d;$$

m is the order of the curve, d, h the numbers of its real and apparent double points, and r its rank (Noether's formulae).

Functions which equated to zero give a surface having C as a simple (or nodal) line are called *simple* (or *nodal*) *terms*.

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*) Hilbert, Math. Ann. 36 (1890), p. 473.

A function is *linearly independent* of another function f if it is not the sum of a multiple of f and nodal terms.

The surface $f = 0$ is denoted by simply f, which means both the surface and the function according to the context.

Points of intersection of curves and surfaces not lying on C are called *free*.

II. Intersection of adjacent curves.

Let C_m, C_m' be adjacent curves of the same character lying upon a surface $f_\varkappa$ having $D_\varkappa$ nodes at simple points of C and d_2 nodes at double points of C. Let C, C' have N' intersections at simple points of C, and let $d_1 = d - d_2$ be the number of double points of C at simple points of f.

Now let C' move on f to coincide with C. Then there are ultimately four intersections of C, C' in the neighbourhood of each of the d_1 points and two in the neighbourhood of each of the d_2 points; the postulation and equivalence of the system C, C', on any surface f_λ of sufficiently high degree, are

$$2 P_1 - N, \qquad 2 E_1 - 2 N,$$

where

$$N = N' + 4 d_1 + 2 d_2.$$

Now project C, C' from any point A into two plane curves c, c' of class r having r^2 common tangents, which are the traces of the common tangent planes to C, C' which pass through A. When C' moves up to and coincides with C, the limiting positions of these r^2 common tangents fall into three classes.

1. The traces of the n tangent planes to $f_\varkappa$ at points of C, which pass through A, where n is the class of the developable formed by all these tangent planes. Considering the intersections of C with the first polar of A with regard to $f_\varkappa$, we have

$$(\varkappa - 1) m = n + D_\varkappa + 2 d_2.$$

2. The $N' + 2d$ tangents to c at the projections of the ultimate intersections of C, C'.

3. The double tangents to c each counted twice. The number of these = the number of double points on the reciprocal of c, whose degree is r and class m,

$$= \frac{1}{2} \{ r(r - 1) - m \}.$$

Therefore

$$r^2 = (\varkappa - 1) m - D_\varkappa - 2 d_2 + N' + 2 d + r(r - 1) - m,$$
$$N = N' + 2 d + 2 d_1 = \varrho + D_\varkappa - \varkappa m,$$

or $\qquad D_\varkappa = N + \varkappa m - \varrho \geqq \varkappa m - \varrho.$

Therefore unless the surface $f_\varkappa$ has at least $\varkappa m - \varrho$ nodes at simple points of C, it cannot be thought of as containing a curve C' adjacent to C.

Now when C, C' coincide, the postulation and equivalence remain unaltered, and hence for the cases of contact which arise in this way, their values are

$$2P_1 - N, \qquad 2E_1 - 2N,$$

where

$$N = \varrho + D_\varkappa - \varkappa m.$$

Now f_λ touches the same developable as $f_\varkappa$ and has the same d_2 nodes:

$$\lambda m - D_\lambda = n + m + 2d_2 = \varkappa m - D_\varkappa,$$

and is independent of λ or $\varkappa$; also $N = \varrho + D_\lambda - \lambda m$, which has a meaning even when it is negative, and can no longer mean the number of intersections of adjacent curves. We shall show that these formulae still hold.

III. Postulation, Noether's method.

1. *Straight line.*

Let the equations of the straight line C be $z = w = 0$, and let the given surface $f_\varkappa$ containing C be

$$f_\varkappa \equiv z\,\Phi_{\varkappa-1}(x, y) + w\,\Phi'_{\varkappa-1}(x, y) + \text{higher powers of } z,\ w,$$

and let it have $D_\varkappa$ nodes on C.

Then the tangent plane to $f_\varkappa$ at any point $(x\,y\,0\,0)$ of C is given by the group of terms in $f_\varkappa$ linear in z, w, if in them x, y are regarded as constants. Now if $(x_1\,y_1\,0\,0)$ is a node of $f_\varkappa$, at this point the tangent plane is indeterminate and Φ, Φ' both vanish, and Φ, Φ' have a common factor $xy_1 - yx_1$, which does not affect the position of the tangent plane at other points and can be removed from their equations. Similarly for all the other nodes.

Therefore Φ, Φ' have a common factor of degree D_λ, whose linear factors vanish one at each node; let the remaining factors of Φ, Φ' be Ψ, Ψ'; then these have no common factor.

Then the surface of degree $\varkappa' = \varkappa - D_\varkappa$, viz.

$$f_{\varkappa'} \equiv z\Psi + w\Psi',$$

has no node on C; it has the same tangent plane as $f_\varkappa$ except at the $D_\varkappa$ points where $f_\varkappa$ is singular, and at these points $f_{\varkappa'}$ touches the limiting position of the tangent planes to $f_\varkappa$ as we approach the node along C.

Now if a surface f_λ of any degree is required to touch $f_\varkappa$ along C, it also touches $f_{\varkappa'}$, and the surface $f_{\lambda'}$, formed from f_λ in the same way, must coincide with $f_{\varkappa'}$.

Therefore

$$\lambda - D_\lambda = \varkappa - D_\varkappa = n + 1,$$

where n is the class of the common tangent developable to f_λ, $f_\varkappa$ along C:

$$f_\lambda \equiv \Phi_{D_\lambda} \cdot f_{\varkappa'} + \text{quadratic and higher powers of } z, w,$$

and f_λ is subject to $3\lambda - D_\lambda$ conditions, i. e. the postulation of contact with $f_\varkappa$ along C is

$$P = 3\lambda - D_\lambda \qquad = 2P_1 - N,$$

where

$$N = 2 + D_\lambda - \lambda = 1 - n.$$

Now this straight line of contact can be thought of as a pair of adjacent straight lines in two cases only, when they do or do not intersect, $N = 1$ or 0, $n = 0$ or 1, $D_\lambda = \lambda - 1$ or $\lambda - 2$. But D_λ can have any value from 0 to $\lambda - 1$ inclusive (if $D_\lambda > \lambda - 1$, then C is a nodal line on f_λ), and there are λ types of contact for surfaces of degree λ along a straight line.

2. *Total intersection.*

Let C_m be the total intersection of $f_\varkappa$ and another surface g_α. The general surface of degree λ containing C is

$$f_\lambda \equiv f_\varkappa \Phi_{\lambda-\varkappa} + g_\alpha \Phi'_{\lambda-a}.$$

If $p_\varkappa$, p_α are the tangent planes to $f_\varkappa$, g_α at any point of C, then the tangent plane to f_λ at the same point is

$$p_\varkappa \Phi + p_\alpha \Phi',$$

where the coordinates of the point of contact are substituted in Φ, Φ'.

Now since this plane coincides with $p_\varkappa$, and in general p_α is a different plane:

$\Phi' \equiv 0$ at all points of C, and Φ' is a simple and $g\Phi'$ a nodal term.

In considering the number of independent terms in f_λ, we must admit

1. nodal and higher terms, in number $\binom{\lambda+3}{3} - (P_2)_\lambda$,

2. multiples of $f_\varkappa$, where in order not to count terms twice, we must exclude all simple and higher terms from the multiplier, leaving $(P_1)_{\lambda-\varkappa}$ independent terms.

Thus the postulation is

$$P = (P_2)_\lambda - (P_1)_{\lambda-\varkappa} = (2\lambda + \varkappa + 4)m - 2\varrho$$
$$= 2P_1 - (\varrho - \varkappa m),$$

and since every node of $f_\varkappa$ on C is a double point of C, we have $D_\varkappa = 0$ and our previous formula (at the end of § II) is established for this case.

3. *Any curve.*

Now let C_m degenerate (the total number of apparent double points remaining unaltered), into two curves $C'_{m'}$, $C''_{m''}$ intersecting one another

in I points, supposed simple points of each, of which I_1 are simple points of $f_{\varkappa}$ and I_2 are nodes of $f_{\varkappa}$, where $I = I_1 + I_2$:

$$m = m' + m'', \qquad \varrho = \varrho' + \varrho'' + 2I,$$

$$P = \{(2\lambda + \varkappa + 4)m' - 2\varrho'\} + \{(2\lambda + \varkappa + 4)m'' - 2\varrho''\} - 4I.$$

But

$$P = P' + P'' - \sigma_1 I_1 - \sigma_2 I_2,$$

where P', P'' are the postulations of contact with $f_{\varkappa}$ along C', C'' separately, and σ_1, σ_2 are the reductions in total postulation due to the two kinds of intersection respectively. Then

$$P' = (2\lambda + \varkappa + 4)m' - 2\varrho' - \frac{1}{2}(4 - \sigma_1)I_1 - \frac{1}{2}(4 - \sigma_2)I_2.$$

Now σ_1, σ_2 depend only on the behaviour of the curves and surfaces in the immediate neighbourhood of the intersection, and not on their general characteristics (degree, genus etc). They are therefore absolute constants and can be determined from the simplest possible cases, viz. when C', C'' are straight lines, and $f_{\varkappa}$ is a plane or a pair of planes. This gives $\sigma_1 = 4$, $\sigma_2 = 2$, and

$$P' = (2\lambda + \varkappa + 4)m' - 2\varrho' - I_2.$$

Now C' is any curve on $f_{\varkappa}$, and I_2 is the number of nodes of $f_{\varkappa}$ at simple points of C'.

If we drop the dashes and write $D_{\varkappa}$ for I_2, we have

$$P = (2\lambda + \varkappa + 4)m - 2\varrho - D_{\varkappa}$$
$$= 2P_1 - (\varrho + D_{\varkappa} - \varkappa m);$$

and this formula is general, provided only that λ is great enough for Noether's formulae for $(P_2)_{\lambda}$, $(P_1)_{\lambda - \varkappa}$ to be applicable.

IV. Postulation, second method.

We shall now give two direct algebraic proofs that if $f_{\varkappa}$ has a node at a simple point A of C, we can construct a surface f' touching f along C, but with a simple point at A and therefore linearly independent of f. Then in the general equation of the family of degree λ touching f along C, we admit not only nodal terms and multiples of f, but also multiples of f'; and we shall prove that the latter give only one independent term.

Also all other nodes of f on C are nodes of f', so if we form another surface f'' corresponding to another node B of f, then f' has a node at B, and f'' has not; f'' is independent of f' as well as of f, and the reduction in postulation is exactly $D_{\varkappa}$.

Now at a node of f at a double point of C, f touches two different planes along the two branches of C, and it is geometrically impossible to construct f' touching f along C, and having a simple point at this node of f, so that such nodes do not reduce the postulation.

This gives a second proof of the formula

$$P = (P_2)_\lambda - (P_1)_{\lambda-\varkappa} - D_\varkappa,$$

which is equivalent to the one obtained in the last paragraph.

We shall first treat as an example the simplest curve which is not a total intersection, viz. the twisted cubic.

1. *Twisted cubic.*

Let the curve C_3 be given parametrically by

$$x : y : z : w = 1 : \theta : \theta^2 : \theta^3,$$

and also as the common intersection of the three quadrics

$$g \equiv xz - y^2, \qquad g' \equiv xw - yz, \qquad g'' = yw - z^2.$$

Let p, p', p'' be the tangent planes to g, g', g'' at the point of C whose parameter is θ.

Now let $f_\varkappa$ be a surface containing C and having a node at $(1, 0, 0, 0)$ i. e. the point $\theta = 0$. Then we may assume

$$f_\varkappa \equiv \sum_{l=1}^{\varkappa-2} x^{\varkappa-l-2}(y, z, w \,\big\rangle^l g, g', g''),$$

in which the general term can also be written

$$x^{\varkappa-l-2}(y, z, w \,\big\rangle y, z, w \,\big\rangle^{l-1} g, g', g'').$$

The tangent plane to $f_\varkappa$ at the point θ is

$$P_\varkappa \equiv \sum(\theta, \theta^2, \theta^3 \,\big\rangle^l p, p', p'')$$
$$= \theta \sum(1, \theta, \theta^2 \,\big\rangle \theta, \theta^2, \theta^3 \,\big\rangle^{l-1} p, p', p'')$$
$$= \theta K \text{ say.}$$

Now K can be interpreted as the tangent plane at the point θ to

$$f'_\varkappa \equiv \sum x^{\varkappa-l-2}(x, y, z \,\big\rangle y, z, w \,\big\rangle^{l-1} g, g', g''),$$

which has a simple point at $\theta = 0$. Since f, f' have identical tangent planes at all points of C except $\theta = 0$ and $\theta = \infty$, where they are respectively singular, they are in contact along C.

Now

$$xf - yf' = \sum x^{\varkappa-l-2}(0, g, g' \,\big\rangle y, z, w \,\big\rangle^{l-1} g, g', g''),$$

which is nodal, and similarly zf', wf' only differ from multiples of f by nodal terms, but xf' is linearly independent of f.

Therefore, if $f_{\lambda-\varkappa}$ is subject to the single condition of passing through the node of $f_\varkappa$, then $f_{\lambda-\varkappa} \cdot f_\varkappa$ is not independent; i. e., the multiples of $f'_\varkappa$ give only one independent term in f_λ.

There is a certain amount of arbitrariness in the way of putting $(y, z, w)^l$ into the form $(y, z, w \mskip\medspace y, z, w)^{l-1}$; but, the various forms of f' that arise differ only by nodal terms.

2. *Any curve. First method.*

Let the curve C_m be given parametrically by

$$x : y : z : w = X\Phi_{l-1}(X, Y, Z) : Y\Phi_{l-1} : Z\Phi_{l-1} : \Phi_l,$$

where

$$\Phi_m(X, Y, Z) = 0;$$

the last equation is that of a plane projection of C. Let C be given also as the common intersection of as many surfaces g_α as may be necessary.

We assume

$$f_\varkappa \equiv \sum f_{\varkappa-\alpha}\, g_\alpha,$$

and its tangent plane at any point of C is

$$P_\varkappa \equiv \sum \Phi_{(\varkappa-\alpha)l} \cdot p_\alpha,$$

where Φ is what f becomes when $xyzw$ are replaced by their values in terms of the parameters of the point, and p is the tangent plane to g at the point. Since these planes p all contain the tangent line to C, they are connected by a certain number of linear identities.

Now let f have a node at the point $X = Y = Z$; then at this point the plane $P_\varkappa$ is indeterminate, and by using if necessary the identities between the p, we can put $P_\varkappa$ into the form

$$P_\varkappa = \sum \{ (X - Y)\Phi_{(\varkappa-\alpha)l-1} + (Y - Z)\Psi_{(\varkappa-\alpha)l-1} \} p_\alpha.$$

Now since $X = Y = Z$ is a point of C, it lies on Φ_m; and

$$\Phi_m \equiv (X - Y)\Phi_{m-1} + (Y - Z)\Psi_{m-1}.$$

We suppose that Φ_{m-1} does not vanish at the point; then

$$\Phi_{m-1}P_\varkappa = (Y - Z)\sum \Phi_{(\varkappa-\alpha)l+m-2}\, p = (Y - Z)K$$

say, where

$$\Phi_{(\varkappa-\alpha)l+m-2} \equiv \Phi_{m-1} \cdot \Psi_{(\varkappa-\alpha)l-1} - \Psi_{m-1}\Phi_{(\varkappa-\alpha)l-1}.$$

Now K can be interpreted as the tangent plane to

$$f'_{\varkappa} \equiv \sum \Phi_{(\varkappa-\alpha)l+m-2}\,(x, y, z)\, g_\alpha,$$

of degree

$$\varkappa' = (\varkappa - \alpha)l + m + \alpha - 2,$$

which touches f along C and has a simple point at the node of f, and

is therefore linearly independent of f; but if $f_{\lambda-\varkappa'}$ passes through the node, its parametric expression has the form

$$(X-Y)\Phi + (Y-Z)\Psi,$$

and the tangent plane to $f_{\lambda-\varkappa'} \cdot f'_{\varkappa'}$ is

$$\Phi_{l-1}^{\varkappa'-\alpha} \cdot (\Phi_{m-1}\Psi - \Psi_{m-1}\Phi) P_{\varkappa},$$

so that $f_{\lambda-\varkappa'} \cdot f'_{\varkappa'}$ is the sum of a multiple of f and nodal terms.

3. *Any curve. Second method.*

If $f'_{\varkappa'}$ touches $f_{\varkappa}$ along C, there must exist an identity

(1) $$f'_{\varkappa'}\varphi_{\alpha-\varkappa'} - f_{\varkappa}\varphi'_{\alpha-\varkappa} \equiv \text{nodal terms,}$$

where φ does not contain C.

Now suppose that φ is quite arbitrary; then (1) imposes $(P_2)_\alpha$ conditions upon the $\binom{\varkappa'+3}{3} + \binom{\alpha-\varkappa+3}{3}$ coefficients of f', φ'. But since

$$\binom{\varkappa'+3}{3} + \binom{\alpha-\varkappa+3}{2} - (P_2)_\alpha$$
$$= \left\{\binom{\varkappa'+3}{3} - (P_2)_{\varkappa'}\right\} + \left\{\binom{\alpha-\varkappa+3}{3} - (P_2)_{\alpha-\varkappa}\right\} + (P_2)_{\varkappa'-\varkappa},$$

the only solutions are the obvious ones in which f', φ' are the sums of nodal terms and the same multiple of f, φ respectively.

But if φ is specialized, some of the $(P_2)_\alpha$ conditions may fall upon the coefficients of f, φ and not on those of f', φ'; in this case there are solutions for f' other than those just enumerated.

Now (1) shows that f' must contain C, and that the nodes of f on C are nodes on $f'\varphi$, and, if φ is general, are nodes on f'; i. e., some of the $(P_2)_\alpha$ necessary conditions express that f' has nodes at these points.

Now subject φ to the single condition of passing through a node A of f on C; then one of the necessary conditions is satisfied by virtue of this specialization and does not fall on the coefficients of f', φ', and in this case there is just one independent solution for f'.

We arrive at the same solution f', whatever be the surface φ through A. For, the same problem, with the same f, φ but with $\varkappa' + \varepsilon$, $\alpha + \varepsilon$ instead of $\varkappa'$, α, is satisfied by $f_\varepsilon \cdot f'$ where f_ε is arbitrary. But we have seen that there is only one such solution; so $f_\varepsilon \cdot f'$ ceases to be independent if f_ε is subject to a certain single condition, which is clearly that it pass through A, i. e. f' is the solution if we start with f_ε instead of φ.

Therefore the existence of the node of $f_\varkappa$ at A reduces by 1 the postulation of contact with $f_\varkappa$ along C, whatever be the degree of the family considered.

V. Equivalence.

1. Cayley's method.*)

Let f_λ, f_μ, f_ν be three surfaces touching one another along C_m. Let them have D_λ, D_μ, D_ν distinct nodes at simple points of C respectively, and d_2 common nodes at double points of C. Let the residual intersection $C'_{\lambda\mu-2m}$ of f_λ, f_μ meet C in I points. These are in general distinct from the nodes; for one of the D_λ points is simple on f_μ, and through it pass two branches of the total intersection, viz. C counted twice; and one of the d_2 points is a common node, and through it pass four branches of the intersection, viz. the two branches of C each counted twice. At each of the I points, C' touches f_ν.

Therefore C' meets f_ν in $\nu(\lambda\mu - 2m) - 2I$ points not lying on C, which are the free intersections of f_λ, f_μ, f_ν; the remaining intersections are absorbed by C; the equivalence is

$$E = \lambda\mu\nu - \{\nu(\lambda\mu - 2m) - 2I\}$$
$$= 2\nu m + 2I.$$

To find I, consider an auxiliary surface f_{ϖ} containing C but not touching f_λ, and having no singularities on C. It meets C' in the I points on C and in $\varpi(\lambda\mu - 2m) - I$ other points, which are the free intersections of f_λ, f_μ, f_{ϖ}.

Now the residual intersection $C''_{\lambda\varpi-m}$ of f_λ, f_{ϖ} passes through the D_λ nodes of f_λ, but not through the d_2 nodes; let it also meet C in I' simple points, at which it touches f_μ.

Therefore C'' meets f_μ in $\mu(\lambda\varpi - m) - D_\lambda - 2I'$ other points which are again the free intersections of f_λ, f_μ, f_{ϖ}. Equating these two values, we have

$$I = (\mu - 2\varpi)m + D_\lambda + 2I'.$$

To find I', consider the locus $f_{\lambda+\varpi-2}$ of points whose polar planes with regard to f_λ, f_{ϖ} meet in a straight line which intersects an arbitrary straight line. This meets C in $(\lambda + \varpi - 2)m$ points, consisting of
the I' intersections of C, C'',⎫
the d_1 double points of C,   ⎬ being points of contact of f_λ, f_{ϖ},
the $D_\lambda + d_2$ nodes of f_λ and
the r points of C at which the tangent line to C meets the arbitrary line.
Therefore:

$$I' = (\lambda + \varpi - 2)m - r - D_\lambda - 2d_1 - 2d_2,$$
$$I = (2\lambda + \mu)m - 2\varrho - D_\lambda,$$
$$E = 2(\lambda + \mu + \nu)m - 4\varrho + 2(\lambda m - D_\lambda)$$
$$= 2E_1 - 2N,$$

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*) Cayley, Phil. Trans. 159, p. 221. = Coll. Math. Papers VI, p, 350, 1869.

where
$$N = \varrho + D_\lambda - \lambda m.$$

Since E must be symmetrical in λ, μ, ν, this proves again that $D_\lambda - \lambda m$ is independent of λ.

2. *Salmon's method.*[*])

The value of I can also be found as follows.

Let C, C' have H apparent intersections:
$$I + H = (\lambda\mu - 2m)m.$$

To find H, consider the equation $F = 0$ obtained by eliminating ε between
$$f_\lambda(\varepsilon x + x', \varepsilon y + y', \varepsilon z + z', \varepsilon w + w') = f_\lambda(x', y', z', w'),$$
$$f_\mu(\varepsilon x + x', \varepsilon y + y', \varepsilon z + z', \varepsilon w + w') = f_\mu(x', y', z', w').$$

It is of degree $\lambda\mu - 1$ in $xyzw$, and of degree $(\lambda - 1)(\mu - 1)$ in $x'y'z'w'$. Let these be the coordinates of two points A, A'. Now if A' is a common point of f_λ, f_μ, then F, regarded as the locus of A, is the cone vertex A' standing on the total intersection of f_λ, f_μ; i. e., if A' is on C', then F consists of:

the cone of degree m standing on C counted twice and
the cone of degree $\lambda\mu - 2m - 1$ standing on C'.

But if A' is on C, then F consists of:

the cone of degree $m - 1$ standing on C counted twice,
the cone of degree $\lambda\mu - 2m$ standing on C' and
the common tangent plane at A' to f_λ, f_μ.

Now let A be fixed, and regard F as the locus of A'. It meets C in $(\lambda - 1)(\mu - 1)m$ points consisting of:

the H feet of straight lines through A meeting C and C',
the $2h$ feet of straight lines through A meeting C twice,
the d_2 common nodes of f_λ, f_μ and
the n points of C at which the common tangent plane to f_λ, f_μ passes through A, where as before
$$n = (\lambda - 1)m - D_\lambda - 2d_2.$$

Also, since when A' is on C, F has a squared factor which vanishes at the $2h$ points, these count twice in the intersection of F, C.

Therefore
$$H = (\lambda - 1)(\mu - 1)m - 4h - 2d_2 - n,$$
$$I = (\lambda + \mu)m - 2\varrho - D_\lambda + \lambda m,$$
as before.

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*) Salmon, Geom. of Three Dim. 4th ed. p. 309, Dublin 1882.

VI. Examples and observations.

1. *Common nodes.*

The family (f_λ) has in general no common nodes other than the d_2 nodes at double points of C, but three particular members of the family may have common nodes, which will absorb some of the free intersections and increase the equivalence. If D of the nodes are common, Cayley's method gives $2D$ as the additional equivalence.

E. g., let C be the twisted cubic, and $f_\varkappa$ the quadric cone g: then $d_2 = 0$, $D_\varkappa = 1$. Now g is touched along C by the cubic family

$$f_3 \equiv wg' - zg'' + f_1 g, \quad \text{where } f_1 \text{ is arbitrary.}$$

This family has four nodes on C all varying with f_1; and $D_3 = 4$, $D = 0$.

Now consider the three quartic surfaces touching g along C viz.

$$\Phi_2 g, \quad \Phi_1 f_3, \quad (g', g'')^2 \quad \text{where } \Phi_2, \ \Phi_1 \text{ are arbitrary.}$$

Here $D_4 = 7$, $D = 0$, $E = 42$, and we verify that the remaining 22 free intersections lie 2 on g, Φ_1; 8 on Φ_2, Φ_1; and 12 on Φ_2, Φ_3. But if Φ_1 passes through the node of g, the three surfaces have a common node at this point, which absorbs the first two of the free intersections; $D = 1$, $E = 44$.

2. *Maxima and minima of P, E.*

If we only know that the surfaces (f_λ) touch one another along a given curve C_m, without knowing what fixed surface they touch, then there are a whole series of values possible for P, E depending on the values of $D_\varkappa - \varkappa m$, where $\varkappa$ is the degree of any surface $f_\varkappa$ containing C, and $D_\varkappa$ is the number of nodes of $f_\varkappa$ at simple points of C. When λ is given, the maximum of P is $(P_2)_\lambda - 1$, when there is only one independent surface f_λ for which C is not a nodal line; then $f_\varkappa \equiv f_\lambda$, and can be the most general surface of degree λ through C, which, if λ is great enough, has no nodes at either simple or double points of C.

The formula for P does not hold, since $\lambda - \varkappa = 0$ and $(P_1)_{\lambda - \varkappa}$ has no meaning; but instead

$$P = P_2 - 1 = 3(\lambda + 2)m - \frac{5}{2}\varrho - 1.$$

To find the equivalence in this case we may suppose that C is nodal on f_μ, f_ν; then their residual intersection meets C in $2(\mu + \nu)m - 4\varrho$ points, and since now $\lambda = \mu = \nu$, we have

$$E = 8\lambda m - 4\varrho.$$

The minima of P, E when λ is given are much more complicated, and depend on the nature of the surfaces of lowest degree through C. Since $\varkappa m - D_\varkappa = m + n + 2d_2$, the absolute minima of P, E for a given

value of m occur when $n = d_2 = 0$ and $f_\varkappa$ is a plane; then C is a plane curve and $\varrho = m(m+1)$, giving

$$P = 2P_1 - m^2, \quad E = 2E_1 - 2m^2,$$

as for two distinct coplanar curves intersecting in m^2 points.

3. *Gaps in the series of values of P.*

When λ, C_m are given, then $\varkappa m - D_\varkappa$ varies between λm and a fixed minimum, but the intermediate values may not all be possible. E. g., consider the C_5 which with a straight line makes up the total intersection of a non-singular quadric g_2 and a cubic surface g_3 having one node on C. No other quadric passes through C, and if $\varkappa = 2$ then $D_\varkappa = 0$; but the general cubic surface through C is

$$f_3 \equiv g_3 + f_1 g_2,$$

for which $D_\varkappa = 0$ or 1 only.

Thus $\varkappa m - D_\varkappa$ cannot lie between 10 and 14.

This is connected with the fact that if two distinct adjacent curves of this nature lie on the same quadric, they meet in 12 points; if they lie on the same cubic surface, they are residual to the same straight line C_1', and meet in 8 or 7 points according as the surface has or has not a node on C_1'; but the number of their intersections cannot lie between 12 and 8.

4. *Application to twisted curves.*

Concerning the residual $C'_{\lambda\mu - 2m}$, we know that it meets C in I points, and the number h' of its apparent double points is given by Salmon's formula (l. c., p. 311), provided we regard C as a pair of curves meeting in N points, i. e., as a curve of order $2m$ with $2h + m^2 - N$ apparent double points, and this holds even when N is negative. E. g., let two cubic surfaces touch along a straight line;

$$I = 5, \quad N = -1, \quad h' = 12,$$

and the residual is a septic of genus 3 with one 5-secant.[*]

Cambridge, January 1912.

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[*] Noether, Zur Grundlegung der Theorie der algebr. Raumkurven, § 17. Abhandlungen d. K. Preuß. Ak. d. Wiss. 1882 (where there are some more examples).