INCIDENCE RELATIONS FOR CREMONA SPACE TRANSFORMATIONS

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1. *Plane incidence relations.*

A Cremona plane transformation is determined by its fundamental system (F-system), that is, the set of σ fixed points O_α of fixed multiplicities i_α, the base of the first homaloidal net (ϕ_n) of curves in the first plane S that $\sim$ (correspond to) the lines of the second plane S'; their degree n is the same as that of the second homaloidal net (ϕ'), in S', that corresponds to the lines of S.

Each F-point O_α corresponds to a principal curve (P-curve) j'_α of degree i_α; these curves together form the Jacobian $J' \equiv \prod_{\alpha=1}^{\sigma} j'_\alpha$ of (ϕ'). Similarly there is the second F-system with accented symbols, and the first P-system $\sum_{\alpha'} j_{\alpha'}$. Let $i_{\alpha\alpha'}$ be the multiplicity of O_α on $j_{\alpha'}$.

If some of the F-points are adjacent, then part of the definition of (ϕ) may be that O_α is a singularity of any assigned nature. For the purpose of the incidence relations, this can always be regarded as the limit of a set of distinct ordinary multiple and simple F-points, and the formulae to be stated hold without modification; they must be properly interpreted when a P-curve breaks up.

Clebsch's incidence relations are

$$(1) \qquad \sigma = \sigma',$$

$$(2) \qquad \sum_{\alpha} i_{\alpha} = 3(n-1),$$

$$(3) \qquad \sum_{\alpha} i_{\alpha}^2 = n^2-1,$$

$$(4) \qquad i_{\alpha\alpha'} = i'_{\alpha'\alpha},$$

$$(5) \qquad \sum_{\alpha} i_{\alpha\alpha'} = 3i'_{\alpha'}-1,$$

$$(6) \qquad \sum_{\alpha} i_{\alpha} i_{\alpha\alpha'} = ni'_{\alpha'},$$

$$(7) \qquad \sum_{\alpha} i_{\alpha\alpha'} i_{\alpha\beta'} = i'_{\alpha'} i'_{\beta'} \quad (\alpha' \neq \beta'),$$

$$(8) \qquad \sum_{\alpha} i_{\alpha\alpha'}^2 = i'^2_{\alpha'}+1,$$

and the similar equations with the planes interchanged.

Of these, (2)–(7) are proved geometrically by considering the determination, genus, and intersections of ϕ and $\Sigma j_{\alpha'}$; then (1) follows from (5), and (8) from (7), by summation. Jung also finds (5), (6) and (8) by applying (2) and (3) to a transformation compounded of two others having one common F-point.

We wish to find analogues of these relations for space transformations. The chief new results of this paper are, first, that (1) holds without change: the total number, properly reckoned, of F-elements of a Cremona space transformation is the same in each space; and, secondly, the proof of the theorem merely stated by Tummarello, that the irrational curves of the two F-systems are associated in a (1, 1) relation in pairs of equal genera.

2. *Reduction of space transformations.*

In space, a Cremona transformation is also determined by the F-system, the base of the homaloidal web (ϕ_n) of surfaces in S that correspond to the planes of S'. The F-elements may be points or curves; contact cannot, in general, be treated as a limit, and the order of contact, as well as the multiplicity, is an essential characteristic of each F-element, as are also the numbers of incidences of the F-curves with each other and with the F-points of higher multiplicities.

Since the multiplicities of a plane F-system are replaced by this host of numbers, any general space relations answering to those of § 1 are ponderous. Our attempt must be restricted to cases when only the simplest types of F-elements are present.

For these, as for singularities of a surface, it is to some extent arbitrary what we consider as simple. Segre calls a singular point of a surface f extraordinary or ordinary, according as the tangent cone has, or has not, a repeated sheet. Levi has shown that, by a series of auxiliary Cremona transformations of well known types, f can be so reduced as to have ordinary multiple points and curves only.

Let O be an intersection of two curves ω_α, ω_β of f, of multiplicities i_α, i_β, where $1 < i_\alpha \leqslant i_\beta$; then O is of multiplicity at least i_β, and if it is not higher, the tangent cone to f at O consists of i_β planes through t_β, the tangent line to ω_β, of which i_α coincide with the plane $p_{\alpha\beta}$ of t_α, t_β; and O is extraordinary. Hence, when f is reduced, all intersections of multiple curves fall at hypermultiple points.

On these lines, a Cremona transformation would be regarded as simple when the F-system consists of ordinary points and curves without contact, and all intersections of F-curves, whether simple or multiple, fall at F-points of higher multiplicity, where the tangent cone to ϕ has no fixed sheet. For this case a few general formulae have been given by Noether and Pannelli.

In Chisini's more recent work, a surface f is reduced to have ordinary multiple curves only, each curve by itself free from singularities, and f has no singular points at all, except the two kinds necessitated by the curves: (i) elementary pinch-points, where two of the distinct branches of the plane section are replaced by a cusp of first species; (ii) elementary intersections of two different multiple curves, where f satisfies no additional condition; the multiplicity of f at each is the greater of those of the two curves through it; such a point is extraordinary, the tangent cone having a repeated plane.

This point of view is artificial; for example, a quadric cone is regarded as requiring reduction. But it has the great advantage of retaining only one type of singular element, the curve, instead of both curve and point.

An immediate extension of Chisini's method reduces any Cremona transformation, by compounding it with well known auxiliary transformations, to a simple type $T_{n-n'}$ in which there are no isolated F-points or contact conditions. The first F-system consists of σ ordinary curves such as ω_α of multiplicity i_α, degree m_α, genus p_α, with d_α intersections with each of the homaloidal curves $c_{n'}$ that correspond to the lines of S'; also ω_α is free from multiple points, and has D_α simple intersections $O_{\alpha\beta}$ with ω_β, no other F-curve passing through $O_{\alpha\beta}$, and here the homaloidal surface ϕ_n satisfies no other condition than those imposed by $\omega_\alpha^{i_\alpha}$, $\omega_\beta^{i_\beta}$. This is an F-point of partial or total contact, i_α sheets of (ϕ) touching the fixed plane $p_{\alpha\beta}$ if $i_\alpha \leqslant i_\beta$, but it does not present any additional contact *conditions.*

The pinch-points on ω_α are variable with ϕ, and are not F-elements. None of these special points lie on the general c.

The second F-system is of the same nature. The degrees n, n' of the homaloids in S, S' are different in general.

Such a transformation will be called *ordinary*.

3. *Known properties of the transformation T.*

An F-curve ω_α is of the first or second species according as $d_\alpha > 0$ or $= 0$. If ω_α is of first species, it corresponds to a P-surface j'_α of degree d_α, whose genus, as given by Noether's formula, is $-p_\alpha$; let j'_α have $i'_{\alpha'\alpha}$ sheets through $\omega'_{\alpha'}$. A point Q_α of ω_α corresponds to a P-curve κ'_α of degree i_α, whose locus is j'_α. The Jacobian of (ϕ') is $J' \equiv \prod_{\alpha=1}^{\sigma} j'_\alpha$, and $\omega'_{\alpha'}$ is $(4i'_{\alpha'}-1)$-fold on J'.

If ω_α is of the second species, it is rational, and corresponds to an F-curve $\omega'_{\alpha'}$, also of the second species, and these are of multiplicities $4i_\alpha$, $4i'_{\alpha'}$ on J, J' respectively. A point Q_α of ω_α corresponds to the whole of $\omega'_{\alpha'}$ taken μ_α times, in the sense that any surface through Q_α corresponds to a surface, μ_α of whose sheets through $\omega'_{\alpha'}$ touch a fixed developable associated with Q_α, where

$$\mu_\alpha = \mu'_{\alpha'} = i_\alpha / m'_{\alpha'} = i'_{\alpha'} / m_\alpha.$$

Since now there is no P-surface which corresponds to ω_α, the symbols $i'_{\alpha'\alpha}$, $i'_{\beta'\alpha}$ are at our disposal, and we arbitrarily define

$$i'_{\alpha'\alpha} = -1, \quad i'_{\beta'\alpha} = 0 \quad (\beta' \neq \alpha'),$$

when ω_α, $\omega'_{\alpha'}$ are corresponding F-curves of the second species. By this device, it is found that all our formulae hold for F-curves of either species without modification.

The free intersection (not absorbed by F-curves) of two general homaloids is one curve c; that of ϕ, $j_{\alpha'}$ is a variable set of $m'_{\alpha'}$ curves $\kappa_{\alpha'}$; a general $\kappa_{\alpha'}$ has no free intersection with ϕ or with $j_{\beta'}$.

4. *Transformation of an intersection of F-curves.*

The free intersection of j'_α, j'_β corresponds to the set of $D_{\alpha\beta}$ points $O_{\alpha\beta}$ of intersection of ω_α, ω_β. Let $i_\alpha \leqslant i_\beta$, and let Q_α, $Q_\beta \to O$ along ω_α, ω_β respectively. The P-curve κ'_α that corresponds to Q_α coincides in the limit with the whole or part of κ'_β, lying on both j'_α and j'_β; and if $i_\alpha < i_\beta$, there is another part $\kappa'_{\beta\alpha}$ of κ'_β, lying on j'_β but not on j'_α, and meeting κ'_α in one point X'.

We assume that X' is not an F-point, and in particular that neither κ'_α nor $\kappa'_{\beta\alpha}$ is coincident with an F-curve $\omega'_{\alpha'}$, that is, adjacent to $\omega'_{\alpha'}$, lying on j'_β. If this occurred, we should compound T with another auxiliary transformation, having $\omega'_{\alpha'}$ as F-curve, when the adjacent elements would be scattered and the peculiarity would disappear.

Both of ω_α, ω_β cannot be of the second species if T is ordinary; for if they were, part of $\kappa'_\beta \equiv \mu_\beta \omega'_{\beta'}$ would coincide with $\kappa'_\alpha \equiv \mu_\alpha \omega'_{\alpha'}$, and $\omega'_{\alpha'}, \omega'_{\beta'}$ would coincide in an F-curve of contact.

As long as Q_α is different from O, a plane p through t_α corresponds to a ϕ' having a double point at one point P' of κ'_α, and any plane through P' corresponds to a ϕ having one sheet touching p at Q_α. This sets up a (1, 1) relation between p, P'. As P' describes κ'_α once, p rotates once about t_α; when P' meets $\omega'_{\alpha'}$, then p touches $j_{\alpha'}$, and conversely: the number of incidences of κ'_α, $\omega'_{\alpha'}$ is $i_{\alpha\alpha'}$, being the number of sheets of $j_{\alpha'}$ through ω_α. These incidences fall at points simple on $\omega'_{\alpha'}$, which has no multiple point; they may be simple or multiple on κ'_α. Similarly, all the sheets of J through ω_α correspond to the F-points on κ'_α.

If ω_α is a line, p always contains the whole of it, including O; the argument holds, p being replaced by a surface of higher degree, touching ω_α at Q_α, but not containing the whole of it.

As $Q_\alpha \to O$, all the tangent planes to ϕ through $t_\alpha \to p_{\alpha\beta}$; the (1, 1) relation between p, P' degenerates: all the points of κ'_α, including F-points, correspond to the one plane $p_{\alpha\beta}$, with the exception of one point Y' of κ'_α, corresponding to every other plane through t_α. Every general plane of the pencil corresponds to a ϕ' having Y' as a double point, and $p_{\alpha\beta}$ corresponds to a ϕ' having κ'_α as a double curve; and there is through ω_α either one or no sheet of J which does not touch $p_{\alpha\beta}$, according as Y' is or is not an F-point.

(i) If $i_\alpha < i_\beta$, any plane through O corresponds to a ϕ' containing both κ'_α and $\kappa'_{\beta\alpha}$, and therefore touching j'_β at X'. If the plane contains t_α, it touches ω_α, and ϕ' touches j'_α along κ'_α and therefore at X'. But j'_α, j'_β do not touch, since ω_α, ω_β do not; hence this ϕ' has a double point at X', which coincides with Y'. Now ϕ has $i_\beta - i_\alpha$ variable tangent planes at O through t_β, in a (1, 1) relation to the points of $\kappa'_{\beta\alpha}$, in which $p_{\alpha\beta}$ answers to X'. A plane through X' corresponds to a ϕ having $i_\alpha + 1$ sheets touching $p_{\alpha\beta}$ at O, but not containing any fixed curve, since X' is assumed not to be an F-point; and no sheet of J through ω_α touches any plane but $p_{\alpha\beta}$. But the behaviour of $j_{\alpha'}$ at O is determined solely by its two multiplicities $i_{\alpha\alpha'}$, $i_{\beta\alpha'}$; hence

$$i_{\alpha\alpha'} \leqslant i_{\beta\alpha'}, \quad \text{if} \quad i_\alpha < i_\beta,$$

and similarly

$$i_{\alpha\alpha'} \geqslant i_{\beta\alpha'}, \quad \text{if} \quad i_\alpha > i_\beta,$$

An analytical proof is given in § 9.

(ii) If $i_\alpha = i_\beta$, the (1, 1) relation degenerates for ω_β also, and there is a point X' corresponding to every plane through t_β. This again $\equiv Y'$, for a general point P near O corresponds to a point near both X' and Y'.

Now O is a point of total contact of (ϕ), the fixed tangent cone being $p_{\alpha\beta}$ taken i_α times. The planes of the star (X') correspond to a net of (ϕ) containing points adjacent to O in general directions; on these therefore O is $(i_\alpha+1)$-fold, the tangent cones having t_α, t_β as i_α-fold edges, and so breaking up into $p_{\alpha\beta}$ taken $i_\alpha-1$ times and a net (q) of quadric cones through t_α, t_β. If ϕ, ψ are two homaloids of this net, their free intersection c touches the two other common generators of q_ϕ, q_ψ, and has an additional double point at O; being already rational, it breaks up, and one branch through O is a fixed P-curve κ, which corresponds to X'. Hence X' is an F-point; it is double on the ϕ' which corresponds to any plane of each of the pencils through t_α, t_β, and therefore double on the net (ϕ') that corresponds to the star (O); hence X' is a point of total contact of the whole web (ϕ'), and is therefore an intersection $O'_{\alpha'\beta'}$ of two F-curves $\omega'_{\alpha'}$, $\omega'_{\beta'}$ of equal multiplicities, and is of the same nature as O; and κ is a common P-curve of $j_{\alpha'}$, $j_{\beta'}$, each of which therefore has one sheet through O not touching $p_{\alpha\beta}$.

Now there is only one such sheet through each of ω_α, ω_β; hence one of $j_{\alpha'}$, $j_{\beta'}$, say $j_{\alpha'}$, has one such sheet through ω_α, and $j_{\beta'}$ has one through ω_β, and no other P-surface has any such sheet:

$$i_\alpha = i_\beta, \quad i_{\alpha\alpha'} = i_{\beta\alpha'}+1, \quad i_{\alpha\beta'} = i_{\beta\beta'}-1, \quad i_{\alpha\gamma'} = i_{\beta\gamma'}.$$

Further, κ is continuous with a series of curves $\kappa_{\alpha'}$ through adjacent points Q_α of ω_α, but not with a series of $\kappa_{\alpha'}$ through Q_β, and with a series of $\kappa_{\beta'}$ through Q_β but not through Q_α. Hence the one sheet of j'_α which does not touch $p'_{\alpha'\beta'}$ passes through $\omega'_{\alpha'}$, and that of j'_β through $\omega'_{\beta'}$; and we have

$$i'_{\alpha'} = i'_{\beta'}, \quad i'_{\alpha'\alpha} = i'_{\beta'\alpha}+1, \quad i'_{\alpha'\beta} = i'_{\beta'\beta}-1, \quad i'_{\alpha'\gamma} = i'_{\beta'\gamma},$$

as a necessary consequence of the other relations.

Hence the pair of intersecting F-curves ω_α, ω_β in S, of equal multiplicities, selects a similar pair $\omega'_{\alpha'}$, $\omega'_{\beta'}$ in S', and of these, ω_α selects $\omega'_{\alpha'}$ and ω_β selects $\omega'_{\beta'}$, and the above relations hold; also the points of intersection of the first pair are in (1, 1) relation to those of the second:

$$D_{\alpha\beta} = D'_{\alpha'\beta'}.$$

If ω_α is the second species, parts of the argument are modified, and the results hold.

None of this applies to F-curves which do not meet; then $D_{\alpha\beta} = 0$ and j'_α, j'_β have no free intersection.

[*Added* 13 *August*, 1926.

5. *Proof of Tummarello's theorem: the genera of the irrational F-curves are the same in each space.*

Instead of an intersection of two distinct F-curves, let O be a d.p. of one F-curve ω_α; then H ceases to be ordinary to this extent. As before, O is an i_α-fold point of total contact of (ϕ), and is an $(i_\alpha+1)$-fold point on a net of (ϕ) which corresponds to the net of planes through a point X', a d.p. of $\Sigma\omega'$; and X' corresponds to a P-curve κ with one branch through O, a double curve on Σj, not touching the plane p of the two tangents to ω_α.

Any $j_{\beta'}$ has $i_{\alpha\beta'}$ sheets through each branch of ω_α at O; if it contains κ, it has an $(i_{\alpha\beta'}+1)$-fold point at O, and its tangent cone consists of p taken $i_{\alpha\beta'}-1$ times and a quadric cone $q_{\beta'}$ through the tangents to ω_α and κ. If κ were an intersection of two different P-surfaces $j_{\beta'}$, $j_{\gamma'}$, it would have two branches through O, touching the common generators of $q_{\beta'}$, $q_{\gamma'}$, other than the tangents to ω_α. Hence κ is not an intersection of $j_{\beta'}$, $j_{\gamma'}$, but a double curve on one P-surface $j_{\beta'}$, and X' is a d.p. on $w'_{\beta'}$; H' ceases to be ordinary to this extent.

Thus if, in an ordinary transformation T, one F-curve ω_α of H acquires a d.p. without breaking up, there is one F-curve $\omega'_{\beta'}$ of H' which also acquires a d.p. without breaking up. Both ω_α and $\omega'_{\beta'}$ are irrational.

Now the equations proved below for an ordinary transformation, and assumed here,

$$\sigma = \sigma', \quad \Sigma p = \Sigma p',$$

hold also for T when modified, as can be proved by reducing it to one of ordinary type. Now σ is unaltered when ω_α acquires a d.p.; hence σ is also unaltered, and no ω' breaks up. Also Σp is lowered by 1, and therefore $\Sigma p'$ also; but p'_β is lowered by 1, hence no other d.p. of $\Sigma\omega'$ can arise.

The two curves ω_α, $\omega'_{\beta'}$ are associated by the fact that if either acquires a d.p., so does the other. This sets up a (1, 1) relation between the sets of irrational curves of H, H'.

Now let ω_α acquire p_α d.p.'s; it does not break up, σ and therefore σ' are unaltered, and ω'_β does not break up when it acquires p_α d.p.'s. Hence $p'_{\beta'} \geqslant p_\alpha$, and similarly $p_\alpha \geqslant p'_{\beta'}$, that is

$$p_\alpha = p'_\beta.$$

In the (1, 1) relation, associated F-curves of H, H' have the same genus, which proves Tummarello's theorem.

Thus the numbers of irrational curves in H, H' are equal, and also the total numbers of curves; it follows that the numbers of rational curves are equal also.

If ω_a is a rational curve of degree greater than 1, it breaks up on acquiring a d.p. This may cause (ϕ) to break up, and lower n; or it may entail an additional F-curve of second species, and lower n'; or it may cause a P-surface to break up, one part j meeting (ϕ) in F-curves only, then j corresponds to an isolated F-point of (ϕ') and T ceases to be ordinary.

If T is still ordinary and of unaltered degrees, σ and therefore σ' are increased by 1, and one ω' breaks up on acquiring a d.p., and is therefore also a rational curve of degree greater than 1. But the particular ω' that breaks up may depend on the particular way in which ω_a breaks up, that is, on the degrees of the two components of ω_a and the distribution between them of the intersections of ω_a with the other F-curves. We do not thus obtain any (1, 1) relation between the rational curves of H, H'.]

6. *Incidence relations for space transformations.*

(1) $\sigma = \sigma'$.

(2) $\sum_a p_a = \sum_{a'} p'_{a'}$.

(3) $\sum_a d_a = 4n'-4$.

(4) $\sum_a i'_{a'a} = 4i'_{a'}-1$.

(5) $\sum_a d_a i_a = nn'-1$.

(6) $\sum_a d_a i_{aa'} = n'd'_{a'}$.

(7) $\sum_a i_a i'_{a'a} = ni'_{a'}$.

(8) $\sum_a i_{aa'} i'_{\beta'a} = d'_{a'} i'_{\beta'} \quad (a' \neq \beta')$.

(9) $\sum_a i_{aa'} i'_{a'a} = d'_{a'} i'_{a'}+1$.

(10) $\sum_a i_a m_a - 4n = \sum_{a'} i'_{a'} m'_{a'} - 4n'$.

(11) $\sum_a i_a^2 m_a = n^2-n'$.

(12) $\sum_a i_a i_{aa'} m_a = nd'_{a'} - i'_{a'} m'_{a'}$.

(13) $\sum_a i_{aa'} i_{a\beta'} m_a = d'_{a'} d'_{\beta'} - i'_{i'} D'_{a'\beta'} \quad (a' \neq \beta')$.

$$(13)' \quad i_l D_{\alpha\beta} = -\sum_{\alpha'} i'_{\alpha'\alpha} i'_{\alpha'\beta} m'_{\alpha'} + d_\alpha d_\beta \quad (\alpha \neq \beta).$$

$$(14) \quad \sum_\beta i_l D_{\alpha\beta} = \sum_{\alpha'} i'_{\alpha'\alpha}(i'_{\alpha'\alpha}+1)\, m'_{\alpha'} + 4 i_\alpha m_\alpha - d_\alpha(d_\alpha+4).$$

$$(15) \quad \sum_\beta i_\beta D_{\alpha\beta} = \sum_{\alpha'} i'_{\alpha'\alpha} m'_{\alpha'} + n m_\alpha - 2 i_\alpha(p_\alpha - 1) - 4 d_\alpha.$$

$$(16) \quad \sum_\beta i_l i_\beta D_{\alpha\beta} = i_\alpha \sum_{\alpha'} i'^2_{\alpha'\alpha} m'_{\alpha'} + n i_\alpha m_\alpha - i_\alpha d_\alpha^2 - d_\alpha.$$

$$(17) \quad \sum_\beta i^2_l D_{\alpha\beta} = 2 i_\alpha \sum_{\alpha'} i'^2_{\alpha'\alpha} m'_{\alpha'} + i_\alpha^2(4m_\alpha + 2p_\alpha - 2) - 2 i_\alpha d_\alpha^2 - d_\alpha.$$

$$(18) \quad \sum_\beta i_{\beta\alpha'} D_{\alpha\beta} - d'_{\alpha'} m_\alpha + 2 i_{\alpha\alpha'}(p_\alpha - 1) = \sum_{\beta'} i'_{\beta'\alpha} D'_{\alpha'\beta'} - d_\alpha m'_{\alpha'} + 2 i'_{\alpha'\alpha}(p'_{\alpha'} - 1).$$

$$(19) \quad \sum_\beta i_\beta i_{\beta\alpha'} D_{\alpha\beta} - \sum_\beta (i_\beta - i_\alpha)(i_{\beta\alpha'} - i_{\alpha\alpha'}) D_{\alpha h}$$
$$= (n i_{\alpha\alpha'} + i_\alpha d'_{\alpha'}) m_\alpha - i_\alpha i_{\alpha\alpha'}(4m_\alpha + 2p_\alpha - 2) - i'_{\alpha'\alpha} m'_{\alpha'}.$$

$$(20) \quad \sum_\beta i_{\beta\alpha'} i_{\beta\beta'} D_{\alpha\beta} - \sum_\beta (i_{\beta\alpha'} - i_{\alpha\alpha'})(i_{\beta\beta'} - i_{\alpha\beta'}) D_{\alpha h}$$
$$= (d'_{\alpha'} i_{\alpha\beta'} + d'_{\beta'} i_{\alpha\alpha'}) m_\alpha - i_{\alpha\alpha'} i_{\alpha\beta'}(4m_\alpha + 2p_\alpha - 2) - i'_{l'\alpha} D'_{\alpha'\beta'} \quad (\alpha' \neq \beta').$$

$$(21) \quad \sum_\beta i^2_{\beta\alpha'} D_{\alpha\beta} - \sum_\beta (i_{\beta\alpha'} - i_{\alpha\alpha'})^2 D_{\alpha h}$$
$$= 2 d'_{\alpha'} i_{\alpha\alpha'} m_\alpha - i^2_{\alpha\alpha'}(4m_\alpha + 2p_\alpha - 2) + d_\alpha m'_{\alpha'} - i'_{\alpha'\alpha}(4m'_{\alpha'} + 2p'_{\alpha'} - 2)$$
$$- \sum_{\beta'} (i'_{\beta'\alpha} - i'_{l'\alpha}) D'_{\alpha'\beta'}$$

$$(22) \quad \sum_\alpha \sum_\beta D_{\alpha\beta} - 4 \sum_\alpha m_\alpha = \sum_{\alpha'} \sum_{\beta'} D'_{\alpha'\beta'} - 4 \sum_{\alpha'} m'_{\alpha'}.$$

$$(23) \quad \sum_\alpha \sum_\beta i_l D_{\alpha\beta} = \sum_\alpha i_\alpha(4m_\alpha - p_\alpha + 1) - 11(n-1).$$

$$(24) \quad \sum_\alpha \sum_\beta i_h D_{\alpha\beta} = \sum_\alpha \{n m_\alpha - i_\alpha(p_\alpha - 1)\} - \sum_{\alpha'} m'_{\alpha'} - 5(n-1).$$

$$(25) \quad \sum_\alpha \sum_\beta i_l^2 D_{\alpha\beta} = \sum_\alpha \sum_{\alpha'} i_\alpha i'^2_{\alpha'\alpha} m'_{\alpha'} - \sum_\alpha \{i_\alpha^2(p_\alpha - 1) + d_\alpha^2 i_\alpha\}$$
$$+ 2n^2 - 4n' + 2.$$

$$(26) \quad \sum_\alpha \sum_\beta i_\alpha i_\beta D_{\alpha\beta} = \sum_\alpha \{n i_\alpha m_\alpha - i_\alpha^2(p_\alpha - 1)\} - 2n^2 + 2.$$

$$(27) \quad 2 \sum_\alpha \sum_\beta i_l^3 D_{\alpha\beta} = 3 \sum_\alpha \sum_{\alpha'} i_\alpha^2 i'^2_{\alpha'\alpha} m'_{\alpha'} + \sum_\alpha \{4 i_\alpha^3(2m_\alpha + p_\alpha - 1) - 3 d_\alpha^2 i_\alpha^2\}$$
$$- n^3 + 1.$$

$$(28) \quad 2 \sum_\alpha \sum_\beta i_l^2 i_h D_{\alpha\beta} = \sum_\alpha \sum_{\alpha'} i_\alpha^2 i'^2_{\alpha'\alpha} m'_{\alpha'} - \sum d_\alpha^2 i_\alpha^2 + n^3 - 2nn' + 1.$$

$$(29) \quad 2 \sum_\alpha \sum_\beta i_l i_h^2 D_{\alpha\beta} = \sum_\alpha \sum_{\alpha'} i_\alpha^2 i'_{\alpha'\alpha}(2 - i'_{\alpha'\alpha})\, m'_{\alpha'}$$
$$- \sum_\alpha \{4 i_\alpha^3(p_\alpha - 1) - i_\alpha^2(d_\alpha - 8)\} + n^3 - 1.$$

$$(30) \quad \sum_\alpha \sum_\beta i_{l\alpha'} D_{\alpha\beta} + \sum_{\beta'} D'_{\alpha' h'} = \sum_\alpha i_{\alpha\alpha'}(4m_\alpha - p_\alpha + 1) + 4m'_{\alpha'} - p'_{\alpha'} + 1 - 11 d'_{\alpha'}.$$

$$(31) \quad \sum_\alpha \sum_\beta i_{h\alpha'} D_{\alpha\beta} + \sum_{\beta'} D'_{\alpha' l'} = \sum_\alpha \{d'_{\alpha'} m_\alpha - i_{\alpha\alpha'}(p_\alpha - 1)\} - p'_{\alpha'} + 1 - 5 d'_{\alpha'}.$$

For an ordinary space transformation, the plane characteristics n, i_α of the F-system are replaced by

$$n, \quad n', \quad i_\alpha, \quad m_\alpha, \quad p_\alpha, \quad D_{\alpha\beta}, \quad d_\alpha;$$

and the incidence numbers $i_{\alpha\alpha'}$ of the P-system by $i_{\alpha\alpha'}$, $i_{\alpha'\alpha}$. We shall prove the above relations, and the corresponding (1)′ ... (31)′ with the spaces interchanged, where

i_l, i_h stand for the lower and higher of i_α, i_β, or both for their common value if they are equal; and similarly $i_{l\alpha'}$, $i_{h\alpha'}$ for the lower and higher of $i_{\alpha\alpha'}$, $i_{\beta\alpha'}$;

$D_{\alpha h}$ means $D_{\alpha\beta}$ if $i_\alpha \leqslant i_\beta$, and means 0 if $i_\alpha > i_\beta$;

$D_{\alpha l}$ means $D_{\alpha\beta}$ if $i_\alpha > i_\beta$, and means 0 if $i_\alpha \leqslant i_\beta$.

7. *Noether's and Pannelli's relations.*

To prove all these we have, in the first place, the simplified forms of Noether's and Pannelli's relations; the first three express postulation, genus and equivalence for (ϕ):

$$(n+1)(n+2)(n+3)-24 = \sum_\alpha i_\alpha(i_\alpha+1)\{(3n+6)m_\alpha-(2i_\alpha+1)(2m_\alpha+p_\alpha-1)\} - \sum_\alpha\sum_\beta i_l(i_l+1)(3i_h-i_l+1)\,D_{\alpha\beta},$$

$$(n-1)(n-2)(n-3) = \sum_\alpha i_\alpha(i_\alpha-1)\{(3n-6)m_\alpha-(2i_\alpha-1)(2m_\alpha+p_\alpha-1)\} - \sum_\alpha\sum_\beta i_l(i_l-1)(3i_h-i_l-1)\,D_{\alpha\beta},$$

$$n^3-1 = \sum_\alpha i_\alpha^2\{3nm_\alpha-2i_\alpha(2m_\alpha+p_\alpha-1)\}-\sum_\alpha\sum_\beta i_l^2(3i_h-i_l)\,D_{\alpha\beta},$$

$$\sum_\alpha(p_\alpha-1) = \sum_{\alpha'}(p'_{\alpha'}-1),$$

$$\sum_\alpha\sum_\beta D_{\alpha\beta}-4\sum_\alpha m_\alpha = \sum_{\alpha'}\sum_{\beta'} D'_{\alpha'\beta'}-4\sum_{\alpha'} m'_{\alpha'},$$

$$4\sum_\alpha\sum_\beta i_l D_{\alpha\beta}-\sum_\alpha i_\alpha(5m_\alpha-4p_\alpha+4) = 4\sum_{\alpha'}\sum_{\beta'} i'_{l'} D'_{\alpha'\beta'}-\sum_{\alpha'} i'_{\alpha'}(5m'_{\alpha'}-4p'_{\alpha'}+4).$$

Next we express the facts that the degree of c is that of ϕ':

$$n^2-\sum_\alpha i_\alpha^2 m_\alpha = n';$$

and that the genera of plane sections of ϕ, ϕ' are equal:

$$\tfrac{1}{2}(n-1)(n-2)-\sum_\alpha \tfrac{1}{2}i_\alpha(i_\alpha-1)\,m_\alpha = \tfrac{1}{2}(n'-1)(n'-2)-\sum_{\alpha'}\tfrac{1}{2}i'_{\alpha'}(i'_{\alpha'}-1)\,m'_{\alpha'}.$$

By combining these, and the same with the spaces interchanged, we can prove some of our formulae, and combinations of others, namely

(10), (11), (22), (23), (26), (2)–(1), $3\times(28)$–(27).

Others can be proved, for curves of first species only, by simple geometrical considerations:

(3) expresses the degree of J;

(4) ,, ,, multiplicity of $\omega'_{a'}$ on J';

(5) ,, ,, number of intersections of ϕ, c;

(6) ,, ,, ,, ,, ,, ,, $j_{a'}$, c;

(7) ,, ,, ,, ,, ,, ,, ϕ, $\kappa_{a'}$;

(8) ,, ,, ,, ,, ,, ,, $j_{a'}$, $\kappa_{\beta'}$;

(12) ,, ,, degree of the intersection of ϕ, $j_{a'}$;

(13) ,, ,, ,, ,, ,, ,, $j_{a'}$, $j_{\beta'}$.

This also holds for (4) when $\omega'_{a'}$ is of second species, and is $4i'_{a'}$-fold on J'; for then $\sum_a i_{a'a}$ contains a term -1, arising from the corresponding F-curve of second species, which does not answer to any sheet of J'.

8. *Transformation of certain sets of points.*

I. Let $k'_{\xi'}$ be a curve of degree ξ' and genus p, having $\eta'_{a'}$ simple intersections with $\omega'_{a'}$; and $f'_{x'}$ a surface of degree x' on which k' is simple and $\omega'_{a'}$ is $y'_{a'}$-fold. Let $f'_{z'}$ be another such surface, of degree z' on which k' is simple and $\omega'_{a'}$ is $v'_{a'}$-fold, and let each of η', y', v' be greater than or equal to 1. If $\omega'_{a'}$ is of the first species, from the homologues of k', $f_{x'}$ there fall away $\eta'_{a'}$ P-curves $\kappa_{a'}$ and $y'_{a'}$ P-surfaces $j_{a'}$ respectively. If $\omega'_{a'}$ is of the second species, the corresponding F-curve ω_a of the second species falls away $\mu'_{a'}\eta'_a$ times from k, reducing its degree by $\mu'_{a'}\eta'_{a'}m_a = \eta'_{a'}i'_{a'}$, and the reduction of η_a is as before. No P-surface is dropped from f_x on account of $\omega'_{a'}$, and the multiplicity of ω_a on f_x is increased by η_a. Now $d'_{a'}=0$, and, with our conventions, $i_{a\beta'}=i'_{a'\beta}=0$, $i_{aa'}=i'_{a'a}=-1$; we have for the homologues

$$\xi=\xi'n'-\sum_{a'}\eta'_{a'}i'_{a'}, \qquad \eta_a=\xi'd_a-\sum_{a'}\eta'_{a'}i'_{a'a},$$

$$x=x'n-\sum_{a'}y'_{a'}d'_{a'}, \qquad y_a=x'i_a-\sum_{a'}y'_{a'}i_{aa},$$

$$z=z'n-\sum_{a'}v'_{a'}d'_{a'}, \qquad v_a=z'i_a-\sum_{a'}v'_{a'}i_{aa'},$$

the sums extending to F-curves of both species.

We can take ξ', x', z' to be large compared with η', y', v'; then all of ξ, η, x, y, z, v are greater than 0.

Now k meets its total residual, in the intersection of f_x, f_z, in

$$\xi(x+z-4)-2(p-1)$$

points; since ω_α is of multiplicity greater than or equal to that of the simple curve k on both f_x and f_z, each of its intersections with k absorbs $y_\alpha+v_\alpha-1$ of these points. The free intersections of k with its residual correspond to the points of the same nature for k', f'_x, f'_z:

$$\xi(x+z-4)-2(p-1)-\sum_\alpha \eta_\alpha(y_\alpha+v_\alpha-1) \\ = \xi'(x'+z'-4)-2(p-1)-\sum_{\alpha'}\eta'_{\alpha'}(y'_{\alpha'}+v'_{\alpha'}-1).$$

Now ξ', η', x', y', z', v' are arbitrary. Substituting for $\xi \dots v$, and equating coefficients, we have the relations (3)–(9).

Sum (9) with regard to α':

$$\sum_{\alpha'}\sum_{\alpha} i_{\alpha\alpha'} i'_{\alpha'\alpha} = nn'-1+\sigma, \quad \text{by (5)}.$$

The double sum and the product nn' are symmetrical in S, S'. Hence (1), which signifies that the total number of F-curves is the same in the two spaces. Since F-curves of the second species occur in corresponding pairs, and therefore in equal numbers in S, S', the number of F-curves of the first species is also the same in the two spaces.

In the process of reducing an arbitrary transformation V to one T of ordinary type, the F-elements of each auxiliary transformation give rise to equal numbers of F-curves of T in S, S'. Hence the equality of F-elements holds for V also, provided that any isolated F-point or F-curve of contact of V is counted as that number of distinct F-elements to which it gives rise when V is reduced to T.

II. Next, let f'_x be such that $y'_{\alpha'} < y'_{\beta'}$ if $i'_{\alpha'} < i'_{\beta'}$. The number of free intersections of $\omega'_{\alpha'}$ with its residual, in the intersection of f'_x, ϕ', is

$$m'_{\alpha'}(x'i'_{\alpha'}+y'_{\alpha'}n')-i'_{\alpha'}y'_{\alpha'}(4m'_{\alpha'}+2p'_{\alpha'}-2) \\ -\sum_{\beta'}(y'_{\alpha'}i'_{\beta'}+y'_{\beta'}i'_{\alpha'}-y'_{\alpha'}i'_{\alpha'})D'_{\alpha'h'}-\sum_{\beta'}y'_{\beta'}i'_{\beta'}D'_{\alpha'l'}.$$

If $\omega'_{\alpha'}$ is of the first species, these points correspond to the free intersections of $j_{\alpha'}$ with a general plane section of f_x, in number

$$xd'_{\alpha'}-\sum_\alpha y_\alpha i_{\alpha\alpha'} m_\alpha.$$

If $\omega'_{a'}$ is of the second species, they correspond to points of the plane section of f_x adjacent to the corresponding F-curve ω_a, in number $y_a m_a$, to which the last expression reduces in this case.

The equality holds for $\omega'_{a'}$ of either species; substituting for x, y, and equating coefficients of x', $y'_{\beta'}$, $y'_{a'}$, we have (12), (13), and (15)′–(14)′.

III. Again, let f_x be an arbitrary surface, with x large compared with each y, and y_a greater than each y_β; let f_z be of the same nature. Then

$$y'_{a'}-y'_{\beta'} = x(i'_{a'}-i'_{\beta'})-\sum_{\gamma=1}^{\sigma} y_\gamma(i'_{\beta'\gamma}-i'_{a'\gamma}).$$

If $i'_{a'} < i'_{\beta'}$, it follows that $y'_{a'} < y'_{\beta'}$.

If $i'_{a'} = i'_{\beta'}$, and ω'_{a}, $\omega'_{\beta'}$ intersect, so that $D'_{a'\beta'} > 0$, then every term in $\sum_{\gamma} y_\gamma(i'_{\beta'\gamma}-i'_{a'\gamma})$ vanishes except the two arising from the two F-curves ω_γ, ω_β, of equal multiplicities $i_\gamma = i_\beta$, selected by $\omega'_{a'}$, $\omega'_{\beta'}$ respectively; and

$$y'_{a'}-y'_{\beta'} = -(y_\gamma-y_\beta).$$

In particular, if $\gamma = a$, we have

$$y_a > y_\beta, \quad y'_{a'} < y'_{\beta'}.$$

In every case, $v'_{a'} >, =,$ or $< v'_{\beta'}$ according as $y'_a >, =,$ or $< y'_{\beta'}$.

The number of free intersections of ω_a with its residual, in the intersection of f_x, f_z, is

$$m_a(xv_a+zy_a)-y_a v_a(4m_a+2p_a-2)-\sum_{\beta} y_\beta v_\beta D_{a\beta}.$$

These points correspond to some of the free intersections of $j'_a, f'_{x'}, f'_{z'}$; also, corresponding to each point $O_{a\beta}$ for which $i_a < i_\beta$, there lies on $f'_{x'}$ a P-curve $\kappa'_{\beta a}$, of multiplicity $y_a - y_\beta$ for $f'_{x'}$, which lies on j'_β and meets j'_a once, at the point X' of § 4; this is (v_a-v_β)-fold on $f'_{z'}$, and it accounts for $(y_a-y_\beta)(v_a-v_\beta)$ free intersections of $j'_a, f'_{x'}, f'_{z'}$.

By Noether's general equivalence formula the remaining number is

$$x'z'd_a-\sum_{a'}\left[(x'v'_{a'}+z'y'_{a'})\,i'_{a'a}m'_{a'}+y'_{a'}v'_{a'}\{d_a m'_{a'}-i'_{a'a}(4m'_{a'}+2p'_{a'}-2)\}\right]$$
$$+\sum_{a'}\sum_{\beta'}\begin{bmatrix} \{(y'_{a'}v'_{\beta'}+y'_{\beta'}v'_{a'})\,i'_{a'a}+y'_{a'}v'_{a'}(i'_{\beta'a}-i'_{a'a})\} & \text{if } y'_{a'} \leqslant y'_{\beta'} \\ \text{or } \{(y'_{a'}v'_{\beta'}+y'_{\beta'}v'_{a'})\,i'_{\beta'a}+y'_{\beta'}v'_{\beta'}(i'_{a'a}-i'_{\beta'a})\} & \text{if } y'_{a'} > y'_{\beta'}\end{bmatrix} D_{a'\beta'}$$
$$-\sum_{\beta}(y_a-y_\beta)(v_a-v_\beta)D_{a\beta} \quad (\text{for } i_a < i_\beta).$$

In the second sum, every pair of curves which has $i'_{\alpha'} < i'_{\beta'}$ has also $y'_{\alpha'} < y'_{\beta'}$, and belongs to the first line; then $i'_{\alpha'\alpha} \leqslant i'_{\beta'\alpha}$ and we can write $i'_{l'\alpha}$ for $i'_{\alpha'\alpha}$. Every pair with $i_\alpha > i_\beta$ belongs to the second line, and for it we can write $i'_{l'\alpha}$ for $i'_{\beta'\alpha}$. If $i'_{\alpha'} = i'_{\beta'}$ and also $i'_{\alpha'\alpha} = i'_{\beta'\alpha}$, the two lines are identical, and it is indifferent to which line we assign the pair.

For all these pairs the summand can be written in the common form

$$\{(y'_{\alpha'}v'_{\beta'}+y'_{\beta'}v'_{\alpha'})\,i'_{l'\alpha}+y'_{\alpha'}v'_{\alpha'}(i'_{\beta'\alpha}-i'_{l'\alpha})+y'_{\beta'}v'_{\beta'}(i'_{\alpha'\alpha}-i'_{l'\alpha})\}\,D'_{\alpha'\beta'}.$$

There remain the pairs for which $i'_{\alpha'} = i'_{\beta'}$ and $i'_{\alpha'\alpha} \neq i'_{\beta'\alpha}$, that is, those selected by ω_α and the curves ω_β of equal multiplicity which meet it. If such a pair belongs to the first line,

$$i'_{\beta'\alpha} = i'_{l'\alpha} = i'_{\alpha'\alpha}-1,$$

$$y'_{\beta'}-y'_{\alpha'} = y_\alpha-y_\beta, \quad v'_{\beta'}-v'_{\alpha'} = v_\alpha-v_\beta, \quad D'_{\alpha'\beta'} = D_{\alpha\beta}.$$

The summand is equal to the common form just given with the correction

$$(y'_{\alpha'}v'_{\beta'}+y'_{\beta'}v'_{\alpha'}-y'_{\alpha'}v'_{\alpha'}-y'_{\beta'}v'_{\beta'})(i'_{\alpha'\alpha}-i'_{\beta'\alpha})\,D'_{\alpha'\beta'} = -(y_\alpha-y_\beta)(v_\alpha-v_\beta)\,D_{\alpha\beta},$$

and similarly for a pair belonging to the second line. These corrections can be included in the third sum if this is extended to curves for which $i_\alpha = i_\beta$ as well as those for which $i_\alpha < i_\beta$; this we denote by writing $D_{\alpha h}$ for $D_{\alpha\beta}$. The second and third sums become

$$\sum_{\alpha'}\sum_{\beta'}\{(y'_{\alpha'}v'_{\beta'}+y'_{\beta'}v'_{\alpha'})\,i'_{l'\alpha}+y'_{\alpha'}v'_{\alpha'}(i'_{\beta'\alpha}-i'_{l'\alpha})+y'_{\beta'}v'_{\beta'}(i'_{\alpha'\alpha}-i'_{l'\alpha})\}\,D'_{\alpha'\beta'}$$
$$-\sum_{\beta}(y_\alpha-y_\beta)(v_\alpha-v_\beta)\,D_{\alpha h}.$$

If ω_α is of the second species, we find, as before, that the formula remains correct. Substituting for x, y, z, v, and equating coefficients, we have (17)$-2\times$(16), 19, 20, 21.

9. *The compound transformation W.*

Let $V_{x-x''}$ be another ordinary transformation, between S and a third space S'', which has in common with T one F-curve in S of the first species, say ω_α of multiplicity y_α, the other F-curves $\Sigma\omega_\delta$ of multiplicities Σy_δ being in as general positions as possible with regard to T, so that ω_δ meets ω_α in $D_{\alpha\delta}$ points, but does not meet any ω_β ($\beta \neq \alpha$). To construct V, we take a monoidal transformation of arbitrary degree, with an arbitrary simple

F-curve ω, reduce this to ordinary type, and identify one of the spaces with S and the homologue of ω with ω_a. We can take x large compared with y_a. Let f_x be the general homaloid of V.

We have thus set up the compound transformation $W \equiv V.T^{-1}$ between S', S'', whose homaloid in S' is the surface $f'_{x'}$, which $\underset{T}{\sim} f_x$ (corresponds to f_x in the transformation T), where

$$x' = xn' - y_a d_a,$$

and the F-system of W consists of four sets of ordinary curves:

(i) An F-curve $\omega'_{a'}$ of T is of degree m'_a, and of multiplicity $xi'_{a'} - y_a i'_{a'a}$ for W.

(ii) The curve ω'_δ which $\underset{T}{\sim} \omega_\delta$ is of degree $nm_\delta - i_a D_{a\delta}$, and of multiplicity y_δ for W; it meets another curve ω'_ϵ of the same set in $D_{\delta\epsilon}$ points, and meets $\omega'_{a'}$ in the $d'_{a'} m_\delta - i_{aa'} D_{a\delta}$ points which $\underset{T}{\sim}$ those intersections of ω_δ, $j_{a'}$ which do not lie on ω_a.

(iii) If $y_a < y_\delta$, each intersection $O_{a\delta}$ of ω_a, ω_δ is a hypermultiple point of f_x on ω_a, and $\underset{T}{\sim}$ a curve κ'_a of degree i_a. Each of the $D_{a\delta}$ such curves is an F-curve of W of second species of multiplicity $y_\delta - y_a$; it meets $\omega'_{a'}$ in $i_{aa'}$ points, and meets once the one ω'_δ with which it is associated.

(iv) If $i_a < i_\beta$, each $O_{a\beta} \underset{T}{\sim}$ a curve κ'_β breaking up into κ'_a, $\kappa'_{\beta a}$; the first component lies on j'_a and is dropped from $f'_{x'}$, but the second remains as an F-curve of W of second species, of degree $i_\beta - i_a$, and of multiplicity y_a. It meets $\omega'_{a'}$ in $i_{\beta a'} - i_{aa'}$ points, and meets no F-curve of W of the other sets.

The F-system of W in S'' is of the same nature, and consists of

$$\Sigma\omega''_{\delta''}, \qquad \Sigma\omega''_\beta, \qquad \Sigma\kappa''_a \ (i_a < i_\beta), \qquad \Sigma\kappa''_{\delta a} \ (y_a < y_\delta),$$

of multiplicities $ny''_{\delta''} - i_a y''_{\delta'' a}$, $\quad i_\beta$, $\quad i_\beta - i_a$, $\qquad i_a$, respectively.

There are two kinds of P-surfaces of W in S''. We have $\omega'_a \underset{W}{\sim}$ the surface $j''_{a'}$ which $\underset{V}{\sim} j_{a'}$; on this, the multiplicities of

$$\omega''_{\delta''}, \qquad \omega''_\beta, \qquad \kappa''_a, \qquad \kappa''_{\delta a}$$

are $\qquad d'_{a'} y''_{\delta''} - i_{aa'} y''_{\delta'' a}$, $\quad i_{\beta a'}$, $\quad i_{\beta a'} - i_{aa'}$ (if positive), $\quad i_{aa'}$ respectively.

Also $\omega'_\delta \underset{W}{\sim} j''_\delta$, on which $\omega''_{\delta''}$ is $y''_{\delta''\delta}$-fold; and, if $y_a < y_\delta$, it contains simply the $D_{a\delta}$ curves $\kappa''_{\delta a}$ associated with ω_δ.

10. *Incidence relations for W.*

Thus W is an ordinary transformation, and we can apply to it the formulae already proved for T. In particular, apply (4) to κ''_α:

$$4(i_\beta - i_\alpha) - 1 = \sum_{\alpha'} (i_{\beta\alpha'} - i_{\alpha\alpha'}) - 1,$$

where on the right the sum extends to those values of α' for which the summand is positive, and the term -1 arises from the corresponding F-curve $\kappa'_{\beta\alpha}$ of the second species.

But, by the same formula for T,

$$4(i_\beta - i_\alpha) = \sum_{\alpha'} (i_{\beta\alpha'} - i_{\alpha\alpha'}),$$

summed for all values of α', whether the summand is positive, zero, or negative. Hence there are no negative terms in the last sum, that is, $i_{\alpha\alpha'} \leqslant i_{\beta\alpha'}$ provided that $i_\alpha < i_\beta$ and ω_α meets ω_β, as was proved geometrically in § 4.

We have here assumed that ω_α is of the first species for T; if it is of the second species, every κ'_α is adjacent to the curve $\omega'_{\alpha'}$ which $\underset{T}{\sim}\ \omega_\alpha$, and which is an F-curve of contact for W. But though W is no longer ordinary, κ''_α is an ordinary F-curve of W, for which (4) still holds.

We shall next apply to W the formula (23), which can be deduced from those already proved. We need to know which has the greater multiplicity of each pair of intersecting F-curves of W.

The difference of multiplicities of $\omega'_{\beta'}$, $\omega'_{\alpha'}$ is

$$x(i''_{\beta'} - i''_{\alpha'}) - y_\alpha(i''_{\beta'\alpha} - i''_{\alpha'\alpha}).$$

Since x is large compared with y_α, we have $\omega'_{\alpha'}$ of lower multiplicity provided that either

$$i''_{\alpha'} < i''_{\beta'}$$

or else $$i''_{\alpha'} = i''_{\beta'} \text{ and also } i''_{\alpha'\alpha} > i''_{\beta'\alpha}.$$

The last alternative occurs if, and only if, $\omega'_{\alpha'}$, $\omega'_{\beta'}$ are a pair of F-curves of T selected by ω_α and an ω_β of equal multiplicity.

The multiplicity for W of ω'_δ is less than that of $\omega'_{\alpha'}$; its relation to another curve ω'_ϵ of the same set as itself is the same as that of ω_δ to ω_ϵ for V. The F-curves of W of the second species are all of lower multiplicities than any F-curves which they meet.

Hence the term in (23) arising from the intersection of $\omega'_{\alpha'}$, $\omega'_{\beta'}$ is

$$(xi''_{l'} - y_\alpha i''_{l'\alpha})\, D'_{\alpha'\beta'} \quad \text{if} \quad i''_{\alpha'} \neq i''_{\beta'}, \quad \text{or} \quad i''_{\alpha'} = i''_{\beta'} \quad \text{and} \quad i''_{\alpha'\alpha} = i''_{\beta'\alpha};$$

and it is $$\{xi''_{l'} - y_\alpha(i''_{l'\alpha} + 1)\}\, D'_{\alpha'\beta} \quad \text{if} \quad i''_{\alpha'} = i''_{\beta'} \quad \text{and} \quad i''_{\alpha'\alpha} \neq i''_{\beta'\alpha},$$

$$= (xi''_{l'} - y_\alpha i''_{l'\alpha})\, D'_{\alpha'\beta'} - y_\alpha D_{\alpha\beta} \quad \text{where} \quad i_\alpha = i_\beta.$$

The equation is therefore :

$$
\begin{aligned}
\sum_{\alpha'}\sum_{\beta'}(x i'_{l'} - y_\alpha i'_{l'\alpha})\, D'_{\alpha'\beta'} - y_\alpha \sum_{\beta} D_{\alpha\beta}\ (\text{for } i_\alpha = i_\beta) + \sum_{\delta}\sum_{\epsilon} y_l D_{\delta\epsilon} \\
+\sum_{\delta}\sum_{\alpha'} y_\delta (d'_{\alpha'} m_\delta - i_{\alpha\alpha'} D_{\alpha\delta}) + \sum_{\beta}\sum_{\alpha'} y_\alpha (i_{\beta\alpha'} - i_{\alpha\alpha'})\, D_{\alpha\beta}\ (\text{for } i_\alpha < i_\beta) \\
+\sum_{\delta}(y_\delta - y_\alpha)(\sum_{\alpha'} i_{\alpha\alpha'} + 1)\, D_{\alpha\delta}\ \ (\text{for } \ y_\alpha < y_\delta) \\
-\sum_{\alpha'}(x i'_{\alpha'} - y_\alpha i'_{\alpha'\alpha})(4m'_{\alpha'} - p'_{\alpha'} + 1) - \sum_{\delta} y_\delta \{ 4(n m_\delta - i_\alpha D_{\alpha\delta}) - p_\delta + 1 \} \\
-\sum_{\beta}\{4(i_\beta - i_\alpha) + 1\}\, y_\alpha D_{\alpha\beta}\ (\text{for } i_\alpha < i_\beta) \\
-\sum_{\delta}(4 i_\alpha + 1)(y_\delta - y_\alpha)\, D_{\alpha\beta}\ (\text{for } y_\alpha < y_\delta) \\
+11(x n' - y_\alpha d_\alpha - 1) = 0.
\end{aligned}
$$

By means of equations already proved, applied to T and to V, this reduces to (30)′. If ω_α is of the second species for T, (30) becomes the same as (21), already proved for either species.

The rest of the formulae of § 5 now follow by combination and summation of those already proved. Many others can be obtained by pursuing the methods further. For example, Noether's formula for the genus $-p'_{\alpha'}$ of $j_{\alpha'}$ gives

$$
\begin{aligned}
(d'_{\alpha'} - 1)(d'_{\alpha'} - 2)(d'_{\alpha'} - 3) + 6p'_{\alpha'} \\
= \sum_{\alpha} i_{\alpha\alpha'}(i_{\alpha\alpha'} - 1)\{(3d'_{\alpha'} - 6)\, m_\alpha - (2i_{\alpha\alpha'} - 1)(2m_\alpha + p_\alpha - 1)\} \\
-\sum_{\alpha}\sum_{\beta} i_{l\alpha'}(i_{l\alpha'} - 1)(3 i_{h\alpha'} - i_{l\alpha'} - 1)\, D_{\alpha\beta}.
\end{aligned}
$$