LINEAR DEPENDENCE OF THE SCHUR QUADRICS OF A CUBIC SURFACE

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LET $a_1 \ldots a_6$, $b_1 \ldots b_6$ be a double six on a cubic surface. Then there exists a Schur quadric with regard to which a_a, b_a are polar reciprocals, for $a = 1, \ldots, 6$. The 36 such quadrics are each harmonically inscribed to the first polar of an arbitrary point; hence their tangential equations satisfy four linear relations, and not more than six of them can be linearly independent†.

We shall show that just six of the tangential equations, and ten of the point equations, are linearly independent, by proving it for the special cubic surface

$$x^3+y^3+z^3+w^3 = 0,$$

for which 18 sets of co-planar lines are concurrent.

Nine of its lines have the form

$$x+\epsilon^a y = z+\epsilon^\beta w = 0,$$

where $\epsilon^3 = 1, \quad a, \beta = 0, 1, \text{ or } 2;$

the rest are formed from these by permuting the coordinates.

For the double six

$$\begin{array}{ll}
x+y = z+\epsilon^2 w = 0, & x+\epsilon^2 y = z+w = 0, \\
x+\epsilon y = z+w = 0, & x+y = z+\epsilon w = 0, \\
x+\epsilon^2 y = z+\epsilon w = 0, & x+\epsilon y = z+\epsilon^2 w = 0, \\
x+z = y+\epsilon w = 0, & x+\epsilon z = y+w = 0, \\
x+\epsilon z = y+\epsilon^2 w = 0, & x+\epsilon^2 z = y+\epsilon w = 0, \\
x+\epsilon^2 z = y+w = 0, & x+z = y+\epsilon^2 w = 0,
\end{array}$$

the tangential equation of the Schur quadric is

$$q_0 \equiv \eta\zeta - \xi\theta = 0.$$

We can write ϵx for x, and therefore $\epsilon^2 \xi$ for ξ, and so on, and also permute the coordinates. This gives nine quadrics. From the trios such as $\eta\zeta - \epsilon^a \xi\theta$ $(a = 0, 1, 2)$, by linear combination, we form the six plane-pairs such as $\eta\zeta$; hence six of the tangential equations are effectively independent.

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† H. F. Baker, *Proc. London Math. Soc.* (2), 11 (1912), 285–301 (300).

For another double six

$$\begin{array}{ll} x+\epsilon y = z+\epsilon w = 0, & x+\epsilon^2 w = z+\epsilon^2 y = 0, \\ x+\epsilon^2 y = z+\epsilon^2 w = 0, & x+\epsilon w = z+\epsilon y = 0, \\ x+\epsilon z = w+\epsilon y = 0, & x+\epsilon^2 y = w+\epsilon^2 z = 0, \\ x+\epsilon^2 z = w+\epsilon^2 y = 0, & x+\epsilon y = w+\epsilon z = 0, \\ x+\epsilon w = y+\epsilon z = 0, & x+\epsilon^2 z = y+\epsilon^2 w = 0, \\ x+\epsilon^2 w = y+\epsilon^2 z = 0, & x+\epsilon z = y+\epsilon w = 0, \end{array}$$

the Schur quadric is

$$q_1 \equiv \eta\zeta+\zeta\xi+\xi\eta+\xi\theta+\eta\theta+\zeta\theta = 0.$$

In this replace ξ, η by $\epsilon\xi, \eta$; $\epsilon\xi, \epsilon\eta$; $\epsilon\xi, \epsilon^2\eta$ in turn. We obtain

$$\eta\zeta+\eta\theta+\zeta\theta+\epsilon\xi(\eta+\zeta+\theta) = 0,$$

$$\zeta\theta+\epsilon(\xi+\eta)(\zeta+\theta)+\epsilon^2\xi\eta = 0,$$

$$\xi\eta+\zeta\theta+\epsilon\xi(\zeta+\theta)+\epsilon^2\eta(\zeta+\theta) = 0.$$

By exchanging ϵ, ϵ^2 and permuting the coordinates we obtain all the $9+1+8+6+12 = 36$ Schur quadrics, all linear functions of the six independent product terms.

The point equations are formed in the same way from

$$yz-xw = 0$$

and
$$x^2+y^2+z^2+w^2-yz-zx-xy-xw-yw-zw = 0,$$

by introducing ϵ and permuting the coordinates; we verify that ten are linearly independent.

Prof. A. L. Dixon has shown that for any cubic surface there are at least 120 linear relations between trios of Schur quadrics, such as the trio associated with three double sixes, each pair of which have six lines in common. Our example shows that there are no other related trios. By substitutions as before, the three types

$$q_0(\xi, \eta, \zeta, \theta)+\epsilon q_0(\epsilon\xi, \eta, \zeta, \theta)+\epsilon^2 q_0(\epsilon^2\xi, \eta, \zeta, \theta) \equiv 0,$$

$$q_1(\xi, \epsilon\eta, \zeta, \theta)-\epsilon q_1(\xi, \eta, \epsilon^2\zeta, \theta)+(1-\epsilon)q_0(\xi, \eta, \zeta, \theta) \equiv 0,$$

$$q_1(\xi, \eta, \zeta, \theta)+\epsilon q_1(\epsilon\xi, \eta, \zeta, \theta)+\epsilon^2 q_1(\epsilon^2\xi, \eta, \zeta, \theta) \equiv 0,$$

give rise to the $3+81+36 = 120$ relations between trios.