ON THE 3–3 BIRATIONAL TRANSFORMATION IN THREE DIMENSIONS*

By HILDA P. HUDSON, M.A.

[Received March 6th, 1910.—Read March 10th, 1910.]

CONTENTS.

1.

In a paper read before the London Mathematical Society in 1869, Cayley† was the first to call attention to birational transformations in three dimensions, as an immediate extension of Cremona's two-dimensional work. In particular he investigated the 3–3 transformation, which is most compactly expressed by three bilinear relations between the two sets of homogeneous coordinates in the two spaces. He shewed that planes in one space correspond to cubic surfaces in the other, and since three planes meet in only one point, three of these cubic surfaces meet in only one variable point, and their remaining intersection is a fixed sextic curve with seven apparent double points. Cayley gave the formulæ of transformation for two cases: (i) when the sextic does not degenerate, and (ii) when it becomes four non-intersecting straight lines and their two transversals. Soon afterwards Cremona‡ and Noether§ both gave many examples proving that the sextic may degenerate in different ways in the two spaces.

The method of inversion, which is a special form of this transformation, had long been known; but since the time of Cremona the general theory

* For the suggestion of this subject I am indebted to Mr. Berry.

† Cayley, *Proc. London Math. Soc.*, Vol. III, p. 127, 1869.

‡ Cremona, *Rend. R. Ist. Lomb.*, T. IV, p. 269, 1871; *Math. Ann.*, Bd. IV, p. 213, 1871; *Ann. di Mat.*, T. V, p. 131, 1872; and other papers.

§ Noether, *Math. Ann.*, Bd. III, p. 547, 1871.

has been greatly developed, and birational transformations are constantly employed in connection with such problems as singularities of surfaces.* In particular, the 3–3 transformation, and some of its special cases, have been very useful, and it seems worth while to attempt a systematic treatment of all its varieties. Many of these have been discussed,† but, as far as I know, with no attempt at completeness.

The following paper deals with the cases where the first principal sextic curve consists of six straight lines with eight simple intersections, or their equivalent in multiple intersections, taking account of all the possible arrangements of these intersections on the lines.

The different types which arise are as follows :—

		No. of Cases.
A	8 simple intersections	5
B	1 triple and 6 simple intersections	7
B′	1 special triple and 5 simple intersections...	3
C	2 triple and 4 simple intersections	4
C′	1 special triple, 1 ordinary triple, and 3 simple intersections	2
C″	2 special triple and 2 simple intersections...	1
D	3 triple and 2 simple intersections	1
E	4 triple intersections	1
F	1 quadruple and 4 simple intersections	2
G	1 quadruple, 1 triple, and 2 simple intersections	1
H	1 quintuple and 2 simple intersections	1
	Total	28

It appears that for the purpose of this paper the following must be taken as equivalent :—

An ordinary triple intersection	with	2	simple	intersections,
A special ,, ,,	,,	3	,,	,,
A quadruple intersection	,,	4	,,	,,
A quintuple ,,	,,	6	,,	,,

* The most recent paper I have seen on birational transformations in general, is by Margherita Beloch, *Ann. di Mat.*, T. XVI, p. 27, 1909.

† v. Krieg, *Zeitschrift für Math. und Phys.*, Bd. XXIX, p. 38, 1884; Ascioni, *Giorn. di Mat.*, T. XXXI, p. 55, 1893; Döhlemann, *Sitzungsber. d. K. Bay. Ak. der Wiss. zu München*, Bd. XXIV, p. 41, 1894 (among many others).

2.

If we start with the three bilinear relations between the two sets of homogeneous coordinates $xyzw$, $XYZW$ of corresponding points in the two spaces, they can be put into either of the forms

$$\left.\begin{array}{l} Xp_1+Yq_1+Zr_1+Ws_1 = 0 \\ Xp_2+Yq_2+Zr_2+Ws_2 = 0 \\ Xp_3+Yq_3+Zr_3+Ws_3 = 0 \end{array}\right\} \quad \text{or} \quad \left.\begin{array}{l} xP_1+yQ_1+zR_1+wS_1 = 0 \\ xP_2+yQ_2+zR_2+wS_2 = 0 \\ xP_3+yQ_3+zR_3+wS_3 = 0 \end{array}\right\},$$

where $p_1 \ldots s_3$ are linear functions of $xyzw$, and $P_1 \ldots S_3$ of $XYZW$. Solving these, $XYZW$ are proportional to the determinants of the matrix

$$\left\| \begin{array}{cccc} p_1 & q_1 & r_1 & s_1 \\ p_2 & q_2 & r_2 & s_2 \\ p_3 & q_3 & r_3 & s_3 \end{array} \right\|,$$

and $xyzw$ to those of

$$\left\| \begin{array}{cccc} P_1 & Q_1 & R_1 & S_1 \\ P_2 & Q_2 & R_2 & S_2 \\ P_3 & Q_3 & R_3 & S_3 \end{array} \right\|,$$

and the first sextic curve arises as the common intersection of the four cubic surfaces in the first space $0 = X = Y = Z = W$.

In what follows this process is reversed. First, I enumerated the particular forms of degenerate sextics to be considered. For each of these I wrote down the most convenient equations, and obtained by inspection four linearly independent cubic surfaces $0 = X = Y = Z = W$ containing them; in every case degenerate surfaces are available. (It is possible to express the conditions that the general cubic surface contains the sextic; these conditions can be used to reduce its twenty coefficients to linear functions of four; these four then are multiplying $XYZW$.) Then I find by inspection the three bilinear relations between $XYZW$. (Again it is possible to write down the general form $Xp+Yq+Zr+Ws$, containing sixteen arbitrary constants, and express that this is identically 0; this must leave three arbitrary constants, each multiplying the left-hand side of one of the bilinear equations sought.) Having obtained these equations I solved them for $x : y : z : w$; these are proportional to four cubic functions of $XYZW$, which equated to 0 give four cubic surfaces in the second space whose common intersection is the second sextic. One of the chief interests of the investigation lies in the extraordinary variety of forms assumed by this curve. The only constant feature, besides the essential one of having seven apparent double points, is that in all the cases here investigated, *i.e.*, where the first sextic is six distinct straight lines, the second sextic consists partly of three straight lines. In twelve of the twenty-eight cases considered it is of the same kind as the first sextic; in two others it is a different set of six straight lines, in ten it is a conic and four straight lines and in the remaining four cases, a cubic and three straight lines.

I have given the work at length in a typical case; for the rest I have stated the bilinear relations in their simplest form and shewn each sextic in a diagram. A closed oval represents a conic and an unclosed curve a cubic. The tetrahedron of reference is shewn in each figure, the vertices being marked 1, 2, 3, 4, and where the directions of three lines that meet in a point are parallel to a plane, this is also marked by a small arc meeting the three. The intersections shewn are the actual intersections; any which would be introduced by producing lines would be apparent intersections.

3.

The distribution of the eight intersections on the six straight lines is restricted by the fact that the system lies on a cubic surface. Thus:

(i) Not more than three can lie in a plane; for if so the four linearly independent cubic functions, which are proportional to the four coordinates in the other space, equated to 0 give degenerate surfaces consisting of this plane and a quadric through the other two lines; the functions have a common linear factor which can be removed, and the transformation is not cubic but quadratic. In particular, there cannot be five intersections between four lines.

(ii) If three of the straight lines lie in a plane, all the other three must meet one or other of them; for, as above, the cubic surfaces cannot meet the plane in any other point.

(iii) Again, the six straight lines cannot all lie in two planes or on a quadric surface; for if so the quadric and an arbitrary plane form a cubic surface with four arbitrary constants containing the system, and is therefore the most general cubic that does so; the four independent cubic functions have a common quadratic factor and the transformation is linear. (The case in which a cubic surface, though containing six generators of a quadric, need not contain the quadric, does not arise, for they are three generators of one system and three of the other, involving nine intersections.)

4.

The general cubic surface will be taken in the form

$$Ax^3+Bx^2y+Cxy^2+Dy^3+z(Ex^2+Fxy+Gy^2)+w(Hx^2+Jxy+Ky^2)$$
$$+z^2(Lx+My)+zw(Nx+Py)+w^2(Qx+Ry)$$
$$+Sz^3+Tz^2w+Uzw^2+Vw^3=0.$$

It meets any straight line in three points, and if it meet it in a fourth it contains the whole line; therefore to contain a straight line imposes four conditions. If the line (a) is $0=z=w$, the four conditions are

$$A=B=C=D=0.$$

To contain two straight lines imposes eight conditions, unless the lines intersect, when one condition is the same for both, viz., that their common point lies on the surface. If the second line (b) is $0=y=w$, the three further conditions are $E=L=S=0$. Similarly for every intersection a condition is lost. Six non-intersecting straight lines would impose twenty-four conditions; but since there are fifteen real or apparent double points, if the set of six straight lines is a particular case of Cayley's sextic curve with seven apparent double points, there are eight intersections, reducing the number of conditions to sixteen, and leaving four independent coefficients. There are therefore four linearly independent cubic surfaces through the six straight lines; these, or four linear functions of them, are proportional to the second set of coordinates.

A. *Eight simple intersections.*

Of the six lines $abcdef$, let a be the one that meets most. It cannot meet them all, for if so there would be three intersections between $bcdef$; one (b) would meet two others (cd) and $abcd$ would lie in a plane.

(1) If a meets $bcde$ (and not f), f may meet $bcde$, or

(2) f may meet bcd and e meet b, or

(3) f may meet bc; then bc cannot meet (for if so $abcf$ lie in a plane), nor can d (or e) meet both of them (for if so $abcd$ lie in a plane); therefore d meets b, and e meets c.

And f cannot meet only b, for if so there are three intersections between $bcde$, one (b) must meet two others (cd) and $abcd$ lie in a plane.

If a meets bcd (and not e or f) (*i.e.*, considering the case where no line meets more than three), then if bcd do not meet each other, either

(4) e and f do not meet, one (e) meets all of bcd and f meets bc, or

(5) *ef* meet, and each meets two of *bcd*; say, *e* meets *bc*; then *f* cannot meet the same two (for if so *bcef* lie in a plane) but meets *bd*.

And if *bc* meet, they lie in a plane with *a*; *def* all meet this plane on one of the lines; but *a* meets *d*, and *bc* are only to meet one more each; therefore *b* meets *e*, and *c* meets *f*, and there are two intersections between *def*. But this configuration is the same as in (5) where *bef* form the triangle.

B. *One triple and six simple intersections.*

If *abc* meet in a point, they may or may not lie in a plane.

If they do not, let their equations be $0 = z = w$, $0 = y = w$, $0 = y = z$. To contain these imposes ten conditions on the cubic

$$A = B = C = D = E = L = S = H = Q = V = 0,$$

so that this singularity counts as two simple intersections and one apparent double point, and it can be obtained from A by letting two simple intersections come together. It is a conical point on all the cubic surfaces.

No one of *def* can meet all three arms of the triple point without passing through the vertex and forming a quadruple point (F below), and no two of *def* can meet the same two arms without both lying in their plane.

(1) If *def* do not meet each other, each meets a different pair of arms.

(2) If *e* meets *f* (and neither meets *d*), since one of them (*e*) must meet two arms (*ac*), *f* can only meet *b* (for if it met *a*, *acef* would lie in a plane), and then *d* must meet *b* and *a* (or *c*).

If *f* meets *de* (but *de* do not meet), there are four intersections between *def* and the arms, either two on two of *def* or two on one and one on each of the others.

(3) If *f* meets *bc*, then *d* and *e* must each meet *a*, for they cannot meet *b* or *c*.

(4) If *f* meets *c* only, one of *d* or *e* must meet *a* and *b*, and the other *a* (or *b*).

(5) If *f* meets no arm, *d* and *e* meet different pairs of arms.

If *def* form a triangle, all three arms meet one or other of them, either:

(6) each arm meets a different side, or

(7) two arms meet one side and the third meets another side.

B′. *One special triple and five simple intersections.*

If the three arms of the triple point lie in a plane, let their equations be

$$0 = z = w, \qquad 0 = y = w, \qquad 0 = w, \; y = z.$$

There are only nine conditions

$$A = B = C = D = E = L = S = F = G+M = 0.$$

The reason for this is that all the cubic surfaces meet the plane ab in a, b and a third straight line. If two other points in the plane lie on all the surfaces, the straight line c joining them is part of the common intersection; so c only gives two more conditions and must be considered as meeting both a and b.

Such a triple intersection will be called *special;* it is not a conical point on the surfaces. It is equivalent to three simple intersections, and can be obtained from A by making a triangle diminish to a point, the sides remaining in a plane.

Now def must all meet the plane abc on one of the arms, and none of them can meet two arms without making a fourth line in the plane; therefore they have two intersections among themselves, say df, ef. Then a cannot meet def, for if so the four lie in a plane; therefore either

(1) a meets de, and f meets b, or

(2) a meets df, and e meets b, or

(3) a meets d, and e meets b, and f meets c.

C. *Two triple and four simple intersections.*

If neither triple point is special, each is a conical point on all the cubic surfaces, the line joining them lies on all the cubics and the triple points have a common arm; let them be abc, ade.

Now b (or c) cannot meet both d and e, for if so $abde$ would lie in a plane; therefore there are at most two intersections between the arms, bd, ce, and at least two on f. Also f cannot meet a, for if so it must also meet another (b); then neither f nor b can meet any other, and c would have to meet both d and e, which is impossible; therefore either

(1) f meets $bcde$, or

(2) f meets bcd, and e meets b (or c), or

(3) f meets bc, one of these (b) meets d, and c meets e, or

(4) f meets bd, b meets e, and c meets d.

These cases can be obtained from B.

C′. *One special triple, one ordinary triple, and three simple intersections.*

The special triple point is not a conical point on the surfaces, and the two need not have a common arm. If they do, let them be (abc), ade; then neither d nor e can meet the plane abc again, all the three intersections must lie on f, viz., one where it meets the plane abc, and two with d, e. But then the whole set lies in the two planes abc, def, and the transformation degenerates; therefore the triple points have not a common arm. Let them be (abc), def; then def each meet one or other of abc. Either

(1) each meets a different one, or

(2) two meet a and one meets b.
For def cannot all meet a without the four lying in a plane.

These cases can be obtained from B or B′.

C″. *Two special triple and two simple intersections.*

The two must have a common arm, for if not the six lines would lie in the two planes. Let them be (abc), (ade). None of these can meet again, for if so all five would lie in a plane; therefore there are two intersections on f, where it meets the two planes. f cannot meet a, for if so it could meet neither plane again; therefore it meets b (or c) and d (or e).

This case can be obtained from B′.

D. *Three triple and two simple intersections.*

With three triple points and only six lines, each pair must have a common arm, and three of the lines are the sides of the triangle of triple points. None of them can be special, for if so there would be a fourth line in the plane of the triangle. There must be two intersections between the remaining three arms.

This case can be obtained from C.

E. *Four triple intersections.*

As in the last case, no one can be special, and the six lines are the six edges of the tetrahedron of triple points.

This case can be obtained from D.

If four straight lines meet in a point, and no three lie in a plane, let their equations be

$$0 = z = w, \qquad 0 = y = w, \qquad 0 = y = z, \qquad y = z = w,$$

giving the twelve conditions

$$A = B = C = D = E = L = S = H = Q = V = F+J+N$$
$$= G+K+M+P+R+T+U = 0,$$

so that this singularity is equivalent to four simple intersections. It cannot, in general, be obtained from B by making a fourth line pass through the triple point, as that would only absorb three intersections, except in the two cases where (i) the triple point was special, (ii) the fourth line met two of the arms; each of these leads to a special quadruple point with three arms in a plane. Now, if three arms lie in a plane, let the equations of the last be replaced by $y = z$, $0 = w$; then the last two conditions are replaced by two others, viz., $F = G+M = 0$, so in this case also the quadruple point is equivalent to four simple intersections, we have a particular case and not a fresh type. The analytical view of this is as follows:—

Each straight line through the vertex (1, 0, 0, 0) imposes one condition on the coefficients of each power of x. Now, there are: one term in x^3, three in x^2, six in x^1, ten in x^0. The first straight line imposes four conditions and abolishes the term in x^3. The second and third impose three conditions each, and abolish the terms in x^2, unless the three lines lie in a plane, when the three conditions on the coefficients of terms in x^2 reduce to two (and the factor multiplying x^2 gives the equation of the plane). With four arms, which cannot all lie in a plane, taking first three which do not, they impose ten conditions and abolish the terms in x^3, x^2. The fourth, whether it lie in a plane with two of the others or not, imposes two conditions, on the coefficients of terms in x^1, x^0 respectively. A fifth and sixth also impose two conditions each, and abolish the terms in x^1 (in which case x has disappeared entirely, the cubic surfaces are all cones, and cannot have a single variable point of intersection: the transformation breaks down); except in the case where the sixth straight line lies on the quadric cone through the first five, given by the factor multiplying x. But in this case the six lines lie on a quadric and do not give rise to a cubic transformation. Therefore a sextuple point need not be considered.

F. *One quadruple and four simple intersections.*

Let the quadruple point be *abcd*. Then neither *e* nor *f* can meet three arms.

(1) If they do not meet each other, *ef* each meet two arms; not the same pair, for if so the four lie in a plane; and not distinct pairs, *ab*, *cd*, for if so the whole lies in the two planes *abe*, *cdf*; therefore one (*e*) meets *ab* and *f* meets *ac*.

(2) If *ef* meet, one (*e*) meets two arms and lies in their plane, and *f* meets a third arm.

These cases cannot be obtained from former ones.

G. *One quadruple, one triple, and two simple intersections.*

Let these be *abcd*, *aef*, for there must be a common arm. Then *e*, *f* each meet a different one of *bcd*. The triple point cannot be special, for if so one intersection would make four lines lie in a plane.

This case can be obtained from F.

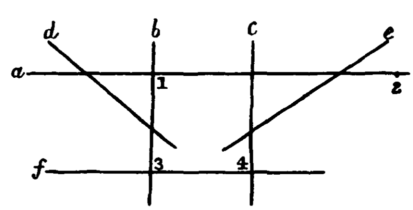
H. *One quintuple and two simple intersections.*

If the quintuple point is *abcde*, then *f* meets two of the arms.

This case cannot be obtained from any former one.

5.

I give a specimen of the work in detail.



With this set of intersections, taking the tetrahedron of reference as shewn, the equations of the six lines can be

$$(a)\ z = w = 0, \qquad (b)\ y = w = 0, \qquad (d)\ w = lx + my + nz = 0,$$

$$(f)\ x = y = 0, \qquad (c)\ z = \alpha x + \beta y = 0, \qquad (e)\ z = px + qy + sw = 0.$$

By inspection four sets of three planes are found to contain these, and we write

$$\frac{X}{yzw} = \frac{Y}{xzw} = \frac{Z}{w(\alpha x + \beta y)(px + qy + sw)} = \frac{W}{yz(lx + my + nz)}.$$

These satisfy the three bilinear relations

$$Xx = Yy, \qquad Zz = (\beta X + \alpha Y)(px + qy + sw), \qquad Ww = X(lx + my + nz).$$

Solving these, we find

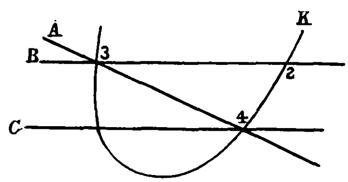
$$\frac{x}{Y[ZW - nsX(\beta X + \alpha Y)]} = \frac{y}{X[ZW - nsX(\beta X + \alpha Y)]} = \frac{z}{(\beta X + \alpha Y)[W(qX + pY) + sX(mX + lY)]}$$
$$= \frac{w}{X[Z(mX + lY) + n(\beta X + \alpha Y)(qX + pY)]}.$$

The denominators equated to 0 give four cubic surfaces whose common intersection is the second sextic. This is found to consist of three straight lines A, B, C and a cubic K, the equations being

$$(A)\ X = Y = 0, \qquad (B)\ X = W = 0, \qquad (C)\ Z = \beta X + \alpha Y = 0,$$

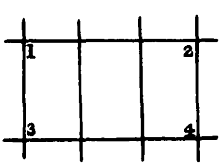
and B, K are the complete intersection of the two quadrics

$$W(qX + pY) + sX(mX + lY) = 0, \qquad ZW = nsX(\beta X + \alpha Y).$$

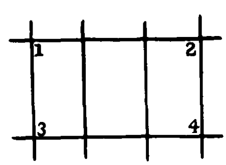


In this, putting $\alpha = 0$, *abc*, *BCK* concur (B1); $p = 0$, *abe* concur and K degenerates (B2); $\alpha = s = 0$, *abc*, *cef* concur (C3), &c.; $n = m = q = \alpha = s = 0$, *abe*, *cef*, *ade*, *bdf* concur (E). In the last case the sextic in each space consists of the six edges of a tetrahedron, and the transformation is the well known one of inversion.

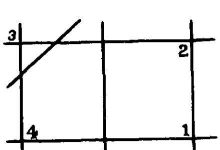
6.

A1. 

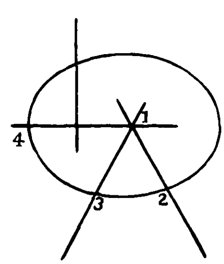
$$X(z + w) = Z(\alpha x + \beta y)$$
$$W(x + y) = Y(\gamma z + \delta w)$$
$$(X + \gamma Y)z = (\alpha Z + W)x$$



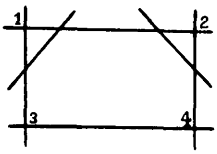
Cayley, *Proc. London Math. Soc.*, Vol. III, p. 174; Noether, *Math. Ann.*, Bd. III, p. 571; Cremona, *ib.* Bd. IV, p. 221; v. Krieg, *Zeitschr. für Math.*, Bd. XXIX, p. 60; Ascioni, *Giorn. di Mat.*, T. XXXI, p. 79.

A 2. 

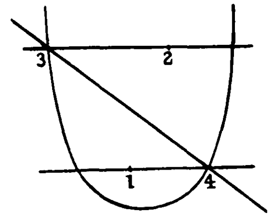
$$Zz = Ww$$
$$Y(y + z) = W(w + \delta x)$$
$$[\delta(rW + sZ) - X]x = (Y - Z)(qy + rz + sw)$$



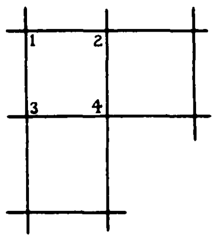
A 3.



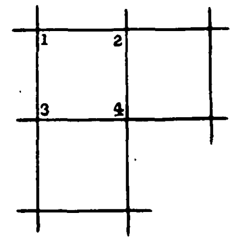
$$Xx = Yy$$
$$Zz = Y(px+qy+sw)$$
$$Ww = X(lx+my+nz)$$



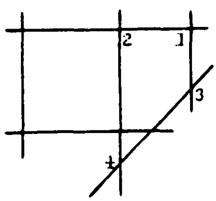
A 4.



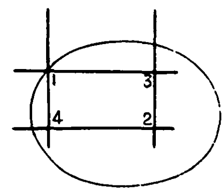
$$(\alpha X-\gamma Z)(x+z)+(\beta Y-\delta W)(y+w) = 0$$
$$(X-Y)(\alpha x+\beta y)+(Z-W)(\gamma z+\delta w) = 0$$
$$X(\alpha z-\beta y)+\beta Y(y+w)-Z(\gamma z+\delta w) = 0$$



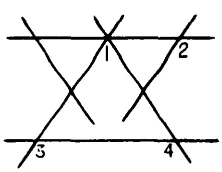
A 5.



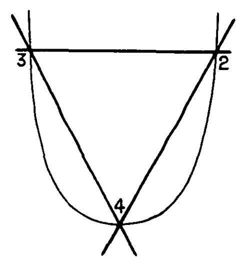
$$Yy = Ww$$
$$Z(z+w) = Y(x+y+z+\delta w)$$
$$Xx+Zy = W(x+y+z+\delta w)$$



B 1.

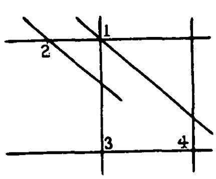


$$Xx = Yy$$
$$Zz = X(px+qy+sw)$$
$$Ww = X(lx+my+nz)$$

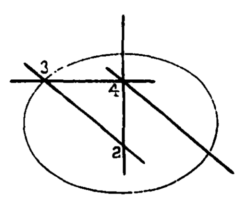


Cremona, *Rend. R. Ist. Lomb.*, T. IV, p. 319.

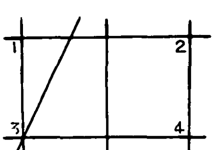
B 2.



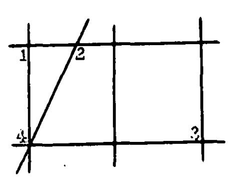
$$Xx = Yy$$
$$Zz = (\beta X+\alpha Y)(y+w)$$
$$Ww = X(x+z)$$



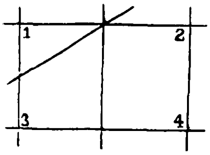
B 3.



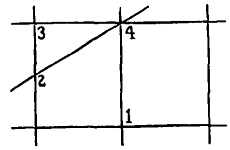
$$Xx = Zz$$
$$Y(x+y) = Z(z+\delta w)$$
$$(\delta pZ-W)w = (Y-X)(px+qy)$$



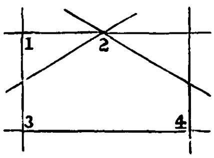
B 4.



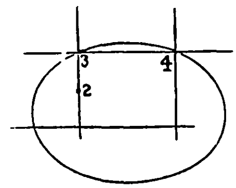
$$Xx = Zz$$
$$Y(x+y) = Z(z+\delta w)$$
$$[\delta(X+Z)+W]w = (Y-X)(x+y+z)$$



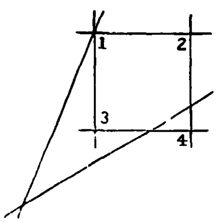
B 5.



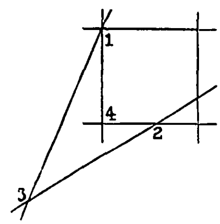
$$Xx = Yy$$
$$Zz = (X+Y)(x+w)$$
$$Ww = X(x+z)$$



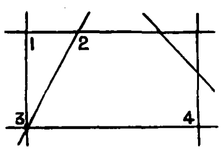
B 6.



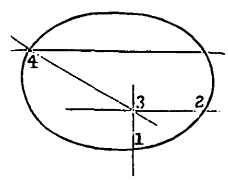
$$Yy = Ww$$
$$Z(z+w) = Y(y+z+\delta w)$$
$$Xx+Zy = W(y+z+\delta w)$$



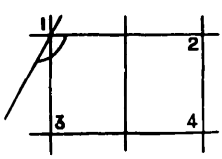
B 7.



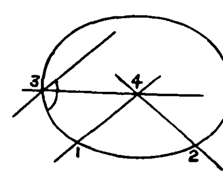
$$Xx = Yy = Ww$$
$$Zz = (X+Y)(ax+y+w)$$



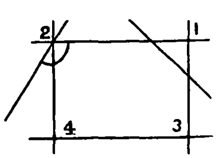
B′ 1.



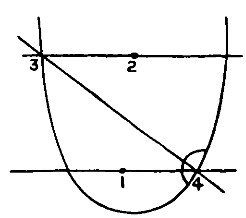
$$Xx = Zz$$
$$Y(x+y) = Z(z+w)$$
$$(X+W)w = (Y-X)(y+z)$$



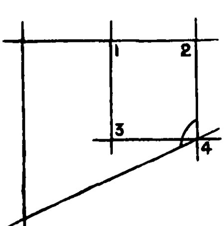
B′ 2.



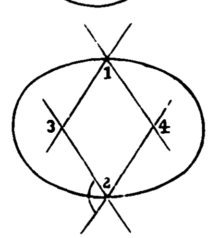
$$Xx = Yy$$
$$Zz = Y(x+w)$$
$$Ww = X(x+y+z)$$



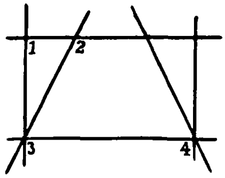
B′ 3.



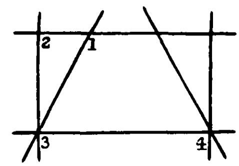
$$Yy = Ww$$
$$Z(z+w) = Y(x+y+z)$$
$$Xx+Zy = W(x+y+z)$$



C 1.

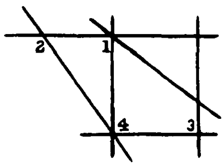


$$Xx = Yy = Ww$$
$$Zz = (X+Y)(px+qy)$$

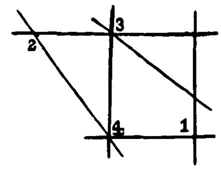


Ascioni, *Giorn. di Mat.*, T. xxxi. p. 69.

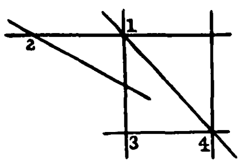
C 2.



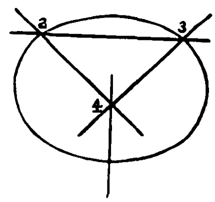
$$Xx = Yy = Zz$$
$$Ww = (X-Y)(y+z)$$



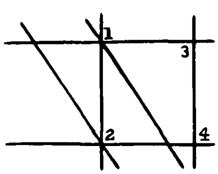
C 3.



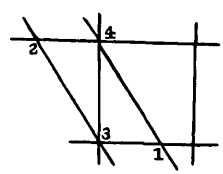
$$Xx = Yy$$
$$Zz = X(x+y)$$
$$Ww = X(x+z)$$



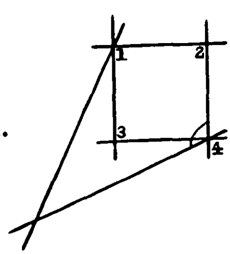
C 4.



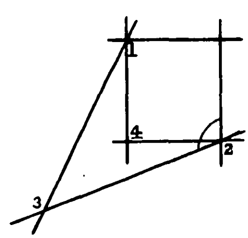
$$Xx = Yy$$
$$Zz = Y(y+w)$$
$$Ww = X(x+z)$$



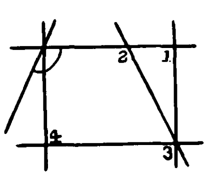
C′ 1.



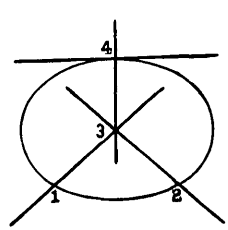
$$Yy = Ww$$
$$Z(z+w) = Y(y+z)$$
$$Xx+Zy = W(y+z)$$



C′ 2.

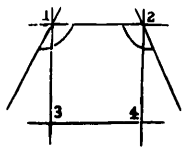


$$Xx = Yy = Ww$$
$$Zz = (X+Y)(x+y+w)$$

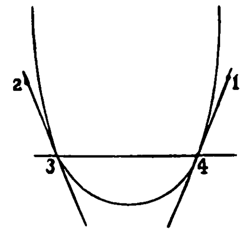


(A straight line touches the conic.)

C″.

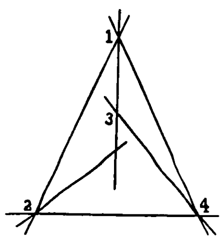


$$Xx = Yy$$
$$Zz = Y(x+w)$$
$$Ww = X(y+z)$$

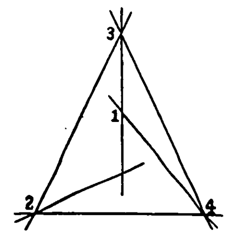


(Two straight lines touch the cubic.)

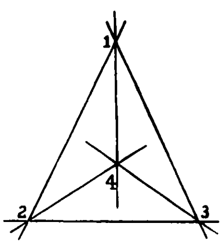
D.



$$Xx = Yy = Zz$$
$$Ww = X(x+z)$$

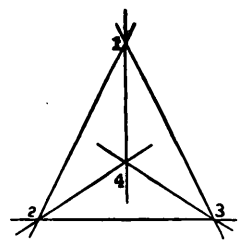


E.

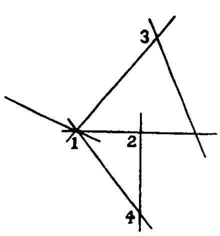


Inversion : passim.

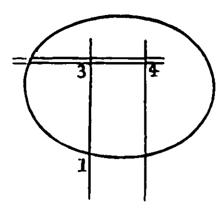
$$Xx = Yy = Zz = Ww$$



F 1.

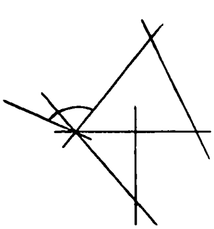


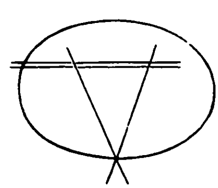
$$Zz = (Y - \beta X)\,x$$
$$Ww = Y(x+y)$$
$$X(y+\beta w) = Y(z+w)$$



(Two straight lines coincide.)

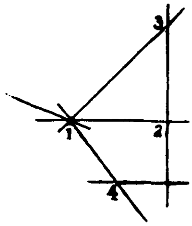
$\beta = 0$



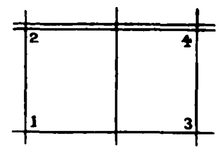


If in the first figure the three lines lie in a plane, in the second two lines meet on the conic.

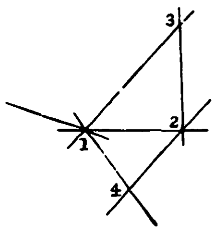
F 2.



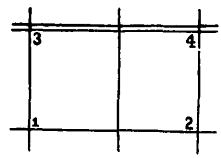
$$Xx = Y(\gamma y + z)$$
$$X(z+w) = Z(y + \beta z)$$
$$Ww = \beta Zx - Y(z+w)$$



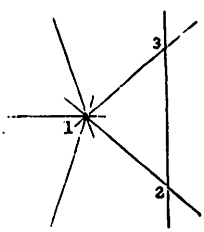
G.



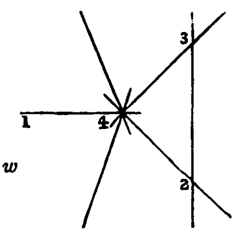
$$Xx = Zz$$
$$Y(y+\beta z) = X(z+w)$$
$$Ww = Z(z+w) - \beta Yx$$



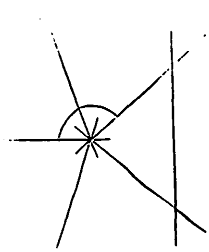
H.

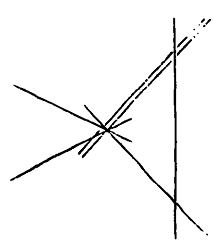


$$Xx = Ww$$
$$Yy = Zz$$
$$X[(1-\gamma)y + (1-\beta)z + (1-\beta\gamma)w]$$
$$= (1-\beta\gamma)Zz + [\beta(1-\gamma)Y + \gamma(1-\beta)Z]w$$



$\gamma = 0$





If in the first figure three lines lie in a plane, in the second two lines coincide.

$\beta = \gamma = 0$