ON THE 3–3 BIRATIONAL TRANSFORMATION IN THREE DIMENSIONS*

(SECOND PAPER.†)

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CONTENTS.

1.

In continuation of the discussion of Cayley's bilinear space-transformation, those cases are here considered in which the first fundamental sextic degenerates into six straight lines, some of which coincide. The geometrical language of actual and apparent double points ceases to have an obvious meaning, and more help is derived from the analytical discussion of the numbers of conditions and of independent surfaces, although it is possible to trace the geometrical meaning in some cases by reference to an adjacent position in which the lines do not coincide. If this adjacent non-coincident pair intersect, there is a distinction between lines in the plane of the pair, which meet both, and other lines which may meet one or neither; so, for the coincident pair which is the limit of this, we expect

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to find a special plane tangent to all the surfaces at all points of the line, and any line in this plane meets the surfaces in two points on the line. If the adjacent non-coincident pair do not intersect, there is still the distinction between lines meeting both and lines meeting one; the former are the sets of generators of all the quadrics containing the pair of lines. So in the limiting case we expect to find a special family of quadrics touching the surfaces all along the line. But the cubic surfaces can touch each other all along the line without so touching any quadric or plane; in this case the adjacent non-coincident pair, if it existed, would have to be treated as intersecting in -1 points.

The fact that certain cases cannot be derived from cases of non-coïncident lines becomes apparent on comparing the diagrams of the second fundamental sextic, given at the end of this paper, with those in the former paper. Some of these involve non-degenerate quartics or pairs of conics, which cannot be derived from any of the former set, which all consisted partly of three straight lines.

If the intersection of two surfaces consists partly of two or more coincident straight lines, the surfaces have contact of first or higher order at every point of that line; analytically, in the general equation of the family, the independent terms or groups of terms, *i.e.*, those by which any two members of the family can differ, are small of the second or higher order in the neighbourhood of the line.

Let the equations of the straight line be $0 = z = w$; the terms independent of z, w are absent from the equation of the general cubic surface containing the line; and the remaining terms fall into three groups according to their degree in z, w, which is in general the same as their order of smallness; these are

linear terms of first order $z(Ex^2+Fxy+Gy^2)+w(Hx^2+Jxy+Ky^2)$,

quadratic terms of second order $z^2(Lx+My)+zw(Nx+Py)+w^2(Qx+Ry)$,

cubic terms of third order $Sz^3+Tz^2w+Uzw^2+Vw^3$.

Two cases are excluded from present consideration, in which

(i) the general surface degenerates; for then the transformation, if it exists, is of lower degree;

(ii) the linear group disappears entirely; then the line is a common nodal line on all the surfaces, and the transformation is of a different type. This is treated separately in § 3.

If no special relations hold, then if two cubic surfaces touch all along the line, their linear terms must be the same; if they have contact of second order, their linear and quadratic terms must be the same; and they cannot have higher contact without coinciding. Although in general the order of smallness of any group of terms is equal to the lowest degree in z, w of any one term of the group, yet the group of linear terms as a whole, though apparently of first order, is really of second order, for it can be equated to quadratic and cubic terms; and the group of linear and quadratic terms together, though apparently of first order, is really of third order. Any group containing either of these groups as a factor is also of higher order of smallness than appears from considering its separate terms; but such a group is more than cubic in x, y, z, w, and cannot enter into the equation. But if the linear or linear+quadratic group falls into factors, one is necessarily finite (since, if both factors were small, there could be no linear terms in the product), and the other contains terms linear in z, w, so that it is apparently of first order, but its order is really that of the group. If this special factor ϕ is quadratic in x, y, z, w, then the cubic functions $x\phi$, $y\phi$ are of the same order as ϕ, and $z\phi$, $w\phi$ are of higher order. If the special factor θ is linear, the same holds of multiples of θ and of θ^2. Again, if the linear+quadratic group has a factor ϕ and also the linear group a factor θ, the product $\theta\phi$ must be considered; again, if θ or θ^2 is also a factor of the cubic group (but not of the quadratic group, for if so the general surface degenerates), then the cubic group is of fourth or higher order, and ϕ is of the same order.

In all these ways certain groups may occur, which are of higher order of smallness than appears from considering their separate terms, and so increase the number of groups that are independent for a given order of contact all along the line, *i.e.*, for a given number of lines of intersection coinciding in $0 = z = w$. Table I (p. 18) gives the terms which may occur in the various forms which the cubic equation can assume from this point of view. In the first column are the functions θ, ϕ, which in each type have orders of smallness higher than appears, reduced to their simplest forms by choice of a frame of reference. The order of smallness is found in the second column. The rest of the table gives the terms or groups that can then occur, arranged in ascending order of smallness. If one of these is not cubic, it will appear in the equation multiplied by an arbitrary function of x, y only, of proper degree. The frames of reference are so chosen that w is always small of first order, θ occurs in the form z, and ϕ in one of the forms ϕ_1, ϕ_2.

To obtain the types of cubic families which have k straight lines of common intersection coinciding in $0 = z = w$, and have therefore contact of order $k-1$ all along this line, we must consider terms or groups of order of smallness k or higher to be independent; such a family

TABLE I.

		Terms or groups that may occur, of order of smallness: 1	2	3	4	5	6	7	8	9	Type
		z, w	z^2, zw, w^2	z^3, z^2w, zw^2, w^3							i
ϕ_1	2		z^2, zw, w^2 ϕ_1	z^3, z^2w, zw^2, w^3 $z\phi_1, w\phi_1$							ii
ϕ_1	3			z^3, z^2w, zw^2, w^3 ϕ_1	$z\phi_1, w\phi_1$						iii
θ	2		w^2 z	w^3 wz	w^2z z^2	wz^2	z^3				iv
θ	3			w^3 z	wz	w^2z	z^2	wz^2		z^3	v
θ ϕ_2	2 3			w^3 wz, ϕ_2	w^2z $z^2, w\phi_2$	wz^2 $z\phi_2$	z^3				vi
θ ϕ_2	2 4				w^2z z^2, ϕ_2	wz^2 $w\phi_2$	z^3 $z\phi_2$				vii
θ ϕ_2	2 5	$\theta = z$ $\phi_1 = zx + wy$ $\phi_2 = zx + w^2$				wz^2 ϕ_2	z^3 $w\phi$	$z\phi_2$			viii
θ ϕ_2	2 6						z^3 ϕ_2	$w\phi_2$	$z\phi_3$		ix

TABLE II.

Number of coincident lines	2	3	4	5	6	7	8	9	Type
Number of independent groups	11	5							i
	12*	7							ii
		8	3						iii
	13	9	6	3	2				iv
		11	8	6	5	3		2	v
		10*	7†	4	2				vi
			8†	5	3				vii
				6	4	2			viii
					5	3	2		ix

* Here the groups $z\phi$, $w\phi$ are counted under their separate terms, and do not give additional groups.

† Here $z\phi$ is counted under its separate terms, but $w\phi$ gives an additional group.

will not arise in a type in which there are no terms of order as high as k, nor in a type in which there are no terms of order as low as k, for then there is necessarily contact of higher order.

Table II (p. 18) gives the numbers of independent surfaces which have any given number of coincident lines in each type.

Since the surfaces touch all along the line, the tangent plane is assigned at each of its points. Now the equation of the tangent plane is given by the linear group, $(x, y, 0, 0)$ being the point of contact; and to assign this plane for all values of $x : y$ is, in general, to assign all the coefficients of the group (type i). But if the group has a factor ψ involving x, y only, this does not affect the position of the plane, and the coefficients contained in the factor remain independent. If ψ is linear (type ii), the surfaces all touch the quadric $\phi_1 = 0$, or any one of the family of quadrics $\phi_1 + az^2 + \beta zw + \gamma w^2 = 0$, all along the line; in type iii the surfaces all osculate the special quadric $\phi_1 = 0$ of this family. If ψ is quadratic, the surfaces touch the plane $\theta = 0$ all along the line (type iv); in type v they osculate the plane $\theta = 0$; in type vi they osculate any one of the family of cones $\phi_2 + \theta(az + \beta w) = 0$; in type vii they have contact of third order with the more restricted family of cones $\phi_2 + a\theta^2 = 0$; in types viii and ix there is contact of fourth and fifth orders with the special cone $\phi_2 = 0$.

Using the notation of the former paper, a line b meeting the coincident pair a will not in general lie in the given tangent plane at the point ab. If it does not, a surface which also contains b must have a conical point; therefore, in calculating the reduction in the number of conditions imposed by b, due to its intersection with a, we must consider separately those lines which do or do not lie in the given tangent planes at their points of intersection with a. In type iii a further particularity attaches to generators of the special quadric; in types vi and vii, to lines through the vertex of the family of cones; and in types viii and ix, to generators of the special cone.

The same idea appears in analytical form as follows. In any one type a certain number of the coefficients, though arbitrary, are not independent, *i.e.*, they are the same for all the members of the family; these are the coefficients that occur in the group of all terms that are not independent in the case considered, and also those contained in the special functions θ, ϕ; such coefficients will be distinguished by circumflexes. Now, if a line b imposes a condition involving these alone, this condition is not to be counted for the purpose of this investigation; for it does not reduce the number of independent surfaces, but merely restricts the generality of the assigned set of tangent planes, causing it to have some particular relation to b.

The 127 transformations investigated are classified as follows, where, *e.g.*, the symbol $1^4.2$ denotes four distinct lines and one coincident pair. The first set are those given in the former paper.

Reference...	A to H	I to M	N to Q	R to S	T	U	V	W	X	Y	Z
Symbol ...	1^6	$1^4.2$	$1^2.2^2$	2^3	$1^3.3$	$1.2.3$	3^2	$1^2.4$	2.4	1.5	6
Number ...	28	26	19	3	14	14	3	8	5	4	3

We have seen that the nature of the sextic curve is not enough to determine the transformation. The fact that a certain number of the six straight lines coincide is capable of interpretation in several ways; we need to know something of the set of tangent planes, osculating quadrics, &c., given all along the coincident lines. In the set of formulæ and diagrams in § 4, the classification according to the form of the sextic has been retained; but in the investigation of § 2 it has been found more convenient to arrange the transformations, to a certain extent, under the different types to which they belong.

In this second paper the interest lies in coincidence of lines and its interpretation, rather than in coincidence of intersections. No separate investigation is given of the cases in which coincident intersections can be deduced from distinct intersections by obvious changes. The references to the deduced transformations are given, and their equations and diagrams will be found in their places in § 4. In the latter part of § 2 certain cases of coincidence of lines are dismissed in the same way. When three or more lines coincide (T to Z), coincident intersections lose their interest still further, since one arm of the triple point must be of higher multiplicity than any of the others; these varieties do not receive separate letters of reference. When distinct transformations arise from the same form of sextic, they are grouped under the same letter and number.

The method of enumerating the possible transformations is as follows. Considering in turn the cases in which 2, 3, 4, 5, or 6 straight lines coincide, the general cubic equation is written down for each of the possible types; the rest of the six lines have then to be arranged so as to reduce the number of independent cubics to four. Every such arrangement gives rise to a transformation.

It is clearly useless to consider a case in which there are fewer than four independent cubics; it is also useless to consider a case in which there are more than four. In Cayley's language, the condition of equivalence implies the condition of postulation. For, in general, let F_1, F_2, F_3, F_4, F_5 be five members of a linear family satisfying the condition of equivalence, *i.e.*, such that any three independent surfaces of the form

$$\sum_1^5 (A_n F_n) = 0$$

meet in one and only one point not common to all the surfaces of the family. Take the general points P', P'', and let F' be the value of F at P', &c. Then the three equations contained in

$$\begin{Vmatrix} F_1 & F_2 & F_3 & F_4 & F_5 \\ F_1' & F_2' & F_3' & F_4' & F_5' \\ F_1'' & F_2'' & F_3'' & F_4'' & F_5'' \end{Vmatrix} = 0$$

represent surfaces which intersect in the two points P', P'' not common to the family. They are therefore not independent, but connected by a linear relation, which is a linear relation connecting F_1, F_2, F_3, F_4, F_5; therefore the family does not contain more than four linearly independent members.

With regard to the connection between the first and second fundamental sextics of any of these transformations, it may be remarked that if one has an ordinary triple point, the other contains a real or degenerate plane cubic, and *vice versa*.

It may also be remarked that the transformations B4 and B5, and those deducible from them, may be obtained as squares of the quadri-quadric transformation; A3 and W as squares of the quadri-cubic transformation; and D as the product of a quadri-quadric and a quadri-cubic transformation.

2.

I to M. *Two coincident and four distinct straight lines.*

A reference to the second table shows that types i, ii, and iv are the only ones to be considered, and that the four distinct lines must give exactly 7, 8, and 9 conditions in the respective types.

Type i.

$$\begin{aligned} & z(\hat{E}x^2+\hat{F}xy+\hat{G}y^2)+w(\hat{H}x^2+\hat{J}xy+\hat{K}y^2) \\ & +z^2(Lx+My)+zw(Nx+Py)+w^2(Qx+Ry) \\ & +Sz^3+Tz^2w+UzW^2+Vw^3 \qquad = 0. \end{aligned}$$

A line b meeting a gives two conditions. (Here, as in all that follows, "condition" is used in the sense of "condition which reduces the number of independent surfaces." Of the four conditions imposed by b, one is satisfied identically, one causes the given tangent plane at the point ab to contain b, and two are effective for our purpose.)

If $bcde$ all meet a, then two of them meet, say, bc; this may be specialized by allowing intersections to coincide [I1; J1; J'1; K1; L1; L'1; M. Since x^3 is absent, and the terms in x^2 are not independent, each line through the vertex $(1, 0, 0, 0)$ gives only two conditions, and a quadruple point does not offer the peculiarity noticed in the former paper (F). An ordinary triple point acd gives a degenerate system unless another arm passes through the point.]

If only bcd meet a, then
either e meets bcd [I2; J2; J3; K2];
or e meets bc, and d meets b [I3; J4; J5; J'2; L'2].

If only bc meet a, the system imposes too many conditions.

Type ii.

$$(zx+wy)(Ex+Fy)+z^2(Lx+My)+zw(Nx+Py)+w^2(Qx+Ry) \\ +Sz^3+Tz^2w+Uzw^2+Vw^3 = 0.$$

A line b meeting a and lying in the given tangent plane at the point ab gives two conditions. A line meeting a and not lying in the given tangent plane gives a relation between E, F and reduces this to a particular case of type i; therefore we need not consider a triangle nor an ordinary triple point on a, for in these cases one line could not lie in its tangent plane. A pair of lines meeting on a and both lying in the same tangent plane also gives a relation between E, F; therefore we need not consider a special triple point on a.

Then $bcde$ may all meet a [I4].

If only bcd meet a, then e meets two of them [I5].

If only bc meet a, then d meets both, and
either e meets bc [I6; J6];
or e meets bd [I1; J'3].

If only b meets a, there are too many conditions.

Type iv.

$$z(Ex^2+Fxy+Gy^2)+z^2(Lx+My)+zw(Nx+Py)+w^2(Qx+Ry) \\ +Sz^3+Tz^2w+Uzw^2+Vw^3 = 0.$$

A line lying in the tangent plane $z=0$ gives two conditions. A line meeting a and not lying in the tangent plane gives three conditions.

There are too many conditions unless there is a line b in the tangent plane; there cannot be more than one.

If only b meets a, there are too many conditions.

If only bc meet a, there must be four more intersections; the configurations are the same as in the corresponding cases of type ii. Taking c to be $0 = w = x$, then $G = 0$, and these are particular cases of those of type ii, obtained by omitting the term wy from the quadratic factor.

If only bcd meet a, we have in the same way particular cases of type i with two relations between E, F, G; except in the case there excluded, in which acd form an ordinary triple point; then

either e meets bc [J7; $L2_1$; bce cannot concur, for if so, c lies in the tangent plane];

or e meets cd; then b must meet one side of the triangle cde, and therefore passes through the vertex acd [$L2_2$].

If $bcde$ all meet a, there must be two more intersections, and b can meet no other line; the system degenerates unless there is a triple point, and gives a particular case of type i unless there is a quadruple point $acde$ [L3]. A quintuple point at (1, 0, 0, 0) causes x to disappear entirely.

N to S. *Two coincident pairs and two distinct straight lines.*

First let the coincident pairs not intersect. Then let them be

$$(a) \quad 0 = z = w, \qquad (b) \quad 0 = x = y.$$

The cubic containing these can be rearranged in the form

$$\begin{aligned} z(Ex^2+Fxy+Gy^2)+w(Hx^2+Jxy+Ky^2) & \\ +x(Lz^2+Nzw+Qw^2) & \\ +y(Mz^2+Pzw+Rw^2) &= 0. \end{aligned}$$

Each of a, b may belong to any one of the types i, ii, or iv.

Type a i, b i gives only two independent surfaces.

Type a ii, b i gives only three.

Type a iv, b i gives four, all containing the line (c) $0 = z = \hat{Q}x+\hat{R}y$, but no other line.

Type a ii, b ii gives four, the general form being

$$(zx+wy)(Ex+Fy)+[x(\hat{N}z+\hat{P}w)+y(\hat{Q}z+\hat{R}w)](Lz+Mw) = 0,$$

all containing c, d the two remaining lines of intersection of the two quadrics; cd both meet ab [N1].

Then cd may coincide, type ii [R].

Type a iv, b ii gives five independent surfaces, the general form being

$$z(Ex^2+Fxy+Gy^2)+(zx+wy)(Lz+Mw) = 0,$$

all containing (c) $0 = y = z$, which meets ab. Now d must give only one condition; but it cannot meet abc, for if so $abcd$ would be in a plane; it cannot lie in the tangent plane along a, for if so the system would degenerate; therefore d meets b and lies in the given tangent plane at the point bd and also meets another line, which cannot be c, but must be a, giving a particular case of N1, in which the first quadric degenerates.

Type a iv, b iv gives six independent surfaces, the general form being

$$z(Ex^2+Fxy+Gy^2)+x(Lz^2+Nzw+Qw^2) = 0,$$

all containing (c) $0 = x = z$, which is the line of intersection of the tangent planes. Then d must give two conditions; but it cannot lie in either tangent plane, and therefore cannot meet ac; therefore d meets ab, giving again a particular case of N1.

[The transformation R is the only one with three pairs of coincident lines which do not concur; this arrangement need not be sought again.]

Next let the coincident pairs intersect. Then let them be

$$(a)\quad 0 = z = w, \qquad (b)\quad 0 = y = w.$$

Type a i, b i.

$$zy(\hat{F}x+\hat{G}y+\hat{M}z)+w(\hat{H}x^2+\hat{J}xy+\hat{K}y^2)+wz(\hat{N}x+\hat{T}z)$$
$$+Pzwy+w^2(Qx+Ry+Uz+Vw) = 0,$$

giving six independent surfaces, unless $\hat{F} = \hat{H} = 0$, in which case there are seven, for then the terms linear in y, w are independent of those linear in z, w.

If this is not so, the surfaces all contain (c) $0 = w = \hat{F}x+\hat{G}y+\hat{M}z$. Then d must give two conditions and therefore meet a only [N2; P1; P2; P′1; Q′1. If cd coincide, there are too few conditions.]

But if $\hat{F} = \hat{H} = 0$, we must have three conditions. A line in the plane ab reduces this to the last case.

If cd meet a, they meet each other; then if the plane of the triangle is $z = 0$, one side is $0 = z = \hat{J}y + Qw$ forming a triple point with ab [$P3_1$; $Q'2$; S_1 (*cd* coincide, type iv, the tangent plane containing a)].

If only c meets a, then d meets bc; the case immediately following includes this [N3].

Type a ii, b i.

$$(zy+wx)(\hat{E}x+Fy)+\hat{M}z^2y+zw(\hat{N}x+\hat{T}z)$$
$$+Pzwy+w^2(Qx+Ry+Uz+Vw) = 0.$$

We have seen that any line meeting a lies in its given tangent plane, and that there are no triangles nor triple points on a. A line meeting either a or b gives two conditions.

If cd both meet a, they cannot meet each other, and there are too many conditions.

If only c meets a, then d meets bc [N3].

If no line meets a, then bcd form a triangle [N4; P'2].

Type a iv, b i.—First let (b) $0 = y = w$ not lie in $z = 0$, the tangent plane along a

$$zy(\hat{F}x+Gy+\hat{M}z)+wz(\hat{N}x+\hat{T}z)+Pwzy+w^2(Qx+Ry+Uz+Vw) = 0.$$

If G is not independent, a is type i; therefore any line meeting a either lies in the plane $z = 0$ (but there can only be one such), or forms a triple point with ab, giving in either case two conditions; therefore

either c meets a, then d meets bc (a particular case of N3);

or bcd form a triangle; then if its plane is $y = 0$, one side is $0 = y = \hat{N}z + Qw$, forming a triple point with ab [$P3_2$];

or there is an ordinary quadruple point [Q_1; S_2 (*cd* coincide, type iv)].

Next let b lie in the tangent plane $w = 0$ along a.

$$w(\hat{H}x^2+\hat{N}xz+\hat{T}z^2)+\hat{M}yz^2$$
$$+wy(Jx+Ky+Pz)+w^2(Qx+Ry+Uz+Vw) = 0.$$

No other line can lie in $w = 0$. A line meeting a gives three conditions and a line meeting b gives two. Therefore

either c meets a, and d meets bc (a particular case of N3);

or cd both meet b [$N5_1$; if *bcd* concur, the system degenerates; $P4_1$; Q_2; if *ca* coincide, we have S_1 with a, c interchanged].

Type a ii, b ii.

$$(zy+wx)(Ex+Fy+Mz)+Pzwy+w^2(Qx+Ry+Uz+Vw)=0.$$

There are no triangles nor triple points. Any line meeting a or b gives two conditions. Therefore

either cd both meet a [N5_2];
or c meets a, and d meets b [N6].

Type a iv, b ii cannot occur, since it requires more terms linear in y, w than are present in either form of type a iv, b i.

Type a iv, b iv. First let (b) $0=y=w$ not lie in $z=0$, the tangent plane along a, nor a in $y=0$, the tangent plane along b.

$$zy(Fx+Gy+Mz+Pw)+w^2(Qx+Ry+Uz+Vw)=0.$$

A line in the plane $z=0$ gives two conditions, but two such lines would make the system degenerate. A line c meeting a, but not lying in the tangent plane, gives a relation between F, G and reduces a to type i, except in the case in which abc form a triple point; then c gives two conditions, but two such lines would cause x to disappear entirely.

Therefore abc form a triple point, and d lies in one of the tangent planes [P4_2].

Next let b lie in $w=0$, the tangent plane along a. Then in the second form of type a iv, b i,

either the tangent plane along b is $y=0$ not containing a; then $\hat{H}=\hat{N}=\hat{T}=0$, and b is type i;
or the tangent plane along b is $w=0$ containing a; then $\hat{M}=0$, and the system degenerates.

T to V. *Three coincident and three distinct straight lines.*

Type i gives five independent cubics, and the three distinct lines give too many conditions.

Type ii.

$$(zx+wy)(\hat{E}x+\hat{F}y+Lz+Rw)+\hat{M}z^2y+zw(\hat{N}x+\hat{P}y)+\hat{Q}w^2x$$
$$+Sz^3+Tz^2w+Uzw^2+Vw^3=0.$$

A line (b) $0=y=w$, not lying in the tangent plane at the point ab, gives two conditions $\hat{E} = L = S = 0$, and then all the surfaces contain (c) $0 = w = \hat{F}x + \hat{M}z$, lying in the tangent plane at the point ac, which is the plane ab. A line b in its tangent plane gives one condition. Therefore

either bcd all meet a, each in its tangent plane [$T1_1$; T7; $U2_1$ (bc coincide, type ii); $U2_2$ (bc coincide, type iv); V_1 (bcd coincide, type iii); V_2 (bcd coincide, type iv)];

or bc meet a, each in its tangent plane and d meets ab [T2; $T4_1$: $U1_1$ and $U5_1$ (bc coincide, type i)].

Type iii.

$$(zx+wy)(Ex+Fy+Lz+Rw)+Sz^3+Tz^2w+Uzw^2+Vw^3 = 0.$$

The surfaces osculate the quadric $zx+wy = 0$; therefore if they contain any other point on the quadric, the generator through that point which meets a meets the surfaces in four points, and therefore lies on them all.

Therefore a line b not meeting a fixes two generators cd and imposes four conditions [$T3_1$; $U3_1$ (cd coincide, type ii)].

A line b meeting a, not a generator, fixes one generator, but reduces a to type ii.

Three generators bcd give too few conditions unless bc coincide, type i, [$U2_3$], or bcd all coincide, type ii [see above; if bcd coincide, type iv, a becomes type ii].

Type iv.

$$z(\hat{E}x^2+\hat{F}xy+\hat{G}y^2)+w^2(\hat{Q}x+\hat{R}y)+z^2(Lx+My)+zw(Nx+Py) \\ +Sz^3+Tz^2w+Uzw^2+Vw^3 = 0.$$

A line in the tangent plane gives one condition; there can only be one such line. A line meeting a and not lying in the tangent plane gives two conditions.

If b lies in the tangent plane,

either cd meet a [$T1_2$; $T5_1$ (acd concur); $T5_2$ (abc concur); T8; $U2_4$ and $U4_1$ (bc coincide, type i); $U2_5$ and $U4_2$ (cd coincide, type iv); if bcd all coincide, a is type iii];

or c meets a, and d meets bc [$T3_2$; $T6_1$; $U3_2$ (bc coincide, type i)].

If no line lies in the tangent plane, bcd all meet a, and c meets d; but this causes the linear terms to vanish entirely unless there is a triple point abc [$T4_2$; $T9$; $U1_2$ and $U5_2$ (bc coincide, type i); if cd coincide, type i, we have a particular case of $U4_2$; V_3 (bcd coincide, type v)].

Type v.

$$z(Ex^2+Fxy+Gy^2)+z^2(Lx+My)+zw(Nx+Py)$$
$$+Sz^3+Tz^2w+Uzw^2+Vw^3=0.$$

The plane $z=0$ meets the surfaces in a thrice, and therefore every other line meets a and does not lie in $z=0$, giving three conditions. There would have to be two intersections between bcd, but then $abcd$ would all lie in a plane.

Type vi.

$$(zx+w^2)(Ex+Fy)+z^2(Lx+My)+wz(Nx+Py)$$
$$+Sz^3+Tz^2w+Uzw^2+Vw^3=0.$$

If E, F are related, this becomes type iv; therefore the only lines to be considered are those which do not meet a, giving four conditions, and those through the vertex of the special cone and not lying in the tangent plane, giving two conditions.

If bcd all pass through this point, y disappears entirely unless bc coincide, type i, but then a is type iii unless further bcd coincide, type iv (see above).

If only bc pass through the vertex, the triple point cannot be special and d meets bc [$T6_2$; $U4_3$ (bc coincide, type i)].

W, X. *Four coincident and two distinct straight lines.*

Type iv.—

$$z(\hat{E}x^2+\hat{F}xy+\hat{G}y^2)+zw(\hat{N}x+\hat{P}y)+w^2(\hat{Q}x+\hat{R}y+\hat{V}w)$$
$$+z^2(Lx+My)+Sz^3+Tz^2w+Uzw^2=0,$$

all containing (b) $0=z=\hat{Q}x+\hat{R}y+\hat{V}w$.

Then c must give two conditions, and therefore meet a [$W1_1$; $W3_1$; if bc coincide, there are too few conditions].

Type v.

$$z(\hat{E}x^2+\hat{F}xy+\hat{G}y^2)+\hat{V}w^3+z^2(Lx+My)+zw(Nx+Py)$$
$$+Sz^3+Tz^2w+Uzw^2 \qquad =0.$$

No other line can lie in the tangent plane: bc must both meet a [$W1_2$; if abc concur, we have a particular case of type vi].

If bc coincide, let them be $0=y=w$; therefore $\hat{E}=L=S=0$, and the terms linear in y, w are $yz(\hat{F}x+Mz)+wz(Nx+Tz)$. Then

either $\hat{F}=0$, so that this group is independent of the terms linear in z, w, and b is type i [X_1];

or $\hat{F}\neq 0$; then b must be type iv, the tangent plane along b containing a, giving a particular case of type vi.

Type vi.

$$(zx+w^2)(\hat{E}x+\hat{F}y+Rw)+wz(\hat{N}x+\hat{P}y)+\hat{V}w^3$$
$$+z^2(Lx+My)+Sz^3+Tz^2w+Uzw^2=0.$$

A line in the tangent plane would make R not independent and reduce a to type iv. A line meeting a and not lying in the tangent plane gives two conditions; but two such lines cause the linear terms to disappear entirely, unless one passes through the vertex of the cone; a second line through the vertex gives only one condition; therefore

either abc form a triangle with one side through the vertex [W2; W4; X_2; X_3 (*bc* coincide, type iv, the tangent plane containing *a*)],

or abc form an ordinary triple point at the vertex of the cone [W3_2].

Type vii.

$$(zx+w^2)(Ex+Fy+Rw)+z^2(Lx+My)+Sz^3+Tz^2w+Uzw^2=0.$$

If E, F are not independent, this reduces to type vi; therefore bc coincide, type i, and pass through the vertex of the cone, but do not lie in the tangent plane [X_4].

Y. *Five coincident straight lines and one distinct.*

Type v.

$$z(\hat{E}x^2+\hat{F}xy+\hat{G}y^2)+zw(\hat{N}x+\hat{P}y)+\hat{V}w^3$$
$$+z^2(Lx+My)+Sz^3+Tz^2w+Uzw^2=0.$$

No other line can lie in the tangent plane; b must meet a, giving the two conditions required [Y_1].

Type vi.

$$(zx+w^2)(\hat{E}x+\hat{F}y+Qz)+z^2(\hat{L}x+\hat{M}y)+zw(\hat{N}x+\hat{P}y)+\hat{U}w^2z+\hat{V}w^3$$
$$+Sz^3+Tz^2w=0,$$

all containing (*b*) $0 = z = \hat{E}x + \hat{F}y + \hat{V}w$, and giving four independent surfaces [Y_2].

Type vii.

$$(zx+w^2)(\hat{E}x+\hat{F}y+Qz+Rw)+z^2(\hat{L}x+\hat{M}y)+\hat{U}zw^2+Sz^3+Tz^2w = 0.$$

If b lies in the tangent plane, R is not independent, giving type vi. There are too many conditions unless b passes through the vertex of the special cone [Y_3].

Type viii.

$$(zx+w^2)(Ex+Fy+Qz+Rw)+Sz^3+Tz^2w = 0.$$

A line in the tangent plane reduces this to type vi. A generator of the special cone gives too few conditions; an arbitrary line through the vertex causes y to disappear entirely. Any other line gives too many conditions.

Z. Six coincident straight lines.

Type viii is the only one giving exactly four independent surfaces [Z].

3.

It remains to consider the case of a common nodal line, reserved (p. 16) for separate treatment. Cayley's investigation excludes this, but the extended formulæ, given without proof at the end of his paper,* show that, besides one nodal and two distinct straight lines, the fundamental system must contain two additional points. The birational transformation which results is not in general bilinear. It will be shown that if it is so, the fundamental points must both lie on the nodal line, *i.e.*, at each of two fixed points of this line one set of sheets of the cubic surfaces touch a fixed plane.

Only a particular case of Cayley's formula is required, the proof of which may be outlined as follows:—

Lemma.—The two distinct lines bc of the sextic must meet the nodal line a, and cannot meet each other unless abc form a triple point.

For the terms linear in z, w are absent, and if b does not meet a, we may take it to be $0 = x = y$, and the terms cubic in z, w vanish. Then $XYZW$ are all formed from terms quadratic in z, w and linear in x, y, and their ratios are functions of the two quantities $z : w$ and $x : y$ only; therefore b and c must meet a. Now any plane through a meets the surface in only one other line, therefore bc cannot lie in a plane through a and so cannot meet each other except on a.

We may therefore assume (*b*) $0 = y = w$, (*c*) $0 = x = z$. Then $XYZW$ are formed from the terms z^2y, w^2x, zwx, zwy, z^2w, zw^2, and may be chosen so that the first two of these terms do not enter into Z, W, which are therefore zwp_1, zwp_2, where p_1, p_2 are linear in x, y, z, w.

* Cayley, *Proc. London Math. Soc.*, Vol. III, p. 179.

Now three of the cubic surfaces can meet in only one point not belonging to the fundamental system; but $Z=0$, $W=0$ meet the general surface in three points lying on the straight line $0=p_1=p_2$; therefore two of these points are fundamental, and either are isolated fundamental points or lie on the fundamental lines. In the former case we can proceed to construct the transformation; the latter case can be obtained from the former by letting the straight line $0=p_1=p_2$ move from its general position to intersect two or coincide with one of the fundamental lines. The surfaces will then be found to have contact at the limiting positions of the isolated fundamental points which have come to lie on the fundamental lines.

If the isolated fundamental points are $(\alpha\beta\gamma\delta)$ and $(\xi\eta\zeta\theta)$, the equations of transformation are

$$\begin{aligned} X:Y:Z:W = {} & w^2x\gamma\zeta(\delta\zeta-\gamma\theta)+zw\left[z\delta\theta(\alpha\zeta-\gamma\xi)-w(\alpha\delta\zeta^2-\gamma^2\xi\theta)\right] \\ & : z^2y\delta\theta(\delta\zeta-\gamma\theta)+zw\left[z(\beta\gamma\theta^2-\delta^2\eta\zeta)-w\gamma\zeta(\beta\theta-\delta\eta)\right] \\ & : zw\left[x(\delta\zeta-\gamma\theta)+z(\alpha\theta-\delta\xi)-w(\alpha\zeta-\gamma\xi)\right] \\ & : zw\left[y(\delta\zeta-\gamma\theta)+z(\beta\theta-\delta\eta)-w(\beta\zeta-\gamma\eta)\right], \end{aligned}$$

which will be found to be birational and symmetrical, but not in general bilinear.

Now, if three bilinear relations exist, and the matrix of coefficients of $XYZW$ is

$$\left\|\begin{matrix} p_1 & q_1 & r_1 & s_1 \\ p_2 & q_2 & r_2 & s_2 \\ p_3 & q_3 & r_3 & s_3 \end{matrix}\right\|,$$

the surfaces $X=0$, $Y=0$ meet in the fundamental sextic and a cubic curve given by

$$\left\|\begin{matrix} r_1 & r_2 & r_3 \\ s_1 & s_2 & s_3 \end{matrix}\right\| = 0.$$

If $P(1, 0, 0, 0)$ is a fundamental point, it lies on all these cubic curves, and its coordinates annihilate all the determinants of order two contained in the first matrix. This implies that the terms in x in any column are proportional to those in any other column, so that the terms in x^3, x^2 vanish in all the four determinants X, Y, Z, W.

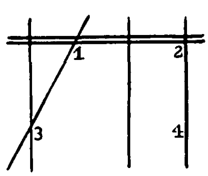
Therefore P is a conical point on all the surfaces, and lies on the common nodal line; for if not the surfaces degenerate, containing the plane aP.

Therefore both the fundamental points lie on the nodal line.

This gives the transformation $W1$. Then bc may meet on a [$W3$], or coincide, type i [X]. Or b may coincide with a; then one sheet of every surface touches a fixed plane all along the line [Y]. Or, finally, bc may both coincide with a; then either both sheets of every surface touch fixed planes all along the line [Z_1], or one sheet osculates a fixed plane [Z_2].

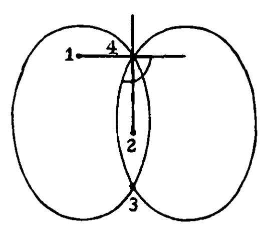
4.

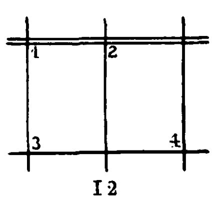
This section contains the actual bilinear equations of all the transformations discovered in the preceding investigation. In each case the frame of reference has been chosen so as to give these equations in their most simple and symmetrical form. All superfluous constants have been absorbed; of those which remain, Greek letters denote constants depending on the position of the straight lines, and Roman letters those depending on the position of the given tangent planes. Coincident lines are represented by parallel lines drawn close together and the conventions explained in the former paper are observed.



I 1

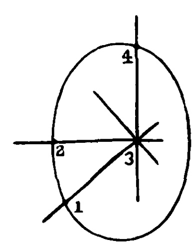
$$Xz = Zx$$
$$Y(w+z) = Z(x+y)$$
$$Ww = (X-Y)(\alpha x + \beta y) + \gamma Xy$$

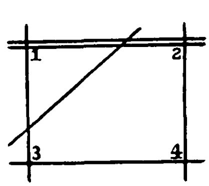




I 2

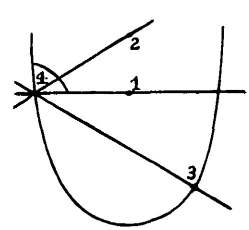
$$(X+Y)w = (W-X)z$$
$$Zz = X[H(x+y)-Fy] + Y[J(x+y)+Fx]$$
$$Zw = W(Hx+Jy) - X[H(x+y)-Fy]$$

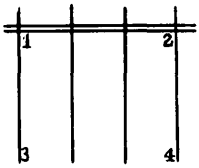




I 3

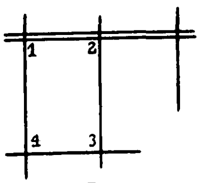
$$Xy = Yx$$
$$Xw = Zz$$
$$Ww = Y(x+y+z) + Z(Hx+Jy)$$





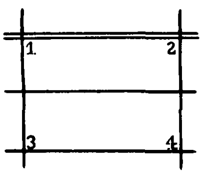
I 4

$$\alpha(Xw - Wx) = \beta[Yz - Z(y+z)] = \gamma(Xz - Wy) = \delta(Zx - Yw) \qquad \text{I 4}$$



I 5

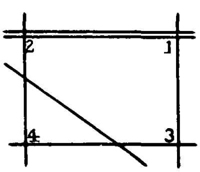
$$Xy = Yx$$
$$X(z+w) = x(Z+W) - Ww \qquad \text{I 5}$$
$$Y(z+w) = y(Z+W) + Zz$$



I 6

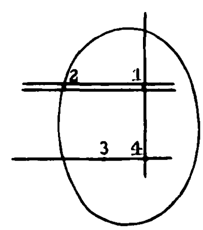
$$Zw = Wz$$
$$\alpha Xy - \beta Yx = Zy + Wx = Yz + Xw \qquad \text{I 6}$$

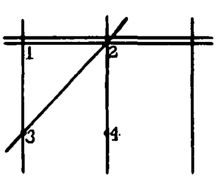
De Paolis, *Rendic. R. Acc. Linc.* (4), I. p. 740, 2°; $\mu = \nu_1 = \nu_2 = 1$.



I 7

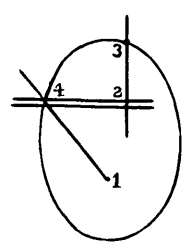
$$Zw = Wz$$
$$Xx = Y(y+z+w)$$
$$Yw = Zy + Wx$$

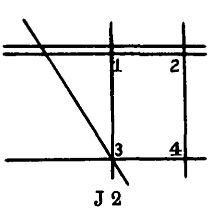




J 1

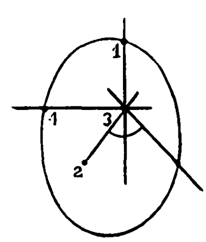
$$Xz = Zx$$
$$Y(z+w) = Z(x+y)$$
$$Ww = X(x+Hy) - Yx$$



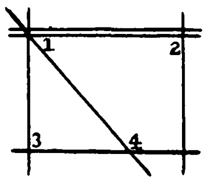


J 2

$$Xy = Yx$$
$$Xw = Wz$$
$$Zw = Y(x+y) + W(Hx+Jy)$$

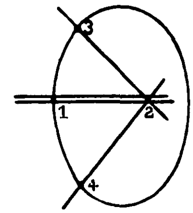


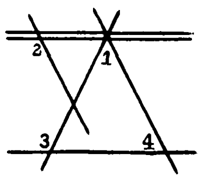
D



J 3

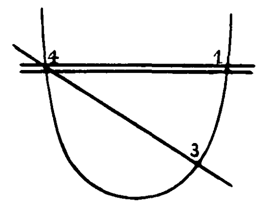
$$Zw = Wz$$
$$Yz = Z(Fx+Gy)+(J-F)Xy$$
$$Yw = W(Jx+Gy)-(J-F)Xy$$

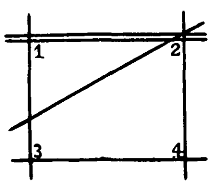




J 4

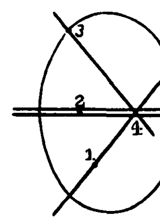
$$Xy = Yx$$
$$Yw = Zz$$
$$Ww = Y(x+z)+Z(Jx+Ky)$$

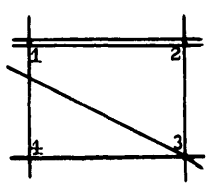




J 5

$$Xy = Yx$$
$$Xw = Zz$$
$$Ww = Y(x+z)+Z(Hx+Jy)$$

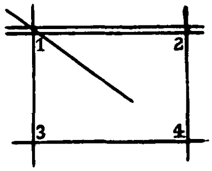




J 6

$$Zw = Wz$$
$$Zx = Xy - Yx = Yw - Wy$$

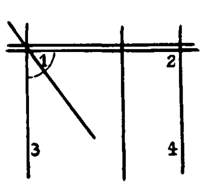
I 6



J 7

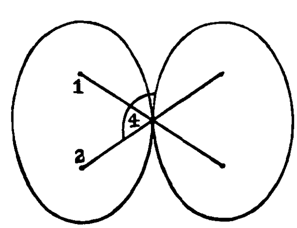
$$Xy = Yx$$
$$Y(z+w) = Z(y+w)$$
$$Zx = X(z+w)+Wz$$

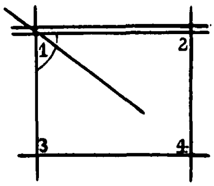
N 3



J'1

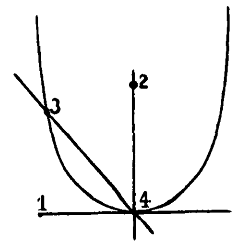
$$Xz = Zx$$
$$Y(z+w) = Z(x+y)$$
$$Ww = HXy+(X-Y)(Hx+y+z)$$

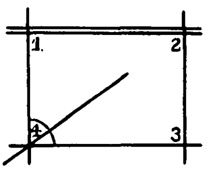




J′2

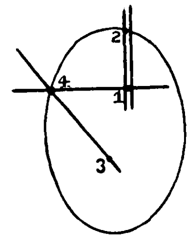
$$Xy = Yx$$
$$Xw = Zz$$
$$Ww = Y(y+z) + Z(Hx+Jy)$$

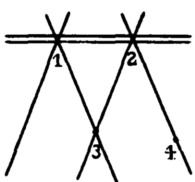




J′3

$$Zw = Wz$$
$$Xy = Y(x+z)$$
$$Yw = Wy + Zx$$

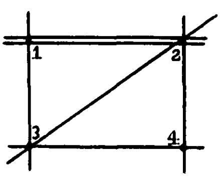




K 1

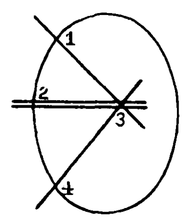
$$Xz = Zx$$
$$Zy = Y(z+w)$$
$$Ww = JXy + Yx$$

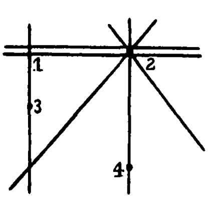
N 3



K 2

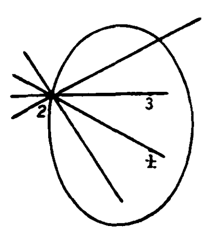
$$Xy = Yx$$
$$= Zw + W(Hx+Jy)$$
$$Xw = Wz$$

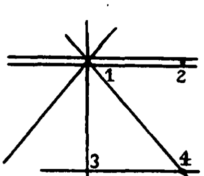




L 1

$$Xw = Wz$$
$$X(z+w) = Zx$$
$$Yw = y(X+Z) + W(Hx+Jy)$$





L 2_1

$$Xy = Yx$$
$$Zw = Wz$$
$$Y(z+w) = Z(y+w)$$

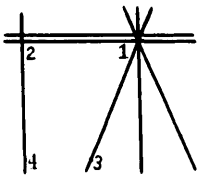
P 3_2

D 2

L 2_2

$$\begin{aligned} Xy &= Yx \\ Zw &= W(z+\delta w) \\ Y(z+w) &= y(Z+W)+Wz \end{aligned}$$

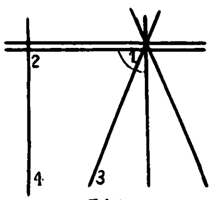
L 2_2



L 3

$$\begin{aligned} Xz &= Zx \\ Y(\gamma z+w) &= W(y+w) \\ Y(z+w) &= y(Z+W) \end{aligned}$$

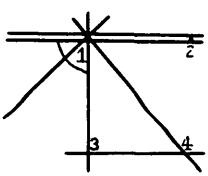
L 3



L′ 1

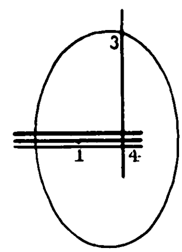
$$\begin{aligned} Xz &= Zx \\ Zy &= Y(z+w) \\ Ww &= J(Xy-Yx)+(\beta y+\gamma z)(Y-Z) \end{aligned}$$

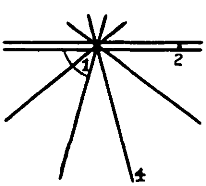
T 1



L′ 2

$$\begin{aligned} Xy &= Yx \\ Yw &= Zz \\ Ww &= Y(y+z)+Z(Jx+Ky) \end{aligned}$$

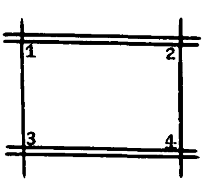




M

$$\begin{aligned} Yy &= Zz \\ Zw &= W(z+w) \\ Xw &= Wx+(Y-Z)(\beta y+\gamma z) \end{aligned}$$

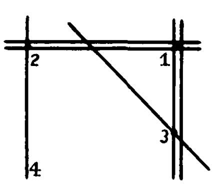
M



N 1

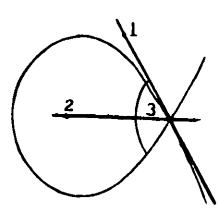
$$\begin{aligned} Xy &= Yx \\ Zw &= Wz \\ Xw+Zy &= JWx+NYz \end{aligned}$$

N 1

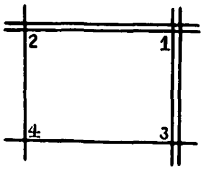


N 2

$$\begin{aligned} Xz &= Wx \\ Yw &= Wy \\ Zw &= Y(x+y)+X(Hx+Jy)+W(Nx+Tz) \end{aligned}$$

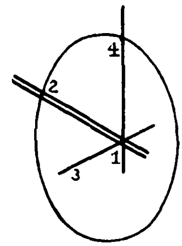


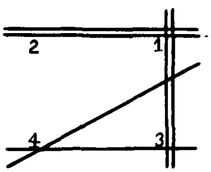
(One line is a tangent at the node of the quartic.)



N 3

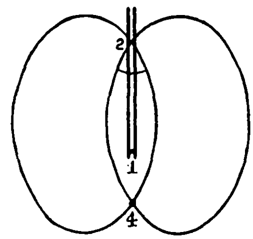
$$
\begin{aligned}
Yw &= Wy \\
Xy &= Y(Ex+Mz)-(M-N)Zx \\
Xw &= W(Ex+Nz)+MZz
\end{aligned}
$$

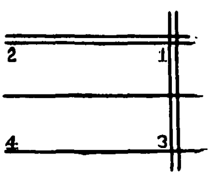




N 4

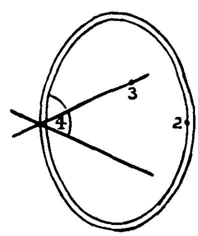
$$
\begin{aligned}
Zw &= Wz \\
Zy &= Xw-Wx \\
Yy &= M(Xz-Zx)+x(X+Z)
\end{aligned}
$$





N 5_1

$$
\begin{aligned}
Xy &= Yx \\
Zy &= Yz+w(X+Z) \\
Ww &= Zz+M(Xz-Zx)
\end{aligned}
$$

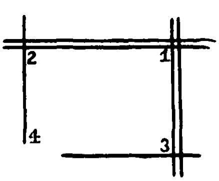


(Two coincident conics.)

N 5_2

$$
\begin{aligned}
Xy &= Yx \\
Zw &= W(z+Nw) \\
Xw &= W(x+Nw)+Zy-Yz
\end{aligned}
$$

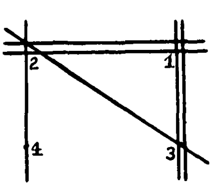
N 5_2



N 6

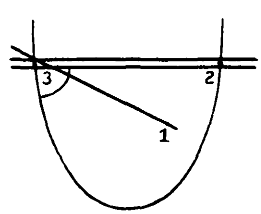
$$
\begin{aligned}
Xz &= Zx \\
Yz &= Zy+Ww \\
Y(x-w) &= y(X-W)
\end{aligned}
$$

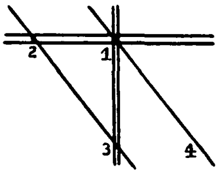
N 6



P 1

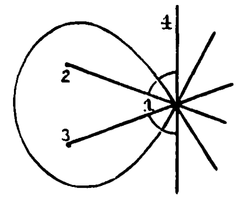
$$
\begin{aligned}
Yw &= Wy \\
Xz &= Wx \\
Zw &= FYx+X(Hx+Jy)+W(Nx+Tz)
\end{aligned}
$$



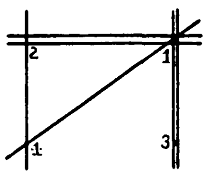


P 2

$$\begin{aligned} Yy &= Zz \\ &= Ww \\ Xw &= Wx + Y(Nx + Tz) + Z(Jx + Ky) \end{aligned}$$

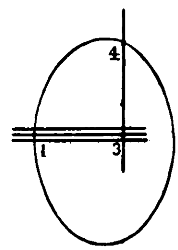


(Plane nodal cubic.)



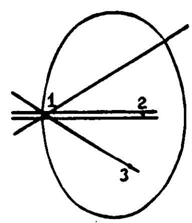
P 3_1

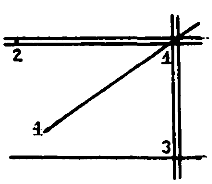
$$\begin{aligned} Yw &= Wy \\ Zy &= MXz - Y(Jx + Tz) \\ Zw &= GYz + W(Nx + Tz) \end{aligned}$$



P 3_2

$$\begin{aligned} Yy &= Ww \\ Zw &= Wz \\ Xz &= x(Y + W) + z(Fx + My) \end{aligned}$$





P 4_1

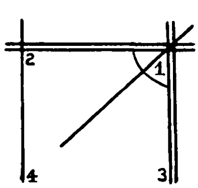
$$\begin{aligned} Xy &= Yx \\ Y(z + w) &= Z(y + w) \\ Ww &= Z(z - x) + X(z + w) \end{aligned}$$

U 2_1

P 4_2

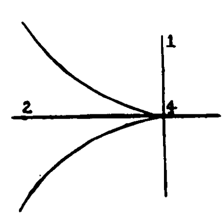
$$\begin{aligned} Xy &= Yx \\ Zw &= W(z + w) \\ Z(y - w) &= z(Y - W) \end{aligned}$$

P 4_2

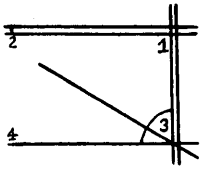


P′ 1

$$\begin{aligned} Xz &= Zx \\ Yw &= Zy \\ Ww &= Y(y + z) + X(Hx + Jy) + Z(Nx + Tz) \end{aligned}$$



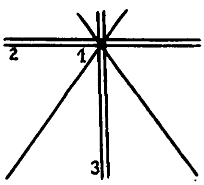
(One line is a tangent at the cusp of the quartic.)



P′ 2

$$Xy = Yx$$
$$Zw = Wz$$
$$Yw = Zy + W(x+w)$$

N 5_1



Q_1

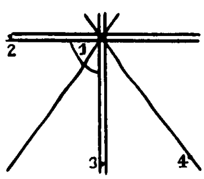
$$Yw = Wy$$
$$Xy = Yx + MZz$$
$$Xw = Wx - MZz + (z+w)(JZ + MW)$$

Q_1

Q_2

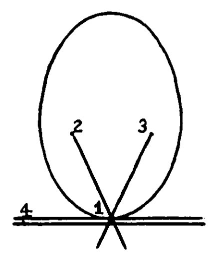
$$Xw = Wx + Zz$$
$$Yw = W(y-w)$$
$$Yz = Z(y+z)$$

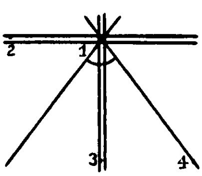
Q_2



Q′ 1

$$Yy = Zz = Ww$$
$$Xw = W(y+z) + Y(Nx + Tz) + Z(Jx + Ky)$$

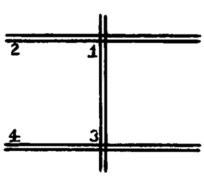




Q′ 2

$$Zw = Wz$$
$$Xw = Wx + Zy$$
$$Xz = Yy + Z(x - z - w)$$

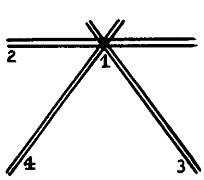
W 3_2



R

$$Xy = Yx$$
$$Zw = Wz$$
$$Xw + Y(z + Nw) = Wx + y\,(Z + FW$$

R



S_1

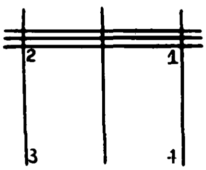
$$Zw = Wz$$
$$Xw = Wx + GZy$$
$$Xz = Zx + M(Yy + Zz)$$

X_2

S₂

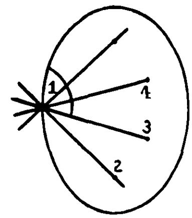
$$\begin{aligned} Yw &= Wy \\ Xy &= Yx + TZz \\ Xw &= W(x+Tz) + Z(Fz+Jw) \end{aligned}$$

S_2



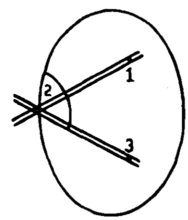
T 1_1

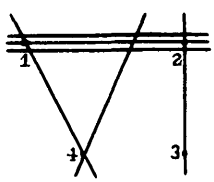
$$\begin{aligned} Zw &= Wz \\ Xz &= Yy + Z(Ex+Fy+w) \\ Xw &= Yx + W(Ex+Fy+z) \end{aligned}$$



T 1_2

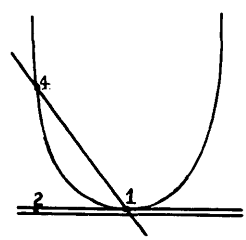
$$\begin{aligned} Xw &= Wx \\ Z(z+w) &= W(x+y) \\ Yz &= X(x+y) - Zx + RWy \end{aligned}$$

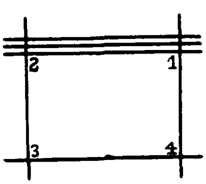




T 2

$$\begin{aligned} Zw &= Wz \\ Yw &= Wy + Zx \\ Xz &= Y(x+y) + Z(Nx+Py) \end{aligned}$$





T 3_1

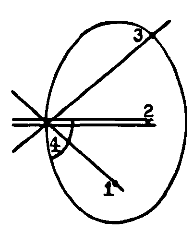
$$\begin{aligned} Xy &= Yx \\ Zw &= Wz \\ Xz &= Zx + Yw - W(y-z) \end{aligned}$$

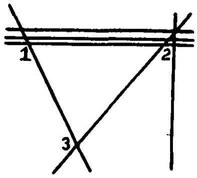
T 3_1

De Paolis, *l.c.*, p. 741, 1°; p. 754, 1°; $\mu = 1$, $\nu = 2$.

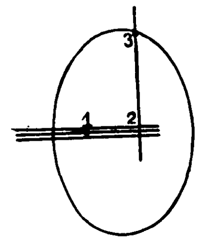
T 3_2

$$\begin{aligned} Xy &= Yx \\ Xz &= Zw \\ Wz &= RYw + Z(Ex+Fy) \end{aligned}$$





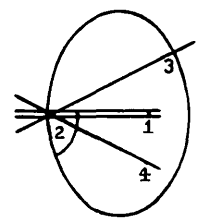
$$\begin{aligned} Zw &= Wz \\ Xz &= Zx + Wy \\ Xy &= Yw + W(Nx + Py) \end{aligned}$$

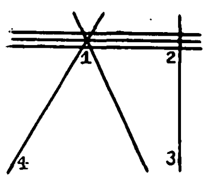


T 4_1

T 4_2

$$\begin{aligned} Zw &= Wz \\ X(w+z) &= Zx \\ X(y+Qw) &= Yw + W(Qx + Ry) \end{aligned}$$





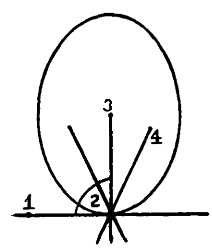
$$\begin{aligned} Xw &= Wx \\ Y(z+w) &= Wy \\ Yw &= Zz + F(Xy - Yx) \end{aligned}$$

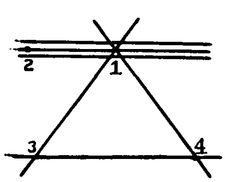
U 2_3

T 5_1

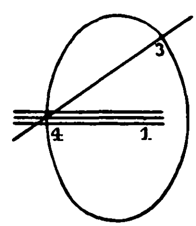
T 5_2

$$\begin{aligned} Xz &= Zw \\ X(z+w) &= Wx \\ Yz &= Z(Ex + Fy) + Wy \end{aligned}$$





$$\begin{aligned} Xy &= Yx \\ Yz &= Zw \\ Yw &= Wz + Z(Fx + Gy) \end{aligned}$$

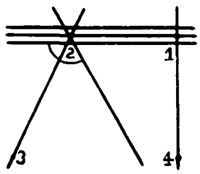


T 6_1

T 6_2

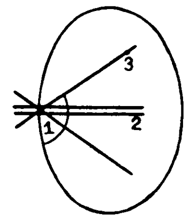
$$\begin{aligned} Xy &= Yx \\ Zw &= Wz \\ Xz &= Zx + (W+Z)w \end{aligned}$$

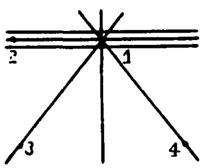
T 6_2



T 7

$$Zw = Wz$$
$$Xz = Zx + Wy$$
$$Yw = EX(x+z) + W(Nx+Py)$$

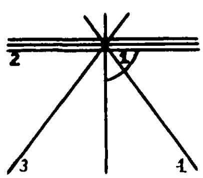




T 8

$$Zw = Wz$$
$$Y(y+w) = W(z+w)$$
$$Xz = Zx + Yy$$

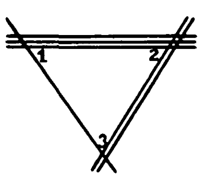
M



T 9

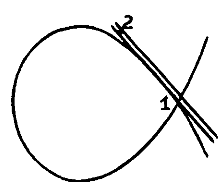
$$Xz = Zx$$
$$Zw = W(z-w)$$
$$X(x+z) = Yw - Wy + Q(Xw - Wx)$$

T 9



U 1_1

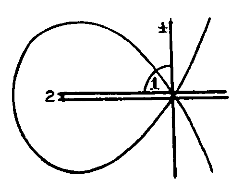
$$Zw = Wz$$
$$Yw = Wy + Zx$$
$$Xw = Yy + QWx + Z(Nx + Py + Tz)$$



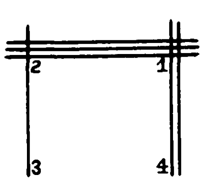
(Two lines coincide in a tangent at the node of the quartic.)

U 1_2

$$Zw = Wz$$
$$Yw = Zy$$
$$Xw = Yx + TZz + W(Qx + Ry)$$

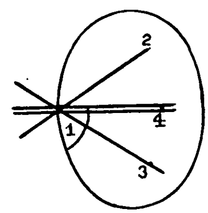


(Plane nodal cubic.)



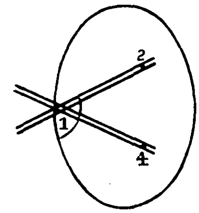
U 2_1

$$Zw = Wz$$
$$Xz = PYy + Z(Ex + Fy)$$
$$Xw = PYx + W(Ex + Fy + Pz)$$



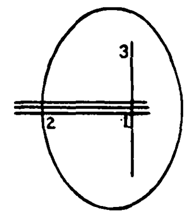
U 2_2

$$\begin{aligned} Zw &= Wz \\ Yw &= Zx + Wy \\ Xz &= Yy + Z(Nx + Py) \end{aligned}$$



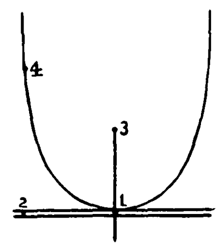
U 2_3

$$\begin{aligned} Yz &= Zy \\ Xy &= Y(Ex + Rw) + Zw + Wx \\ Xz &= Z(Ex + Rw) + Ww \end{aligned}$$



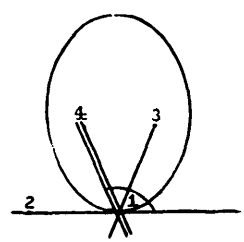
U 2_4

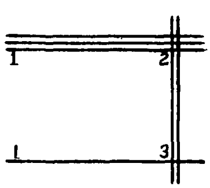
$$\begin{aligned} Yz &= Wy \\ Zw &= Wx \\ Xz &= Yw + Z(Ex + Fy + Nw) \end{aligned}$$



U 2_5

$$\begin{aligned} Yz &= Zy \\ Yw &= Wz \\ Xw &= Z(x + Tw) + W(Jx + Ky) \end{aligned}$$





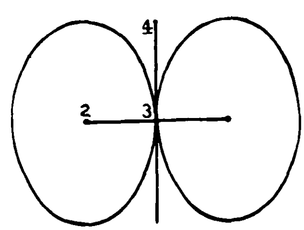
U 3_1

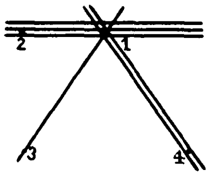
$$\begin{aligned} Xy &= Yx \\ Zw &= Wz \\ Xz &= Yw + Zx - W(y - w) \end{aligned}$$

U 3_1

U 3_2

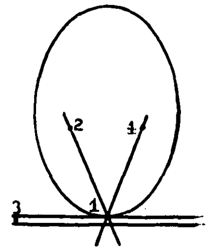
$$\begin{aligned} Xy &= Yx \\ Yz &= Zw \\ Zz &= Ww + X(Hx + Nz) + Y(Jx + Ky) \end{aligned}$$





U 4_1

$$\begin{aligned} Yw &= Wy \\ &= Zz \\ Xz &= Zw + Y(Fx + Gy) + W(Nx + Uw) \end{aligned}$$



Q

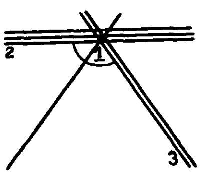
U 4_2

$$\begin{aligned} Xz &= Zx + RYw \\ Zw &= z(FY + QW) \\ Xw &= Y(Fx + Rw) + W(Qx - Ry) \end{aligned}$$

U 4_3

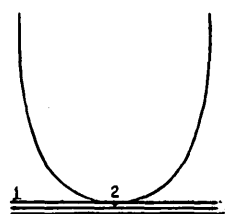
$$\begin{aligned} Yw &= Wy \\ Xw &= Wx + TZz \\ Yx &= Xy + T(Z - W)(z + w) \end{aligned}$$

U 4_3



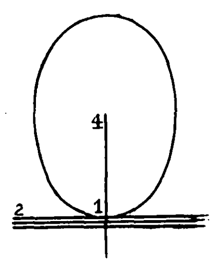
U 5_1

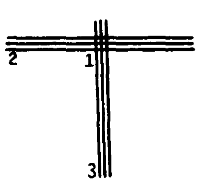
$$\begin{aligned} Zw &= Wz \\ Xw &= Wx + Zy \\ Yw &= GX(y + z) + RWy + Z(Nx + Py + Tz) \end{aligned}$$



U 5_2

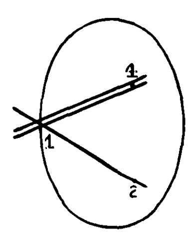
$$\begin{aligned} Zw &= Wz \\ Yw &= Zy \\ Xw &= GY(y + z) + Z(Nx + Tz) + W(Qx + Ry) \end{aligned}$$





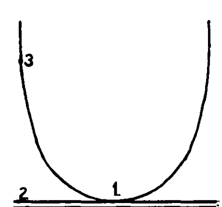
V_1

$$\begin{aligned} Zw &= Wz \\ Xz &= MYy + Z(Ex + Fy) \\ Xw &= M(Zz - Yx) + W(Ex + Fy) \end{aligned}$$



V_2

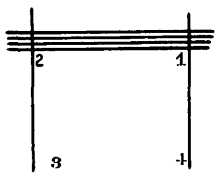
$$\begin{aligned} Zw &= Wz \\ Yz &= Zx + Wy \\ Xz &= Yy + W(Nx + Uw) \end{aligned}$$



V_3

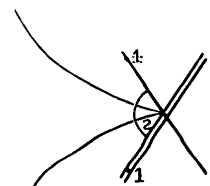
$$\begin{aligned} Yw &= Wy \\ Xy &= Yx + UZw \\ Xw &= Wx + Uz\,(Z - W) \end{aligned}$$

$U\,4_3$



W 1_1

$$\begin{aligned} Zw &= Wz \\ Xw &= Zx \\ Yz &= x\,(EX + NZ) + y\,(FX + PZ + RW) \end{aligned}$$

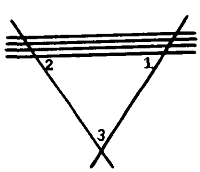


(Plane cuspidal cubic.)

W 1_2

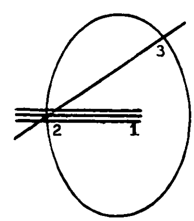
$$\begin{aligned} Yw &= Wy \\ X\,(z + w) &= Wx \\ Zz &= Ww + F\,(Xy - Yx) \end{aligned}$$

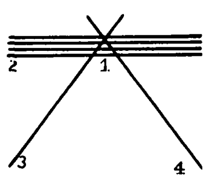
V_2



W 2

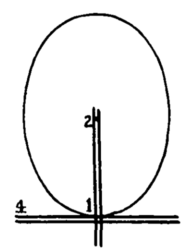
$$\begin{aligned} Zw &= Wz \\ Xz &= Zx + Ww \\ Xy &= Yw + V\,(Xw - Wx) \end{aligned}$$



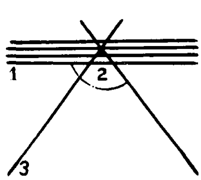


W 3_1

$$\begin{aligned} Zw &= Wz \\ Yw &= Zy \\ Xz &= Wy + Y\,(Fx + Gy) + Z\,(Nx + Py) \end{aligned}$$



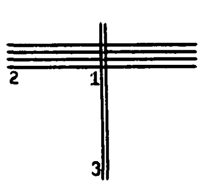
W 3_2 (see Q′ 2)



W 4

$$\begin{aligned} Zw &= Wz \\ Xz &= Zx + Ww \\ Yw &= Wy + x\,(X + Z) \end{aligned}$$

W 4



X_1

$$\begin{aligned} Yw &= Wy \\ Xw &= Wx + Yz \\ Zz &= Ww + G\,(Xy - Yx) \end{aligned}$$

Z

X_2 (see S_1)

X_3

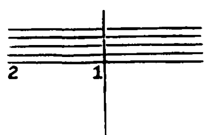
$$Zw = Wz$$
$$Yz = Zy + Ww$$
$$Xw = Wx + Yy$$

X_3

X_4

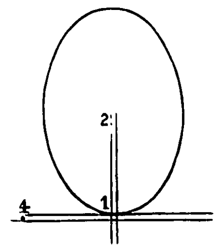
$$Yw = Wy$$
$$Xw = Wx + T2z$$
$$Xy = Yx + T(Zw - Wz)$$

X_4



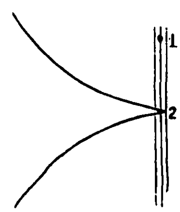
Y_1

$$Zw = Wz$$
$$Yw = Zy$$
$$Xz = Ww + Y(Fx + Gy) + Z(Nx + Py)$$



Y_2

$$Zw = Wz$$
$$Xw = Zz + Wx$$
$$Xy = Yw + Z(Nx + Py) + W(Qx + Ry)$$

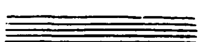


(Plane cuspidal cubic.)

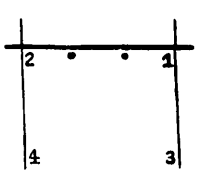
Y_3

$$Zw = Wz$$
$$Xz = Zx + Yw$$
$$Xw = Y(y + Pw) + W(x + z)$$

U 4_2



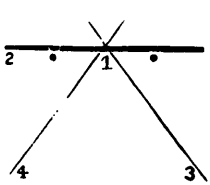
Z (see X_1)



$W1$

$$Zw = Wz$$
$$Xz = Wx + Z(Ax + By)$$
$$Yw = Zy + W(Cx + Dy)$$

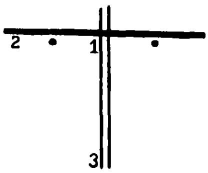
$W1$



$W2$

$$Zw = Wz$$
$$Xz = Wy + Z(Ax + By)$$
$$Yw = Zy + W(Cx + Dy)$$

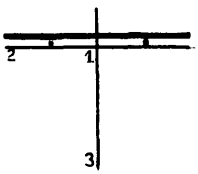
$W1$



X

$$
\begin{aligned}
Zw &= Wz\\
Xw &= Yz + Z(Ax + By) + W(Ex + Fy)\\
Yw &= Zy + W(Cx + Dy)
\end{aligned}
$$

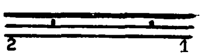
$W\,1$



Y

$$
\begin{aligned}
Zw &= Wz\\
Xz &= Ww + Z(Ax + By)\\
Yw &= Zy + W(Cx + Dy)
\end{aligned}
$$

Y



Z_1

$$
\begin{aligned}
Zw &= Wz\\
Xz &= Zz + Ww\\
Yw &= Wy + Zz
\end{aligned}
$$

Z_1

De Paolis, *l.c.*, p. 739, 1*a*; $\mu = 3$.

Z_2

$$
\begin{aligned}
Zw &= Wz\\
Xz &= Ww + Z(Ax + By)\\
Yz &= Xw + Z(Cx + Dy) + W(Ex + Fy)
\end{aligned}
$$

Y

Berry, *Camb. Phil. Soc. Trans.*, Vol. XIX, p. 275; Vol. XX, pp. 109. 111.