PLANE HOMALOIDAL FAMILIES OF GENERAL DEGREE

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1. *History and Summary.*

A homaloidal family of curves is one that can be transformed into the straight lines of another plane by a Cremona transformation. In the memoir that in 1865 laid the foundation of the plane theory, Cremona gave 49 types of homaloidal families, of which 44 are essentially distinct, for general values of the degree n of the form $x\mu+\nu$, $\nu = 0, 1, \ldots, \mu-1$, with $x = 1, 2, 3$, and 4. We shall quote these as C. 1, ..., C. 49. Ruffini's formulæ (R. 4, ..., R. 7) contain as particular cases a few of these and five more for $x = 2$, of which Larice rediscovered two, and his general formula (R. m) gives one more for each of the forms of n with $x = 3, 4$.

By compounding two de Jonquières transformations, with all possible varieties of coincidence of the fundamental points in the intermediate space, Bianchi obtained a very general solution (B.), the form of n involving 3 parameters; the simplest case with no coincidences was given later by de Jonquières. These do not produce any new types with $x \leqslant 4$. Palatini repeated Bianchi's work and extended it. In his formula (P.) for the composition of three de Jonquières transformations, with arbitrary coincidences of the fundamental points, n involves 6 parameters. From this there can be deduced all the remaining types with $x = 3$, and many with $x = 4$. All these general formulæ will be found in § 7 below. Palatini also gives a heavy formula, with 16 parameters, for the composition of four de Jonquières transformations, which has not been analysed here.

In most of these researches the new general types are obtained by compounding simpler known types. By this method, there is no guarantee of completeness, and none of the writings quoted contain all the types of the kind considered. On the other hand, there is a guarantee of the existence of such types as are found, as geometrical transformations and not merely as arithmetical solutions of the equations of condition.

In this paper the equations are attacked from the arithmetical point of view, so as to ensure completeness; it is shown that all the types for $x = 1$, 2, and 3 are given or hidden in one or other of the memoirs. They are collected and classified in §§ 3, 4, and 5. We make use of several general conditions that the families do not break up in specified ways: the final proof of existence is to transform each family either into one of a general type already proved to exist, or into one for which $n \leqslant 15$, of which more or less accurate tables have been published in scattered places. A complete table for $n \leqslant 11$ is given by Montesano (*loc. cit. Atti*, p. 34).

This method does not lead to the pairing of conjugate solutions; but by giving this up we are able to tabulate the types, at the end of § 5, in an order much more convenient for use.

For $x = 4$, the number of types is much larger; in § 6 they are classified, and a specimen of each set is given. The same methods could be applied to any numerical value of x. In § 7 there are given the types which exist for a general value of x.

There prove to be 2 quite general solutions ($x = 1$); 6 for n even and 5 for n odd ($x = 2$); 18 for n a multiple of three, and 16 in each of the contrary cases ($x = 3$). The inequality of these numbers is remarkable.

2. *General Conditions.*

A plane homaloidal family of degree n is completely determined by its F-system, that is, the set of fundamental or F-points common to all the curves of the family together with their multiplicities. Let the F-system consist of a_i points of multiplicity $i (i = 1, 2, \ldots, n-1)$, and let it be written Σa_i^i. Cremona showed that the positive integers i, a_i satisfy the two equations:

$$\Sigma i a_i = 3n-3 \quad (\mathrm{I}), \qquad \Sigma i^2 a_i = n^2-1 \quad (\mathrm{II}).$$

We have also certain inequalities expressing that the family is not wholly degenerate, so that no fixed curve of degree y, determined by the

F-points, meets one of the family in more than yn points. Thus

(straight lines)	$i_1+i_2 \leqslant n$,
(conics)	$i_1+i_2+\ldots+i_5 \leqslant 2n$,
(nodal cubics)	$2i_1+i_2+\ldots+i_7 \leqslant 3n$,
(quartics with triple point)	$3i_1+i_2+\ldots+i_9 \leqslant 4n$,
(quartics with 3 double points)	$2(i_1+i_2+i_3)+i_4+\ldots+i_8 \leqslant 4n$,

where i_1, i_2, ... are the highest multiplicities.

More complicated tests are needed when the degrees of both components depend on n.

A set of numbers i, α_i, which satisfy I and II but violate any of these conditions, or for which any α_i is negative, form an arithmetical but not a geometrical solution; that is, a solution of the equations of condition to which there does not correspond a homaloidal family of proper curves.

The following known properties will also be used:

$$(a) \quad i_1 \leqslant n-1.$$

$$\left.\begin{array}{ll}(b) & i_1+i_2+i_3 > n \\ (c) & i_1 > n/3\end{array}\right\} \quad \text{(Noether).}$$

$$(d) \quad \Sigma\alpha_i \leqslant 2n-1 \quad \text{(Montesano).}$$

All those solutions which differ from each other only in the part independent of n, that is, which have values of α_i differing by definite integers or zero when i is independent of n, but equal values of α_i when i involves n, can be quickly obtained from each other by the following device.

If y is any given positive integer, then on any solution of (I), (II) we can superpose any positive or negative multiple of

$$E_y \equiv 1^y\{-\tfrac{1}{2}y(y-1)\}^2\{y(y-2)\}^1,$$

and obtain another arithmetical solution, which is also a geometrical solution provided that, in the result, every $\alpha_i \geqslant 0$ and that the family does not break up. Hence, if at first we admit negative values of α_1 and α_2 we can put $\alpha_3 = \alpha_4 = \ldots 0$, in the part independent of n, in all the work until the last stage of all, when from this arithmetical solution or *base* we can deduce the corresponding set of geometrical solutions, by systematically superposing proper multiples of

$$E_3 \equiv 1^3(-3)^2 3^1, \qquad E_4 \equiv 1^4(-6)^2 8^1,$$

$$E_5 \equiv 1^5(-10)^2 15^1, \quad E_6 \equiv 1^6(-15)^2 24^1, \quad \&c.$$

3. $x = 1$.

In any solution for a quite general value of n, both i and a_i are polynomials in n. Since the solution is to hold for n large, the properties (a) and (d) of § 2 show that these polynomials are of degrees 0 or 1; and further, (a) shows that i has one of the forms $n-\sigma$ or σ, where σ is a positive integer, and (d) shows that a_i has one of the forms $2n-\sigma$, $n\pm\sigma$, σ. In every case where n occurs, its coefficient is positive.

Since (I) and (II) are satisfied for all values of n, we equate coefficients of its powers. From the terms in n^2 in (I), or in n^3 in (II), we see that i and a_i cannot both contain n for the same value of i. We therefore assume an F-system denoted, with a slight change of notation, by

$$\sum_{\sigma_a=1}^{\sigma_0} a^{n-\sigma_a} \sum_{i=1}^{i_0} (\epsilon_i n+\eta_i)^i,$$

separating the groups of points for which i involves n from the groups in which it does not; in the latter only, a_i may involve n, if $\epsilon_i > 0$.

The upper limits σ_0, i_0 are positive integers to be determined; a, σ_a, i, ϵ_i are positive integers or zero, and $\epsilon_i \leqslant 2$; the η_i are integers which may be negative; if $\epsilon = 0$, then $\eta_i \geqslant 0$; if $\epsilon_i = 1$, then $\eta_i \gtreqless 0$; and if $\epsilon_i = 2$, then $\eta_i < 0$.

Now equate coefficients of powers of n in (I), (II), in the order most convenient for future use.

$$\begin{array}{lrcll} (\mathrm{II}\, n^2) & \Sigma a & = & 1, & (1) \\ (\mathrm{I}\, n) & \Sigma a \ + \Sigma i\, \epsilon_i & = & 3, & (2) \\ (\mathrm{II}\, n) & -2\Sigma a\sigma_a + \Sigma i^2 \epsilon_i & = & 0, & (3) \\ (\mathrm{I}\, 1) & -\Sigma a\sigma_a + \Sigma i\, \eta_i & = & -3, & (4) \\ (\mathrm{II}\, 1) & \Sigma a\sigma_a^2 + \Sigma i^2 \eta_i & = & -1. & (5) \end{array}$$

From (1), the first sum has only one term, with $a = 1$, arising from a single F-point; from (2), the second has only one term involving n, that is, with $\epsilon_i > 0$, which arises from two or one F-points, giving the two cases:

(1) $\epsilon_1 = 2$, all other $\epsilon_i = 0$; (2) $\epsilon_2 = 1$, all other $\epsilon_i = 0$.

In either case we have the condition

(straight lines) $\quad n-\sigma_a+i \leqslant n, \;$ that is, $\; i \leqslant \sigma_a$.

Case 1.—Since $\quad \epsilon_1 = 2, \;$ hence $\; \Sigma i^2 \epsilon_i = 2$;

from (3) $\quad \sigma_a = 1, \;$ whence $\quad i = 1$.

Thus there is only one term in the second sum. Then (4), (5) are both satisfied by $\eta_1 = -2$; all the constants are now determined, and we have the well-known de Jonquières solution, which is Cremona's first general type:

C. 1 $$1^{n-1}(2n-2)^1.$$

Case 2.—Here $\quad \epsilon_2 = 1, \quad \Sigma i^2 \epsilon_i = 4$.

From (3) $\quad \sigma_a = 2, \;$ whence $\; i \leqslant 2$,

and there are just two terms in the second sum. Now (4), (5) give

$$\eta_1 = 3, \qquad \eta_2 = -2,$$

and we have the second solution

C. 2 $$1^{n-2}(n-2)^2 3^1.$$

These two, both given by Cremona, are therefore the only types possible for a quite general value of n.

In both cases a quadratic transformation *applied to* the three highest F-points of the family, that is, with its three F-points coinciding with them, reduces it to one of the same type and of lower degree. A series of such quadratic transformations reduces either family to straight lines or conics; since these do not break up, neither do the two general families.

4. $x = 2$.

To find the solutions depending on whether n is odd or even, we write $n = 2\mu+\nu$, $\nu = 0$ or 1, and assume i, a_i to be polynomials in μ, instead of in n, with coefficients depending on ν. As before, by (a), (d)

$$i \leqslant 2\mu+\nu-1, \qquad a_i \leqslant 4\mu+2\nu-1,$$

and i, a_i do not involve μ both at once. The F-system has the form

$$\Sigma \alpha^{n-\sigma_a} \Sigma \beta^{\mu-\sigma_\beta} \Sigma(\epsilon_i \mu+\eta_i)^i \quad (\epsilon_i = 0,\ 1,\ 2,\ 3, \text{ or } 4).$$

Here σ_β may be negative, but not σ_a.

Equate coefficients of powers of μ in (I), (II).

$$4\Sigma a \quad + \Sigma\beta \quad = 4, \tag{1$_2$}$$

$$2\Sigma a \quad + \Sigma\beta + \Sigma i\,\epsilon_i = 6, \tag{2$_2$}$$

$$-4\Sigma a(\sigma_a - \nu) - 2\Sigma\beta\sigma_\beta + \Sigma i^2\epsilon_i = 4\nu, \tag{3$_2$}$$

$$-\Sigma a(\sigma_a - \nu) - \Sigma\beta\sigma_\beta + \Sigma i\,\eta_i = 3\nu - 3, \tag{4$_2$}$$

$$\Sigma a(\sigma_a - \nu)^2 + \Sigma\beta\sigma_\beta^2 + \Sigma i^2\eta_i = \nu^2 - 1. \tag{5$_2$}$$

Since a, β are positive integers, $(1)_2$ gives two alternatives:

1. The first sum has one term, with $a = 1$, the second sum is absent;
2. The first sum is absent, the second has not more than four terms, arising from exactly four F-points.

Case 1.— $a = 1, \quad \beta = 0.$

From $(2)_2$, $(3)_2$ $\qquad \Sigma i\epsilon_i = 4, \quad \Sigma i^2\epsilon_i = 4\sigma_a.$

The first of these gives five possibilities:

$$\epsilon_1 = 4\,; \quad \epsilon_1 = 2,\ \epsilon_2 = 1\,; \quad \epsilon_2 = 2\,; \quad \epsilon_1 = 1,\ \epsilon_3 = 1\,; \quad \epsilon_4 = 1.$$

In the first and third of these, the only ϵ_i is even; in the whole F-system, μ only appears in the form 2μ or 4μ, and we are shut up to types already met in § 3. In the second and fourth, $\Sigma i^2\epsilon_i$ is not a multiple of 4, so these are excluded by $(3)_2$. We are left with the last case:

$$\epsilon_4 = 1, \quad \sigma_a = 4,$$

with the further condition

(straight lines) $\qquad n - 4 + i \leqslant n$, that is, $i \leqslant 4$.

It remains to satisfy $(4)_2$ and $(5)_2$, which become

$$\eta_1 + 2\eta_2 + 3\eta_3 + 4\eta_4 = 2\nu + 1,$$

$$\eta_1 + 4\eta_2 + 9\eta_3 + 16\eta_4 = 8\nu - 17.$$

We first assume $\eta_3 = \eta_4 = 0$, and obtain the base

$$\eta_1 = 19 - 4\nu, \qquad \eta_2 = -9 + 3\nu.$$

Considering the last set of F-points, we see that only η_4 can be negative. We therefore superpose negative multiples of E_4, and positive multiples of E_3, for each value of ν, in every possible way that leaves $\eta_1, \eta_2 \geqslant 0$.

We can, however, exclude many cases and limit the work by another consideration. Write down the condition that the family does not break up, one component being of degree y, determined by a $(y-1)$-fold point at the $(n-4)$-fold F-point, and $2y$ simple points at the next highest F-points, that is, by $1^{y-1}(2y)^1$. Take

$$2y = \mu+\eta_4+\eta_3+\eta_2-\rho, \qquad \rho \leqslant \eta_2,$$

where ρ is chosen to make this expression even. If our type exists for both odd and even values of μ, we can take $\rho = 0$.

We have the condition

$$yn \geqslant (y-1)(n-4)+4(\mu+\eta_4)+3\eta_3+2(\eta_2-\rho),$$

whence
$$2\eta_4+\eta_3 \leqslant \nu-4.$$

This excludes all but four cases, C. 22, 18, 28, 26, valid for all values of $\mu \geqslant -\eta_4$; they are shown in the table. In certain other cases where $\eta_2 = 0$, and therefore $\rho = 0$, the value assumed for y is only possible if μ is odd (or even, as the case may be), and must be replaced in the opposite case by

$$2y = \mu+\eta_4+\eta_3-\rho, \qquad 0 < \rho \leqslant \eta_3,$$

$$yn \geqslant (y-1)(n-4)+4(\mu-\eta_4)+3(\eta_3-\rho),$$

whence
$$2\eta_4+\eta_3 \leqslant \nu-4+\rho.$$

This leads to some types valid if n is of the form $4\mu+\nu$ instead of $2\mu+\nu$; we return to these in § 6.

For each existing type, we verify that it does not break up in some other way by transforming it into a known family of lower degree, by quadratic transformations applied to the three highest F-points.

Case 2.—Let $\sigma_{\beta_1}, \ldots, \sigma_{\beta_4}$, in descending order, refer to the four points of the second sum, where these σ's can be equal to each other. We have the conditions

(straight lines) $\mu-\sigma_{\beta_1}+\mu-\sigma_{\beta_2} \leqslant 2\mu+\nu$, that is, $\sigma_{\beta_1}+\sigma_{\beta_2} \geqslant -\nu$;

(conics) $4\mu-\Sigma\sigma_{\beta_1}+i \leqslant 4\mu+2\nu$, that is, $i \leqslant 2\nu+\Sigma\sigma_{\beta_1}$.

From $(2)_2$
$$\Sigma i\epsilon_i = 2,$$

giving two sub-cases, as for n general,

$$\text{(i)} \quad \epsilon_1 = 2, \qquad \text{(ii)} \quad \epsilon_2 = 1.$$

Case 2 i.— $a = 0, \quad \Sigma\beta = 4, \quad \epsilon_1 = 2.$

From $(3)_2$ $\quad \sigma_{\beta_1}+\sigma_{\beta_2}+\sigma_{\beta_3}+\sigma_{\beta_4} = 1-2\nu,$ whence $i \leqslant 1.$

But $\quad \sigma_{\beta_1}+\sigma_{\beta_2} \geqslant -\nu,$

whence $\quad \sigma_{\beta_3}+\sigma_{\beta_4} \leqslant 1-\nu,$

and similarly for any other pair. Sum the three inequalities of each type that involve σ_{β_1}, and use $(3)_2$:

$$-1-\nu \leqslant 2\sigma_{\beta_1} \leqslant 2-\nu,$$

and similarly for σ_{β_2}, σ_{β_3}, σ_{β_4}. The only solution of $(3)_2$ compatible with these is

$$\sigma_{\beta_1} = -\nu, \quad \sigma_{\beta_2} = \sigma_{\beta_3} = 0, \quad \sigma_{\beta_4} = 1-\nu.$$

Now, since $i \leqslant 1$, the last sum has only one term; from $(4)_2$, $(5)_2$

$$\eta_1 = \nu-2, \quad \eta_1 = -\nu^2+2\nu-2,$$

which are compatible since $\nu = 0$ or 1. We have the pair of solutions C. 3, 5.

Case 2ii.— $a = 0, \quad \Sigma\beta = 4, \quad \epsilon_2 = 1.$

A similar argument leads to $\Sigma i^2\epsilon_i = 4$, $\Sigma\sigma_{\beta_1} = 2-2\nu$, whence $i \leqslant 2$,

$$-\nu \leqslant \sigma_{\beta_1}+\sigma_{\beta_2} \leqslant 2-\nu,$$

$$-2-\nu \leqslant 2\sigma_{\beta_1} \leqslant 4-\nu.$$

Here are three possibilities:

$$\sigma_{\beta_1} = -1, \quad \sigma_{\beta_2} = \sigma_{\beta_3} = 1-\nu, \quad \sigma_{\beta_4} = 1;$$

or $$\sigma_{\beta_1} = \sigma_{\beta_2} = 0, \quad \sigma_{\beta_3} = \sigma_{\beta_4} = 1-\nu;$$

or $$\sigma_{\beta_1} = \sigma_{\beta_2} = \sigma_{\beta_3} = 0, \quad \sigma_{\beta_4} = 2-2\nu.$$

Each gives two types with $\nu = 0$ or 1; the last two are the same if $\nu = 1$, but different if $\nu = 0$. It is this circumstance that causes the numbers of solutions for n odd and n even to be unequal.

Now $i \leqslant 2$, and η_1, η_2 are given by $(4)_2$, $(5)_2$; we have Ruffini's five solutions, three for n even and two for n odd.

This exhausts the possibilities with n of the form $2\mu+\nu$.

5. $x = 3$.

Next let $n = 3\mu+\nu$, $\nu = 0$, 1, or 2. The F-system is

$$\Sigma \alpha^{n-\sigma_\alpha} \Sigma \beta^{2\mu-\sigma_\beta} \Sigma \gamma^{\mu-\sigma_\gamma} \Sigma (\epsilon_i\mu+\eta_i)^i,$$

and the constants satisfy

$$9\Sigma\alpha \quad +4\Sigma\beta + \Sigma\gamma \quad = 9, \tag{1$_3$}$$

$$3\Sigma\alpha \quad +2\Sigma\beta + \Sigma\gamma + \Sigma i\epsilon_i = 9, \tag{2$_3$}$$

$$-6\Sigma\alpha(\sigma_\alpha-\nu) - 4\Sigma\beta\sigma_\beta - 2\Sigma\gamma\sigma_\gamma + \Sigma i^2\epsilon_i = 6\nu, \tag{3$_3$}$$

$$-\Sigma\alpha(\sigma_\alpha-\nu) - \Sigma\beta\sigma_\beta - \Sigma\gamma\sigma_\gamma + \Sigma i\eta_i = 3\nu-3, \tag{4$_3$}$$

$$\Sigma\alpha(\sigma_\alpha-\nu)^2 + \Sigma\beta\sigma_\beta^2 + \Sigma\gamma\sigma_\gamma^2 + \Sigma i^2\eta_i = \nu^2-1. \tag{5$_3$}$$

From $(1)_3$ there are four possibilities:

$$\alpha = 1,\ \beta = \gamma = 0; \qquad \alpha = 0,\ \Sigma\beta = 2,\ \gamma = 1;$$

$$\alpha = 0,\ \beta = 1,\ \Sigma\gamma = 5; \qquad \alpha = \beta = 0,\ \Sigma\gamma = 9.$$

The second is excluded (straight lines), and we have three cases.

Case 1.—From $(2)_3$, $(3)_3$ $\Sigma i\epsilon_i = 6$, $\Sigma i^2\epsilon_i = 6\sigma_\alpha$,

(straight lines) $\qquad n-\sigma_\alpha+i \leqslant n$, that is, $i \leqslant \sigma_\alpha$.

Now every $\epsilon_i \geqslant 0$; if we relax this condition, and then put $\epsilon_3 = \epsilon_4 = \ldots = 0$, we have the base for the ϵ_i,

$$\epsilon_1 = 12-6\sigma_\alpha, \quad \epsilon_2 = -3+3\sigma_\alpha.$$

It follows that the geometrical possibilities are

$$\sigma_\alpha = 1,\ \epsilon_1 = 6; \quad \sigma_\alpha = 2,\ \epsilon_2 = 3; \quad \sigma_\alpha = 3,\ \epsilon_3 = 2; \quad \sigma_\alpha = 6,\ \epsilon_6 = 1.$$

In the first two of these, μ only appears in the form 3μ, and we have types already met in § 3. There remain two sub-cases.

Case 1i.— $\alpha = 1$, $\beta = \gamma = 0$, $\sigma_\alpha = 3$, $\epsilon_3 = 2$, $i \leqslant 3$.

Here η_3 can be negative, but not η_1, η_2. From $(4)_2$, $(5)_2$ we have the base

$$\eta_1 = 10-2\nu, \quad \eta_2 = -5+2\nu, \quad \eta_3 = 0,$$

whence six types, C. 8, 6, 12, 10, 14, 16, valid for $\mu \geqslant -\eta_3$.

Case 1ii.— $\alpha = 1,\quad \beta = \gamma = 0,\quad \sigma_\alpha = 6,\quad \epsilon_6 = 1,\quad i \leqslant 6.$

Only η_6 can be negative. The base is

$$\eta_1 = 43-8\nu,\quad \eta_2 = -20+5\nu,\quad \eta_3 = \eta_4 = \eta_5 = \eta_6 = 0.$$

We can exclude many arithmetical solutions by the test of § 4, Case 1. That argument leads to

$$3\eta_6+2\eta_5+\eta_4 \leqslant -6+\nu.$$

A quadratic transformation applied to the three highest F-points reduces each of these types to a similar one with μ lowered by 2. To prove the proper existence of each we have therefore to check it for $\mu = -\eta_3$ and $-\eta_3+1$. Three are found to fail, breaking up in ways other than those specified so far. The remaining geometrical solutions are tabulated; again the number depends on the form of n, being 8 for $\nu = 0$ and 6 for $\nu = 1$ or 2.

A general formula for this set can be obtained from (P.) by putting

$$N = 6,\quad n_1 = 3,\quad 2n_3-l' = \mu;$$

the remaining parameters l'', l''', n_2 taking all sets of values that make all the $a_i > 0$.

Case 2.—Here we have the conditions:

$$\begin{array}{lll}
\text{(straight lines)} & 2\mu-\sigma_\beta+\ \mu-\sigma_{\gamma_1} & \leqslant 3\mu+\nu, \\
\text{(conics)} & 2\mu-\sigma_\beta+4\mu-(\sigma_{\gamma_2}+\sigma_{\gamma_3}+\sigma_{\gamma_4}+\sigma_{\gamma_5}) & \leqslant 6\mu+2\nu, \\
\text{(nodal cubics)} & 2(2\mu-\sigma_\beta)+5\mu-\Sigma\sigma_{\gamma_1}+i & \leqslant 9\mu+3\nu,
\end{array}$$

that is,
$$-\sigma_\beta-\nu \leqslant \sigma_{\gamma_1} \leqslant \sigma_\beta+2\nu+\Sigma\sigma_{\gamma_1},$$
$$i \leqslant 2\sigma_\beta+3\nu+\Sigma\sigma_{\gamma_1}.$$

Now, from $(2)_8$ we have $\Sigma i\epsilon_i = 2$, giving the usual two sub-cases.

Case 2i.— $\alpha = 0,\quad \beta = 1,\quad \Sigma\gamma = 5,\quad \epsilon_1 = 2,\quad \Sigma i^2\epsilon_i = 2.$

From $(3)_8$ $\qquad \Sigma\sigma_{\gamma_1} = 1-2\sigma_\beta-3\nu$, whence $i \leqslant 1$.

Now $$-\sigma_\beta-\nu \leqslant \sigma_{\gamma_1} \leqslant 1-\sigma_\beta-\nu,$$

hence each of the five numbers σ_{γ_1} is either $-\sigma_\beta-\nu$ or $1-\sigma_\beta-\nu$; from $(3)_8$ we find that $4-3\sigma_\beta-2\nu$ have the former value and $1+3\sigma_\beta+2\nu$ the latter.

Sum the inequalities for $\sigma_{\gamma_1}, \sigma_{\gamma_2}, \ldots$, substitute for $\Sigma\sigma_{\gamma_1}$ and rearrange:

$$-1-2\nu \leqslant 3\sigma_\beta \leqslant 4-2\nu.$$

For each value of ν there are two possible values of σ_β, and for each of these the σ_γ are determined. From $(4)_3$ we have $\eta_1 = -2-\sigma_\beta$, and then $(5)_3$ is also satisfied. This leads to six more types, C. 7, 9, 13, 11, 15, 17.

Case 2 ii.— $\alpha = 0, \quad \beta = 1, \quad \Sigma\gamma = 5, \quad \epsilon_2 = 1, \quad \Sigma i^2\epsilon_i = 4.$

The same argument leads to

$$\Sigma\sigma_{\gamma_1} = 2-2\sigma_\beta-3\nu, \quad i \leqslant 2,$$

$$-\sigma_\beta-\nu \leqslant \sigma_{\gamma_1} \leqslant 2-\sigma_\beta-\nu,$$

$$-2-2\nu \leqslant 3\sigma_\beta \leqslant 8-2\nu.$$

We find twenty-one sets of values for σ_γ; then η_1, η_2 are determined by $(4)_3$, $(5)_3$, and three cases must be rejected because $\eta_1 < 0$. Of the eighteen geometrical solutions, three are particular cases of (R.m). A general formula for the set arises from (P.) with $n_1 = 2$, $n_3 = 3$.

A quadratic transformation, applied to the $(2\mu-\sigma_\beta)$-fold point and any two points of the next set, reduces the degree of the family to

$$2(3\mu+\nu)-(2\mu-\sigma_\beta)-(2\mu-\sigma_{\gamma_1}-\sigma_{\gamma_2}) = 2\mu+2\nu+\sigma_\beta+\sigma_{\gamma_1}+\sigma_{\gamma_2},$$

which therefore belongs to § 4. Conversely, we can get all the types of the present Case 2 i and 2 ii by compounding those of § 4 with a quadratic transformation, letting its F-points coincide in any way with the simple and double F-points of the earlier general type. But except when there is only one value of σ_γ, the later type arises several times over.

Case 3.— $\alpha = \beta = 0, \quad \Sigma\gamma = 9.$

From $(2)_3$, $(3)_3$, $(4)_3$, $\Sigma i\epsilon_i = 0$, $\Sigma\gamma\sigma_\gamma = -3\nu$, $\Sigma i\eta_i = -3$.

The first of these requires that every $\epsilon_i = 0$, and therefore every $\eta_i \geqslant 0$, contradicting the last; there are no solutions in this case, and we have exhausted the possibilities with n of the form $3\mu+\nu$.

6. $x = 4$.

The case $x = 4$ is treated on the same lines. We assume $n = 4\mu+\nu$, $\nu = 0, 1, 2$, or 3; the F-system is

$$\Sigma\alpha^{n-\sigma_\alpha}\Sigma\beta^{3\mu-\sigma_\beta}\Sigma\gamma^{2\mu-\sigma_\gamma}\Sigma\delta^{\mu-\sigma_\delta}\Sigma(\epsilon_i\mu+\eta_i)^i,$$

§ 3.	n general		
Case 1.	C. 1	1^{n-1}	$(2n-2)^1$
Case 2.	C. 2	1^{n-2} $(n-2)^2$	3^1

§ 4.	$n = 2\mu$						$n = 2\mu+1$					
			α_4	α_3	α_2	α_1			α_4	α_3	α_2	α_1
Case 1.	C. 22 $\equiv$ 38	1^{n-4}	$\mu-2$	0	3	3	C. 28 $\equiv$ 46	1^{n-4}	$\mu-2$	1	3	2
	C. 18 $\equiv$ 36	1^{n-4}	$\mu-3$	2	3	1	C. 26 $\equiv$ 42	1^{n-4}	$\mu-3$	3	3	0
Case 2 i.	C. 3	3^{μ} $1^{\mu-1}$				$2\mu-2$	C. 5	$1^{\mu+1}$ 3^{μ}				$2\mu-1$
Case 2 ii.	R. 6	$1^{\mu+1}$ $3^{\mu-1}$			$\mu-2$	3	R. 7	$1^{\mu+1}$ 2^{μ} $1^{\mu-1}$			$\mu-1$	2
	R. 4	2^{μ} $2^{\mu-1}$			$\mu-1$	1	R. 4	4^{μ}			μ	0
	R. 7	3^{μ} $1^{\mu-2}$			$\mu-2$	3						

§ 5.	$n = 3\mu$							
			α_6	α_5	α_4	α_3	α_2	α_1
Case 1 i.	C. 8	1^{n-3}				$2\mu-2$	1	4
	C. 6	1^{n-3}				$2\mu-3$	4	1
Case 1 ii.	P.	1^{n-6}	$\mu-2$	0	0	3	1	4
		1^{n-6}	$\mu-2$	0	0	2	4	1
		1^{n-6}	$\mu-3$	1	1	3	0	3
		1^{n-6}	$\mu-3$	1	1	2	3	0
		1^{n-6}	$\mu-3$	0	3	2	1	1
		1^{n-6}	$\mu-4$	3	0	3	1	1
		1^{n-6}	$\mu-4$	1	4	2	0	0
		1^{n-6}	$\mu-5$	4	1	3	0	0
Case 2 i.	C. 9	$1^{2\mu}$		4^{μ}	$1^{\mu-1}$			$2\mu-2$
	C. 7	$1^{2\mu-1}$	$1^{\mu+1}$	4^{μ}				$2\mu-3$
Case 2 ii.	R. m	$1^{2\mu}$		4^{μ}		$1^{\mu-2}$	$\mu-2$	3
	P.	$1^{2\mu}$		3^{μ}	$2^{\mu-1}$		$\mu-1$	1
		$1^{2\mu-1}$	$2^{\mu+1}$	1^{μ}	$2^{\mu-1}$		$\mu-2$	2
		$1^{2\mu-1}$	$1^{\mu+1}$	3^{μ}	$1^{\mu-1}$		$\mu-1$	0
		$1^{2\mu-2}$	$1^{\mu+2}$	4^{μ}			$\mu-3$	3
		$1^{2\mu-2}$	$2^{\mu+1}$	3^{μ}			$\mu-2$	1

§ 5.	$n = 3\mu+1$							
			α_6	α_5	α_4	α_3	α_2	α_1
Case 1 i.	C. 12	1^{n-3}				$2\mu-1$	0	5
	C. 10	1^{n-3}				$2\mu-2$	3	2
Case 1 ii.	P.	1^{n-6}	$\mu-2$	0	1	3	0	4
		1^{n-6}	$\mu-2$	0	1	2	3	1
		1^{n-6}	$\mu-3$	2	0	3	1	2
		1^{n-6}	$\mu-3$	1	2	2	2	0
		1^{n-6}	$\mu-3$	0	4	2	0	1
		1^{n-6}	$\mu-4$	3	1	3	0	1
Case 2 i.	C. 13	$1^{2\mu+1}$		5^{μ}				$2\mu-1$
	C. 11	$1^{2\mu}$	$2^{\mu+1}$	3^{μ}				$2\mu-2$
Case 2 ii.	R. m	$1^{2\mu+1}$		4^{μ}	$1^{\mu-1}$		$\mu-1$	2
	P.	$1^{2\mu}$	$3^{\mu+1}$		$2^{\mu-1}$		$\mu-2$	3
		$1^{2\mu}$	$2^{\mu+1}$	2^{μ}	$1^{\mu-1}$		$\mu-1$	1
		$1^{2\mu-1}$	$1^{\mu+2}$	$1^{\mu+1}$ 3^{μ}			$\mu-2$	2
		$1^{2\mu-1}$	$3^{\mu+1}$	2^{μ}			$\mu-1$	0
		$1^{2\mu-2}$	$5^{\mu+1}$				$\mu-3$	3

§ 5.	$n = 3\mu+2$							
			α_6	α_5	α_4	α_3	α_2	α_1
Case 1 i.	C. 14	1^{n-3}				$2\mu-1$	2	3
	C. 16	1^{n-3}				$2\mu-2$	5	0
Case 1 ii.	P.	1^{n-6}	$\mu-2$	1	0	3	1	3
		1^{n-6}	$\mu-2$	1	0	2	4	0
		1^{n-6}	$\mu-2$	0	2	2	2	1
		1^{n-6}	$\mu-3$	2	1	3	0	2
		1^{n-6}	$\mu-3$	1	3	2	1	0
		1^{n-6}	$\mu-4$	4	0	3	1	0
Case 2 i.	C. 15	$1^{2\mu+1}$		$3^{\mu+1}$	2^{μ}			$2\mu-1$
	C. 17	$1^{2\mu}$		$5^{\mu+1}$				$2\mu-2$
Case 2 ii.	R. m	$1^{2\mu+2}$			5^{μ}		$\mu-1$	3
	P.	$1^{2\mu+1}$		$3^{\mu+1}$	1^{μ}	$1^{\mu-1}$	$\mu-1$	2
		$1^{2\mu+1}$		$2^{\mu+1}$	3^{μ}		μ	0
		$1^{2\mu}$	$2^{\mu+2}$		3^{μ}		$\mu-2$	3
		$1^{2\mu}$	$1^{\mu+2}$	$2^{\mu+1}$	2^{μ}		$\mu-1$	1
		$1^{2\mu-1}$	$1^{\mu+2}$	$4^{\mu+1}$			$\mu-2$	3

and the constants satisfy

$$16\Sigma\alpha \quad +9\Sigma\beta \; +4\Sigma\gamma \; + \; \Sigma\delta \qquad\qquad = 16, \tag{1$_4$}$$

$$4\Sigma\alpha \quad +3\Sigma\beta \; +2\Sigma\gamma \; + \; \Sigma\delta \; +\Sigma i\epsilon_i = 12, \tag{2$_4$}$$

$$-8\Sigma\alpha(\sigma_\alpha-\nu) -6\Sigma\beta\sigma_\beta-4\Sigma\gamma\sigma_\gamma-2\Sigma\delta\sigma_\delta+\Sigma i^2\epsilon_i = 8\nu, \tag{3$_4$}$$

$$-\Sigma\alpha(\sigma_\alpha-\nu) - \Sigma\beta\sigma_\beta- \Sigma\gamma\sigma_\gamma- \Sigma\delta\sigma_\delta+\Sigma i\eta_i = 3\nu-3, \tag{4$_4$}$$

$$\Sigma\alpha(\sigma_\alpha-\nu)^2+ \Sigma\beta\sigma_\beta^2+ \Sigma\gamma\sigma_\gamma^2+ \Sigma\delta\sigma_\delta^2+\Sigma i^2\eta_i = \nu^2-1. \tag{5$_4$}$$

From $(1)_4$, the possibilities are

$\alpha=1$	0	0	0	0	0	0	0
$\beta=0$	1	1	0	0	0	0	0
$\Sigma\gamma=0$	1	0	4	3	2	1	0
$\Sigma\delta=0$	3	7	0	4	8	12	16.

The second is excluded (straight lines), and the last by condition (*c*) of § 2. There remain six cases to examine.

Case 1.— $\alpha = 1, \quad \beta = \gamma = \delta = 0.$

From $(2)_4$, $(3)_4$ $\quad \Sigma i\epsilon_i = 8, \quad \Sigma i^2\epsilon_i = 8\sigma_\alpha,$

(straight lines) $\quad i \leqslant \sigma_\alpha.$

The base for the ϵ_i is $\epsilon_1 = 16-8\sigma_\alpha, \quad \epsilon_2 = -4+4\sigma_\alpha.$

We consider these for $\sigma_\alpha = 1, 2, \ldots,$ and deduce the admissible sets of values with each $\epsilon_i \geqslant 0$ and $i \leqslant \sigma_\alpha$. In every case except one, every ϵ is even, and the types have occurred already in §§ 3, 4.

Case 1 i.—In § 4, Case 1, we met types with $\sigma_\alpha = 4$, $\eta_4 = 2$ in the notation of this section, which arose there but really belong here. By the methods shown there, we find eight types, C. 24, 20, 32, 30, 40, 34, 48, 44.

A specimen of the set is

C. 24. $\quad n = 4\mu, \quad 1^{n-4}(2\mu-2)^4 1^3 6^1,$

which is compounded of C. 1 and a quartic transformation with the F-system $1^3 6^1$. Since 2μ is even, this can be reduced, by a series of quadratic transformations, to the type with $\mu = 1$,

$$n = 4, \quad 1^3 6^1,$$

which is proper. But if 2μ is replaced by an odd number, $2\mu+1$, we obtain

$$n = 4\mu+2, \quad 1^{n-4}(2\mu-1)^4 1^3 6^1,$$

which can be reduced to the type with $\mu = 1$,

$$n = 6, \quad 1^4 1^3 1^2 6^1,$$

which breaks up, one component being the straight line joining the two highest F-points.

Case 1ii.—The only other sub-case to pursue is

$$\sigma_a = 8, \quad \epsilon_1 = \epsilon_2 = \ldots = \epsilon_7 = 0, \quad \epsilon_8 = 1, \quad i \leqslant 8;$$

we can also obtain the condition

$$4\eta_8+3\eta_7+2\eta_6+\eta_5 \leqslant -8+\nu.$$

From $(4)_4$, $(5)_4$ the base for the η is

$$\eta_1 = 75-12\nu, \quad \eta_2 = -35+7\nu, \quad \eta_3 = \ldots = \eta_8 = 0,$$

whence a large number of geometrical solutions. All the types of this set involve μ in the same way, the F-systems beginning with $1^{n-8}\mu^8$, and differing only in the integers $\eta_8, \ldots, \eta_1$ that follow; η_8 is always negative. Each is reduced, by a quadratic transformation, to one of the same type with μ lower by 2, and the process repeated brings μ down to $-\eta_8$ or $-\eta_8+1$.

If in (P.) we put $N = 8$, $n_1 = 4$, $\mu = 2n_3-l'$, we obtain thirty-two different types of this set, in all of which, however, $\eta_2 = 0$. An example other than these, given at the end of this section, is compounded of C. 2 and a quartic transformation with $3^2 3^1$.

Case 2.—From $(2)_4$, $\Sigma i\epsilon_i = 2$, giving the usual two sub-cases.

Case 2i.—

$$a = \gamma = 0, \quad \beta = 1, \quad \Sigma\delta = 7, \quad \epsilon_1 = 2, \quad \sigma_\beta+\Sigma\delta\sigma_\delta = -4\nu+1.$$

By straight lines, nodal cubics, and quartics with a triple point, we arrive at the limits

$$-1-3\nu \leqslant 4\sigma_\beta \leqslant 6-3\nu,$$

$$-\sigma_\beta-\nu \leqslant \sigma_{\delta_1} \leqslant 1-\sigma_\beta-\nu.$$

For each of $\nu = 0, 1, 2, 3$, there are two possible values for σ_β, and for each of these, one set of values of $\sigma_{\delta_1}, \ldots, \sigma_{\delta_7}$. We find eight solutions,

C. 25, 21, 33, 31, 41, 35, 49, 45, the first being compounded of a quartic transformation with $1^3 6^1$ and C. 1.

Case 2 ii.—With $\epsilon_2 = 1$, a similar argument leads to a set with $i \leqslant 2$, which is given by (P.) when $n_1 = 2$, $n_3 = 4$; there are twenty-six types, seven for each of the even forms of n and six for each of the odd forms. The first is also contained in (R.m), and is compounded of a quartic transformation with $1^3 6^1$ and C. 2.

All the types of Case 2 i and 2 ii arise in many ways from § 4, Case 2, by compounding with a quadratic transformation.

Case 3.— $\alpha = \beta = \delta = 0, \quad \Sigma\gamma = 4, \quad \Sigma i\epsilon_i = 4, \quad \Sigma i^2\epsilon_i = 8\nu + 4\Sigma\gamma\sigma_\gamma.$

Again excluding the cases where every ϵ_i is even, we are left with $\epsilon_4 = 1$, $i \leqslant 4$. The specimen of this set is compounded of a quadratic transformation and C. 22. All these can be obtained in many ways from § 4, Case I. None of them is included in (P.).

Case 4.—Again $\Sigma i\epsilon_i = 2$, and we subdivide.

Case 4 i.—

$$\alpha = \beta = 0, \quad \Sigma\gamma = 3, \quad \Sigma\delta = 4, \quad \epsilon_1 = 2, \quad 2\Sigma\gamma\sigma_\gamma + \Sigma\delta\sigma_\delta = 1 - 4\nu.$$

Straight lines, nodal cubics, and conics give limits for σ_γ and σ_δ, and quartics with three double points give $i \leqslant 1$. We find eight types, C. 23, 19, 29, 27, 39, 37, 47, 43. Each of these is compounded of a quadratic transformation and one of § 4, Case 2 i, the simple F-points of the two coinciding in any way. The resolution is unique.

Case 4 ii.— $\alpha = \beta = 0, \quad \Sigma\gamma = 3, \quad \Sigma\delta = 4, \quad \epsilon_2 = 1.$

In the same way, each of these arises, once only, by compounding a quadratic transformation with one of § 4, Case 2 ii; there are forty such.

Case 5.— $\alpha = \beta = 0, \quad \Sigma\gamma = 2, \quad \Sigma\delta = 8.$

A quadratic transformation applied to the three highest F-points would reduce this to § 5, Case 3, which we have seen does not exist; neither does the present case.

Case 6.— $\alpha = \beta = 0, \quad \gamma = 1, \quad \Sigma\delta = 12.$

From $(2)_4$, $\Sigma i\epsilon_i = -2$, which is impossible, and Case 6 is also excluded. We have therefore discussed all the solutions with $n = 4\mu + \nu$. There are seven kinds, of which the following types are specimens, with $n = 4\mu$.

Case 1 i.	C. 24.	$1^{n-4}(2\mu-2)^4 1^3 6^1$.
1 ii.		$1^{n-8}(\mu-2)^8 3^4 3^2 3^1$.
2 i.	C. 25.	$1^{3\mu} 6^{\mu} 1^{\mu-1}(2\mu-2)^1$.
2 ii.	R. *m*.	$1^{3\mu} 6^{\mu} 1^{\mu-2}(\mu-2)^2 3^1$.
3.		$3^{2\mu} 1^{2\mu-4}(\mu-2)^4 3^2 3^1$.
4 i.	C. 23.	$3^{2\mu} 3^{\mu} 1^{\mu-1}(2\mu-2)^1$.
4 ii.		$3^{2\mu} 1^{\mu+1} 3^{\mu-1}(\mu-2)^2 3^1$.

7. *x general.*

Let $n = x\mu+\nu$, and let us search for solutions valid for general values of x, μ. We assume i, a_i to be multinomials in x, μ, with coefficients depending on ν. The conditions

$$i \leqslant n-1, \quad \Sigma a_i \leqslant 2n-1$$

show that the multinomials are linear in each of x, μ. The conditions

$$i_1 > n/3, \quad i_1+i_2 \leqslant n$$

show that there is one and only one point for which i has a term in $x\mu$, and the coefficient is 1. We assume provisionally an F-system of the form

$$1^{x\mu+ax+b\mu+c}\,\Sigma(\alpha x\mu+\beta x+\gamma\mu+\delta)^{dx+e\mu+f}.$$

Here a, d, $e \geqslant 0$, and if $\alpha = 0$, then β, $\gamma \geqslant 0$.

Equate coefficients of certain powers of x, μ in II:

$$\begin{aligned} (x^3\mu) \quad & \Sigma\alpha d^2 = 0, \\ (x\mu^3) \quad & \Sigma\alpha e^2 = 0, \\ (x^3) \quad & \Sigma\beta d^2 = 0, \\ (\mu^3) \quad & \Sigma\gamma e^2 = 0. \end{aligned}$$

Hence for every term in which $\alpha > 0$, we have $d = e = 0$.

" " " $\alpha = 0$, $\beta > 0$, " $d = 0$.

" " " $\alpha = 0$, $\gamma > 0$, " $e = 0$.

Slightly changing the notation, we have the F-system more closely represented by

$$1^{x\mu+ax+b\mu+c}\,\Sigma(\alpha\mu+\beta)^{dx+e}\,\Sigma(\gamma x+\delta)^{f\mu+g}\,\Sigma(\epsilon x\mu+\eta x+\zeta\mu+\theta)^{i}.$$

Here $a, b \leqslant 0$; $d, f > 0$ (for if either $= 0$, the term would be absorbed in the last sum); $\alpha, \gamma, \epsilon \geqslant 0$; $\epsilon = 0, 1,$ or 2; if $\epsilon = 0$, then $\eta, \zeta \geqslant 0$.

Equate coefficients of the remaining powers of x, μ in (I), (II):

$$
\begin{array}{llllllll}
\text{(I)} & (x\mu) & 1 & +\Sigma\alpha d & +\Sigma\gamma f & +\Sigma\epsilon i & = 3, \\
 & (x) & a & +\Sigma\beta d & +\Sigma\gamma g & +\Sigma\eta i & = 0, \\
 & (\mu) & b & +\Sigma\alpha e & +\Sigma\delta f & +\Sigma\zeta i & = 0, \\
 & (1) & c & +\Sigma\beta e & +\Sigma\delta g & +\Sigma\theta i & = 3\nu-3. \\
\text{(II)} & (x^2\mu) & 2a & +\Sigma\alpha d^2 & & & = 0, \\
 & (x\mu^2) & 2b & & +\Sigma\gamma f^2 & & = 0, \\
 & (x\mu) & \multicolumn{4}{l}{2c+2ab+\Sigma 2\alpha de+\Sigma 2\gamma fg+\Sigma\epsilon i^2} & = 2\nu, \\
 & (x^2) & a^2 & +\Sigma\beta d^2 & & & = 0, \\
 & (\mu^2) & b^2 & & +\Sigma\delta f^2 & & = 0, \\
 & (x) & 2ac & +\Sigma 2\beta de & +\Sigma\gamma g^2 & +\Sigma\eta i^2 & = 0, \\
 & (\mu) & 2bc & +\Sigma\alpha e^2 & +\Sigma 2\delta fg & +\Sigma\zeta i^2 & = 0, \\
 & (1) & c^2 & +\Sigma\beta e^2 & +\Sigma\delta g^2 & +\Sigma\theta i^2 & = \nu^2-1.
\end{array}
$$

In (I $x\mu$), every term $\geqslant 0$; also from (II $x^2\mu$) we have an even value for $\Sigma\alpha d^2$; thus we cannot have $\Sigma\alpha d = 1$, which would give, say, $\alpha_1 = d_1 = 1$, every other $\alpha = 0$, and therefore $\Sigma\alpha d^2 = 1$. Similarly for the other sums. Hence one of these three sums $= 2$, and the other two $= 0$.

If $$\Sigma\epsilon i = 2, \quad \Sigma\alpha d = \Sigma\gamma f = 0,$$

then from (II $x^2\mu$, $x\mu^2$, x^2, μ^2, I x, μ) we find

$$a = b = \beta = \delta = \eta = \zeta = 0.$$

Make the substitutions $c = \nu - \sigma_\alpha$, $\theta_i = \epsilon_i\nu + \eta_i$;

then x, μ enter only in the combination $x\mu + \nu \equiv n$, and the remaining equations are equivalent to the set of § 3. We are shut up to the two types there found, valid for any form of n.

The other two possibilities are obtained from each other by interchanging x, μ; we need only investigate one. We therefore take

$$\Sigma\alpha d = 2, \quad \Sigma\gamma f = \Sigma\epsilon i = 0,$$

that is, every γ, $\epsilon = 0$, and therefore every δ, η, $\zeta \geqslant 0$. We have also

(II $x\mu^2$) $\qquad b = 0,$

(μ^2) $\qquad \Sigma\delta f^2 = 0$, hence every $\delta = 0$ as well as every γ, and the second sum is entirely absent.

(μ) $\quad \Sigma a e^2 + \Sigma\zeta i^2 = 0$, hence $e = 0$ if $a > 0$, Σae, Σade, $\Sigma ae^2 = 0$, and also every $\zeta = 0$. Then (I μ) is satisfied.

($x\mu$) $\qquad c = \nu.$

(Straight lines) $\qquad (a+d)\mu + e \leqslant 0$

for all values of μ, hence $\qquad d \leqslant -a.$

Since $\Sigma ad = 2$, we have the following cases:

$$\left.\begin{array}{ll} \textit{Case 1.—} & a_1 = 2, \quad d_1 = 1 \\ \textit{Case 2.—} & a_1 = 1, \quad d_1 = 2 \end{array}\right\} \text{ all other } a = 0.$$

The apparent possibility of two terms with $a = d = 1$ does not arise; for $e = 0$ whenever $a > 0$, the two terms would have the same index μ, and would coalesce into a single term $a = 2$, $d = 1$, reducing this to Case 1.

Case 1.—Here $\qquad \Sigma ad^2 = 2.$

(II $x^2\mu$) $\qquad a = -1.$

But $\qquad 0 < d \leqslant -a$, hence every $d = 1$.

(II x^2) $\qquad \Sigma\beta = -1.$

(I x) $\qquad \Sigma\eta i = 2,$

giving two sub-cases

$$\text{(i)} \quad \eta_1 = 2, \quad \Sigma\eta i^2 = 2; \qquad \text{(ii)} \quad \eta_2 = 1, \quad \Sigma\eta i^2 = 4,$$

with every other $\eta = 0$ in each case.

Case 1 i.—

(II x) $\qquad \Sigma\beta e = \nu - 1.$

Then using a curve of degree μ, determined by $1^{\mu-1}(2\mu)^1$, we find

$$i \leqslant 1;$$

hence the last sum has only one term, with $i = 1$.

Using a curve of degree $\mu-1$, we find

$$e \geqslant -1.$$

(Straight lines) $$e \leqslant 0,$$

hence $e=0$ or -1, and the first sum has only two terms. The F-system is reduced to

$$1^{n-x}(2\mu+\beta_1)^x \beta_2^{x-1}(2x+\theta_1)^1.$$

(II x^2) $$\left.\begin{aligned}\beta_1+\beta_2 &= -1\\ -\beta_2 &= \nu-1\end{aligned}\right\} \quad \text{hence } \begin{aligned}\beta_1 &= \nu-2,\\ \beta_2 &= 1-\nu,\end{aligned}$$

(II x)

and $\nu \leqslant 1$ since $\beta_2 \geqslant 0$.

(I 1) $$\theta_1 = \nu-2$$

and (II 1) is satisfied. The solution is therefore

(B) $$1^{n-x}(2\mu+\nu-2)^x(1-\nu)^{x-1}(2x+\nu-2)^1,$$

valid if $\nu \leqslant 1$, $2x \geqslant 2-\nu$. If $\nu \leqslant 0$, this is equivalent to Bianchi's formula for the composition of two de Jonquières transformations (C. 1) with $-\nu$ of the simple F-points of the two transformations coinciding in the intermediate space.

Case 1 ii.— $\eta_2 = 1, \quad \Sigma\eta i^2 = 4.$

The same argument gives

$$\Sigma\beta e = \nu-2, \quad i \leqslant 2, \quad e = 0,\ -1, \text{ or } -2;$$

there are three terms in the first sum and two in the last. The F-system is

$$1^{n-x}(2\mu+\beta_1)^x \beta_2^{x-1} \beta_3^{x-2}(x+\theta_2)^2 \theta_1^1.$$

(II x^2) $$\beta_1+\beta_2+\beta_3 = -1,$$

(II x) $$-\beta_2-2\beta_3 = \nu-2,$$

(I 1) $$\theta_1+2\theta_2 = \nu-1,$$

(II 1) $$\beta_2+4\beta_3+\theta_1+4\theta_2 = -1.$$

We can express the solution in terms of one of the constants as parameter, say β_3,

$$\beta_1 = -3+\nu+\beta_3, \quad \theta_1 = 1+\nu+2\beta_3,$$

$$\beta_2 = 2-\nu-2\beta_3, \quad \theta_2 = -1-\beta_3.$$

All the solutions are derived from the common base

$$1^{n-x}(2\mu-3+\nu)^x(2-\nu)^{x-1}(x-1)^2(1+\nu)^1,$$

by superposing a suitable multiple β_3 of

$$1^x(-2)^{x-1}\,1^{x-2}(-1)^2 2^1.$$

The conditions for β_3 are $\qquad \beta_3 \geqslant 0,$

$$\beta_2 \geqslant 0, \quad \text{that is,} \quad 2\beta_3 \leqslant 2-\nu, \quad \text{whence} \quad \nu \leqslant 2,$$

$$\theta_1 \geqslant 0, \quad \text{that is,} \quad 2\beta_3 \geqslant -1-\nu.$$

If $\nu = 2$ or 1, then $\beta_3 = 0$; if $\nu \leqslant 0$, there are always two values for β_3, their form depending on whether ν is even or odd.

Since ν can take all negative values, but only two positive values, it is convenient to write

$$-\nu = \rho = 2\sigma+\tau, \quad \tau = 0, 1;$$

then $\sigma \geqslant -1$ and $\beta_3 = \sigma+1$ or σ, provided these $\geqslant 0$.

We thus have two solutions, given in the table. In the first put

$$\sigma = 0,\ \tau = 0; \quad n = \mu x; \qquad 1^{n-x}(2\mu-2)^x\,1^{x-2}(x-2)^2\,3^1.$$

$$\sigma = -1,\ \tau = 0; \quad n = \mu x+2; \quad 1^{n-x}(2\mu-1)^x \qquad (x-1)^2\,3^1.$$

$$\sigma = -1,\ \tau = 1; \quad n = \mu x+1; \quad 1^{n-x}(2\mu-2)^x\,1^{x-1}(x-1)^2\,2^1.$$

These can be expressed in the following single formula, equivalent to Ruffini's, with $n = x\mu+\nu$,

$$(\text{R}.\,m) \quad 1^{n-x}(2\mu-2)^x\,1^{x+\nu-2}\left\{x-\tfrac{1}{2}(\nu^2-3\nu+4)\right\}^2(\nu^2-2\nu+3)^1, \quad \nu = 0, 1, 2.$$

Case 2.— $\qquad a_1 = 1, \quad d_1 = 2, \quad \Sigma a d^2 = 4.$

(II $x^2\mu$) $\qquad a = -2.$

(Straight lines) $\qquad d = 1$ or 2, and if $d = 2$, then $e \leqslant 0$.

The first sum now consists of three sorts of terms:

one with $a_1 = 1, \quad d_1 = 2, \quad e = 0, \quad \beta = \beta_1,$ say;

a set with $a = 0, \quad d = 2, \quad e < 0$; let Σ_2 refer to these;

a set with $a = 0, \quad d = 1$; let Σ_1 refer to these.

Using a curve of degree $\frac{1}{2}(\mu+\beta_1+\Sigma_2\beta)$, we find

$$\beta_1+\Sigma_2\beta \leqslant -2.$$

(II x^2) $\quad 4\beta_1+4\Sigma_2\beta+\Sigma_1\beta = -4,$

(I x) $\quad 2\beta_1+2\Sigma_2\beta+\Sigma_1\beta+\Sigma\eta i = 2,$

whence $\quad \beta_1+\Sigma_2\beta = -3+\frac{1}{2}\Sigma\eta i \geqslant -3.$

Therefore $\quad \beta_1+\Sigma_2\beta = -3$ or -2.

In the first case, every $\eta = 0$, every $\theta \geqslant 0$; further,

(II x) $\quad 4\Sigma_2\beta e+2\Sigma_1\beta e = 4\nu,$

(I 1) $\quad \Sigma_2\beta e+\Sigma_1\beta e+\Sigma\theta i = 2\nu-3,$

$$-\Sigma_2\beta e \qquad +\Sigma\theta i = -3.$$

But in every term of Σ_2 we have $\beta \geqslant 0$, $e<0$; in the last equation, every term on the left $\geqslant 0$, which is impossible. Hence we must have

$$\beta_1+\Sigma_2\beta = -2, \quad \Sigma\eta i = 2, \quad \Sigma_1\beta = 4.$$

Now use a curve of degree $\mu-1$ determined by $1^{\mu-3}(\mu-3)^2 5^1$, we find that in any term of Σ_2,

$$e \geqslant -2.$$

Hence Σ_2 consists of two terms, say $\beta_2^{2x-1}\beta_3^{2x-2}$.

Now use a curve of degree $y = \frac{1}{2}(\mu-2+p)$, $p = 0$, 1, 2, 3, or 4, determined by $1^{y-1}(2y)^1$. This leads to

$$\Sigma_2\beta e+\underset{(p)}{\Sigma} e \leqslant \nu,$$

where $\underset{(p)}{\Sigma} e$ is 0 or the sum of the values of e at any set of p out of the four points of Σ_1.

(II x) $\quad 4\Sigma_2\beta e+2\Sigma_1\beta e+\Sigma\eta i^2 = 4\nu.$

Use a curve of degree μ determined by $1^{\mu-2}(\mu-2)^2 5^1$:

$$i \leqslant \tfrac{1}{2}\Sigma\eta i^2.$$

Since $\Sigma\eta i = 2$, there are the usual two sub-cases.

Case 2 i.— $\quad \eta_1 = 2, \quad \Sigma\eta i^2 = 2, \quad i = 1.$

$$\text{(II } x) \qquad -4\beta_3-2\beta_2+\Sigma_1\beta e \qquad = 2\nu-1,$$

$$\text{(I 1)} \qquad -2\beta_3-\ \beta_2+\Sigma_1\beta e+\theta_1 = 2\nu-3,$$

$$\text{(II 1)} \qquad 4\beta_3+\ \beta_2+\Sigma_1\beta e^2+\theta_1 = -1,$$

$$2\beta_3 \qquad +\Sigma_1\beta e^2 \qquad = 1.$$

Hence $\beta_3 = 0$, and in Σ_1, for one of the four points, $e^2 = 1$, $e = \pm 1$, and for the other three, $e = 0$.

With the upper sign, the equations give

$$\beta_2 = 1-\nu, \quad \beta_1 = -3+\nu, \quad \theta_1 = -3+\nu,$$

and with the lower,

$$\beta_2 = -\nu, \quad \beta_1 = -2+\nu, \quad \theta_1 = -2+\nu.$$

The two solutions are given in the table.

Case 2 ii.— $\quad \eta_2 = 1, \quad \Sigma\eta i^2 = 4, \quad i = 1$ or 2.

$$\text{(II } x) \qquad 2\Sigma_2\beta e+\Sigma_1\beta e = 2\nu-2.$$

Now we have shown

$$\Sigma_2\beta e \leqslant \nu, \quad \Sigma_2\beta e+\Sigma_1\beta e \leqslant \nu, \quad \text{and also} \quad \Sigma_2\beta e \leqslant 0.$$

Hence there are three possibilities:

$$\Sigma_2\beta e = \nu-y, \quad \Sigma_1\beta e = 2y-2, \quad y = 0,\ 1,\ 2,$$

provided $\nu-y \leqslant 0$.

We have also proved $\qquad \Sigma_2\beta e+\underset{(p)}{\Sigma} e \leqslant \nu,$

whence $\qquad \underset{(p)}{\Sigma} e \leqslant y, \quad p = 1,\ 2,\ 3,$ or 4.

Thus for each value of y there are two sets of values of e in Σ_1. The simultaneous values are given in the following list:

y	e_1	e_2	e_3	e_4	$\Sigma_1\beta e^2$
0	0	0	0	-2	4
0	0	0	-1	-1	2
1	1	0	0	-1	2
[1	0	0	0	0	0]
2	2	0	0	0	4
2	1	1	0	0	2.

We can now determine all the remaining constants in terms of one of them, say β_3, from the relations

$$\beta_2+\ \beta_3=-2-\beta_1,$$

$$-\beta_2-2\beta_3=\nu-y,$$

$$\text{(I 1)}\qquad -\beta_2-2\beta_3+\Sigma_1\beta e+\theta_1+2\theta_2=2\nu-3,$$

$$\text{(II 1)}\qquad \beta_2+4\beta_3+\Sigma_1\beta e^2+\theta_1+4\theta_2=\ -1,$$

The solution is
$$\beta_1=\ \nu-2-y+\ \beta_3,$$

$$\beta_2=-\nu\quad +y-2\beta_3,$$

$$\theta_1=\ \nu-1-y+2\beta_3+\ \Sigma_1\beta e^2,$$

$$\theta_2=\qquad\qquad -\ \beta_3-\tfrac{1}{2}\Sigma_1\beta e^2.$$

Now we must have
$$\beta_2\geqslant 0,\quad \theta_1\geqslant 0,$$

which give
$$-\nu+y\geqslant 2\beta_3\geqslant -\nu+y+1-\Sigma_1\beta e^2;$$

and also, by addition, $\quad \Sigma_1\beta e^2-1\geqslant 0,$

which excludes one line of the list.

Since $\beta_3\geqslant 0$, we see again that $\nu-y\leqslant 0$. We can take any arbitrary value of $\nu\leqslant y$; there are then either two or one value of β_3 according as $\Sigma_1\beta e^2$ is 4 or 2, the form of β_3 in terms of ν depending on whether ν is even or odd. As before, let

$$-\nu=\rho=2\sigma+\tau,\quad \tau=0 \text{ or } 1\,;$$

in the five surviving cases of the list we find

$$\beta_3=\sigma \text{ or } \sigma-1,\quad \sigma,\quad \sigma+\tau,\quad \sigma+1 \text{ or } \sigma,\quad \sigma+1,$$

which give the seven types tabulated below.

We can prove that they actually exist as follows.

Certain quadratic transformations have the effect of altering the values of the constants, in a way depending on the multiplicities of the points used, as follows:

Multiplicities.			*Effect.*		
$n-2x$	$2x$	$2x$	$\mu-2$ for μ		
$n-2x$	0	0	$\mu+2$ for μ		
$n-2x$	$2x$	$2x-1$	$\mu-2$ for μ,	$\nu+1$ for ν	
$n-2x$	1	0	$\mu+2$ for μ,	$\nu-1$ for ν	
$n-2x$	$2x$	$2x-2$	$\mu-2$ for μ,	$\nu+2$ for ν,	β_3-1 for β_3
$n-2x$	2	0	$\mu+2$ for μ,	$\nu-2$ for ν,	β_3+1 for β_3

provided none of these changes introduces a negative number of points. By combining these, we can independently vary

$$\mu \text{ by } \pm 2, \quad \nu \text{ by } \pm 1, \quad \beta_3 \text{ by } \pm 1,$$

and the existence of each type depends on that of the particular case in which these parameters have their extreme values,

$$\mu = 2 \text{ or } 3, \quad \nu = y, \quad \beta_3 = 0,$$

and these are ten types found in §§ 3, 4.

This exhausts the possible types with $n = x\mu+\nu$ where x, μ can take all values $>$ certain lower limits depending on ν. In every case, ν has an upper limit $\leqslant 2$, but can take an infinite number of negative values.

Many types can be constructed, with n depending on three or more parameters, by compounding two or more of the foregoing types. The parameters of the new type are those which already occur in the components and a new set determining the coincidences of F-points. Thus (R. m) is compounded of a C. 1 of degree x and a C. 2 of degree $\mu+\nu$, the $(x-1)$-fold point of the C. 1 being of multiplicity ν for the C. 2, where $\nu = 0$, 1, or 2. Since ν has just three different values, the numbers of simple and double F-points in the result can be expressed as quadratic functions of ν.

Another example is Palatini's formula for compounding three de Jonquières transformations, frequently referred to above:

(P) $$n = n_3N - l'n_1 - l''(n_1-1) - l''', \qquad N = n_1n_2 - l,$$

$$1^{n-N}(2n_3-2-l'-l''-l''')^N \, l'''^{N-1} \, l''^{N+1-n_1} (l'+1)^{N-n_1}$$
$$(2n_2-2-l-l')^{n_1}(l+l''+1)^{n_1-1}(2n_1-2-l-l''')^1.$$

Owing to its large number of parameters, this includes, as particular cases, a great many of the types which we have been examining, but as it admits only seven different values of i other than unity, it is not exhaustive, for at least one of our types has eight.

$$n = x\mu - \rho, \quad \rho = 2\sigma + \tau.$$

1 i.	$1^{n-x}(2\mu-\rho-2)^{x}$	$(\rho+1)^{x-1}$		$(2x-\rho-2)^1$
1 ii.	$1^{n-x}(2\mu-\sigma-\tau-2)^{x}\,\tau^{x-1}$	$(\sigma+1)^{x-2}$		$(x-\sigma-2)^2(3-\tau)^1$
	$1^{n-x}(2\mu-\sigma-\tau-3)^{x}(\tau+2)^{x-1}\,\sigma^{x-2}$			$(x-\sigma-1)^2(1-\tau)^1$
2 i.	$1^{n-2x}(\mu-\rho-3)^{2x}$	$(\rho+1)^{2x-1}$	$1^{x+1}\,3^{x}$	$(2x-\rho-3)^1$
	$1^{n-2x}(\mu-\rho-2)^{2x}$	ρ^{2x-1}	$3^{x}\,1^{x-1}$	$(2x-\rho-2)^1$
2 ii.	$1^{n-2x}(\mu-\sigma-\tau-2)^{2x}\,\tau^{2x-1}$	σ^{2x-2}	$3^{x}\,1^{x-2}$	$(x-\sigma-2)^2(3-\tau)^1$
	$1^{n-2x}(\mu-\sigma-\tau-3)^{2x}(\tau+2)^{2x-1}(\sigma-1)^{2x-2}$		$3^{x}\,1^{x-2}$	$(x-\sigma-1)^2(1-\tau)^1$
	$1^{n-2x}(\mu-\sigma-\tau-2)^{2x}\,\tau^{2x-1}$	σ^{2x-2}	$2^{x}\,2^{x-1}$	$(x-\sigma-1)^2(1-\tau)^1$
	$1^{n-2x}(\mu-\sigma-3)^{2x}$	$(1-\tau)^{2x-1}(\sigma+\tau)^{2x-2}$	$1^{x+1}\,2^{x}\,1^{x-1}$	$(x-\sigma-\tau-1)^2\,\tau^1$
	$1^{n-2x}(\mu-\sigma-\tau-3)^{2x}\,\tau^{2x-1}$	$(\sigma+1)^{2x-2}$	$1^{x+2}\,3^{x}$	$(x-\sigma-3)^2(3-\tau)^1$
	$1^{n-2x}(\mu-\sigma-\tau-4)^{2x}(\tau+2)^{2x-1}\,\sigma^{2x-2}$		$1^{x+2}\,3^{x}$	$(x-\sigma-2)^2(1-\tau)^1$
	$1^{n-2x}(\mu-\sigma-\tau-3)^{2x}\,\tau^{2x-1}$	$(\sigma+1)^{2x-2}$	$2^{x+1}\,2^{x}$	$(x-\sigma-2)^2(1-\tau)^1.$