THE CREMONA TRANSFORMATIONS OF A CERTAIN PLANE SEXTIC

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[Received March 2nd, 1916.—Read March 9th, 1916.]

For references see: Sturm, *Die Lehre von den geometrischen Verwandtschaften*, Leipzig, 1909.

I. *Degenerate Homaloids.*

A plane Cremona transformation of degree n transforms the straight lines of the first plane into a homaloidal family in the second plane; that is, into a doubly infinite family of rational curves of degree n, such that two general curves of the family meet in one and only one variable point of intersection. The remaining intersections are all accounted for by the base points common to the whole family. The straight lines of the second plane are transformed into a homaloidal family in the first plane, also with a set of base points.

To every point p in the first plane there corresponds a unique point P in the second plane: except that to a base point a there corresponds not a single point but a curve J. The set of curves J corresponding to the whole set of base points, taken with proper multiplicities, constitute the Jacobian of the second homaloidal family.

To a straight line l passing through a there corresponds a homaloid which breaks up into J, corresponding to the single point a, and a rational curve ϕ of degree less than n, corresponding to all the rest of l. Then J meets ϕ in one point other than base points; J is exactly determined by its passages through base points, and ϕ has one degree of freedom, corresponding to the one degree of freedom of the straight line l through a. We have thus a singly infinite family (ϕ) of rational curves, which are transformed into straight lines by a Cremona transformation of degree higher than that of ϕ.

If l joins the base points a_1, a_2, the corresponding homaloid breaks up into J_1, J_2, ϕ, where J_1, J_2 are Jacobian curves, and ϕ corresponds properly to l. Each of J_1, J_2 meets ϕ in one point other than base points, and ϕ is exactly determined by its passages through base points. If three or more of the base points a lie on a straight line l, to this there corresponds properly a rational curve ϕ, which is more than determined by its passages through base points, which means that the conditions presented to it, by the required multiplicities at the base points, are not independent: but are related to one another in the same way as are the conditions presented to the straight line l by the collinear points a.

Now, if we are given the rational curve ϕ, determined or over-determined by the conditions of having certain base points of assigned multiplicities, the question arises whether we can always regard it as the effective part of a degenerate homaloid of higher degree. The answer is negative, and in the third part of this paper an example is given of a rational curve which cannot be transformed into a straight line by a Cremona transformation of however high degree.

Assume that ϕ is the effective part of a degenerate homaloid of degree $n+\nu$, the remaining part consisting of an aggregate J_ν of degree ν of k curves J belonging to the Jacobian of the homaloidal family. Then J_ν has no multiple points except the base points of the family, and no two curves J intersect elsewhere; each J meets ϕ at one other point. Let Σ extend to all the base points of the family, including all the assigned multiple points of ϕ and possibly other points. Let ϕ, J_ν have r, ρ branches respectively through a specimen base point; for the additional points introduced into Σ, the value of r is 0 or 1. There are a series of geometrical facts expressed by equations between these numbers.

ϕ is rational: $$\Sigma \tfrac{1}{2} r(r-1) = \tfrac{1}{2}(n-1)(n-2).$$

ϕ can be transformed into a straight line l passing through the k base points in the other plane which correspond to the k curves J; if all the conditions, presented to ϕ by its assigned multiple points, were independent, it would have $2-k$ degrees of freedom. We shall say that the *apparent freedom* is $2-k$; if $k>2$, the conditions cannot be independent:

$$\Sigma \tfrac{1}{2} r(r+1) = \tfrac{1}{2} n(n+3)-2+k.$$

The multiplicity $r+\rho$ of $\phi . J_\nu$ at each base point is the same as that of a homaloidal family of degree $n+\nu$; this family is rational:

$$\Sigma \tfrac{1}{2}(r+\rho)(r+\rho-1) = \tfrac{1}{2}(n+\nu-1)(n+\nu-2),$$

and has two degrees of freedom:

$$\Sigma \tfrac{1}{2}(r+\rho)(r+\rho+1) = \tfrac{1}{2}(n+\nu)(n+\nu+3)-2.$$

J_ν is an aggregate of k rational curves, and is therefore of genus

$-(k-1)$; all its singularities occur in Σ:

$$\Sigma \tfrac{1}{2}\rho(\rho-1) = \tfrac{1}{2}(\nu-1)(\nu-2)-1+k;$$

and J_ν is exactly determined:

$$\Sigma \tfrac{1}{2}\rho(\rho+1) = \tfrac{1}{2}\nu(\nu+3).$$

All but k of the $n\nu$ intersections of ϕ, J_ν fall at base points:

$$\Sigma r\rho = n\nu - k.$$

These are equivalent to five independent equations:

$$\Sigma r^2 = n^2-1+k, \qquad \Sigma \rho^2 = \nu^2+k,$$

$$\Sigma r = 3n-3+k, \qquad \Sigma \rho = 3\nu-k,$$

$$\Sigma r\rho = n\nu-k.$$

Now in the case studied below, of a sextic with ten double points,

$$n=6, \quad r=2, 1 \text{ or } 0, \quad \Sigma r\rho \leqslant 2\Sigma\rho,$$

$$6\nu-k \leqslant 2(3\nu-k), \qquad k \leqslant 0,$$

which is impossible if k is positive. It will be shewn that this curve actually exists; and it cannot be transformed into a straight line by any plane Cremona transformation whatever.

It can, of course, be mapped on a straight line by a Riemann transformation; for example, a quartic curve of the singly infinite family, determined by the ten double points and three fixed simple points of the sextic, meets the sextic in one variable point P, whose coordinates are therefore rational functions of the parameter of the family, which may be taken as the coordinate, on a straight line, of a variable point P' corresponding to P.

The sextic can also be transformed into a curve of lower degree by a Cremona space transformation; the sextic is the projection of a twisted sextic with one actual double point, which is the residual intersection of a sextic cone with ten double edges, standing on the plane curve, and a cubic monoid, with the same vertex as the cone, passing through nine of these edges. The twisted sextic is projected from its double point into a plane quartic. This succession of projections can be brought about by a single Cremona space transformation, which does not, however, transform the rest of the plane of the sextic into the plane of the quartic.

II. *Extension of Noether's Theorem.*

Noether* has proved that every Cremona transformation is the result of compounding a series of quadratic transformations; because the sum of the three highest multiplicities of a homaloidal family at base points is always greater than the degree of the family, and a quadratic transformation, with these three points as fundamental triad, lowers the degree of the family. So instead of seeking a Cremona transformation of higher degree, which transforms ϕ into a straight line, we may seek a quadratic transformation which lowers the degree of the curve.

Noether's proof rests on a manipulation of the equations which express that a homaloidal family is rational:

$$\Sigma \tfrac{1}{2} r(r-1) = \tfrac{1}{2}(n-1)(n-2),$$

and that it has two degrees of freedom:

$$\Sigma \tfrac{1}{2} r(r+1) = \tfrac{1}{2} n(n+3)-2,$$

or the equivalent pair

$$\Sigma r^2 = n^2-1, \qquad \Sigma r = 3n-3,$$

where n is the degree of the family, Σ extends to all its base points, and r is the multiplicity of one of them.

Assume that there are at least three multiple base points; the other cases are trivial, and require separate treatment. Let x, y, z be the three highest values of r, and let Σ' denote a summation from which these three points are excluded. Let

$$x = z+\lambda, \qquad y = z+\mu;$$

then

$$\lambda \geqslant \mu \geqslant 0, \quad z \geqslant r, \quad z > 1,$$

$$\Sigma' r^2 = n^2-1-x^2-y^2-z^2,$$

$$\Sigma' r = 3n-3-x-y-z.$$

Multiplying the second of these by z and subtracting the first from it,

$$\Sigma' r(z-r) = z(3n-3-x-y-z)-(n^2-1-x^2-y^2-z^2),$$

which can be put in the form, obtained by Prof. Baker in a more general

* *Math. Annalen*, Bd. 3, p. 167; Bd. 5, p. 635.

case,

$$(x+y+z-n)(n+\lambda+\mu) = \Sigma' r(z-r)+2z(\lambda+\mu)+2\lambda\mu+3(z-1)+2.$$

Now $n+\lambda+\mu$, $z-1$, 2 are positive; r, $z-r$, λ, μ are positive or zero; the right-hand side of the last equation is positive, and

$$x+y+z > n.$$

We generalize this as follows: for any curve ϕ of degree n having assigned multiplicities r at the points included in Σ, let

$$\Sigma \tfrac{1}{2}r(r-1) = \tfrac{1}{2}(n-1)(n-2)-p,$$

$$\Sigma \tfrac{1}{2}r(r+1) = \tfrac{1}{2}n(n+3)-f,$$

whence

$$\Sigma r^2 = n^2+1-p-f = n^2-i \text{ say},$$

$$\Sigma r = 3n-1+p-f.$$

Then p is the genus of ϕ; it cannot be negative if ϕ is a proper curve; f is the apparent freedom of the family having the same multiple points as ϕ; if the conditions which these points present are not independent, f can be negative. If there are at least two curves of the family, $i=f+p-1$ is the number of their intersections which do not fall at base points; it can be negative if there is only one curve ϕ having the assigned multiple points.

As before, let x, y, z be the three highest values of r, and let these three points be excluded from Σ'. We find the relation

$$(x+y+z-n)(n+\lambda+\mu)$$
$$= \Sigma' r(z-r)+2z(\lambda+\mu)+2\lambda\mu+(f-p+1)(z-1)+2-2p. \quad \text{(I)}$$

The conclusion $x+y+z > n$ follows in a great variety of cases besides that of the homaloidal family, $p=0$, $f=2$, just considered. It applies to:

any family of curves of genus 1, $\quad p=1,\ f \geqslant 1$;

any family of rational curves, $\quad p=0,\ f=1$ or 2;

any rational curve exactly determined, $p=0,\ f=0$;

any rational curve for which one of the conditions is a consequence of the others, so that the apparent freedom is -1,

$$p=0,\ f=-1.$$

It also applies, but not so obviously, when $p = 0$, $f = -2$. The right-hand side of (I) is then

$$\Sigma' r(z-r) + z(2\lambda + 2\mu - 1) + 2\lambda\mu + 3,$$

and is certainly positive unless $\lambda = \mu = 0$, $x = y = z$. Now Σ' extends to multiple points for which $z \geqslant r \geqslant 2$. The terms for which $r = z$ are each zero; let β be the number of effective terms, for which $r < z$; then $\beta \geqslant 0$, and since

$$r(z-r) = (r-1)(z-r-1) + (z-1),$$

therefore $$\Sigma' r(z-r) \geqslant \beta(z-1),$$

and the right-hand side of (I)

$$\geqslant (\beta - 1)(z-1) + 2,$$

and is positive unless $\beta = 0$, $z \geqslant 3$. Then $r = z = x = y$, and all the multiple points are of the same order z; let their number be a; the original equations give

$$az^2 = n^2 + 3, \qquad az = 3n + 1;$$

therefore $$z = \frac{n^2+3}{3n+1}, \qquad z - 3 = \frac{n(n-9)}{3n+1}.$$

Since n and $3n+1$ have no common factor, $3n+1$ is a factor of $|n-9|$, which could only be if either

$$\text{(i)} \quad n = 1 \text{ or } 2, \qquad z = 1,$$

which is impossible since the base points are multiple, or

$$\text{(ii)} \quad n = 9, \qquad z = 3,$$

but then a is not integral.

If $p = 0$, $f = -3$, the right-hand side of (I) is

$$\Sigma' r(z-r) + 2z(\lambda + \mu - 1) + 2\lambda\mu + 4;$$

if this is not positive, $\lambda = \mu = 0$, and it becomes

$$\Sigma' r(z-r) - 2z + 4 \geqslant (\beta - 2)(z-1) + 2,$$

which is positive unless either

$$\text{(i)} \quad \beta = 0, \quad x = y = z = r \geqslant 2,$$

or $$\text{(ii)} \quad \beta = 1, \quad x = y = z \geqslant 3.$$

(i) If $\beta = 0$, as before,

$$z = \frac{n^2+4}{3n+2}, \qquad z-2 = \frac{n(n-6)}{3n+2},$$

and since n, $3n+2$ can only have the factor 2 in common, $3n+2$ is a factor of $2\,|\,n-6\,|$. This can only be if $n = 1$, 2 (which do not apply) or 6; hence $n = 6$, $z = 2$, $a = 10$, and we are led again to the sextic with ten double points.

(ii) If $\beta = 1$, there are $a-1$ ($\geqslant 3$) base points of multiplicity z($\geqslant 3$), and one of lower multiplicity r ($\geqslant 2$), then

$$(a-1)\,z^2+r^2 = n^2+4 \geqslant 31, \qquad n \geqslant 6, \qquad r^2 \leqslant n^2-23,$$

$$(a-1)\,z+r = 3n+2\,;$$

therefore $$z = \frac{n^2+4-r^2}{3n+2-r}.$$

If
$$\left.\begin{array}{ll} n = 6, & r = 2 \text{ or } 3 \\ \phantom{n = {}}7, & \phantom{r = {}}2 \ldots 5 \\ \phantom{n = {}}8, & \phantom{r = {}}2 \ldots 6 \\ \phantom{n = {}}9, & \phantom{r = {}}2 \ldots 7 \\ \phantom{n = {}}10, & \phantom{r = {}}2 \ldots 8 \end{array}\right\}$$
, and the only case in which z is an integer above 2 is

$$n = 9, \; r = 2, \; z = 3, \; a = 10.$$

If $f = -3$, then whatever the genus,

$$\Sigma \tfrac{1}{2} r(r+1) = \tfrac{1}{2} n(n+3)+3$$
$$= 3a_2+6a_3+10a_4+15a_5+21a_6+28a_7+\ldots,$$

where a_r is the number of base points of multiplicity r. If, as before, $x+y+z \leqslant n$, then $n \geqslant 6$, since $z \geqslant 2$.

If $n = 6$, $x = y = z = 2$, $a_2 = 10$, the case already referred to.

If $n = 7$ or 8, $r \leqslant 7$, and casting out the threes, $a_4+a_7 \equiv 2 \pmod 3$; there are at least two fourfold or higher points, and $x+y+z > 8$.

Thus the rational sextic with ten double points is the simplest curve, as regards genus, degree and freedom, which cannot be transformed into a curve of lower degree by any plane Cremona transformation. The next curve is of degree 9 at least.

III. *The Sextic with Ten Double Points.*

The total intersection of a quadric and a cubic surface is a twisted sextic curve with six apparent double points. If the two surfaces touch at four points the sextic has, in addition, four actual double points, and its projection from any point of space on to any plane is a plane sextic with ten double points. This way of generating the curve was suggested by Mr. Richmond.

Take homogeneous coordinates $x_1x_2x_3x_4$, the four points of contact $A_1A_2A_3A_4'$ being the corners of the tetrahedron of reference. The general quadric through these points is

$$Q \equiv a_1x_2x_3+a_2x_3x_1+a_3x_1x_2+a_1'x_1x_4+a_2'x_2x_4+a_3'x_3x_4=0, \tag{1}$$

or, say, $\quad 2Q \equiv x_1u_1+x_2u_2+x_3u_3+x_4u_4,$

where $\quad u_1 \equiv a_3x_2+a_2x_3+a_1'x_4, \; \ldots, \; u_4 = a_1'x_1+a_2'x_2+a_3'x_3,$

and $u_1 \ldots u_4$ are the tangent planes to Q at $A_1 \ldots A_4'$.

The general cubic surface touching the same planes at the same points is

$$C' \equiv c_1x_1^2u_1+c_2x_2^2u_2+c_3x_3^2u_3+c_4x_4^2u_4+b_1x_2x_3x_4+b_2x_1x_3x_4+b_3x_1x_2x_4 + b_4x_1x_2x_3=0, \tag{2}$$

and the total intersection is a twisted sextic S with four actual double points $A_1 \ldots A_4'$.

(i) S is in general a *proper* sextic. It also lies on the surface

$$C'-Q\,(c_1x_1+\ldots+c_4x_4) \equiv b_1'x_2x_3x_4+\ldots+b_4'x_1x_2x_3=0, \tag{3}$$

where $b_1' \ldots b_4'$ are other constants, and its projection from A_4' on the plane $A_1A_2A_3$ is given by eliminating x_4 between (1) and (3). We find

$$(b_1'x_2x_3+b_2'x_3x_1+b_3'x_1x_2)(a_1x_2x_3+a_2x_3x_1+a_3x_1x_2)-b_4'x_1x_2x_3u_4=0,$$

which is a proper plane quartic if $a_1 \ldots b_4'$ have general values. Hence, if S broke up, it could only be into one or two straight lines through A_4' and a proper residual of degree 5 or 4; but by the same argument, the straight lines would have to pass through each of $A_1 \ldots A_4'$, which is impossible; therefore S does not break up.

(ii) S has *six apparent double points*. Take any two points $Y(y_1, y_2, y_3, y_4)$, Z, of space, and express the condition that the straight line YZ shall meet S in X; then X lies upon Q and C'. We may put $x=\lambda y+\mu z$ for all suffixes; let the result of substitution be

$$\begin{aligned} \lambda^2Q_{11}+\lambda\mu Q_{12}+\mu^2Q_{22} &= 0, \\ \lambda^3C_{111}'+\lambda^2\mu C_{112}'+\lambda\mu^2C_{122}'+\mu^3C_{222}' &= 0. \end{aligned}$$

Eliminate $\lambda : \mu$; this gives the equation

$$F \equiv \begin{vmatrix} 0, & 0, & Q_{11}, & Q_{12}, & Q_{22} \\ 0, & Q_{11}, & Q_{12}, & Q_{22}, & 0 \\ Q_{11}, & Q_{12}, & Q_{22}, & 0, & 0 \\ 0, & C'_{111}, & C'_{112}, & C'_{122}, & C'_{222} \\ C'_{111}, & C'_{112}, & C'_{122}, & C'_{222}, & 0 \end{vmatrix} = 0,$$

of degree 6 in y, 6 in z. If Y is fixed, this equation in z represents the sextic cone projecting S from Y and *vice versa*.

If Y is a point of S, then $Q_{11} = C'_{111} = 0$: we can divide by μ before eliminating, and obtain

$$K' \equiv \begin{vmatrix} 0, & Q_{12}, & Q_{22} \\ Q_{12}, & Q_{22}, & 0 \\ C'_{112}, & C'_{122}, & C'_{222} \end{vmatrix} = 0,$$

of degree 2 in y, 5 in z. If Y is fixed, this equation in z represents the quintic cone projecting S from Y. If Z is fixed, this equation in y represents a quadric, meeting S in twelve points Y, which are such that each of the cones projecting S from these points passes through Z; in other words, such that each of the straight lines YZ meets S again, in another of the twelve points of intersection of the quadric $K'(y)$ with S. These twelve points therefore fall into six pairs, each collinear with Z; as viewed from the arbitrary point Z, the curve S has six apparent double points.

(iii) The six apparent double points of S are *in addition to* its four actual double points. If the coordinates (1, 0, 0, 0) of A_1 are substituted for y,

$$\begin{aligned} &Q_{12} \text{ becomes } a_3 z_2 + a_2 z_3 + a'_1 z_4 \equiv u_z \text{ say}, \\ &C'_{112} \quad ,, \quad c_1 u_z, \\ &C'_{122} \quad ,, \quad 2c_1 z_1 u_z + c_2 a_3 x_2^2 + c_3 a_2 x_3^2 + b_2 z_3 z_4 + b_3 z_2 z_4 + b_4 z_2 z_3, \end{aligned}$$

and K' breaks up into u_z and another factor, but does not vanish. Hence $K'(y)$, which contains the ends of the six chords of S through Z, does not contain the four actual double points, which are therefore in addition to the apparent double points.

(iv) The six chords of S through any point *lie on a quadric cone.* If we put $y = z$, Q_{12} becomes $2Q_{22}$, and C'_{112}, C'_{122} each become $3C'_{222}$; then

$$K' = Q_{22}^2 C'_{222} \begin{vmatrix} 0, & 2, & 1 \\ 2, & 1, & 0 \\ 3, & 3, & 1 \end{vmatrix} = -Q_{22}^2 C'_{222},$$

which does not vanish; $K'(y)$ does not in general pass through Z, and the six chords of S are not generators of $K'(y)$, unless Z lies either on Q or on C'.

This investigation holds if instead of C' we take $C \equiv C' + Q.P$, where P is any plane, for C is another cubic surface through S, and touches the same planes as C at $A_1 \dots A'_4$. We have thus a whole family of quadrics (K) such as K' through the same twelve points of S, which is, in fact, the family $K' + \lambda Q$. In particular, if P is chosen so that C passes through Z, we have $C_{222} = 0$, and K vanishes when we put $y = z$. This quadric K passes through Z, meets each of the six chords of S through Z in three points, and therefore contains them altogether. Hence K is a quadric cone on which the six chords lie.

Since now $C_{222} = 0$, we have identically

$$K = Q_{22} \begin{vmatrix} Q_{12}, & Q_{22} \\ C_{112}, & C_{122} \end{vmatrix} = Q_{22}.K_2, \text{ say},$$

$$F = Q_{22} \begin{vmatrix} 0, & Q_{11}, & Q_{12}, & Q_{22} \\ Q_{11}, & Q_{12}, & Q_{22}, & 0 \\ 0, & C_{111}, & C_{112}, & C_{122} \\ C_{111}, & C_{112}, & C_{122}, & 0 \end{vmatrix} = Q_2.F_6, \text{ say},$$

where
$$F_6 = K_2(Q_{12}C_{111} - Q_{11}C_{112}) + (Q_{11}C_{122} - Q_{22}C_{111})^2$$
$$= K_2.K_4 + K_3^2, \text{ say},$$

the suffixes shewing the degrees in y. Here F_6, K_2 are cones containing the six chords of S through Z, which are double edges on F_6; this identity shews that they also lie on the cubic surface K_3. They are therefore double on K_3^2 as well as on F_6; therefore, by the same identity, they are double on $K_2.K_4$. Since they are simple but not double edges on K_2, they lie on K_4.

The four actual double points $A_1 \dots A'_4$ of S are double points on F_6; they lie on Q, C, and their coordinates make Q_{11}, C_{111} vanish; they therefore lie on K_3, K_4, but not on K_2. They are double on $F_6 - K_3^2$, and therefore double on $K_2.K_4$; since they do not lie on K_2, they are double on K_4.

The point Z lies on C but not on Q. It is six-fold on F_6, and double on K_2; the substitution of z for y make C_{111}, C_{112}, C_{222} all vanish, so Z lies on K_3 and K_4. It is therefore triple at least on $F_6 - K_2 . K_4$, and therefore triple on K_3^2; this can only be if it is double on K_3. It is then fourfold on $F_6 - K_3^2$, and therefore fourfold on $K_2 . K_4$; since it is double on K_2, it is double on K_4.

These multiplicities can be set out in a table:

Surface.	The Six Chords.	$A_1 \ldots A_4'$	Z
F_6	2	2	6
K_2	1	0	2
K_3	1	1	2
K_4	1	2	2

Take the sections of these four surfaces by the plane $A_1 A_2 A_3$, by putting $y_4 = 0$. Let the six chords meet the plane in $B_1 \ldots B_6$, and let $A_4' Z$ meet it in A_4. The four plane curves are:

a sextic f_6 with ten double points at A_1, A_2, A_3; A_4; $B_1 \ldots B_6$; this will be indicated by $f_6(A_1 \ldots A_4 B_1 \ldots B_6)^2$;

a conic $k_2(B_1 \ldots B_6)^1$;

a cubic $k_3(A_1 A_2 A_3 B_1 \ldots B_6)^1$;

a quartic $k_4(A_1 A_2 A_3)^2 (B_1 \ldots B_6)^1$;

with the relation $\qquad f_6 \equiv k_2 . k_4 + k_3^2.$

The existence of f_6 presents three conditions to the ten points $A_1 \ldots B_6$; k_2 gives a single condition, among $B_1 \ldots B_6$: we shall call it a *condition curve*; k_3 gives no condition, it is exactly determined by its nine simple points; k_4 is another condition curve, for it is exactly determined by its three double points $A_1 A_2 A_3$ and five simple points $B_1 \ldots B_5$: there is therefore one condition that B_6 should lie on this quartic.

There exists another cubic k_3' determined by passing through the nine points $A_2 A_3 A_4 B_1 \ldots B_6$. Then $f_6 + \lambda k_3'^2$ is a family of sextics $k_6(A_2 A_3 A_4 B_1 \ldots B_6)^2$, and each meets k_2 in $B_1 \ldots B_6$ counted twice and no other point. If we use λ to make k_6 pass through another point of k_2, it must break up into k_2, which passes through $B_1 \ldots B_6$ once and not through $A_2 A_3 A_4$, and a quartic $k_4'(A_2 A_3 A_4)^2 (B_1 \ldots B_6)^1$. Similarly there exist two other condition quartics $k_4''(A_1 A_3 A_4)^2 (B_1 \ldots B_6)^1$ and $k_4'''(A_1 A_2 A_4)^2 (B_1 \ldots B_6)^1$: the point A_4 is on exactly the same footing as $A_1 A_2 A_3$.

By employing other exactly determined curves instead of these cubics, we can prove that there exist an unlimited nnmber of condition curves of higher degree determined by the ten double points of the sextic: see below. Each of these curves, whose existence gives a single condition, is of apparent freedom -1.

To obtain the actual equation of the plane sextic, put $y_4=0$ in the determinant F_6. In order to simplify the expressions, take the coordinates of Z to be (1, 1, 1, 1); there is no loss of generality, as Q, C have no metrical properties. Also when we replace C' by $C \equiv C'+Q.P$, we can use the four coefficients of P to make the new cubic surface not only pass through Z, but also have double points at $A_1A_2A_3$. The effect of this is the same as if in C' we assume

$$c_1=c_2=c_3=0, \quad c_4(a_1'+a_2'+a_3')+b_1+b_2+b_3+b_4=0,$$

and the plane curve is

$$f_6 \equiv \begin{vmatrix} 0, & q_{11}, & q_{12}, & q_{22} \\ q_{11}, & q_{12}, & q_{22}, & 0 \\ 0, & c_{111}, & c_{112}, & c_{122} \\ c_{111}, & c_{112}, & c_{122}, & 0 \end{vmatrix} = 0,$$

where

$$q_{11} \equiv a_1y_2y_3+a_2y_3y_1+a_3y_1y_2,$$

$$q_{12} \equiv y_1(a_2+a_3+a_1')+y_2(a_3+a_1+a_2')+y_3(a_1+a_2+a_3'),$$

$$q_{22} \equiv a_1+a_2+a_3+a_1'+a_2'+a_3',$$

$$c_{111} \equiv b_4y_1y_2y_3,$$

$$c_{112} \equiv y_2y_3(b_1+b_4)+y_3y_1(b_2+b_4)+y_1y_2(b_3+b_4),$$

$$c_{122} \equiv y_1(c_4a_1'+b_2+b_3+b_4)+y_2(c_4a_2'+b_3+b_1+b_4)+y_3(c_4a_3'+b_1+b_2+b_4).$$

The equation of f_6 contains the eleven constants $a_1a_2a_3a_1'a_2'a_3'b_1b_2b_3b_4c_4$, subject to one condition; as the first ten enter homogeneously, there are nine independent parameters. In the plane, we have fixed the coordinate system by taking $A_1A_2A_3$ as the triangle of reference and assigning the coordinates (1, 1, 1) to A_4. The twelve coordinates of the remaining six double points $B_1 \dots B_6$ are functions of the nine parameters, and are therefore connected by three independent relations. It is probable that the existence of three condition curves, for example k_2, k_4, k_4', ensures the existence of f_6; but I have no proof that this is so.

IV. *Transformations of the Sextic.*

A quadratic transformation, with any three of the ten double points A, B as fundamental triad, transforms f_6 into another plane sextic with ten double points, but these may be differently grouped. If the fundamental triad is $B_4 B_5 B_6$,

$$\begin{array}{lll} f_6(A_1 \ldots A_4 B_1 \ldots B_6)^2 & \text{becomes} & h_6(A_1 \ldots A_4 B_1 \ldots B_6)^2, \\ k_2(B_1 \ldots B_6)^1 & ,, & h_1(B_1 B_2 B_3)^1, \\ k_3(A_1 A_2 A_3 B_1 \ldots B_6)^1 & ,, & h_3(A_1 A_2 A_3 B_1 \ldots B_6)^1, \\ k_4(A_1 A_2 A_3)^2 (B_1 \ldots B_6)^1 & ,, & h_5(A_1 A_2 A_3 B_4 B_5 B_6)^2 (B_1 B_2 B_3)^1, \end{array}$$

and for these new curves there is the identity $h_6 \equiv h_1 h_5 + h_3^2$. Apparent freedom is not altered by the transformation; h_3 is exactly determined, and h_1, h_5 are condition curves. The ten double points of h_6 are grouped into three collinear points $B_1 B_2 B_3$, and seven others, instead of into six points on a conic, and four others.

There are seven cubics such as h_3, each exactly determined by passing through the three points of the first group and six out of the seven points of the second group; four of these cubics arise by transformation from the four curves such as k_3; the other three, such as $h'_3(A_1 \ldots A_4 B_1 \ldots B_5)^1$, arise from the exactly determined quartics such as

$$k_4(B_4 B_5)^2 (A_1 \ldots A_4 B_1 B_2 B_3 B_6)^1.$$

As above, a discussion of the family $h_6 + \lambda h_3^2$ proves that one of its members breaks up into h_1 and h_5; and so we can prove the existence of the four condition curves such as h_5; in just the same way, a discussion of $h_6 + \lambda h_3'^2$ proves the existence of three condition curves h'_5, passing once through $B_1 B_2 B_3$, and twice through the remaining sets of six points out of the seven of the second group. $h'_5(A_1 \ldots A_4 B_4 B_5)^2 (B_1 B_2 B_3)^1$ arises from $k'_6(B_4 B_5)^3 (A_1 \ldots A_4)^2 (B_1 B_2 B_3 B_6)^1$, which is a condition curve connected with the original sextic, and whose existence could have been proved independently. Since k'_6 divides the six B's into two sets of two and four points, there are fifteen such curves; the remaining twelve give by transformation other condition curves, connected with the second type of sextic which has three collinear double points. These two principles, of transformation and of symmetry, lead to an infinite number of condition curves connected with either of the two types of sextic with ten double points.

Instead of the triad BBB in the auxiliary quadratic transformation, we can use a triad ABB, AAB, or AAA, and we are led to other types of sextics, each with ten double points grouped in some way, and a series of condition curves connected with each. In particular, a transformation using $A_1A_2B_1$, followed by another using $A_3A_4B_1$ (equivalent to a single cubic transformation) transforms the original $k_2(B_1 \dots B_6)^1$ into a $k_4(B_1)^3(A_1 \dots A_4 B_2 \dots B_6)^1$, and transforms the original k_4's into other quartics with triple points; there is now no grouping of the ten double points, which are all on the same footing for this type of sextic.

If we apply quadratic transformations systematically, using every possible variety of triads, to every type of sextic that arises, we obtain five different types and no more, given by the following table.

Reference Number.	Grouping of Double Points.	Simplest Condition Curves.	Some other Condition Curves.
1	A_1A_2; $B_1 \dots B_8$	$k_0(A_1A_2)^1$	$k_6(A_1^3A_2)(B_1 \dots B_7)^2$
			$k_8(B_1B_2)^4(A_1^3A_2)(B_3 \dots B_8)^2$
2	$A_1A_2A_3$; $B_1 \dots B_7$	$k_1(A_1A_2A_3)^1$	$k_5(B_1 \dots B_6)^2(A_1A_2A_3)^1$
3	$A_1 \dots A_4$; $B_1 \dots B_6$	$k_2(B_1 \dots B_6)^1$	$k_4(A_1A_2A_3)^2(B_1 \dots B_6)^1$
			$k_6(B_1B_2)^3(A_1 \dots A_4)^2(B_3 \dots B_6)^1$
			$k_8(A_1)^4(B_1 \dots B_4)^3(A_2A_3A_4)^2(B_5B_6)^1$
			$k_{10}(B_1)^5(A_1A_2)^4(B_2 \dots B_5)^3(A_3A_4)^2(B_6)^1$
4	$A_1A_2A_3$; $B_1 \dots B_7$	$k_3(A_1)^2(B_1 \dots B_7)^1$ (three such)	$k_5(B_1)^3(A_1A_2A_3)^2(B_2 \dots B_7)^1$
5	$A_1 \dots A_{10}$	$k_4(A_1)^3(A_2 \dots A_{10})^1$ (ten such)	

In the first type, what is for the sake of uniformity called k_0, a curve of zero degree, is a single point. Two of the double points, A_1A_2, coincide, and the sextic has a tacnode there; we may think of A_2 as the point adjacent to A_1 along the tacnodal tangent. Each of the condition curves k_6, k_8 has three branches through A_1, one of which passes through A_2, that is, touches the tacnodal tangent.

The existence of any of the condition curves in the last column can be proved, if we assume the sextic and any other condition curve of the same row, by considering a suitable curve exactly determined, corresponding to k_3 in the original discussion, and obtained from it by some series of transformations. For example, in the original case (3), to prove the existence of k_6 we assume k_2 and consider the exactly determined quartic $k_4(B_1B_2)^2(A_1 \dots A_4B_3 \dots B_6)^1$. Then $f_6 . [k_1(B_1B_2)^1]^2 + \lambda k_4^2$ is a singly infinite family of octavics $k_8(B_1B_2)^4(A_1 \dots A_4B_3 \dots B_6)^2$, each meeting k_2 in the equivalent of sixteen points at $B_1 \dots B_6$, and in no other point. One member of the family therefore breaks up into k_2 and another condition curve $k_6(B_1B_2)^3(A_1 \dots A_4)^2(B_3 \dots B_6)^1$. All the properties of any one type follow from the existence of any one of its condition curves; if there exists a condition quartic with three double points, the type is (3), and so on.

In order to prove that a certain quadratic transformation changes a certain type into a certain other type, we only have to prove that it changes the simplest condition curve of the first type into some condition curve of the second.

Applied to type (1), a quadratic transformation using $B_1B_2B_3$ does not alter the type; using $A_1B_1B_2$, it gives type (2), from which (1) was originally derived. A specialized transformation, with two fundamental points coinciding at A_1A_2, does not alter the type.

Whatever the triad, the k_1 and k_2 of types (2) and (3), and suitable ones of the k_3's of type (4), and of the k_4's of type (5), are all transformed into rational curves of degree $\leqslant 4$. If the degree is 0, 1, 2, or 3, the type obtained is (1), (2), (3), or (4); while the two kinds of rational quartics, with three double points or one triple point, give types (3) and (5) respectively.

All these condition curves are rational; the only other curves of degree 6 or less, of any genus determined or over-determined by ten or fewer base points, are such that if we identified the base points with the double points of f_6, the number of intersections with f_6 would exceed the maximum, and f_6 would either break up or coincide with the condition curve; as, for example, $k_3(10)^1$ or $k_6(9)^2(1)^1$.

The curve $k_6(1)^3(7)^2(1)^1$ only appears in the particular case when the simple point is adjacent to the triple point, type (1). This can be proved directly: assume the existence of $f_6(AB_1 \dots B_7CD)^2$ and $k_6(A)^3(B_1 \dots B_7)^2(C)^1$. Then $f_6+\lambda[k_3(AB_1 \dots B_7C)^1]^2$ is a family of sextics $k_6'(AB_1 \dots B_7C)^2$, each meeting k in 36 points. If we use λ to make k' contain any other point of k, it coincides with it entirely. For this value of λ therefore,

$$k \equiv f_6+\lambda k_3^2 \equiv k'(AB_1 \dots B_7C)^2,$$

shewing that k has a double point at C and not merely a simple point as assumed. But k is already rational, and cannot have an additional double point; therefore C coincides with one of the other singularities, either A or B, where f_6 has a tacnode, two distinct branches touching the same tangent line l, and k_3 touches l also; while all the family k', including k, have two branches touching l at C. If C coincided with B, then k would have a tacnode, which reduces the genus by two, so that k would break up; hence C coincides with A. Then k has a triple point with one branch touching l; the other two branches form a curve having a double point at A, which curve must be considered as touching any straight line through A, and therefore touching l; and k must be considered as having two branches through A touching l. But the triple point is not specialized, and does not reduce the genus by more than before.

The equation of the singly infinite family k' having a tacnode at $A(1, 0, 0)$, both branches touching $x_2 = 0$, has the form:

$$(a+b\lambda)\, x_1^4 x_2^2 + x_1^3 x_2 \left\{(c_1+d_1\lambda)\, x_2^2 + (c_2+d_2\lambda)\, x_2 x_3 + (c_3+d_3\lambda)\, x_3^2\right\}$$

$$\text{terms of lower degree in } x_1 = 0,$$

and k is the member of the family for which $\lambda = -a/b$, the first group of terms is absent, and k has an ordinary triple point, one branch touching x_2.

There probably exist other types of the sextic, which are such that no combination of the three conditions among the ten double points expresses the existence of a curve. But we can shew, generally, that if a condition curve exists, of any degree, genus or negative freedom, then the sextic is of one of the five types enumerated.

Let a proper curve k_ν of degree ν be over-determined by passing ρ_k times through the point A_k ($k = 1 \ldots 10$, $\rho \geqslant 0$); let its genus be p and its freedom f:

$$\Sigma \tfrac{1}{2}\rho(\rho-1) = \tfrac{1}{2}(\nu-1)(\nu-2) - p \qquad (p \geqslant 0),$$

$$\Sigma \tfrac{1}{2}\rho(\rho+1) = \tfrac{1}{2}\nu(\nu+3) - f \qquad (f \leqslant -1).$$

Double and subtract:

$$\Sigma\, 2\rho \qquad = 6\nu + 2(p-f-1).$$

If the A's are the double points of $f_6(10)^2$, then $\Sigma 2\rho$ is the number of intersections of k_ν, f_6 which fall at A's; if the curves do not break up nor coincide,

$$\Sigma\, 2\rho \leqslant 6\nu, \qquad p-f-1 \leqslant 0,$$

which can only be so if

$$p = 0, \qquad f = -1,$$

and k_ν is a rational curve of freedom -1; and as we saw in Part I, it can be transformed into a straight line $k_1(3)^1$ by a series of quadratic transformations, using triads of its multiple points, which changes k_6 into another sextic with ten double points, which is of type (2) because of the existence of k_1. It was therefore formerly of one of the types which arise from (2) by quadratic transformations, which are the five types enumerated above.