Труды Матем. Инст. Стеклова
200 (1991)

Proceedings of the Steklov
Institute of Mathematics
1993, Issue 2

GENERATORS IN THE TWO-DIMENSIONAL CREMONA GROUP OVER A NONCLOSED FIELD

UDC 512.7

V. A. ISKOVSKIKH

1

Let k be a perfect field, $\mathbf{P}_k^2$ the projective plane over k, and $k(\mathbf{P}^2) = k(x, y)$ the field of rational functions in two variables over k. The Cremona group $\mathrm{Cr}_2(k)$ is, by definition, the group of birational automorphisms of $\mathbf{P}_k^2$ over k: it is isomorphic to the group $\mathrm{Aut}_k k(x, y)$. In the case when the field is algebraically closed, $k = \overline{k}$, Noether's well-known theorem (see, for example, [1], [6], or [7]) asserts that $\mathrm{Cr}_2(\overline{k}))$ is generated by quadratic birational and nonsingular linear transformations of the plane $\mathbf{P}_k^2$. A system of generators of $\mathrm{Cr}_2(k)$ over a nonclosed field k $(k = \mathbf{Q})$ was described in the classical paper of Kantor [8].

In this paper we describe a different (in general) system of generators of $\mathrm{Cr}_2(k)$. It emerges naturally (almost algorithmically) as a result of simplifying an arbitrary birational automorphism $\chi\colon \mathbf{P}_k^2 \dashrightarrow \mathbf{P}_k^2$. This process is based on the action of $\mathrm{Cr}_2(k)$ on the limit group of cycles $Z^*(\mathbf{P}_k^2)$, and the appropriate formalism can be found in Chapter V of Manin's book [5]. We will use it liberally, including the notation. Note that the study of the Cremona group over a nonclosed field is also of interest because of its connections with the problem of describing the groups of birational automorphisms of three-dimensional algebraic varieties over algebraically closed fields (see [2]).

Let $l \in Z^*(\mathbf{P}_k^2) = \mathrm{Pic}\,\mathbf{P}_k^2 + Z^0(\mathbf{P}_k^2)$ be the class of a line on $\mathbf{P}_k^2$, and $\varphi\colon \mathbf{P}^2 \dashrightarrow V$ a birational map over k onto a smooth projective surface V.

Let $z = \varphi(l) \in Z^*(V) = \mathrm{Pic}\,V + Z^0(V)$ be the image of the element l. Then

$$z = D - \sum n_i x_i, \qquad D \in \mathrm{Pic}\,V, \ n_i \in \mathbf{Z}, \ n_i \geq 0,$$

where the $x_i \in E(V)$ are closed points of the bubble space $E(V)$ over k (see [5], Chapter V, §3). Unlike [5], we use the notation $Z^0(V)$ for the free abelian group generated by the closed points (and not geometric points, as in [5]) of $E(V)$. As a birational image of the class $l \in \mathbf{P}_k^2$, the element z satisfies the following conditions:

$$(1) \qquad \begin{gathered} (z \cdot z) := (D \cdot D) - \sum \deg x_i n_i^2 = 1, \\ -\Omega(z) := (-K_V \cdot D) - \sum \deg x_i n_i = 3, \end{gathered}$$

where K_V is the canonical class on V. Other needed properties of the element z will be given in our first two lemmas.

Lemma 1. (i) *In the above notation, let V be a del Pezzo surface (i.e., $-K_V$ is ample, see [5]), and let $h \in \mathrm{Pic}\,V$ be an element such that $rh = -K_V$ ($r = 1, 2,$ or 3 is*

1991 *Mathematics Subject Classification.* Primary 14E07.

the index of V). *Suppose that* $z = \varphi(l) = ah - \sum n_i x_i \in Z^*(V)$, *with* $a \geq 1$ *when* $r = 1$ *or* 2 *and* $a \geq 2$ *when* $r = 3$. *Then there exists a point* $x_j \in V$ *with* $n_j > a/r$; *if* $x_{j1}, \dots, x_{jm}$ *are all the points of* $E(V)$ *with* $n_{js} > a/r$, $s = 1, \dots, m$ (*including the infinitesimally close points*), *then*

$$\sum_{s=1}^{m} \deg x_{js} < r^2 h^2 \leq 9.$$

(ii) *Let* $V = \mathbf{F}_N$ *be the standard ruled surface with exceptional section* S_N, *let* $(S_N \cdot S_N) = -N$, *and let* $z = \varphi(l) = af_N + bs_N - \sum n_i x_i$, *where* f_N *is the class of the fiber and* s_N *is the class of* S_N *in* $\operatorname{Pic} \mathbf{F}_N$. *Suppose that* $b \geq 1$ *and* $n_i \leq b/2$ *for all* i. *Then* $N = 0$ *or* 1, $b > 2a/(2+N)$, *and there exists a point* $x_j \in \mathbf{F}_N$ *such that*

$$n_j > \begin{cases} a/2 & \text{if } N = 0, \\ a/3 & \text{if } N = 1, \end{cases}$$

and in the case $N = 1$ *the point* x_j *does not lie on the exceptional section* S_1.

If $N = 1$ *and* $x_{j1}, \dots, x_{jm}$ *are all the points of* $E(V)$ *with* $a_{js} > a/3$, *then* $\sum_{s=1}^{m} \deg x_{js} \leq 8$.

(iii) *Let* $\pi \colon V \to \mathbf{P}^1$ *be a pencil of conics with* $\operatorname{Pic} V = \pi^* \operatorname{Pic} \mathbf{P}^1 + \mathbf{Z} K_V$ *and* $z = \varphi(l) = af_\pi - bK_V - \sum n_i x_i$, *where* f_π *is the class of the fiber of the morphism* π (*i.e., the positive generator of the group* $\pi^* \operatorname{Pic} \mathbf{P}^1$), *and let* $b \geq 1$. *If* $n_i \leq b$ *for all* i, *then* $a < 0$; *if, in addition,* $(K_V \cdot K_V) = d > 0$, *then* V *is a del Pezzo surface.*

Remark 1. The assertion of Lemma 1 about the existence of points of maximal multiplicity (we shall call them *maximal* points for z) are analogs of the familiar Noether inequality for $\mathbf{P}^2_{\bar{k}}$: if $z = \varphi(l) = al - \sum n_i x_i$ in $Z^*(\mathbf{P}^2_{\bar{k}})$ over an algebraically closed field $\bar{k}$, $a > 1$, and $n_1 \geq n_2 \geq \cdots$, then $n_1 + n_2 + n_3 > a$ (see, for example, [1], [6], or [7]).

Proof. Let

$$\text{(2)} \qquad \begin{array}{ccc} V_n & & \\ \sigma_n \downarrow & \searrow & \\ \vdots & \psi & \\ \sigma_1 \downarrow & & \searrow \\ V & \overset{\varphi^{-1}}{\dashrightarrow} & \mathbf{P}^2_k \end{array}$$

be the diagram of a resolution of the birational map $\varphi^{-1} \colon V \dashrightarrow \mathbf{P}^2_k$.

(i) If V is a del Pezzo surface and $-K_V = rh$, then $K_{V_n} = -r\sigma^* h + \sum \sigma^{-1} x_i$ is the canonical class of the surface V_n, where $\sigma = \sigma_n \cdots \sigma_1$ is the composition of all the blow-ups. Then

$$\begin{aligned} rl + aK_{\mathbf{P}^2} &= \psi(r\psi^*(l) + aK_{V_n}) \\ &= \psi_*\left(r\left(a\sigma^* h - \sum n_i \sigma^{-1} x_i\right) + a\left(-r\sigma^* h + \sum \sigma^{-1} x_i\right)\right) \\ &= \psi_*\left(\sum (a - rn_i)\sigma^{-1} x_i\right). \end{aligned} \qquad (3)$$

Since $a \geq 1$ when $r = 1$ or 2 and $a \geq 2$ when $r = 3$, we have that the left-hand side of (3) is negative and therefore the divisor $\sum (a - rn_i)\sigma^{-1} x_i$ cannot be effective. Hence there exists a maximal point x_j with $n_j > a/r$. Because multiplicities are monotone under resolutions, we have that x_j lies on the surface V (i.e., it is not infinitesimally close to some other point; see [5], Chapter V, 3.11). The first assertion of (i) now follows. The second follows immediately from (1) and the fact that $r^2h^2 \leq 9$ for del Pezzo surfaces.

Indeed, it follows from the first inequality of (1) that

$$a^2h^2 - \frac{a^2}{r^2}\sum_{s=1}^{m} \deg x_{js} \geq 0. \tag{4}$$

Cancelling a^2, we have the desired inequality.

(ii) Let $V = \mathbf{F}_N$ be the standard ruled surface. Then in diagram (2) we have

$$K_{V_n} = \sigma^*(-N+2)f_N - 2s_N) + \sum \sigma^{-1}x_i,$$

$$\begin{aligned} 2l + bK_{\mathbf{P}^2} &= \psi_*(\psi^*(2l) + bK_{V_n}) \\ &= \psi_*\left(\left(2af_N + 2bs_N - \sum n_i\sigma^{-1}x_i\right) - (N+2)bf_N - 2bs_N + b\sum\sigma^{-1}x_i\right) \\ &= \psi_*((2a - (N+2)b)f_N) + \psi_*\left(\sum(b - 2n_i)\sigma^{-1}x_i\right). \end{aligned} \tag{5}$$

The left-hand side of (5) is negative, and the second summand in the right-hand side is nonnegative by assumption. Therefore $b > 2a/(2+N)$. Moreover, since $z = \varphi(l)$, we have that the linear system $|z|$ has no fixed components. Therefore $0 \leq (z \cdot s_N) = a - bN$. Combining this inequality with the previous one, we have $N < 2$, i.e., $N = 0$ or 1.

Suppose now that $N = 0$. Exchanging f_0 and s_0 and taking into account that $b > a$, we deduce from the above that the inequalities $n_i \leq a/2$ $\forall i$ are impossible, i.e., there exists an index j such that $x_j \in F_0$ and $n_j > a/2$.

Let $N = 1$, and let $\nu \colon \mathbf{F}_1 \to \mathbf{P}^2$ be the blow-down of the exceptional section S_1. Then

$$\nu(z) = al - (a-b)\nu(S_1) - \sum n_i\nu(x_i).$$

As shown above, here we have $a \geq b \geq 1$ and $a - b < a/3$. Therefore, by (i), there exists $\nu(x_j) \in \mathbf{P}^2$, $\nu(x_j) \neq \nu(S_1)$, with $n_j > a/3$. But now $x_j \in \mathbf{F}_1$ satisfies the requirements of the lemma. The last assertion follows from inequality (4) for $\nu(x_{js})$ with $h = l$ and $r = 3$.

(iii) Let $\pi \colon V \to \mathbf{P}^1$ be a pencil of conics as in the lemma above, and let $z = af_\pi - bK_V - \sum n_ix_i$ with $b \geq 1$ and $n_i \leq b$ for all i. We have $K_{V_n} = \sigma^*K_V + \sum\sigma^{-1}x_i$ and

$$\begin{aligned} l + bK_{\mathbf{P}^2} &= \psi_*(\psi^*(l) + bK_{V_n}) \\ &= \psi_*(a\sigma^*f_\pi - b\sigma^*K_V - \sum n_i\sigma^{-1}x_i + b\sigma^*K_V + b\sum\sigma^{-1}x_i) \\ &= \psi_*(a\sigma^*f_\pi + \sum(b - n_i)\sigma^{-1}x_i). \end{aligned} \tag{6}$$

When $b \geq 1$, the left-hand side of (6) is negative, and the second summand in the right-hand side is nonnegative by assumption. Therefore, $a < 0$. To prove that $-K_V$ is ample in the case $K_V^2 = d > 0$ it suffices, by the numerical criterion for ampleness, to show that $(-K_V \cdot X) = 0$ for any curve $X \subset V$. If $(-K_V \cdot X) \leq 0$, then

$$(af_\pi - bK_V \cdot X) = a(f_\pi \cdot X) - b(K_V \cdot X) < 0,$$

because $a < 0$. But this is impossible, since the linear system

$$|af_\pi - bK_V| \supset \left|af_\pi - bK_V - \sum n_ix_i\right|$$

has no fixed components. The lemma is proved.

Lemma 2. *Under the assumptions of parts* (i) *and* (ii) *of Lemma* 1, *the maximal points* $x_1, \dots, x_m$ *are in general position in the sense that the blow-up* $\sigma: V' \to V$ *of any number of those points is again a del Pezzo surface.*

Proof. The assertion of the lemma is invariant under extension of the field k. Thus we may assume that $k = \overline{k}$ is algebraically closed, $V = \overline{V}$, and the x_{js} are geometric points of $E(\overline{V})$. Consider the case when V is a del Pezzo surface different from $\mathbf{P}^1 \times \mathbf{P}^1$. Let $\delta: V \to \mathbf{P}^2$ represent V as the blow-up of $\mathbf{P}^2$ at $9 - (K_V \cdot K_V)$ points $x_{m+1}, \dots, x_n$. Then $\delta(z) = 3al - ax_{m+1} - \dots - ax_n - \sum n_j \delta(x_j)$ when $r = 1$, and $\delta(z) = z - al - \sum n_i x_i$ when $r = 3$. We have to verify that the points $x_1, \dots, x_m, x_{m+1}, \dots, x_n$, where, by Lemma 1, (i), $n \leq 8$, are in general position, i.e., no three of them lie on one line, no six of them lie on one conic, and all the eight points cannot lie at the same time on a cubic such that one of those points is a double point (see [5], Chapter V; [3], §3).

Note that the points $x_{m+1}, \dots, x_n$ are in general position, because their blow-up yields the del Pezzo surface V. Suppose, for example, that some six points $x_{j1}, \dots, x_{j6}$ from $x_1, \dots, x_n$ containing at least one point of $x_1, \dots, x_m$ lie on a conic. Then $2l - x_{j1} - \dots - x_{j6} \in Z_+^*(\mathbf{P}^2)$ in the notation of [5], Chapter V, i.e., this element is represented by an effective cycle on some model (on the blow-up of $\mathbf{P}^2$ at $x_{j1}, \dots, x_{j6}$). Because $\delta(z) \in Z^{+*}(\mathbf{P}^2)$ (the element $\delta(z)$ is represented by a moving linear system without fixed components), we have that

$$0 \leq (2l - x_{j1} - \dots - x_{j6} \cdot \delta(z)) \leq 6a - n_{j1} - \dots - n_{j6}$$

when $r = 1$ and

$$0 \leq (2l - x_{j1} - \dots - x_{j6} \cdot \delta(z)) \leq 2a - n_{j1} - \dots - n_{j6}$$

when $r = 3$. Since among $x_{j1}, \dots, x_{j6}$ there is at least one point of the set $\{x_1, \dots, x_m\}$, we have, by Lemma 1, (i),

$$6a - n_{j1} - \dots - n_{j6} < 0 \quad \text{when } r = 1,$$
$$2a - n_{j1} - \dots - n_{j6} < 0 \quad \text{when } r = 3.$$

These contradictions show that the points $x_{j1}, \dots, x_{j6}$ cannot all lie on a conic. The remaining cases, including the case of infinitely close points, can be argued similarly. A similar argument can also be applied to the case $V = \mathbf{P}^1 \times \mathbf{P}^1$ and the case of Lemma 1, (ii).

This finishes the proof of Lemma 2.

2. The involutions of Bertini and Geiser

In this paper we shall describe a set of some specific birational automorphisms over a nonclosed perfect field k, which, together with linear projective transformations from the group $\mathrm{PGL}_3(k)$, generates the whole Cremona group $\mathrm{Cr}_2(k)$. The argument is standard: if $\chi: \mathbf{P}^2 \to \mathbf{P}^2$ is an arbitrary birational transformation and $z = \chi(l) = al - \sum n_i x_i$ is the image of the class of lines l in the group $Z^*(\mathbf{P}_k^2)$, then, under the assumption $a = 1$, we have $n_i = 0$ for all i, and χ is a linear projective transformation. If $a > 1$, then, by Lemma 1, (i), there exist maximal points and the number of geometric maximal points does not exceed 8. Then, for each closed maximal (over k) point, we find an appropriate birational transformation such that when applied to z it lowers the coefficient a of l. Iterating this process we eventually get $a = 1$, i.e., the result is a linear automorphism. Thus we represent χ as a composition of birational transformations from some standard set, and therefore this set is a system of generators of $\mathrm{Cr}_2(k)$. We shall construct birational transformations associated with maximal points as compositions of appropriate blow-ups and blow-downs; we shall also indicate their representations in the group $Z^*(\mathbf{P}^2_k)$ and interpret them as transformations of $\mathbf{P}^2$ in the language of linear systems with base conditions.

We begin by a description of the classical involutions of Bertini and Geiser. In the plane $\mathbf{P}^2_{\overline{k}}$ over an algebraically closed field $\overline{k}$ we choose 8 arbitrary points $\overline{x}_1, \dots, \overline{x}_8$ in general position (i.e., no three of them lie on the same line, no six on the same conic, and all eight cannot lie on the same cubic for which one of those points is a double point (see [5], Chapter V; [3], §3)). There exists a pencil of elliptic (cubic) curves passing through $\overline{x}_1, \dots, \overline{x}_8$. Let $\overline{x}_0$ be the ninth base point of that pencil. Notice that, since the points $\overline{x}_1, \dots, \overline{x}_8$ are in general position, each curve of that pencil is irreducible and nonsingular at $\overline{x}_1, \dots, \overline{x}_8$.

We choose $\overline{x}_0$ as the zero for the group structure on the generic fiber of the chosen pencil of cubics. Then the operation of taking the inverse in the group can be extended to a birational involution $\beta = \beta(\overline{x}_1, \dots, \overline{x}_8)$ of the plane $\mathbf{P}^2_k$ with $\overline{x}_1, \dots, \overline{x}_8$ as points of indeterminacy (see [7]). Let $\sigma\colon V_1 \to \mathbf{P}^2$ be the blow-up of the points $\overline{x}_1, \dots, \overline{x}_8$. Then V_1 is a del Pezzo surface of degree $(K_{V_1} \cdot K_{V_1}) = 1$, on which the involution β (or, more precisely, its image on V_1) acts biregularly. As a consequence, it preserves the canonical class

$$K_{V_1} = -3\sigma^* l + \sum_{i=1}^{8} \sigma^{-1}\overline{x}_i .$$

The birational transformation $\beta\colon \mathbf{P}^2 \dashrightarrow \mathbf{P}^2$ and its image on V_1, which we shall also denote by β, is called the *Bertini involution* associated with the points $\overline{x}_1, \dots, \overline{x}_8$.

One can give another, more invariant, description of the Bertini involution. On the surface V_1 the linear system $|-2K_{V_1}|$ gives rise to a morphism $\psi\colon V_1 \to Q^* \subset \mathbf{P}^3$ of degree 2 onto a quadratic cone Q^* in $\mathbf{P}^3$. The involution β is an automorphism of the double covering $\psi\colon V_1 \to Q^*$ permuting the points in the fibers. The morphism ψ birationally maps any exceptional curve $X \subset V_1$ onto a conic in Q^*. Therefore $\beta(X) = -2K_{V_1} - X$. From this we obtain a formula for the action of the Bertini involution on $Z^*(\mathbf{P}^2_{\overline{k}})$:

$$\begin{aligned} \beta(l) &= 17l - 6\overline{x}_1 - \dots - 6\overline{x}_8, \\ \beta(\overline{x}_i) &= 6l - 2\overline{x}_1 - \dots - 2\overline{x}_{i-1} - 3\overline{x}_i - 2\overline{x}_{i+1} - \dots - 2\overline{x}_8, \\ \beta(\overline{x}_0) &= \overline{x}_0, \qquad i = 1, \dots, 8. \end{aligned} \tag{7}$$

This means that on $\mathbf{P}^2_{\overline{k}}$ the involution β is given by a linear system of curves of degree 17 passing through points $\overline{x}_1, \dots, \overline{x}_8$ with multiplicity 6. Note that over a nonclosed field k the Bertini involution also makes sense and can be defined over k if the 0-cycle $\sum_{i=1}^{\infty} \overline{x}_i$ is defined over k. Moreover, if the del Pezzo surface V_1 is defined over k, then its Bertini involution is also defined over k.

The *Geiser involution* is defined similarly and is related to seven points $\overline{x}_1, \dots, \overline{x}_7$ which are in general position in $\mathbf{P}^2_{\overline{k}}$. A linear system of cubics passing through all those points gives rise to a rational map $\eta\colon \mathbf{P}^2 \to \mathbf{P}^2$ of degree 2 that is not defined only at $\overline{x}_1, \dots, \overline{x}_7$. Let $\sigma\colon V_2 \to \mathbf{P}^2$ be the blow-up of the points $\overline{x}_1, \dots, \overline{x}_7$. Then V_2 is a del Pezzo surface of degree $(K_{V_2} \cdot K_{V_2}) = 2$. The linear system $|-K_{V_2}|$ gives rise to a morphism $\varphi\colon V_2 \to \mathbf{P}^2$ of degree 2 with branching along a nonsingular quartic in $\mathbf{P}^2$. The automorphism of this double covering permuting the points in the fibers of φ is the Geiser involution $\gamma = \gamma(\overline{x}_1, \dots, \overline{x}_7)$, which acts on $\mathbf{P}^2$ as a birational transformation of order 2. On $Z^*(\mathbf{P}^2_{\bar{k}})$ the involution γ acts by the formulas

$$(8)\qquad \begin{gathered} \gamma(l) = 8l - 3\overline{x}_1 - \dots - 3\overline{x}_7, \\ \gamma(\overline{x}_i) = 3l - \overline{x}_1 - \dots - \overline{x}_{i-1} - 2\overline{x}_i - \overline{x}_{i+1} - \dots - \overline{x}_7, \qquad i = 1, \dots, 7. \end{gathered}$$

Thus the Geiser involution γ on $\mathbf{P}^2_{\bar{k}}$ is given by a linear system of curves of degree 8 passing through $\overline{x}_1, \dots, \overline{x}_7$ three times (see [7]).

If V_2 is a del Pezzo surface of degree 2 over k, then the biregular involution of the double covering $\varphi_{-K}: V_2 \to \mathbf{P}^2$ is defined over k, and we shall also call it the Geiser involution and denote it by γ. As in the case of the Bertini involution, if the cycle $\sum_{i=1}^{7} \overline{x}_i$ is defined over k, then γ is also defined over k.

3. Birational transformations of $\mathbf{P}^2$ associated to points of degree one

Let $\chi: \mathbf{P}^2 \to \mathbf{P}^2$ be a birational transformation, $z = \chi(l) = al - \sum n_i x_i$, and $a \geq 2$. By Lemma 1, (i), there exists a closed point, say x_1, with multiplicity $n_1 > a/3$ and $\deg x_1 \leq 8$. For each such point we shall construct one or several birational transformations of $\mathbf{P}^2$ which, upon their application to z, lower the "degree", i.e., the coefficient a of l. Let us consider separately each possible value of $\deg x_1$.

The case $\deg x_1 = 1$. Let $\sigma_1: \mathbf{F}_1 \to \mathbf{P}^2$ be the blow-up of x_1. Then $z_1 = \sigma_1^{-1}(z) = af_1 + bs_1 - \sum_{i\geq 2} n_i x_i$ in $Z^*(\mathbf{F}_1)$, where f_1 and s_1 are the classes of the fiber and exceptional section of the ruled surface $\mathbf{F}_1$, and $b = a - n_1 > 0$ by (1). Since $n_1 > a/3$ we have that $b < 2a/3$, and, by Lemma 1, (ii), there exists a point, say $x_2 \in \mathbf{F}_1$, of multiplicity $n_2 > b/2$. The geometric fiber $\overline{\mathbf{F}}_{\bar{t}}$, $\bar{t} \in \mathbf{P}^1_{\bar{k}}$, $\deg \bar{t} = 1$, cannot contain more than one geometric point with center x_2, as otherwise the curve $\overline{\mathbf{F}}_{\bar{t}}$ would be a fixed component of the linear system $|z_1|$, which is impossible (because $|z_1|$ is a proper birational image of the linear system $|l|$ of lines in $\mathbf{P}^2$). Therefore, for some integer $m \geq 0$ an elementary transformation $\varepsilon_{x_2}: \mathbf{F}_1 \to \mathbf{F}_m$ with center x_2 is defined (this elementary transformation is the blow-up of x_2 followed by the blow-down of the proper transform of the fiber of the ruled surface $\mathbf{F}_1$ containing x_2, $\deg x_2 = d \geq 1$).

The sequence of elementary transformations with centers at points of multiplicity greater than $b/2$, which we denote by $\varepsilon_1: \mathbf{F}_1 \to \mathbf{F}_N$ maps $\mathbf{F}_1$ birationally onto $\mathbf{F}_N$ for some integer $N \geq 1$. Thus for

$$\varepsilon_1(z_1) = \varepsilon_1 \sigma_1^{-1} \chi(l) = a_N f_N + b_N s_N - \sum_{i\geq 2} n_{N_i} x_{N_i},$$

the following is true in $Z^*(\mathbf{F}_N)$:

$$b_N = b; \quad n_{N_i} \leq b/2 \quad \text{for all } i \geq 2.$$

By Lemma 1, (ii), we have the following two possibilities:

(a) $N = 1$ and $a_1 < 3b/2$.

(b) $N = 0$ and $a_0 < b$.

In case (a) the image of the element $\varepsilon_1(z_1)$ in $Z^*(\mathbf{P}^2)$ goes, under the blow-down $\sigma_1: \mathbf{F}_1 \to \mathbf{P}^2$, to

$$\sigma_1 \varepsilon_1 \sigma_1^{-1}(z) = \sigma_1 \varepsilon_1 \sigma_1^{-1} \chi(l) = a_1 l - (a_1 - b) x_1 - \cdots.$$

We have $a_1 < 3b/2 = 3(a - n_1)/2 < a$, i.e., the birational transformation

$$\alpha_1 = \sigma_1 \varepsilon_1 \sigma_1^{-1} \colon \mathbf{P}^2 \to \mathbf{P}^2$$

lowers the coefficient a of l in the birational automorphism χ. In the classical literature (see, for example, [7]), transformations of the type α_1 are called *de Jonquières transformations.*

In case (b) we have $\mathbf{F}_0 \simeq \mathbf{P}^1 \times \mathbf{P}^1$, and on $\mathbf{F}_0$ there exists a biregular involution $\tau \colon \mathbf{F}_0 \to \mathbf{F}_0$ that permutes the factors $\mathbf{P}^1$. Applying τ to $\varepsilon_1 \sigma_1^{-1}(z)$, we see that

$$\tau \varepsilon_1 \sigma_1^{-1}(z) = b f_0 + a s_0 - \sum_{i \geq 2} n_i \tau(x_i).$$

By Lemma 1, (ii), there exists a point, say $\tau(x_2)$, with $n_2 > a_0/2$. Applying elementary transformations, the composition of which we denote by ε_2, we can make all the multiplicities no greater than $a_0/2$, and, by Lemma 1, (ii), we should end up with the surface $\mathbf{F}_1$ or $\mathbf{F}_0$. If it is $\mathbf{F}_1$, then, descending to $\mathbf{P}^2$ by σ_1, we have

$$\sigma_1 \varepsilon_2 \tau \varepsilon_1 \sigma_1^{-1}(z) = a_2 l - (a_2 - a_0) x_1 - \cdots ,$$

where $a_2 < 3a_0/2 < 3b/2 < a$. If we end up with $\mathbf{F}_0$, then we again apply τ and continue the process of "untwisting" χ until we end up with $\mathbf{F}_1$. As a result, we have

$$\sigma_1 \varepsilon_n \tau \sigma_{n-1} \tau \cdots \tau \varepsilon_1 \sigma_1^{-1}(z) = a_n l - \cdots ,$$

where $a_n < a$, i.e., the birational transformation

$$\alpha_n = \sigma_1 \varepsilon_n \tau \varepsilon_{n-1} \cdots \tau \varepsilon_1 \sigma_1^{-1} \tag{9}$$

lowers the coefficient a of l.

If we choose a point $x_0 \in \mathbf{F}_0$ with $\deg x_0 = 1$ and $\tau(x_0) = x_0$, and denote by $\varepsilon_0 \colon \mathbf{F}_0 \to \mathbf{F}_1$ the elementary transformation with center x_0, then the transformation α_n can be rewritten as

$$\alpha_n = \sigma_1 \varepsilon_n \varepsilon_0^{-1} \sigma_1^{-1} \sigma_1 \varepsilon_0 \tau \cdots \varepsilon_2 \varepsilon_0^{-1} \sigma_1^{-1} \sigma_1 \varepsilon_0 \tau \varepsilon_0^{-1} \sigma_1^{-1} \sigma_1 \varepsilon_0 \varepsilon_1 \sigma_1^{-1}. \tag{9'}$$

This means that α_n is a composition of the de Jonquières transformations $\sigma_1 \varepsilon_0 \varepsilon_1 \sigma_1^{-1}$ and $\sigma_1 \varepsilon_0 \varepsilon_i \varepsilon_0^{-1} \sigma_1^{-1}$, $2 \leq i \leq n$, and the transformation $\sigma_1 \varepsilon_0 \tau \varepsilon_0^{-1} \sigma_1^{-1}$. The latter, as is easily checked, is a linear projective transformation of $\mathbf{P}_k^2$. Thus we have proved the following fact.

Assertion 1. *If $z = \chi(l) = al - n_1 x_1 - \cdots$, $a \geq 2$, has a maximal point x_1 of degree 1, then the coefficient a can be decreased by applying de Jonquières transformations of type* (9) (*or* (9′)) *and projective transformations of* $\mathbf{P}_k^2$.

4. Birational automorphisms of $\mathbf{P}_k^2$ associated with points of degree two

The case $\deg x_1 = 2$. Let $\delta \colon \mathbf{P}^2 \to Q$ denote the birational transformation consisting of blowing up point x_1 and blowing down the inverse image of the line on $\mathbf{P}^2$ passing through the geometric points $x_1 \otimes \overline{k} = (\overline{x}_{11}, \overline{x}_{12}) \in \mathbf{P}_{\overline{k}}^2$. This line is defined over k, and therefore so is δ. The image of δ is a smooth quadric $Q \in \mathbf{P}_k^3$ with $\operatorname{Pic} Q = \mathbf{Z}h$, where h is the class of the sheaf $\mathscr{O}_Q(1)$. We have $-K_Q = 2h$, $h^2 = 2$, and

$$\delta(l) = h - x_0, \qquad \delta(x_1) = h - 2x_0,$$

where x_0 is the image of the line $l - x_1$ being blown down. Then

$$\delta(z) = \delta\chi(l) = (a - n_1)h - (a - 2n_1)x_0 - \sum_{i \geq 2} n_i x_i .$$

We have $(a - 2n_1) < (a - n_1)/2$, because $n_1 > a/3$ and $a - n_1 \geq 1$. Therefore it follows from Lemma 1, (i), that there exists a maximal point, say x_2, with $n_2 > (a - n_1)/2$, and then $\deg x_2 \leq 7$. We have to consider all the cases, based on the values $\deg x_2 = 1, \dots, 7$. We may assume that $n_1 \geq n_2$.

The case $\deg x_1 = 2$, $\deg x_2 = 7$. By Lemma 2, the geometric points $\overline{x}_{21}, \dots, \overline{x}_{27}$ are in general position on $\widetilde{Q} = Q \otimes \overline{k}$. Then the blow-up $\sigma_2 \colon V_1 \to Q$ of x_2 gives rise to a del Pezzo surface V_1 of degree 1. Applying the Bertini involution β on V_1 and utilizing the formulas

$$\beta(K_{V_1}) = K_{V_1}, \qquad \beta(X_2) = -14K_{V_1} - X_2, \quad \text{where } X_2 = \sigma_2^{-1}(x_2),$$

we find that

$$\beta\sigma_1^{-1}(h) = 15\sigma_2^{-1}(h) - 8X_2, \qquad \beta\sigma_2^{-1}(x_2) = 28\sigma_2^{-1}(h) - 15X_2 .$$

Therefore

$$\sigma_2\beta\sigma_2^{-1}\delta(z) = (15a - 15n_1 - 28n_2)h - (a - 2n_1)x_0 - (8a - 8n_1 - 15n_2)x_2 - \cdots .$$

We now set

$$15a - 15n_1 - 28n_2 = a' - n_1', \qquad a - 2n_1 = a' - 2n_1' .$$

We then have

$$n' = 14a - 13n_1 - 28n_2 = 14(a - n_1) - 28n_2 + n_1 < n_1 ,$$

because $n_2 > (a - n_1)/2$. Descending to $\mathbf{P}_k^2$ via the birational transformation $\delta^{-1} \colon Q \to \mathbf{P}_k^2$, we see that

$$\delta^{-1}\sigma_2\beta\sigma_2^{-1}\delta(z) = a'l - \cdots ,$$

where $a' = 28(a - n_1) - 56n_2 + a < a$, because $a - n_1 < 2n_2$.

We have thus obtained

Assertion 2. *In the case of* $\deg x_1 = 2$ *and* $\deg x_2 = 7$ *the transformation* $\gamma_{2,7} = \delta^{-1}\sigma_2\beta\sigma_2^{-1}\delta$ *lowers the coefficient* a. *On* $\mathbf{P}_k^2$ *this transformation is given by*

$$(10) \quad l \to 29l - 14x_1 - 8x_2, \quad x_1 \to 28l - 13x_1 - 8x_2, \quad x_2 \to 56l - 28x_1 - 15x_2 .$$

The case $\deg x_1 = 2$, $\deg x_2 = 6$. As in the previous case, the geometric points $\overline{x}_{21}, \dots, \overline{x}_{26}$ on $\overline{Q}$ are in general position, according to Lemma 2. Let $\sigma_2 \colon V \to Q$ be the blow-up of x_2. Then V_2 is a del Pezzo surface of degree 2 and we can utilize the Geiser involution on V_2. We have

$$\gamma\sigma_2^{-1}(h) = 7\sigma_2^{-1}(h) - 4X_2, \qquad \gamma\sigma_2^{-1}(x_2) = 12\sigma_2^{-1}(h) - 7x_2 ,$$

where $X_2 = \sigma_2^{-1}(x_2)$. Therefore

$$\sigma_2\gamma\sigma_2^{-1}\delta(z) = (7a - 7n_1 - 12n_2)h - (a - 2n_1)x_0 - (4a - 4n_1 - 7n_2)x_2 - \cdots .$$

Setting, as above, $7a - 7n_1 - 12n_2 = a' - n_1'$ and $a - 2n_1 = a' - 2n_1'$, we have

$$n_1' = 6a - 5n_1 - 12n_2 = 6(a - n_1) - 12n_2 + n_1 < n_1 ,$$
$$a' = 12(a - n_1) - 24n_2 + a < a .$$

Assertion 3. *In the case of* $\deg x_1 = 2$ *and* $\deg x_2 = 6$ *the transformation* $\gamma_{2,6} = \delta^{-1}\sigma_2\gamma\sigma_2^{-1}\delta$ *lowers the coefficient* a. *On* $\mathbf{P}_k^2$ *this transformation is given by the linear system*

$$(11) \qquad l \to 13l - 6x_1 - 4x_2, \quad x_1 \to 12l - 5x_1 - 4x_2, \quad x_2 \to 24l - 12x_1 - 7x_2.$$

The case $\deg x_1 = 2$, $\deg x_2 = 5$. On the quadric $\overline{Q} = Q \otimes \overline{k}$ there exists a pair of conjugate curves from the class $3h - 2x_2$ defined over k. Let $\sigma_2 \colon V_3 \to Q$ be the blow-up of x_2. Then V_3 is a smooth cubic surface (i.e., a del Pezzo surface of degree 3). Let $X_2 = \sigma_2^{-1}(x_2)$ and $X_2' = 3\sigma_2^{-1}(h) - 2X_2$ be the proper transform of the pair of conjugate curves from the class $3h - 2x_2$. Then X_2' is a pair of disjoint conjugate lines on the cubic surface V_3. We have that $\sigma_2^{-1}(h) = -2K_{V_3} - X_2'$ and $X_2 = -3K_{V_3} - 2X_2'$, and the image of z on V_3 can be represented in the following form:

$$\begin{aligned}\sigma_2^{-1}\delta(z) &= (a - n_1)\sigma_2^{-1}(h) - n_2X_2 - (a - 2n_1)x_0 - \cdots \\ &= (a - n_1)(-2K_{V_3} - X_2') - n_2(-3K_{V_3} - 2X_2') - (a - 2n_1)x_0 - \cdots \\ &= -(2a - 2n_1 - 3n_2)K_{V_3} - (a - n_1 - 2n_2)X_2' - (a - 2n_1)x_0 - \cdots.\end{aligned}$$

Let $\tau_2 \colon V_3 \to V_5$ be the blow-down of X_2' to a point x_2' on the del Pezzo surface V_5 of degree 5. In the group $Z^*(V_5)$ the image of z is of the form

$$\tau_2\sigma_2^{-1}\delta(z) = -(2a - 2n_1 - 3n_2)K_{V_5} - (3a - 3n_1 - 5n_2)x_2' - (a - 2n_1)x_0 - \cdots.$$

Since $n_2 > (a - n_1)/2$, we have

$$(3a - 3n_1 - 5n_2) < (2a - 2n_1 - 3n_2).$$

If $n_2 > a/3$, then the element z has on $\mathbf{P}^2$ a maximal point of degree 5, a case to be considered later (the case $\deg x_1 = 5$). If $n_2 < a/3$, then $(a - 2n_1) \le 2a - 2n_4 - 3n_2$. Then, by Lemma 1, (i), there exists a maximal point, say x_3, lying on V_5 with $n_3 > 2a - 2n_1 - 3n_2$ and $\deg x_3 \le 4$. Again we have to consider several possibilities. We may assume that $n_1 \ge n_2 \ge n_3$.

The case $\deg x_1 = 2$, $\deg x_2 = 5$, $\deg x_3 = 4$. By Lemma 2, the geometric points $\overline{x}_{31}, \ldots, \overline{x}_{34}$ are in general position on $\overline{V}_5$. Let $\sigma_3 \colon V_1 \to V_5$ be the blow-up of x_3. Then V_1 is a del Pezzo surface of degree 1, and we can utilize the Bertini involution β. We have:

on V_1: $\quad X_3 = \sigma^{-1}(x_3)$, $\beta(X_3) = -8K_{V_1} - X_3$;

on V_5:
$$\begin{aligned}\sigma_3\beta\sigma_3^{-1}(x_3) &= -8K_{V_5} - 9x_3, \\ \sigma_3\beta\sigma_3^{-1}(K_{V_5}) &= 9K_{V_5} + 10x_3, \\ \sigma_3\beta\sigma_3^{-1}\delta(z) &= -(18a - 18n_1 - 27n_2 - 8n_3)K_{V_5} - (a - 2n_1)x_0 \\ &\quad - (3a - 3n_1 - 5n_2)x_2' \\ &= (20a - 20n_1 - 30n_2 - 9n_3)x_3 - \cdots,\end{aligned}$$

on Q:
$$\begin{aligned}\sigma_2\tau_2^{-1}\sigma_3\beta\sigma_3^{-1}\tau_2\sigma_2^{-1}\delta(z) &= (81a - 81n_1 - 120n_2 - 40n_3)h - (a - 2n_1)x_0 \\ &\quad - (48a - 48n_1 - 71n_2 - 24n_3)x_2 - (20a - 20n_1 - 30n_2 - 9n_3)x_3 - \cdots,\end{aligned}$$

on $\mathbf{P}_k^2$:
$$\begin{aligned}\delta^{-1}\sigma_2\tau_2^{-1}\sigma_3\beta\sigma_3^{-1}\tau_2\sigma_2^{-1}\delta(z) &= (161a - 160n_1 - 240n_2 - 80n_3)l - \cdots \\ &= a'l - \cdots.\end{aligned}$$

Here

$$\begin{aligned}a' &= (161a - 161n_1 - 240n_2 - 80n_3) \\ &< 160a - 160n_1 - 240n_2 - 80(2a - 2n_1 - 3n_2) = a.\end{aligned}$$

Assertion 4. *In the case when* $\deg x_1 = 2$, $\deg x_2 = 5$ *(with* $n_2 \le a/3$*) and* $\deg x_3 = 4$ *the coefficient* a *of* l *can be lowered with the help of the transformation* $\gamma_{2,5,4} = \delta^{-1}\sigma_2\tau_2^{-1}\sigma_3\beta\sigma_3^{-1}\tau_2\sigma_2^{-1}\delta$. *On* $\mathbf{P}_k^2$ *it is given by*

$$l \to 161l - 80x_1 - 48x_2 - 20x_3. \tag{12}$$

The case $\deg x_1 = 2$, $\deg x_2 = 5$, $\deg x_3 = 3$. Here the algorithm is the same as in the previous case, but instead of the Bertini involution we should use the Geiser involution on V_2, where V_2 is a del Pezzo surface of degree 2 which is the blow-up $\sigma_3 \colon V_2 \to V_5$ of the point $x_3 \in V_5$. We have:

$$\begin{aligned}
\text{on } V_2:&\quad X_3 = \sigma_3^{-1}(x_3),\ \gamma(X_3) = -3K_{V_2} - X_3;\\
\text{on } V_5:&\quad \sigma_3\gamma\sigma_3^{-1}(x_3) = -3K_{V_5} - 4x_3,\ \sigma_3\gamma\sigma_3^{-1}(K_{V_5}) = 4K_{V_5} + 5x_3,\\
&\quad \sigma_3\gamma\sigma_3^{-1}\tau_2\sigma_2^{-1}\delta(z) = -(8a - 8n_1 - 12n_2 - 3n_3)K_{V_5} - (a - 2n_1)x_0\\
&\qquad - (3a - 3n_1 - 5n_2)x_2' - (10a - 10n_1 - 15n_2 - 4n_3)x_0 - \cdots;\\
\text{on } \mathbf{P}_k^2:&\quad \delta^{-1}\sigma_2\tau_2^{-1}\sigma_3\gamma\sigma_3^{-1}\tau_2\sigma_2^{-1}\delta(z) = (61a - 60n_1 - 90n_2 - 30n_3)l - \cdots\\
&\qquad = a'l - \cdots.
\end{aligned}$$

Here

$$a' = 61a - 60n_1 - 90n_2 - 30n_3 < 61a - 60n_1 - 90n_2 - 30(2a - 2n_1 - 3n_2) = a.$$

Assertion 5. *In the case* $\deg x_1 = 2$, $\deg x_2 = 5$, $\deg x_3 = 3$ *the coefficient* a *can be lowered by the transformation*

$$\gamma_{2,5,3} = \delta^{-1}\sigma_2\tau_2^{-1}\sigma_3\gamma\sigma_3^{-1}\tau_2\sigma_2^{-1}\delta.$$

On $\mathbf{P}_k^2$ *it is of the form*

$$l \to 61l - 30x_1 - 18x_2 - 10x_3. \tag{13}$$

The case $\deg x_1 = 2$, $\deg x_2 = 5$, $\deg x_3 = 2$. Let $\sigma_3 \colon V_3 \to V_5$ be the blow-up of x_3. Then V_3 is a del Pezzo surface of degree 3, i.e., the cubic surface in $\mathbf{P}_k^3$ with a pair of disjoint conjugate lines $(\overline{X}_{31}, \overline{X}_{32}) = X_3 \otimes \overline{k}$, where $X_3 = \sigma_3^{-1}(x_3)$. On V_3 there exists a quintuple of pairwise disjoint conjugate lines from the class $-3K_{V_3} - 2X_3$. They comprise a curve X_3' irreducible over k. Let $\sigma_2' \colon V_3 \to Q'$ be the blow-down of X_3' to a point x_3' on the quadric Q'. We then have:

$$\begin{aligned}
\text{on } Q':&\quad \sigma_2'\sigma_3^{-1}\tau_2\sigma_2^{-1}\delta(z) = (10a - 10n_1 - 15n_2 - 3n_3)h' - (a - 2n_1)x_0'\\
&\qquad - (3a - 3n_1 - 5n_2)x_2' - (6a - 6n_1 - 9n_2 - 2n_3)x_3' - \cdots;\\
\text{on } \mathbf{P}_k^2:&\quad \delta'^{-1}\sigma_2'\sigma_3^{-1}\tau_2\sigma_2^{-1}\delta(z) = (19a - 18n_1 - 30n_2 - 6n_3)l - \cdots = a'l - \cdots.
\end{aligned}$$

Here

$$\begin{aligned}
a' &= 19a - 18n_1 - 30n_2 - 6n_3 < 19a - 18n_1 - 30n_2 - 6(2a - 2n_1 - 2n_2)\\
&= 7a - 6n_1 - 12n_2 < 7a - 6n_1 - 6(a - n_1) = a.
\end{aligned}$$

Assertion 6. *In the case* $\deg x_1 = 2$, $\deg x_2 = 5$, $\deg x_3 = 2$ *the coefficient* a *can be lowered by the transformation* $\gamma_{2,5,2} = \delta'^{-1}\sigma_2'\sigma_3^{-1}\tau_2\sigma_2^{-1}\delta$ *given by the linear system*

$$l \to 19l - 9x_1 - 6x_2 - 3x_3. \tag{14}$$

The case $\deg x_1 = 2$, $\deg x_2 = 5$, $\deg_3 = 1$. Here we can use the transformation from Assertion 3 (i.e., from the case $\deg x_1 = 2$, $\deg x_2 = 6$). Indeed, as is easily seen, the points x_2 and x_3 are in general position on Q and $\deg x_2 + \deg x_3 = 6$. Applying this transformation, we see that

$$a - 2n_1 = a' - 2n_1', \qquad 7a - 7n_1 - 10n_2 - 2n_3 = a' - n_1'.$$

Therefore

$$n' = 6a - 5n_1 - 10n_2 - 2n_3$$

and

$$\begin{aligned} a' &= 13a - 12n_1 - 20n_2 - 4n_3 < 13a - 12n_1 - 20n_2 - 4(2a - 2n_1 - 3n_2) \\ &= 4(a - n_1) - 8n_2 + a < a. \end{aligned}$$

On $\mathbf{P}^2$ this transformation is given by the linear system

$$(11') \qquad l \to 13l - 6x_1 - 4x_2 = 4x_3.$$

The case $\deg x_1 = 2$, $\deg x_2 = 4$. In this case, as in the case $\deg x_1 = 6$ below, the coefficient a can be lowered by the transformation $\mu \colon \mathbf{P}^2_k \to \mathbf{P}^2_k$ given by the linear system of degree 5 curves passing twice through x_1 and x_2:

$$(15) \qquad \mu \colon l \to 5l - 2x_1 - 2x_2.$$

By Lemma 2, the geometric points $\overline{x}_{11}, \overline{x}_{12}, \overline{x}_{21}, \dots, \overline{x}_{24}$ are in general position, and therefore

$$\dim|5l - 2x_1 - 2x_2| = 6 \cdot 7/2 - 1 - 6 - 2 \cdot 3/2 = 2$$

and

$$(5l - 2x_4 - 2x_2 \cdot 5l - 2x_1 - 2x_2) = 25 - 6 \cdot 4 = 1.$$

Thus the transformation (15) is indeed birational. Applying it to z, we have $\mu(z) = \mu(\chi(l)) = a'l - \cdots$, where $a' = 5a - 4n_1 - 4n_2 < 5a - 4n_1 - 4a + 4n_1 = a$.

Assertion 7. *In the case* $\deg x_1 = 2$, $\deg x_2 = 4$ *the transformation* μ *lowers the coefficient* a.

The case $\deg x_1 = 2$, $\deg x_2 = 3$. On the quadric Q there exists a unique hyperplane section $X_2' \sim h - x_2$ passing through x_2. Let $\sigma_2 \colon V_5 \to Q$ be the blow-up of x_2 and $\tau_2 \colon V_5 \to V_6$ the blow-down of the inverse image of the curve X_2' on V_5 to a point $x_2' \in V_6$. Then V_6 is a del Pezzo surface of degree 6, and $\operatorname{rk}\operatorname{Pic} V_6 = 1$ over k. We have $X_2 = \sigma_2^{-1}(x_2)$, $-K_{V_5} = 2\sigma_2^{-1}(h) - X_2$, $X_2' = \sigma_2^{-1}(h) - X_2$, and

$$\tau_2\sigma_2^{-1}\delta(z) = -(a - n_1 - n_2)K_{V_6} - (a - 2n_1)x_0 - (2a - 2n_1 - 3n_2)x_2' - \cdots.$$

Since $2a - 2n_1 - 3n_2 < a - n_1 - n_2$ and $a - 2n_1 \le a - n_1 - n_2$, we have, by Lemma 1, (i), that there exists a maximal point, say x_3, in V_6 with $n_3 > a - n_1 - n_2$ and $\deg x_3 \le 5$. We may assume that $n_1 \ge n_2 \ge n_3$. Again we have several possibilities.

The case $\deg x_1 = 2$, $\deg x_2 = 3$, $\deg x_3 = 5$. Let $\sigma_3 \colon V_1 \to V_6$ be the blow-up of x_3. Then, by Lemma 2, the surface V_1 is a del Pezzo surface of degree 1. Applying the involution β, we have:

on V_1: $\quad X_3 = \sigma_3^{-1}, \ \beta(X_3) = -10K_{V_1} - X_3;$

on V_6: $\quad \sigma_3\beta\sigma_3^{-1}\tau_2\sigma_2^{-1}\delta(z) = -(11a - 11n_1 - 11n_2 - 10n_3)K_{V_6} - (a - 2n_1)x_0 - (2a - 2n_1 - 3n_2)x_2' - (12a - 12n_1 - 12n_2 - 11n_3)x_3 - \cdots;$

on Q: $\quad \sigma_2\tau_2^{-1}\sigma_3\beta\sigma_3^{-1}\tau_2\sigma_2^{-1}\delta(z) = (31a - 31n_1 - 30n_2 - 30n_3)h - (a - 2n_1)x_0 - (20a - 20n_1 - 19n_2 - 20n_3)x_2 - (12a - 12n_1 - 12n_2 - 11n_3)x_3 - \cdots;$

on $\mathbf{P}^2$: $\quad \delta^{-1}\sigma_2\tau_2^{-1}\sigma_3\beta\sigma_3^{-1}\tau_2\sigma_2^{-1}\delta(z) = (61a - 60n_1 - 60n_2 - 60n_3)l - \cdots = a'l - \cdots.$

Here

$$a' = 61a - 60n_1 - 60n_2 - 60n_3 < 61a - 60n_1 - 60n_2 - 60(a - n_1 - n_2) = a.$$

Assertion 8. *In the case* $\deg x_1 = 2$, $\deg x_2 = 3$, $\deg x_3 = 5$ *the coefficient* a *can be lowered by the above-constructed transformation*

$$\gamma_{2,3,5} = \delta^{-1}\sigma_2\tau_2^{-1}\sigma_3\beta\sigma_3^{-1}\tau_2\sigma_2^{-1}\delta,$$

given on $\mathbf{P}^2$ *by the linear system*

$$l \to 61l - 30x_1 - 20x_2 - 12x_3. \tag{16}$$

The case $\deg x_1 = 2$, $\deg x_2 = 3$, $\deg x_3 = 4$. In this case the algorithm is the same as in the previous one, except that the Bertini involution on V_1 should be replaced with the Geiser involution on V_2. We have:

$$\begin{aligned}
\text{on } V_6:\quad & \sigma_3\gamma\sigma_3^{-1}\tau_2\sigma_2^{-1}\delta(z) = -(5a - 5n_1 - 5n_2 - 4n_3)K_{V_6} - (a - 2n_1)x_0 \\
& \quad - (2a - 2n_1 - 3n_2)x_2' - (6a - 6n_1 - 6n_2 - 5n_3)x_3 - \cdots; \\
\text{on } Q:\quad & \sigma_2\tau_2^{-1}\sigma_3\gamma\sigma_3^{-1}\sigma_2^{-1}\delta(z) = (13a - 13n_1 - 12n_2 - 12n_3)h - (a - 2n_1)x_0 \\
& \quad - (8a - 8n_1 - 7n_2 - 8n_3)x_2 - (6a - 6n_1 - 6n_2 - 5n_3)x_3 - \cdots; \\
\text{on } \mathbf{P}_k^2:\quad & \delta^{-1}\sigma_2\tau_2^{-1}\sigma_3\gamma\sigma_3^{-1}\tau_2\sigma_2^{-1}\delta(z) = (25a - 24n_1 - 24n_2 - 24n_3)l - \cdots \\
& \quad = a'l.
\end{aligned}$$

Here

$$a' = (25a - 24n_1 - 24n_2 - 24n_3) < (25a - 24n_1 - 24n_2) - 24(a - n_1 - n_2) = a.$$

Assertion 9. *In the case* $\deg x_1 = 2$, $\deg x_2 = 3$, $\deg x_3 = 4$ *the coefficient* a *can be lowered by the transformation*

$$\gamma_{2,3,4} = \delta^{-1}\sigma_2\tau_2^{-1}\sigma_3\gamma\sigma_3^{-1}\tau_2\sigma_2^{-1}\delta,$$

given on $\mathbf{P}^2$ *by the linear system*

$$l \to 25l - 12x_1 - 8x_2 - 6x_3. \tag{17}$$

The case $\deg x_1 = 2$, $\deg x_2 = 3$, $\deg x_3 = 3$. Here we can use a transformation of type (11) or (11′):

$$l \to 13l - 6x_1 - 4x_2 - 4x_3. \tag{11″}$$

The case $\deg x_1 = 2$, $\deg x_2 = 3$, $\deg x_3 = 2$. Let $\sigma_3\colon V_4 \to V_6$ be the blow-up of x_3. Then V_4 is a del Pezzo surface of degree 4 containing a pair of disjoint conjugate lines from the class $X_3 = \sigma_3^{-1}(x_3)$. Let $X_3' \sim -K_{V_4} - X_3$ be the complementary pair of disjoint conjugate lines on V_4, and let $\sigma_3'\colon V_4 \to V_6'$ be the blow-down of X_3' to a point $x_3' \in V_6'$ with $\deg x_3' = 2$ and $\operatorname{Pic} V_6' \simeq \mathbf{Z}$. We have:

$$\begin{aligned}
\text{on } Q':\quad & \sigma_2'\tau_2'^{-1}\sigma_3^1\gamma_3^{-1}\tau_2\sigma_2^{-1}\delta(z) = (4a - 4n_1 - 3n_2 - 3n_3)h' - (a - 2n_1)x_0' \\
& \quad - (2a - 2n_1 - n_2 - 2n_3)x_2' - (3a - 3n_1 - 3n_2 - 2n_3)x_3 - \cdots; \\
\text{on } \mathbf{P}^2:\quad & \delta'^{-2}\sigma_2'\tau_2'^{-1}\sigma_3'\sigma_3^{-1}\tau_2\sigma_2^{-1}\delta(z) = (7a - 6n_1 - 6n_2 - 6n_3)l - \cdots = a'l - \cdots.
\end{aligned}$$

Here

$$a' = 7a - 6n_1 - 6n_2 - 6n_3 < 7a - 6n_1 - 6n_2 - 6(a - n_1 - n_2) = a.$$

Assertion 10. *In the case* $\deg x_1 = 2$, $\deg x_2 = 3$, $\deg x_3 = 2$ *the coefficient* a *can be lowered by the transformation*

$$\gamma_{2,3,2} = \delta'^{-1}\sigma_2'\tau_2'^{-1}\sigma_3'\sigma_3^{-1}\tau_2\sigma_2^{-1}\delta,$$

given on $\mathbf{P}^2$ *by*

$$l \to 7l - 3x_1 - 2x_2 - 3x_3. \tag{18}$$

The case $\deg x_1 = 2$, $\deg x_2 = 3$, $\deg x_1 = 1$. Here, as in the case $\deg x_1 = 2$, $\deg x_2 = 4$ (or the case $\deg x_1 = 6$, below), the coefficient a can be lowered by the transformation

(15′) $$\mu\colon l \to 5l - 2x_1 - 2x_2 - 2x_3 .$$

Here

$$a' = 5a - 4n_1 - 6n_2 - 2n_3 < 3a - 2n_1 - 4n_2 < a .$$

The case $\deg x_1 = 2$, $\deg x_2 = 2$. Let $\sigma_2\colon V_6 \to Q$ be the blow-up of x_2. Then V_6 is a del Pezzo surface of degree 6 on which there is a pencil of rational curves without base points and fixed components in the class $\sigma_2^{-1}(h) - X_2$, where $X_2 = \sigma_2^{-1}(x_2)$. Let f be the class of the fiber of that pencil. Then

$$-K_{V_6} = 2\sigma_2^{-1}(h) - X_2, \qquad X_2 = -2f - K_{V_6}, \qquad \sigma_2^{-1}(h) = -f - K_{V_6}.$$

We have

$$\sigma_2^{-1}\delta(z) = (2n_2 + n_1 - a)f - (a - n_1 - n_2)K_{V_6} - (a - 2n_1)x_0 - \cdots .$$

Here $(a - n_1 - n_2) > 0$, because $0 < (\sigma_{-1}^{-1}\delta(z) \cdot f) = 2(a - n_1 - n_2)$. Therefore, by Lemma 1, (iii), there exist maximal points of multiplicity $> (a - n_1 - n_2)$. Using elementary transformations along the pencil, we can birationally map V_6 onto a (possibly different) del Pezzo surface V_6' of degree 6 with a pencil $|f'|$ of rational curves such that the multiplicities of the base points of the image of z on V_6' would not exceed $(a - n_1 - n_2)$. Then, by Lemma 1, (iii), the coefficient a' of f' will be negative. Blowing down $X_2' \sim -2f' - K_{V_6'}$ (see [4]), we obtain a quadric Q'. Let $\sigma_2'\colon V_6' \to Q'$ be that blow-down and $\varepsilon\colon V_6 \to V_6'$ the composition of the aforementioned elementary transformations. We then have:

$$\text{on } Q'\colon \quad \sigma_2'\varepsilon\sigma_2^{-1}\delta(z) = (2a - 2n_1 - 2n_2 - a')h - (a - 2n_1)x_0' - (a - n_1 - n_2 + a')x_2' - \cdots ;$$

$$\text{on } \mathbf{P}^2\colon \quad \delta'^{-1}\sigma_2'\varepsilon\sigma_2^{-1}\delta(z) = (3a - 2n_1 - 4n_2 + 2a')l - \cdots .$$

Here

$$3a - 2n_1 - 4n_2 + 2a' < a,$$

because $a' < 0$ and $n_2 > (a - n_1)/2$.

Assertion 11. *In the case* $\deg x_1 = 2$, $\deg x_2 = 2$ *the coefficient* a *can be lowered by the transformation* $\gamma_{2,2,\varepsilon} = \delta'^{-1}\sigma_2'\varepsilon\sigma_2^{-1}\delta$, *which is in fact an analog of the de Jonquières transformation: it maps the pencil of conics on* $\mathbf{P}_k^2$ *with base points* x_1 *and* x_2, *i.e. the pencil* $2l - x_1 - x_2$, *into itself.*

The case $\deg x_1 = 2$, $\deg x_2 = 1$. Here it suffices to apply the quadratic transformation $\kappa_{2,1}\colon l \to 2l - x_1 - x_2$. Indeed,

$$\kappa_{2,1}(z) = (2a - 2n_1 - 2n_2)l - \cdots = a'l - \cdots$$

and

$$a' = 2a - 2n_1 - n_2 < 2a - 2n_1 - \tfrac{1}{2}(a - n_1) < a,$$

because $n_1 > a/3$.

Assertion 12. *In the case* $\deg x_1 = 2$, $\deg x_2 = 1$ *the coefficient* a *can be lowered by the quadratic transformation*

(19) $$\kappa_{2,1}\colon l \to 2l - x_1 - x_2 .$$

5. Birational automorphisms of $\mathbf{P}^2$ associated with closed points of degree d, $3 \le d \le 8$

The case $\deg x_1 = 3$.

Assertion 13. *In this case the coefficient* a *can be lowered by the quadratic transformation*

(19′) $$\kappa_3 : l \to 2l - x_1 .$$

The case $\deg x_1 = 4$. The only pencil of conics passing through $x_1 \in \mathbf{P}^2$ is $2l - x_1$. Let $\sigma_1 : V_5 \to \mathbf{P}^2$ be the blow-up of x_1. Let V_5 is a del Pezzo surface of degree 5 with a pencil of rational curves. If f is the class of the fiber of that pencil, then $f = 2\sigma_1^{-1}(l) - X_1$, where $X_1 = \sigma_1^{-1}(x_1)$, and we have

$$\sigma_1^{-1}(z) = (3n_1 - a)f - (a - 2n_1)K_{V_5} - \cdots .$$

By Lemma 1, (iii), there exist maximal points of multiplicity $> (a - 2n_1)$, because $(3n_1 - a) > 0$. Let $\varepsilon : V_5 \to V_5'$ be a composition of elementary transformations along the pencil, which lowers all the multiplicities to values not exceeding $a - 2n_1$. Then

$$\varepsilon\sigma_1^{-1} = a'f' - (a - 2n_1)K_{V_5'} - \cdots ,$$

where $a' < 0$. On the del Pezzo surface V_5' there exists (see [4]) a quadruple of disjoint conjugate lines from the class $X_1' \sim -3f' - K_{V_5'}$. Let $\sigma_1' : V_5' \to \mathbf{P}^2$ be the blow-down of X_1'. Then

(20) $$\sigma_1'\varepsilon\sigma_1^{-1}(z) = (3a - 6n_1 + 2a')l - \cdots .$$

Here $(3a - 6n_1 + 2a') < a$, because $a' < 0$ and $n_1 > a/3$.

Assertion 14. *In the case* $\deg x_1 = 4$ *the coefficient* a *can be lowered by the aforementioned de Jonquières-type transformation* $\gamma_{4,\varepsilon} = \sigma'\varepsilon\sigma_1^{-1}$ *mapping the pencil of conics* $2l - x_1$ *into itself.*

The case $\deg x_1 = 5$. Let $\delta : \mathbf{P}^2 \to V_5$ be the birational transformation consisting of blowing up x_1 and blowing down the inverse image of the only conic on $\mathbf{P}^2$ passing through x_1. Then

$$\delta(z) = -(a - 2n_1)K_{V_5} - (2a - 5n_1)x_0 - \cdots ,$$

where x_0 is the image of the conic being blown down. Since $n_1 > a/3$, we have $2a - 5n_1 < a - 2n$, and, by Lemma 1, (i), there exists a point, say x_2, with $n_2 > a - 2n_1$ and $\deg x_2 \le 4$. We may assume that $n_1 \ge n_2$. Again, we have several possibilities.

The case $\deg x_1 = 5$, $\deg x_2 = 4$. Here we use the Bertini involution. Let $\sigma_2 : V_1 \to V_5$ be the blow-up of x_2, and let $X_2 = \sigma_2^{-1}(x_2)$. Then

$$\sigma_2^{-1}\delta(z) = -(a - 2n_1)(K_{V_1} - X_2) - n_2X_2 - (2a - 5n_1)x_0 - \cdots .$$

Applying the Bertini involution, we have

$$\beta\sigma_2^{-1}\delta(z) = -(9a - 18n_1 - 18n_2)K_{V_1} - (a - 2n_1 - n_2)X_2 - (2a - 5n_1)x_0 - \cdots ,$$

and, on $\mathbf{P}^2$,

$$\delta^{-1}\sigma_2\beta\sigma_2^{-1}\delta(z) = (41a - 80n_1 - 40n_2)l - \cdots .$$

Here

$$41a - 80n_1 - 40n_2 < 41a - 80n_1 - 40a - 80n_1 = a .$$

Assertion 15. *In the case* $\deg x_1 = 5$, $\deg x_2 = 4$ *the coefficient* a *can be lowered by the transformation* $\gamma_{5,4} = \delta^{-1}\sigma_2\beta\sigma_2^{-1}\delta$ *given by*

$$l \to 41l - 16x_1 - 10x_2. \tag{21}$$

The case $\deg x_1 = 5$, $\deg x_2 = 3$. This case is similar to the previous one except that the Geiser involution γ should be used in place of the Bertini involution. We thus have, in the previous notation,

$$\text{on } V_2: \quad \gamma\sigma_2^{-1}\delta(z) = -(4a-8n_1-3n_2)K_{V_2} - (a-2n_1-n_2)X_2 - (2a-5n_1)x_0 - \cdots;$$
$$\text{on } \mathbf{P}^2: \quad \delta^{-1}\sigma_2\gamma\sigma_2^{-1}\delta(z) = (16a - 30n_1 - 15n_2)l - \cdots.$$

Here
$$16a - 30n_1 - 15n_2 < 16a - 30n_1 - 15a + 30n_1 = a.$$

Assertion 16. *In the case* $\deg x_1 = 5$, $\deg x_2 = 3$ *the coefficient* a *can be lowered by the transformation* $\gamma_{5,3} = \delta^{-1}\sigma_2\gamma\sigma_2^{-1}\delta$ *given by*

$$l \to 16l - 6x_1 - 5x_2. \tag{22}$$

The case $\deg x_1 = 5$, $\deg x_2 = 2$. Here it suffices to apply the Geiser involution
$$\gamma(z) = (8a - 15n_1 - 6n_2)l - \cdots,$$
because
$$8a - 15n_1 - 6n_2 < 8a - 15n_1 - 6a + 12n_1 = a.$$

The case $\deg x_1 = 5$, $\deg x_2 = 1$. As in the case $\deg x_1 = 6$ below, here the result is obtained by applying the transformation

$$\mu: l \to 5l - 2x_1 - 2x_2. \tag{15''}$$

Here $\mu(z) = (5a-10n_1-2n_2)l - \cdots = a'l$, where $a' = 5a-10n_1-2n_2 < 3a-6n_1 < a$.

The case $\deg x_1 = 6$. As already mentioned, we can use the transformation

$$\mu: l \to 5l - 2x_1,\ x_1 \to 12l - 5x_1. \tag{15'''}$$

Here $a' = 5a - 12n_1 < a$, as desired.

The case $\deg x_1 = 7$. Here a direct application of the Geiser involution γ yields the desired result: $a' = 8a - 21n_1 < a$.

The case $\deg x_1 = 8$. Here the Bertini involution β is applicable and yields the result. We have $a' = 17a - 48n_4 < a$.

Assertion 17. *In the case* $\deg x_1 = 8$ *the coefficient* a *can be lowered by the Bertini involution, and in the cases* $\deg x_1 = 7$ *and* $\deg x_1 = 5$, $\deg x_2 = 2$ *it can be lowered by the Geiser involution.*

Thus we have proved the following theorem.

Theorem. *The Cremona group* $\mathrm{Cr}_2(k)$ *of birational automorphisms of the projective plane* $\mathbf{P}_k^2$ *over a perfect field* k *is generated by linear projective transformations and the series of birational transformations of types* (7)–(22).

Bibliography

1. I. R. Shafarevich (editor), *Algebraic surfaces*, Trudy Mat. Inst. Steklov. **75** (1965); English transl., Proc. Steklov Inst. Math. **75** (1965).
2. V. A. Iskovskikh, *Birational automorphisms of three-dimensional algebraic varieties*, Itogi Nauki i Tekhniki: Sovremennye Problemy Mat., vol. 12, VINITI, Moscow, 1979, pp. 159–236; English transl. in J. Soviet Math. **13** (1980), no. 6.

3. ———, *Minimal models of rational surfaces over arbitrary fields*, Izv. Akad. Nauk SSSR Ser. Mat. **43** (1979), 19–43; English transl. in Math. USSR Izv. **14** (1980).

4. ———, *Rational surfaces with a pencil of rational curves and with positive square of the canonical class*, Mat. Sb. **83(125)** (1970), 90–119; English transl. in Math. USSR Sb. **12** (1970).

5. Yu. I. Manin, *Cubic forms: Algebra, geometry, arithmetic*, "Nauka", Moscow, 1972; English transl., North-Holland, Amsterdam, 1973.

6. Masayoshi Nagata, *Rational surfaces*. II, Mem. Coll. Sci. Univ. Kyoto Ser. A Math. **33** (1960), 271–293.

7. Hilda P. Hudson, *Cremona transformations in plane and space*, Cambridge Univ. Press, Cambridge, 1927.

8. S. Kantor, *Rationale Zerlegung der birationalen Transformationen in ihre Primfactoren*, Monatsh. Math. Phys. **10** (1899), 54–74.

Translated by A. MARTSINKOVSKY