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Recent Work on Cremona Transformations

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This survey article contains a very biased treatment of work done in the area of Cremona transformations during the past decade or so. As in classical times, this field is still rich in special cases with pretty descriptions, and divides into very different topics— no general theory has yet emerged. I have merely chosen the subjects that are most interesting to me. Most of these topics will be treated very briefly. A little more time will be taken up by a discussion of *special* Cremona transformations, i.e. those whose base scheme Y is smooth and irreducible. The connection to the theme of this conference is that certain rational mappings can be demonstrated to be Cremona transformations by using the syzygy matrix of the forms defining the mapping. This idea is due to Schreyer.

I have refrained from considering the more general subject of birational mappings between arbitrary varieties, especially in dimension 3. One instance of this is recent work extending the Iskovskikh-Manin approach to non-rationality of varieties (i.e. showing that there are few birational automorphisms). I will settle for merely listing two references here, the original Iskovskikh-Manin paper [IM] and its extension by Pukhlikov [P]. Another interesting topic not considered here is the work on Hanamura [H1,H2] which constructs the *scheme* $\mathrm{Bir}(X)$ of birational automorphisms of X, and shows that it behaves nicely if X is not uniruled. This uses some techniques related to the minimal model program (MMP). To my knowledge, the MMP has never been applied to the study of Cremona transformations.

I. Generalities

Definition. A *Cremona transformation* is a birational mapping from $\mathbf{P}^r$ to $\mathbf{P}^r$. The set of all Cremona transformations of $\mathbf{P}^r$ forms a group, denoted Cr_r.

Let $\Phi = (f_0, \cdots, f_r) : \mathbf{P}^r - - \rightarrow \mathbf{P}^r$ denote a Cremona transformation, with f_i forms of degree n on $\mathbf{P}^r$. The *base locus* X of Φ is the scheme of zeros of the f_i. We can blow up X (or more conveniently, perform a sequence of blow ups with smooth centers) to get a variety $\widetilde{\mathbf{P}}^r$ and a morphism

$\widetilde{\Phi} : \widetilde{\mathbf{P}}^r \to \mathbf{P}^r$ such that the following diagram commutes.

$$\begin{array}{ccccc} E & \subset & \widetilde{\mathbf{P}}^r & & \\ \downarrow & & \downarrow \pi & \searrow \widetilde{\Phi} & \\ X & \subseteq & \mathbf{P}^r & \dashrightarrow & \mathbf{P}^r \end{array}$$

Here, $E = \cup E_i$ denotes the exceptional divisor, with irreducible components E_i. Let H denote the hyperplane class on $\mathbf{P}^r$. Then $\widetilde{\Phi}^* H = n\pi^* H - \Sigma m_i E_i$ for some $m_i \geq 0$.

Lemma 1 [CK1] $\dim |n\pi^* H - \Sigma m_i E_i| = r$, *i.e.* $\widetilde{\Phi}$ *is defined by a complete linear series.*

Proof: If the conclusion of the lemma were false, then Φ would factor through a projection $\mathbf{P}^{r'} \to \mathbf{P}^r$, with $r' > r$. The image of the factorization map must have degree 1, hence be linear. But mappings defined by complete linear systems are nondegenerate, a contradiction. Q.E.D.

The moral is that alternatively, one can describe Cremona transformations by specifying a base locus X, degree n, and multiplicities m_i. The geometry of X then translates to information about the structure of Φ. For instance, if $C \subseteq \mathbf{P}^r$ is a curve whose proper transform $\widetilde{C} \subseteq \widetilde{\mathbf{P}}^r$ satisfies $\widetilde{C} \cdot (n\pi^* H - \Sigma m_i E_i) = 0$, then $\Phi(C)$ is a point. The nicest case is when $\widetilde{\Phi}$ is a blow down mapping (it is not true in dimension ≥ 3 that every birational mapping is the composition of blow ups and blow downs; counterexamples appear in [H], [O], and [C]). Two elementary examples come to mind.

1. The quadratic transformation $\Phi = (x_1x_2,\ x_2x_0,\ x_0x_1) : \mathbf{P}^2 \dashrightarrow \mathbf{P}^2$.

   Here $X = \{p_1,\ p_2,\ p_3\} = \{(1,0,0),\ (0,1,0),\ (0,0,1)\}$. and $\widetilde{\Phi}^* H = 2\pi^* H - E_1 - E_2 - E_3$ if L_{ij} is the line joining p_i and p_j, then $\widetilde{L}_{ij} = \pi^* H - E_i - E_j$. Since $\widetilde{\Phi}^* H \cdot \widetilde{L}_{ij} = 0$, it follows that L_{ij} is blown down to a point. Hence Φ^{-1} is of the same type as Φ (in fact, $\Phi^2 = \mathrm{Id}$).

2. The second example is just as elementary, but less well known. Let $C \subseteq \mathbf{P}^3$ be an irreducible plane conic curve, with p a point of $\mathbf{P}^3$ not contained in the supporting plane P of C. The linear system of quadrics containing p and C define a Cremona transformation Φ. Let $L \subseteq \mathbf{P}^3$ be a line joining p to a point of C, or a line contained in P. Then $\widetilde{\Phi}^*(H) \cdot \widetilde{L} = 0$. Hence Φ blows up a point and a conic, while blowing down P to a point, and the cone over C with vertex p gets blown down to a conic. Thus Φ^{-1} is of the same type as Φ. As an example, take $\Phi = (x_1x_3 - x_2^2,\ x_0x_1,\ x_0x_2,\ x_0x_3)$. Here $p = (1,0,0,0)$, and C is defined by $x_0 = x_1x_3 - x_2^2 = 0$.

II. Special Cremona Transformations

In a series of papers ([K], [CK1], [CK2]), Crauder and Katz have initiated the classification of Cremona transformations with smooth, irreducible base locus X. In the notation of §1, $\widetilde{\Phi}$ is given by the linear system $n\pi^* H - mE$. Let $L \subseteq \mathbf{P}^r$ denote a general line. Then we have

$$(n\pi^* H - mE)^r = 1, \text{ genus } \widetilde{\Phi}^{-1}(L) = 0$$

The genus of $\widetilde{\Phi}^{-1}(L)$ can be calculated by the adjunction formula. This yields a collection of Diophantine equations relating n, m, and the invariants of X. In low dimension, these equations can be solved. Some of the solutions are extraneous, in the sense that it can be shown by geometric reasoning that no X exists with the desired invariants. In the remaining cases, X, n, and m can be constructed, and usually even classified completely. Often, the most difficult part of the problem is to show that a rational mapping constructed from X, n, and m is indeed a Cremona transformation. A new technique for this part – the use of syzygies – has been developed in [HKS]. This technique will be discussed later. The other results in [HKS], together with those of [CK1] and [CK2] may be summarized as follows:

Theorem 1

I. *A Cremona transformation of* $\mathbf{P}^r$ *which becomes a morphism after the blow-up of* C, *a smooth curve of degree* d *and genus* g, *is given by the complete linear series of* n*-tics containing* C *where*

   i. $r = 3$, $d = 6$, $g = 3$ *and* $n = 3$ *or*

   ii. $r = 4$, $d = 5$, $g = 1$ *and* $n = 2$.

   Conversely, if C *is a general curve of degree 6 and genus 3 in* $\mathbf{P}^3$ *or a nondegenerate smooth curve of degree 5 and genus 1 in* $\mathbf{P}^4$, *then the linear series above produce Cremona transformations.*

II. *If* Φ *is a Cremona transformation of* $\mathbf{P}^r$ *which becomes a morphism after the blow-up of* S, *a smooth surface, then* Φ *is given by the complete linear series of* n*-tics containing* S *where*

A. *$r = 4$, $n = 3$ and S is a quintic elliptic scroll where $S = \mathbf{P}_C(\mathcal{E})$ with $e(\mathcal{E}) = -1$,*

B. *$r = 4$, $n = 4$ and S is a degree 10 determinantal surface given by the vanishing of the 4×4 minors of a 4×5 matrix of linear forms.*

C. *$r = 5$, $n = 2$ and S is the Veronese surface,*

D_7. *$r = 6$, $n = 2$ and S is a septic elliptic scroll where $S = \mathbf{P}_C(\mathcal{E})$ with $e(\mathcal{E}) = -1$, or*

D_8. *$r = 6$, $n = 2$ and S is $\mathbf{P}^2$ blown up at eight points and embedded in $\mathbf{P}^6$ as an octic surface by quartic curves passing simply through all eight points.*

Conversely, in case A-D_7, any such smooth S gives a Cremona transformation. That of A is inverse to the quintic elliptic curve transformation of $\mathbf{P}^4$ given in Theorem I. ii, while B and C are self-dual. In case D_8, if no 4 points are colinear, and no 7 lie on a conic, then the resulting morphism Φ is a Cremona transformation.

III. *If Φ is a Cremona transformation of $\mathbf{P}^r$ which becomes a morphism after the blow up of X, a smooth threefold, then Φ is given by the complete linear series of n-tics containing X where*

A. *$r = 5$ and X is given by the 5×5 minors of a 5×6 matrix of linear forms,*

B. *$r = 6$ $n = 3$, or*

C. *$r = 8$ $n = 2$*

Remarks:

1. Cases II D_7 and II D_8 had been previously investigated by Semple and Tyrell [ST1], [ST2] by different methods.

2. In case IIIB, if X is the variety defined by the Pfaffians of the principal 6×6 submatrices of a 7×7 skew-symmetric matrix, then these 7 Pfaffians define a Cremona transformation.

3. Let Y be Fano threefold of index 1 and degree 14, so that $Y = G(2,6) \cap \mathbf{P}^9 \subset \mathbf{P}^{14}$. Let $X \subseteq \mathbf{P}^8$ be the projection of Y from a point on Y. If X is smooth, then the quadrics containing X define a Cremona transformation, giving an example of case IIIC.

4. In [CK2], the consequences of Hartshorne's conjecture on complete intersections were investigated. It turns out that in this case, $m = 1$,

which greatly simplifies the classification. There are many examples of possible invariants of varieties X which can only be eliminated by Hartshorne's conjecture. So one can fantasize that constructive methods for finding Cremona transformations with given invariants can be developed (some methods are given in [ESB] and [HKS]), leading to a counterexample to Hartshorne's conjecture.

Problem: Complete the classification of Cremona transformations whose base locus is a smooth threefold.

Special Cremona transformations have also been considered by Ein and Shepherd-Barron in [ESB]. They consider the following

Definition. A representation V of the semisimple group G is *regular prehomogeneous* if there is an invariant $P \in k[V]$ such that $\mathbf{P}(V^*) - (P = 0)$ is homogeneous under G, and the determinant of the Hessian of P is not identically zero.

Theorem 2 *If $V = \mathbf{C}^n$ is an irreducible regular prehomogeneous representation of the semisimple group G, then there is a coordinate system $(x_1, \ldots, x_n)$ on V such that if P is the unique irreducible invariant, then either*

(i) the rational map $T : \mathbf{P}^{n-1} \to \mathbf{P}^{n-1}$ given by

$$\left(\frac{\partial P}{\partial x_1}, \ldots, \frac{\partial P}{\partial x_n}\right)$$

is a Cremona transformation with $T^2 = 1$, or

(ii) $n = 2v$ is even and the map T given by

$$\left(\frac{\partial P}{\partial x_{v+1}}, \ldots, \frac{\partial P}{\partial x_{2v}}, -\frac{\partial P}{\partial x_1}, \ldots, -\frac{\partial P}{\partial x_v}\right)$$

is a Cremona transformation with $T^2 = 1$.

Recall that a smooth irreducible non-degenerate variety $X \subseteq \mathbf{P}^r$ is called a *Severi variety* if $\dim X = \frac{2r-4}{3}$ and its secant variety is not all of $\mathbf{P}^r$. There are only 4 Severi varieties, the Veronese in $\mathbf{P}^5$, the Segre embedding of $\mathbf{P}^2 \times \mathbf{P}^2$ in $\mathbf{P}^8$, the Plücker embedding of $G(2,6)$ in $\mathbf{P}^{14}$, and the E_6 variety in $\mathbf{P}^{26}$.

Theorem 3 *Let Φ be a Cremona transformation with smooth base locus X. Then Φ and Φ^{-1} are defined by quadrics if and only if X is a Severi variety.*

Proof (sketch): The Diophantine equations mentioned above are easily solved to show that X is a Severi variety. On the other hand, if X is a Severi variety, the secant variety is a cubic, defined by a polynomial P corresponding to a regular prehomogeneous vector space structure, and the partial derivatives of P cut out X scheme theoretically (since $X = Sing(Sec(x))$). Q.E.D.

Note that if Φ is defined by forms of degree n, then Φ blows down the n-secant variety of the base locus (the variety swept out by the n-secant lines of the base locus).

Problem (Crauder): Develop a theory of varieties with degenerate trisecant varieties analogous to the theory of Severi varieties. Relate this to the classification of Cremona transformations Φ such that Φ and Φ^{-1} are defined by cubics.

Ein and Shepherd-Barron have also proven the following.

Theorem 4 *Let $\Phi : \mathbf{P}^r - - \to \mathbf{P}^r$ be a Cremona transformation with smooth base locus X, codim $X = 2$. Then Φ is given by the complete linear series of n-tics containing X where*

A. *$n = r = 3$, 4, or 5, and X is defined by the maximal minors of an $(n \times n+1)$ matrix of linear forms, or*

B. *$r = 4$, $n = 3$, and X is a quintic elliptic scroll.*

The final topic considered on special Cremona transformations is Schreyer's insight that syzygies can be used to detect Cremona transformations [HKS].

Consider a map $\Phi = (f_0, \ldots, f_r) : \mathbf{P}^r - - \to \mathbf{P}^r$ as before, with base locus X and resolution $\tilde{\Phi} : \tilde{\mathbf{P}}^r \longrightarrow \mathbf{P}^r$. Let $Q = (Q_{ij})(0 \le i \le r)$ be any matrix of syzygies, i.e. $\sum_i f_i Q_{ij} = 0$ for all columns j. Let $a = (a_i) \in \mathbf{P}^r$ Let Q_a be the "generalized row" $(Q_a)_j = \sum_i a_i Q_{ij}$ of Q. Let $Z_a \subseteq \mathbf{P}^r$ be the scheme of zeros of the entries of Q_a. Let $f_a^\perp = \{\sum b_i f_i | \sum b_i a_i = 0\}$ be the pullback under Φ of the set of hyperplanes of $\mathbf{P}^r$ passing through a. Thus $V(f_a^\perp) = \pi(\tilde{\Phi}^{-1}(a)) \cup X$.

Theorem 5 *Suppose that for some $a \in \mathbf{P}^r$, Z_a is just $\{p\}$ for some $p \notin X$. Then the following are equivalent:*

1. *Rank $Q(p) = r$*
2. *$p \in V(f_a^\perp)$*
3. *Φ is dominant*

Furthermore, if any of these conditions hold, then Φ is a Cremona transformation.

Example. Here is how it was shown in [HKS] that the quadrics through any elliptic scroll of degree 7 and invariant $e = -1$ in $\mathbf{P}^6$ define a Cremona transformation. First it is shown by geometry and the Schrödinger representation of the Heisenberg group that any such scroll is projectively equivalent to an "H_7 equivariantly embedded elliptic scroll in $\mathbf{P}^6$." This means the following. Let $(x_0, \ldots, x_6)$ be homogeneous coordinates on $\mathbf{P}^6$, with the subscripts understood to be taken mod 7. Let $\sigma \in PGL_7$ be the automorphism inducing $\sigma(x_i) = x_{i+1}$. Then the H_7 equivariantly embedded elliptic scrolls are precisely the varieties cut out by the 7 quadrics $\{\sigma^i Q\}$, where

$$Q = -a_0a_1a_2x_0^2 + a_0^2a_1x_1x_6 - a_1^2a_2x_2x_5 + a_2^2a_0x_3x_4$$

where (a_0, a_1, a_2) satisfy $a_0^3a_1 - a_1^3a_2 - a_2^3a_0 = 0$, $a_0a_1a_2 \neq 0$.

The quadrics $\sigma^i Q$ have the following matrix of syzygies, where the order of the quadrics is shown to the left.

$$\begin{matrix} Q \\ \sigma Q \\ \sigma^2 Q \\ \sigma^3 Q \\ \sigma^4 Q \\ \sigma^5 Q \\ \sigma^6 Q \end{matrix} \begin{pmatrix} 0 & -a_1x_2 & a_2x_4 & a_0x_6 & -a_0x_1 & -a_2x_3 & a_1x_5 \\ -a_1x_5 & a_2x_0 & a_0x_2 & -a_0x_4 & -a_2x_6 & a_1x_1 & 0 \\ a_2x_3 & a_0x_5 & -a_0x_0 & -a_2x_2 & a_1x_4 & 0 & -a_1x_1 \\ a_0x_1 & -a_0x_3 & -a_2x_5 & a_1x_0 & 0 & -a_1x_4 & a_2x_6 \\ -a_0x_6 & -a_2x_1 & a_1x_3 & 0 & -a_1x_0 & a_2x_2 & a_0x_4 \\ -a_2x_4 & a_1x_6 & 0 & -a_1x_3 & a_2x_5 & a_0x_0 & -a_0x_2 \\ a_1x_2 & 0 & -a_1x_6 & a_2x_1 & a_0x_3 & -a_0x_5 & -a_2x_0 \end{pmatrix}$$

Apply theorem 5 to $a = (1, 0, \ldots, 0)$. Then $Z_a = \{(1, 0, \ldots, 0)\}$, and $Q(1, 0, \ldots, 0)$ has rank 6. Hence Φ is a Cremona transformation for the Heisenberg invariant scrolls, hence for all scrolls of the type considered.

For certain types of varietes, their equations can be reproduced from their syzygy matrices. Determinantal varieties are the most obvious example of this. This observation can be used to construct varieties with "many" linear syzygies ([HKS]). Theorem 5 now can be used to conclude that they give rise to Cremona transformations, if the syzygy matrices are properly chosen.

Problem: Find more families of Cremona transformations constructed from a linear syzygy matrix.

III. The Cremona Group Cr_r.

An old problem is to find generators and relations for Cr_r. The answer is known only for $r = 2$.

Let V be any rational variety. A rational mapping $V - - \to \mathbf{P}^r$ induces an isomorphism $Bir(V) \simeq Cr_r$ of the Cremona group with the group of birational automorphisms of V. Here, two answers for $r = 2$ are reviewed.

First, it was known classically that Cr_2 is generated by the group PGL_3 of automorphisms of $\mathbf{P}^2$, together with the quadratic transformations.

The relations have been studied by Gizatullin [G], who found that the relations are generated by relations of the form $g_1g_2g_3 = 1$. Both this fact, and the classical fact about quadratic transformations and automorphisms generating Cr_2 follow from an argument in combinatorial group theory applied to a graph which encodes the data of all possible birational maps of a rational variety to $\mathbf{P}^2$, together with the relations between these maps arising from composition with a quadratic transformation.

Iskovskikh [I] has found a different solution to the problem. Let F_0 denote the rational ruled surface $\mathbf{P}^1 \times \mathbf{P}^1$. He first proves that $Bir(F_0)$ is generated by the "switch involution" $\tau(x,y) = (y,x)$, the biregular automorphisms $PGL(2)\times PGL(2)$, and the involution $e(((u_0 : u_1),\ (v_0 : v_1))) = ((u_0 : u_1),\ (u_0v_1, u_1v_0))$. He then shows that the relations are somewhat more explicit than the relations between the quadratic transformations mentioned above.

Problem: Find generators and relations for Cr_3.

In [U1], [U2], [U3], and [U4], Umemura has found the maximal connected algebraic subgroups of Cr_3.

Let X be a rational threefold, and G an algebraic group acting effectively on X. A birational map $f : X - - \to \mathbf{P}^3$ naturally determines an embedding $G \hookrightarrow Cr_3$; hence the action of G on X determines a conjugacy class of a subgroup of Cr_3, whose members are isomorphic to G. Both the action of G and the resulting conjugacy class in Cr_3 will be referred to as algebraic operations. The conjugacy classes of maximal subgroups will be described as algebraic operations.

Before stating the result, some names are needed for certain rational varieties.

Let $F'_m = Spec\left(\bigoplus_{k\geq 0} \mathcal{O}_{\mathbf{P}^1}(-km)\right)$ be the total space of the line bundle of degree m over $\mathbf{P}^1$.

Let $J'_m = Spec\left(\bigoplus_{k\geq 0} \mathcal{O}_{\mathbf{P}^2}(km)\right)$ be the total space of the line bundle of degree m over $\mathbf{P}^2$.

Let $L'_{m,n} = Spec\left(\bigoplus_{k\geq 0} \mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}(-km, -kn)\right)$ be the total space of the line bundle of bidegree (m,n) over $\mathbf{P}^1 \times \mathbf{P}^1$.

Let $F'_{m,n} = Spec\left(Sym(\mathcal{O}_{\mathbf{P}^1}(-n) \bigoplus \mathcal{O}_{\mathbf{P}^1}(-n)\right)$ be the total space of the vector bundle $\mathcal{O}_{\mathbf{P}^1}(m) \bigoplus \mathcal{O}_{\mathbf{P}^1}(n)$ over $\mathbf{P}^1$.

E'^l_m denotes a particular $\mathbf{A}^1$ bundle over F'_m, depending on l. See [U4] for details.

Theorem 6 (I) *Let G be a connected algebraic group in Cr_3. Then G is contained in the conjugacy class of one of the following algebraic operations:*

(P1) (PGL_4, $\mathbf{P}_3$),
(P2) (PSO_5, *quadric* $\subset \mathbf{P}_4$).
(E1) (PGL_2, PGL_2/Γ), Γ *is an octahedral subgroup of* PGL_2.
(E2) (PGL_2, PGL_2/Γ), Γ *is an icosahedral subgroup of* PGL_2.
(J1) ($\mathrm{PGL}_3 \times \mathrm{PGL}_2$, $\mathbf{P}^2 \times \mathbf{P}^1$).
(J2) ($\mathrm{PGL}_2 \times \mathrm{PGL}_2 \times \mathrm{PGL}_2$, $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$).
(J3) ($\mathrm{PGL}_2 \times \mathrm{Aut}^0\mathrm{F}'_m$, $\mathbf{P}^1 \times \mathrm{F}'_m$) *where m is an integer* ≥ 2.
(J4) (PGL_3, PGL_3/B) *where B is a Borel subgroup of* PGL_3.
(J5) (PGL_2, $\mathrm{PGL}_2/\mathrm{D}_{2n}$) *where n is an integer* ≥ 4.
(J6) $(G, G/H_{m,n})$ *where* $G = \mathbf{G}_m \times \mathrm{SL}_2 \times \mathrm{SL}_2$

$$H_{m,n} = \left\{\left(t_1^m t_2^n, \begin{pmatrix} t_1 & x \\ 0 & t_1^{-1} \end{pmatrix}, \begin{pmatrix} t_2 & y \\ 0 & t_2^{-1} \end{pmatrix}\right) \middle| t_1, t_2 \in k^*, x, y \in k\right\}$$

and m,n are integers with $m > 2$, $-2 > n$.
(J7) ($\mathrm{Aut}^0\mathrm{J}'_m$, J'_m) *where m is an integer* $m \geq 2$.
(J8) ($\mathrm{Aut}^0\mathrm{L}'_{m,n}$ $\mathrm{L}'_{m,n}$) *where m,n are integers with* $m \geq n \geq 1$.
(J9) ($\mathrm{Aut}^0\mathrm{F}'_{m,n}$, $\mathrm{F}'_{m,n}$) *where m,n are integers with* $m > n \geq 2$.
(J10) ($\mathrm{Aut}^0\mathrm{F}'_{m,m}$, $\mathrm{F}'_{m,m}$) *where m is an integer* ≥ 2.

(J11) $(\mathrm{Aut}^0\mathrm{E}_m^{\prime l}, \mathrm{E}_m^{\prime l})$ *where l,m are integers with* $m \geq 2$, $l \geq 2$ *or* $m = 1$, $1 \geq 3$.

(J12) *Generically intransitive operation* (PGL_2, X_π) *with general orbits isomorphic to* $(\mathrm{PGL}_2, \mathrm{PGL}_2/\mathrm{G}_m)$, *where* $\pi : C_1 \rightarrow C_2$ *is an étale 2-covering of a rational curve* C_2 *with genus* $(C_1) \geq 1$. *(These operations are effectively parametrized by the moduli space of nonsingular elliptic or hyperelliptic curves of genus* ≥ 1*).*

(II) *The (conjugacy classes of) algebraic subgroups of* Cr_3 *determined by the above operations* (P1), (P2), (E1), (E2), (J1), $\cdots$,(J12) *are maximal (conjugacy classes of) algebraic subgroups of* Cr_3.

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