IS THE AFFINE SPACE DETERMINED BY ITS AUTOMORPHISM GROUP?

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ABSTRACT. In this note we study the problem of characterizing the complex affine space \mathbb{A}^n via its automorphism group. We prove the following. Let X be an irreducible quasi-projective *n*-dimensional variety such that $\operatorname{Aut}(X)$ and $\operatorname{Aut}(\mathbb{A}^n)$ are isomorphic as abstract groups. If X is either quasi-affine and toric or X is smooth with Euler characteristic $\chi(X) \neq 0$ and finite Picard group $\operatorname{Pic}(X)$, then X is isomorphic to \mathbb{A}^n .

The main ingredient is the following result. Let X be a smooth irreducible quasi-projective variety of dimension n with finite $\operatorname{Pic}(X)$. If X admits a faithful $(\mathbb{Z}/p\mathbb{Z})^n$ -action for a prime p and $\chi(X)$ is not divisible by p, then the identity component of the centralizer $\operatorname{Cent}_{\operatorname{Aut}(X)}((\mathbb{Z}/p\mathbb{Z})^n)$ is a torus.

1. INTRODUCTION

In 1872, Felix Klein suggested as part of his Erlangen Programm to study geometrical objects through their symmetries. In the spirit of this program it is natural to ask to which extent a geometrical object is determined by its automorphism group. For example, a smooth manifold, a symplectic manifold or a contact manifold is determined by its automorphism group, see [Fil82, Ryb95, Ryb02].

We work over the field of complex numbers \mathbb{C} . For a variety X we denote by $\operatorname{Aut}(X)$ the group of regular automorphisms of X. As the automorphism group of a variety is usually quite small, it almost never determines the variety. However, if $\operatorname{Aut}(X)$ is large, like for the affine space \mathbb{A}^n , this might be true. Our guiding question is the following.

Question. Let X be a variety. Assume that Aut(X) is isomorphic to the group $Aut(\mathbb{A}^n)$. Does this imply that X is isomorphic to the affine space \mathbb{A}^n ?

This question cannot have a positive answer for all varieties X. For example, $\operatorname{Aut}(\mathbb{A}^n)$ and $\operatorname{Aut}(\mathbb{A}^n \times V)$ are isomorphic for any complete variety V with a trivial automorphism group. Similarly, the automorphism group of \mathbb{A}^n does not change if one forms the disjoint union with a variety with a trivial automorphism group. Thus we have to impose certain assumptions on X. Moreover, we assume that $n \geq 1$, since there exist many affine varieties with a trivial automorphism group.

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In [Kra17], the first author studies the problem of characterizing the affine space \mathbb{A}^n by its automorphism group regarded as a so-called ind-group. It is shown that if X is a connected affine variety such that $\operatorname{Aut}(X)$ and $\operatorname{Aut}(\mathbb{A}^n)$ are isomorphic as ind-groups, then X and \mathbb{A}^n are isomorphic as varieties. For some generalizations of this result we refer to [Reg17].

In dimension two, a generalization of our guiding question is studied in [LRU17]. For an irreducible normal affine surface X it is shown that if $\operatorname{Aut}(X)$ is isomorphic to $\operatorname{Aut}(Y)$ for an affine toric surface Y, then X is isomorphic to Y.

In order to state our main result, let us introduce the following notation. For a variety X we denote by $\chi(X)$ the Euler characteristic and by $\operatorname{Pic}(X)$ the Picard group.

Main Theorem. Let X be an irreducible quasi-projective variety such that $\operatorname{Aut}(X) \simeq \operatorname{Aut}(\mathbb{A}^n)$. Then $X \simeq \mathbb{A}^n$ if one of the following conditions holds.

- (1) X is smooth, $\chi(X) \neq 0$, $\operatorname{Pic}(X)$ is finite, and $\dim X \leq n$;
- (2) X is toric, quasi-affine, and dim $X \ge n$.

As a direct application of this result we get that $\operatorname{Aut}(\mathbb{A}^n \setminus S)$ and $\operatorname{Aut}(\mathbb{A}^n)$ are non-isomorphic as abstract groups for every non-empty closed subset Sin \mathbb{A}^n with Euler characteristic $\chi(S) \neq 1$.

Let us give an outline of the proof. Let θ : Aut $(\mathbb{A}^n) \xrightarrow{\sim}$ Aut(X) be an isomorphism. First, we prove that if a maximal torus of Aut (\mathbb{A}^n) is mapped onto an algebraic group via θ and X is quasi-affine, then $X \simeq \mathbb{A}^n$ (see Proposition 23). Our main result to achieve this condition is the following.

Theorem 1. Let X and Y be irreducible quasi-projective varieties such that $\dim Y \leq \dim X =: n$. Assume that the following conditions are satisfied:

- (1) X is quasi-affine and toric;
- (2) Y is smooth, $\chi(Y) \neq 0$, and $\operatorname{Pic}(Y)$ is finite.

If θ : Aut $(X) \xrightarrow{\sim}$ Aut(Y) is an isomorphism, then dim Y = n, and for each *n*-dimensional torus $T \subseteq$ Aut(X), the identity component of the image $\theta(T)^{\circ}$ is a closed torus of dimension *n* in Aut(Y). Furthermore, *Y* is quasi-affine.

For the definition of the topology on $\operatorname{Aut}(X)$, the definition of the identity component G° of a subgroup $G \subseteq \operatorname{Aut}(X)$ and the definition of algebraic subgroups of $\operatorname{Aut}(X)$ we refer to section 2.2.

For the proof of Theorem 1 we first remark that every torus $T \subseteq \operatorname{Aut}(X)$ of maximal dimension $n = \dim X$ is self-centralizing (Lemma 10). For any prime p the torus T contains a unique subgroup μ_p which is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^n$. In particular, $T \subseteq \operatorname{Cent}_{\operatorname{Aut}(X)}(\mu_p)$, and thus the image of Tunder an isomorphism θ : $\operatorname{Aut}(X) \to \operatorname{Aut}(Y)$ is mapped to a subgroup of the centralizer of $\theta(\mu_p)$ inside $\operatorname{Aut}(Y)$. Our strategy is then to prove that the identity component of the centralizer $\operatorname{Cent}_{\operatorname{Aut}(Y)}(\theta(\mu_p))$ is an algebraic group. Our main result in this direction is the following generalization of [KS13, Proposition 3.4].

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Theorem 2. Let X be a smooth, irreducible, quasi-projective variety of dimension n with finite Picard group $\operatorname{Pic}(X)$. Assume that X is endowed with a faithful $(\mathbb{Z}/p\mathbb{Z})^n$ -action for some prime p which does not divide $\chi(X)$. Then the centralizer $G := \operatorname{Cent}_{\operatorname{Aut}(X)}((\mathbb{Z}/p\mathbb{Z})^n)$ is a closed subgroup of $\operatorname{Aut}(X)$, and the identity component G° is a closed torus of dimension $\leq n$.

For the proof of Theorem 2 we use the fact that p does not divide $\chi(X)$ to find a fixed point of the $(\mathbb{Z}/p\mathbb{Z})^n$ -action on X (Proposition 16), and the smoothness of X to show that the fixed point variety $X^{(\mathbb{Z}/p\mathbb{Z})^n}$ is finite.

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2. Preliminary results

Throughout this note we work over the field \mathbb{C} of complex numbers. A variety will be a reduced separated scheme of finite type over \mathbb{C} .

2.1. Quasi-affine varieties. Let us recall some well-known results about quasi-affine varieties. The first lemma is known for affine varieties and can be reduced to this case by using open affine coverings.

Lemma 3. Let X, Y be varieties. Then the natural homomorphism

$$\mathcal{O}(X) \otimes_{\mathbb{C}} \mathcal{O}(Y) \to \mathcal{O}(X \times Y)$$

is an isomorphism of \mathbb{C} -algebras.

Lemma 4. Let X be a quasi-affine variety. Then the canonical morphism $\eta: X \to \operatorname{Spec} \mathcal{O}(X)$ is a dominant open immersion of schemes.

Proof. Let $f: \operatorname{Spec} \mathcal{O}(X) \to \mathbb{A}^1$ be a morphism which vanishes on $\eta(X)$. Since f can be understood as a regular function \tilde{f} on X, we get the following commutative diagram



which shows that $\tilde{f} = 0$. This implies that η is dominant.

We have an open immersion $\iota\colon X\to Y$ where Y is affine, and therefore a decomposition of ι

$$\iota\colon X \xrightarrow{\eta} \operatorname{Spec} \mathcal{O}(X) \xrightarrow{\alpha} Y.$$

In particular, η is injective. For any nonzero $f \in \mathcal{O}(Y)$ such that $Y_f \subseteq \iota(X)$ we see that ι induces an isomorphism $X_{\iota^*(f)} \xrightarrow{\sim} Y_f$, hence the composition

$$\iota' \colon X_{\alpha^*(f)} \xrightarrow{\eta'} \operatorname{Spec} \mathcal{O}(X)_{\alpha^*(f)} = \alpha^{-1}(Y_f) \xrightarrow{\alpha'} Y_f$$

is an isomorphism where η' and α' are the restrictions of η and α respectively. Since ι' is an isomorphism $X_{\alpha^*(f)}$ is affine and thus η' is an isomorphism, because $\mathcal{O}(X)_{\alpha^*(f)} = \mathcal{O}(X_{\alpha^*(f)})$. Therefore, η is a local isomorphism, hence an open immersion.

Lemma 5. Let X be a quasi-affine variety and Y a variety. Then every morphism $Y \times X \to X$ extends uniquely to a morphism $Y \times \operatorname{Spec} \mathcal{O}(X) \to$ $\operatorname{Spec} \mathcal{O}(X)$ via $X \to \operatorname{Spec} \mathcal{O}(X)$. In particular, every regular action of an algebraic group on X extends to a regular action on $\operatorname{Spec} \mathcal{O}(X)$.

Proof. We can assume that Y is affine. By Lemma 3 we have $\mathcal{O}(Y \times X) = \mathcal{O}(Y) \otimes_{\mathbb{C}} \mathcal{O}(X)$. Hence $Y \times X \to X$ induces a homomorphism of \mathbb{C} -algebras $\mathcal{O}(X) \to \mathcal{O}(Y) \otimes_{\mathbb{C}} \mathcal{O}(X)$ which in turn gives the desired extension $Y \times \operatorname{Spec} \mathcal{O}(X) \to \operatorname{Spec} \mathcal{O}(X)$.

2.2. Algebraic structure on the group of automorphisms. In this subsection, we recall some basic results about the automorphism group $\operatorname{Aut}(X)$ of a variety X. The survey [Bla16] and the article [Ram64] will serve as references. Recall that a *morphism* $\nu: A \to \operatorname{Aut}(X)$ is a map from a variety A to $\operatorname{Aut}(X)$ such that the associated map

$$\tilde{\nu} \colon A \times X \to X, \quad (a, x) \mapsto \nu(a)(x)$$

is a morphism of varieties. We get a topology on $\operatorname{Aut}(X)$ by declaring a subset $F \subset \operatorname{Aut}(X)$ to be *closed*, if the preimage $\nu^{-1}(F)$ under every morphism $\nu: A \to \operatorname{Aut}(X)$ is closed in A. Similarly, a morphism $\nu = (\nu_1, \nu_2): A \to \operatorname{Aut}(X) \times \operatorname{Aut}(X)$ is a map from a variety A into $\operatorname{Aut}(X) \times \operatorname{Aut}(X)$ such that ν_1 and ν_2 are morphisms. Thus we get analogously as before a topology on $\operatorname{Aut}(X) \times \operatorname{Aut}(X)$. Note that for morphisms $\nu, \nu_1, \nu_2: A \to \operatorname{Aut}(X)$ the following maps are again morphisms

$$A \to \operatorname{Aut}(X), a \mapsto \nu_1(a) \circ \nu_2(a)$$

 $A \to \operatorname{Aut}(X), a \mapsto \nu(a)^{-1}$

and that $\nu^{-1}(\Delta)$ is closed in A where $\Delta \subset \operatorname{Aut}(X) \times \operatorname{Aut}(X)$ denotes the diagonal. From these properties, one can deduce that $\operatorname{Aut}(X)$ behaves a bit like an algebraic group:

Lemma 6. For any variety X, the maps

$$\operatorname{Aut}(X) \times \operatorname{Aut}(X) \to \operatorname{Aut}(X), \ (\varphi_1, \varphi_2) \mapsto \varphi_1 \circ \varphi_2$$
$$\operatorname{Aut}(X) \to \operatorname{Aut}(X), \ \varphi \mapsto \varphi^{-1}$$

are continuous and the diagonal Δ is closed in $\operatorname{Aut}(X) \times \operatorname{Aut}(X)$.

Example 1. For any set $S \subseteq Aut(X)$ the centralizer Cent(S) is a closed subgroup of Aut(X). This is a consequence of Lemma 6.

For a subset $S \subseteq \operatorname{Aut}(X)$ its dimension is defined by

$$\dim S := \sup \left\{ d \mid \text{ there exists a variety } A \text{ of dimension } d \text{ and an in-} \\ \text{jective morphism } \nu \colon A \to \operatorname{Aut}(X) \text{ with image in } S \right\}.$$

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The following lemma generalizes the classical dimension estimate to morphisms $A \to \operatorname{Aut}(X)$.

Lemma 7. If $\nu: A \to \operatorname{Aut}(X)$ is a morphism, then $\dim \nu(A) \leq \dim A$.

Proof. Let $\eta: B \to \operatorname{Aut}(X)$ be an injective morphism such that $\eta(B) \subseteq \nu(A)$. The statement follows if we prove dim $B \leq \dim A$. Since η is injective, there exist points $x_1, \ldots, x_k \in X$ such that the map

$$\eta' \colon B \to \underbrace{X \times \cdots \times X}_{n\text{-times}}, \quad b \mapsto (\eta(b)(x_1), \dots, \eta(b)(x_k))$$

is injective, see e.g. [Ram64, Lemma 1]. Let

$$\nu' \colon A \to \underbrace{X \times \cdots \times X}_{n\text{-times}}, \quad a \mapsto (\nu(a)(x_1), \dots, \nu(a)(x_k)).$$

Since $\eta(B) \subseteq \nu(A)$, we get $\eta'(B) \subseteq \nu'(A)$ and thus dim $B = \dim \overline{\eta'(B)} \leq \dim \overline{\nu'(A)} \leq \dim A$.

For a subgroup $G \subseteq \operatorname{Aut}(X)$, the *identity component* $G^{\circ} \subseteq G$ is defined by

$$G^{\circ} = \left\{ \begin{array}{c|c} g \in G \end{array} \middle| \begin{array}{c} \text{there exists an irreducible variety } A \text{ and a morphism} \\ \nu \colon A \to \operatorname{Aut}(X) \text{ with image in } G \text{ such that } g, e \in \nu(A) \end{array} \right\} \,.$$

We call a subgroup $G \subseteq \operatorname{Aut}(X)$ connected if $G = G^{\circ}$. In the next proposition, we list several properties of the identity component of a subgroup of $\operatorname{Aut}(X)$.

Proposition 8. Let X be a variety and let $G \subseteq Aut(X)$ be a subgroup. Then the following holds:

- (1) G° is a normal subgroup of G;
- (2) The cosets of G° in G are the equivalence classes under:

 $g_1 \sim g_2 :\iff \begin{cases} \text{there exists an irreducible variety } A \\ \text{and a morphism } \nu \colon A \to \operatorname{Aut}(X) \\ \text{with image in } G \text{ such that } g_1, g_2 \in \nu(A) \,; \end{cases}$

- (3) For each morphism ν: A → Aut(X) with image in G, the preimage ν⁻¹(G°) is closed in A. In particular, if G is closed in Aut(X), then G° is closed in Aut(X);
- (4) If X is quasi-projective and G is closed in Aut(X), then the index of G° in G is countable.

Proof. (1): One can directly see, that G° is a normal subgroup of G.

(2): We first show that "~" defines an equivalence relation on G. Reflexivity and symmetry are obvious. For proving the transitivity, let $g \sim h$ and $h \sim k$. By definition there exist irreducible varieties A, B, morphisms $\nu: A \to \operatorname{Aut}(X), \eta: B \to \operatorname{Aut}(X)$ with image in G and $a_1, a_2 \in A, b_1, b_2 \in B$ such that $\nu(a_1) = g, \nu(a_2) = h, \eta(b_1) = h, \eta(b_2) = k$. Then

$$A \times B \to \operatorname{Aut}(X), \quad (a,b) \mapsto \nu(a) \circ h^{-1} \circ \eta(b)$$

is a morphism with image in G that maps (a_1, b_1) onto g and (a_2, b_2) onto k. Thus $g \sim k$, which proves the transitivity. In particular, G° is the equivalence class with respect to \sim which contains the neutral element e. This implies the statement

(3): Let

$$\bigcup_{i=1}^k B_i = \overline{\nu^{-1}(G^\circ)} \subseteq A$$

be the decomposition of the closure of $\nu^{-1}(G^{\circ})$ into irreducible components B_1, \ldots, B_k . Thus $B_i \cap \nu^{-1}(G^{\circ})$ is non-empty. Since ν has image in G it follows from the transitivity of "~" that $\nu(B_i) \subseteq G^{\circ}$. Thus $B_i \subseteq \nu^{-1}(G^{\circ})$ for all *i*. Hence $\nu^{-1}(G^{\circ})$ is closed in A.

(4): Let $\nu: A \to \operatorname{Aut}(X)$ be a morphism. Since $\nu^{-1}(G) \subseteq A$ is closed, it has only finitely many irreducible components. This implies that its image $\nu(A)$ meets only finitely many cosets of G° in G. The claim follows if we show that there exist countably many morphisms of varieties into $\operatorname{Aut}(X)$ whose images cover $\operatorname{Aut}(X)$. We will show this.

Since X is quasi-projective, there exists a projective variety \overline{X} and an open embedding $X \subseteq \overline{X}$. For each polynomial $p \in \mathbb{Q}[x]$ we denote by Hilb^p the Hilbert scheme of $\overline{X} \times \overline{X}$ associated to the Hilbert polynomial p, and denote by $\mathcal{U}^p \subseteq \operatorname{Hilb}^p \times \overline{X} \times \overline{X}$ the universal family, which is by definition flat over Hilb^p . By [Gro95, Theorem 3.2], Hilb^p is a projective scheme over \mathbb{C} . For i = 1, 2 consider the following morphisms

$$q_i: (\operatorname{Hilb}^p \times X \times X) \cap \mathcal{U}^p \to \operatorname{Hilb}^p \times X, \quad (h, x_1, x_2) \mapsto (h, x_i)$$

which are defined over Hilb^{*p*}. By [Gro66, Proposition 9.6.1], the points $h \in H$ where the restriction

$$q_i|_{\{h\}} \colon (\{h\} \times X \times X) \cap \mathcal{U}^p \to \{h\} \times X$$

is an isomorphism, form a constructible subset S^p of Hilb^{*p*}. Now choose locally closed subsets S_j^p , $j = 1, \ldots, k_p$ of Hilb^{*p*} that cover S^p . We equip each S_j^p with the underlying reduced scheme structure of Hilb^{*p*}. Note that (Hilb^{*p*} × X × X) $\cap \mathcal{U}^p$ and Hilb^{*p*} × X are both flat over Hilb^{*p*}. Therefore, we can apply [Gro71, Proposition 5.7] and we get that q_i restricts to an isomorphism over S_j^p . Thus for each *j* we get a morphism of varieties

$$S_j^p \times X \xrightarrow{(q_1|_{S_j^p})^{-1}} (S_j^p \times X \times X) \cap \mathcal{U}^p \xrightarrow{q_2|_{S_j^p}} S_j^p \times X \longrightarrow X$$

which defines a morphism $S_j^p \to \operatorname{Aut}(X)$. For each automorphism φ in $\operatorname{Aut}(X)$, the closure in $\overline{X} \times \overline{X}$ of the graph $\Gamma_{\varphi} \subseteq X \times X$ defines a (closed) point in the Hilbert scheme Hilb^p for a certain rational polynomial p, which belongs to S^p . Thus the images of the morphisms $S_j^p \to \operatorname{Aut}(X)$ cover $\operatorname{Aut}(X)$. Since there are only countably many rational polynomials, the claim follows. \Box

We say that G is an algebraic subgroup of $\operatorname{Aut}(X)$ if there exists a morphism $\nu: H \to \operatorname{Aut}(X)$ of an algebraic group H onto G which is a homomorphism of groups.

The next result gives a criterion for a subgroup of Aut(X) to be algebraic. The main argument is due to Ramanujam [Ram64].

Theorem 9. Let X be an irreducible variety and let $G \subseteq Aut(X)$ be a subgroup. Then the following statements are equivalent:

- (1) G is an algebraic subgroup of Aut(X);
- (2) there exists a morphism of a variety into Aut(X) with image G;
- (3) dim G is finite and G° has finite index in G;
- (4) there is a unique structure of an algebraic group on G such that for each irreducible variety A we have a bijective correspondence

$$\left\{\begin{array}{c} morphisms \ of\\ varieties \ A \to G \end{array}\right\} \xleftarrow{1:1} \left\{\begin{array}{c} morphisms \ A \to \operatorname{Aut}(X)\\ with \ image \ in \ G \end{array}\right.$$

given by $f \mapsto \iota \circ f$ where $\iota \colon G \to \operatorname{Aut}(X)$ denotes the canonical inclusion.

Proof. The implication $(1) \Rightarrow (2)$ is obvious.

Assume that (2) holds, i.e. there is a morphism $\eta: A \to \operatorname{Aut}(X)$ with image equal to G. By Lemma 7 we get dim $G \leq \dim A$ and hence dim Gis finite. Since A is a variety and thus has only finitely many irreducible components, it follows from Proposition 8 (2) that G° has finite index in G. This proves (2) \Rightarrow (3).

The implication (3) \Rightarrow (4) is done in [Ram64, Theorem p.26] in case $G = G^{\circ}$ for irreducible A. Thus in the general case, G° carries the structure of an algebraic group with the required property. Since G° has finite index in G we obtain a unique structure of an algebraic group on G extending the given structure on G° . It remains to see that the required property holds for G. Note that the canonical inclusion $\iota: G \to \operatorname{Aut}(X)$ is a morphism and thus each morphism of varieties $A \to G$ yields a morphism $A \to \operatorname{Aut}(X)$ by composing with ι . For the reverse, let $\nu: A \to \operatorname{Aut}(X)$ be a morphism with image in G. Since A is irreducible, by Proposition 8 (2) there is $g \in G$ such that the image of ν lies in gG° . Note that the composition of ν with $\theta_{g^{-1}}: \operatorname{Aut}(X) \to \operatorname{Aut}(X), \varphi \mapsto g^{-1} \circ \varphi$ is a morphism with image in G° . Thus $\theta_{g^{-1}} \circ \nu$ corresponds to a morphism $A \to G^{\circ}$ of varieties. Using that $G \to G$, $h \mapsto gh$ is an isomorphism of varieties, we get that ν corresponds to a morphism $A \to G^{\circ}$ of varieties.

The implication $(4) \Rightarrow (1)$ is obvious.

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2.3. Ingredients from toric geometry. Recall that a toric variety is a normal irreducible variety X together with a regular faithful action of a torus of dimension dim X. For details concerning toric varieties we refer to [Ful93]. The first two lemmas are certainly well-known; for the convenience of the reader we present for both a short proof.

Lemma 10. Let X be a toric variety, and let T be a torus of dimension $\dim X$ that acts faithfully on X. Then the centralizer of T in $\operatorname{Aut}(X)$ is equal to T. In particular, the image of T in $\operatorname{Aut}(X)$ is closed.

Proof. Let $g \in \operatorname{Aut}(X)$ such that gt = tg for all $t \in T$. By definition, there is an open dense *T*-orbit in *X*, say *U*. Since $gU \cap U$ is non-empty, there exists $x \in U$ such that $gx \in U$. Using that U = Tx we find $t_0 \in T$ with $gx = t_0x$. Thus for each $t \in T$ we get

$$gtx = tgx = tt_0x = t_0tx.$$

Using that U = Tx is dense in X, we get $g = t_0$.

Lemma 11. Let X be a toric variety. Then the coordinate ring $\mathcal{O}(X)$ is finitely generated and integrally closed.

Proof. This follows from [Kno93]. Here is a short direct proof. Let N be the lattice of one-parameter groups corresponding to the torus that acts on X and let Σ be the fan in $N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$ that corresponds to X. Now, $\mathcal{O}(X)$ is generated as a \mathbb{C} -algebra by the intersection of the finitely generated semigroups $\sigma^{\vee} \cap M$, where $M = \operatorname{Hom}(N,\mathbb{Z})$ is the dual lattice to N, σ is a cone in Σ and σ^{\vee} denotes the dual cone of σ inside $M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q}$. As the intersection of the semi-groups $\sigma^{\vee} \cap M$ is the intersection of the convex rational polyhedral cone $\cap_{\sigma \in \Sigma} \sigma^{\vee}$ with M, it is a finitely generated semigroup (Gordon's Lemma, see e.g. [Ful93, Proposition 1 in §1.2]). This proves the first claim.

Since X is normal, every local ring $\mathcal{O}_{X,x}$ is integrally closed, and $\mathcal{O}(X) = \bigcap_{x \in X} \mathcal{O}_{X,x}$. Hence $\mathcal{O}(X)$ is also integrally closed. \Box

The next proposition is based on the study of homogeneous \mathbb{G}_a -actions on affine toric varieties in [Lie10]. Recall that a group action $\nu: G \to \operatorname{Aut}(X)$ on a toric variety is called *homogeneous* if the torus normalizes the image $\nu(G)$. Note that for any homogeneous \mathbb{G}_a -action ν there is a well-defined character $\chi: T \to \mathbb{G}_m$, defined by the formula

$$t \nu(s) t^{-1} = \nu(\chi(t) \cdot s)$$
 for $t \in T, s \in \mathbb{C}$.

Proposition 12. Let X be a n-dimensional quasi-affine toric variety. If X is not a torus, then there exist homogeneous \mathbb{G}_a -actions

$$\eta_1, \ldots, \eta_n \colon \mathbb{G}_a \times X \to X$$

such that the corresponding characters χ_1, \ldots, χ_n are linearly independent.

The proof needs some preparation. Denote by Y the spectrum of $\mathcal{O}(X)$. By Lemma 11, the variety Y is normal, and the faithful torus action on X extends uniquely to a faithful torus action on Y, by Lemma 5.

The following notation is taken from [Lie10]. Let N be a lattice of rank n, $M = \operatorname{Hom}(N, \mathbb{Z})$ be its dual lattice and $N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$, $M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q}$. Thus, we have a natural pairing $M_{\mathbb{Q}} \times N_{\mathbb{Q}} \to \mathbb{Q}$, $(m, n) \mapsto \langle m, n \rangle$. Let $\sigma \subset N_{\mathbb{Q}}$ be the strongly convex polyhedral cone that describes Y and let σ_M^{\vee} be the intersection of the dual cone σ^{\vee} in $M_{\mathbb{Q}}$ with M. Thus $Y = \operatorname{Spec} R$, where

$$R := \mathbb{C}[\sigma_M^{\vee}] = \bigoplus_{m \in \sigma_M^{\vee}} \mathbb{C}\chi^m \subseteq \mathbb{C}[M] \,.$$

For each extremal ray $\rho \subset \sigma$, denote by ρ^{\perp} the elements $u \in M_{\mathbb{Q}}$ with $\langle u, v \rangle = 0$ for all $v \in \rho$. Moreover, let $\tau_M = \rho^{\perp} \cap \sigma_M^{\vee}$ and let

$$S_{\rho} = \{ e \in M \mid e \notin \sigma_{M}^{\vee}, \ e + m \in \sigma_{M}^{\vee} \text{ for all } m \in \sigma_{M}^{\vee} \setminus \tau_{M} \}.$$

By [Lie10, Remark 2.5] we have $S_{\rho} \neq \emptyset$, and $e + m \in S_{\rho}$ for all $e \in S_{\rho}$ and all $m \in \tau_M$. Let us recall the description of the homogeneous locally nilpotent derivations on R.

Proposition 13 ([Lie10, Lemma 2.6, Theorem 2.7]). Let ρ be an extremal ray in σ and let $e \in S_{\rho}$. Then

$$\partial_{\rho,e} \colon R \to R, \quad \chi^m \mapsto \langle m, \rho \rangle \chi^{e+m}$$

is a homogeneous locally nilpotent derivation of degree e, and every homogeneous locally nilpotent derivation of R is a constant multiple of some $\partial_{\rho,e}$.

Proof of Proposition 12. Since X is not a torus, Y is also not a torus. Thus σ contains extremal rays, say ρ_1, \ldots, ρ_k and $k \geq 1$. Recall that associated to these extremal rays, there exist torus-invariant divisors $V(\rho_1), \ldots, V(\rho_k)$ in Y. Again, since X is not a torus, one of these divisors does intersect X. Let us assume that $\rho = \rho_1$ is an extremal ray such that $V(\rho) \cap X$ is non-empty. Then using the orbit-cone correspondence, one can see that $Y \setminus X$ is contained in the union $Z = \bigcup_{i=2}^k V(\rho_i)$, see [Ful93, §3.1]. Let $e \in S_\rho$ be fixed. We claim that the \mathbb{G}_a -action on Y associated to the locally nilpotent derivation $\partial_{\rho,e+m'}$ of Proposition 13 fixes Z for all $m' \in \tau_M \setminus \bigcup_{i>2} \rho_i^{\perp}$.

Let us fix $m' \in \tau_M$ with $\langle m', v \rangle > 0$ for all $v \in \bigcup_{i \geq 2} \rho_i$. Note that the fixed point set of the \mathbb{G}_a -action on Y corresponding to $\partial_{\rho,e+m'}$ is the zero set of the ideal generated by the image of $\partial_{\rho,e+m'}$. The divisor $V(\rho_i)$ is the zero set of the kernel of the canonical \mathbb{C} -algebra surjection

$$p_i \colon \mathbb{C}[\sigma_M^{\vee}] \to \mathbb{C}[\sigma_M^{\vee} \cap \rho_i^{\perp}], \quad \chi^m \mapsto \begin{cases} \chi^m, & \text{if } m \in \rho_i^{\perp} \\ 0, & \text{otherwise} \end{cases}$$

see [Ful93, §3.1]. Thus we have to prove that for all i = 2, ..., k the composition

$$\mathbb{C}[\sigma_M^{\vee}] \xrightarrow{\partial_{\rho,e+m'}} \mathbb{C}[\sigma_M^{\vee}] \xrightarrow{p_i} \mathbb{C}[\sigma_M^{\vee} \cap \rho_i^{\perp}]$$

is the zero map. Since, by definition, $\partial_{\rho,e+m'}$ vanishes on $\tau_M = \rho^{\perp} \cap \sigma_M^{\vee}$, we have only to show that for all $m \in \sigma_M^{\vee} \setminus \tau_M$ the following holds:

$$\langle e+m'+m,v\rangle > 0$$
 for all $v \in \rho_i, i = 2, \dots, k$.

This is satisfied, because $\langle m', v \rangle > 0$ and $\langle e + m, v \rangle \ge 0$ (note that $e \in S_{\rho}$ implies $e + m \in \sigma_M^{\vee}$). This proves the claim.

Since τ_M spans a hyperplane in M and $e \notin \tau_M$, we can choose $m'_1, \ldots, m'_n \in \tau_M \setminus \bigcup_{i\geq 2} \rho_i^{\perp}$ such that $e + m'_1, \ldots, e + m'_n$ are linearly independent in $M_{\mathbb{Q}}$. Hence, the homogeneous locally nilpotent derivations

$$\partial_{\rho,e+m'_i}, \quad i=1,\ldots,n$$

define \mathbb{G}_a -actions on Y that fix Z and thus restrict to \mathbb{G}_a -actions on X. Moreover, the character of $\partial_{\rho,e+m'_i}$ is $\chi_i = \chi^{e+m'_i}$. In particular, χ_1, \ldots, χ_n are linearly independent, finishing the proof of Proposition 12.

2.4. Some topological ingredients. For the convenience of the reader, we insert the following well-known statement.

Lemma 14. For a complex variety X, the rational singular (co)homology groups are finitely generated.

Proof. Using the universal coefficient theorem for cohomology, it is enough to prove this for the homology groups. If X is affine, then X has the homotopy type of a finite CW-complex (see [Kar79] or [HM97, Theorem 1.1]), and hence all homology groups are finitely generated. Since every variety can be covered by finitely many open affine subvarieties and since intersections of open affine subvarieties are again affine, the lemma follows from the Mayer-Vietoris exact sequence.

For a variety X, the Euler characteristic is defined by

$$\chi(X) = \sum_{i \ge 0} (-1)^i \dim_{\mathbb{Q}} H^i(X, \mathbb{Q}) \,,$$

where $H^i(X, \mathbb{Q})$ denotes the *i*-th singular cohomology group with rational coefficients. We will use the following properties of the Euler characteristic, see [KP85, Appendix].

Lemma 15. The Euler characteristic has the following properties.

- (1) If X is a variety and $Y \subseteq X$ a closed subvariety, then $\chi(X) = \chi(Y) + \chi(X \setminus Y)$.
- (2) If $X \to Y$ is a fiber bundle which is locally trivial in the étale topology with fiber F, then $\chi(X) = \chi(Y)\chi(F)$.

2.5. Results on the fixed point variety. If G is a group, acting on a variety X, then we denote by X^G the fixed point variety of X by G.

The first result gives us a criterion for the existence of fixed points for a p-group action.

Proposition 16. Let G be a finite p-group for a prime p and let X be a quasi-projective G-variety. If p does not divide the Euler characteristic $\chi(X)$, then the fixed point variety X^G is non-empty.

Proof. Assume that X^G is empty, i.e. every *G*-orbit has cardinality p^k for some k > 0. We prove by induction on the dimension of X that p divides $\chi(X)$. Since X is quasi-projective, the same is true for the smooth locus X^{sm}

and thus the geometric quotient $\pi: X^{\mathrm{sm}} \to X^{\mathrm{sm}}/G$ exists, see [ByB02, Theorem 4.3.1]. By generic smoothness [Har77, Corollary 10.7, Chp. III] there exists an open dense subset U in X/G such that $q := \pi|_{\pi^{-1}(U)}: \pi^{-1}(U) \to U$ is étale. Since $\pi: X \to X/G$ is finite, q is also finite and thus q is an étale locally trivial fiber bundle, see [Gro71, Corollaire 5.3]. Since each fiber of qis a G-orbit, it follows by Lemma 15 that the Euler characteristic of $\pi^{-1}(U)$ is divisible by p. By assumption, G acts without fixed point on $X \setminus \pi^{-1}(U)$ and thus by induction hypothesis, $\chi(X \setminus \pi^{-1}(U))$ is divisible by p. Using

$$\chi(X) = \chi(\pi^{-1}(U)) + \chi(X \setminus \pi^{-1}(U))$$

(see Lemma 15) we get that p divides $\chi(X)$.

The second result is a consequence of a theorem of Fogarty [Fog73].

Proposition 17. Let G be a reductive group acting on a variety X. Assume that X is smooth at some point $x \in X^G$. Then X^G is smooth at x and the tangent space satisfies $T_x(X^G) = (T_x X)^G$.

Proof. Let us denote by $X^{(G)} \subseteq X$ the largest closed subscheme which is fixed under G, see [Fog73, §2]) for details. It then follows that $X^G = (X^{(G)})_{red}$. For $x \in X$ we denote by $C_x X$ the tangent cone in x, i.e. $C_x X =$ Spec gr $\mathcal{O}_{X,x}$ where gr $\mathcal{O}_{X,x} := \bigoplus_{i\geq 0} \mathfrak{m}_x^i/\mathfrak{m}_x^{i+1}$ is the associated graded algebra with respect to the maximal ideal $\mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$. By definition, there is a closed immersion $\mu_x : C_x X \hookrightarrow T_x X$, and X is smooth at x if and only if μ_x is an isomorphism. If $x \in X^G$ is a fixed point we obtain the following commutative diagram of schemes where all morphisms are closed immersions

$$C_x(X^{(G)}) \xrightarrow{\subseteq} T_x(X^{(G)})$$
$$\subseteq \downarrow \qquad \subseteq \downarrow$$
$$(C_xX)^{(G)} \xrightarrow{\subseteq} (T_xX)^{(G)}.$$

It is shown in [Fog73, Theorem 5.2] that for a reductive group G we have $C_x(X^{(G)}) = (C_x X)^{(G)}$. If X is smooth at x we get $(C_x X)^{(G)} = (T_x X)^{(G)}$. Hence all morphisms in the diagram above are isomorphisms. In particular $X^{(G)}$ is smooth at x and thus $X^G = (X^{(G)})_{red}$ is smooth at x. Moreover, we get $T_x(X^G) = (T_x X)^G$.

Remark 2. Assume that $(\mathbb{Z}/p\mathbb{Z})^n$ acts faithfully on a smooth quasi-projective variety X. If p does not divide $\chi(X)$, then dim $X \ge n$.

In fact, by Proposition 16 there is a fixed point $x \in X$, and the action of $(\mathbb{Z}/p\mathbb{Z})^n$ on the tangent space T_xX is faithful [KS13, Lemma 2.2], hence $n \leq \dim T_xX = \dim X$.

3. Proof of Theorem 1 and Theorem 2

Let us introduce the following terminology for this section. Let X be a variety and let $M \subseteq \operatorname{Aut}(X)$ be a subset. A map $\eta: M \to Z$ into a variety Z is called *regular* if for every morphism $\nu: A \to \operatorname{Aut}(X)$ with image in M, the composition $\eta \circ \nu: A \to Z$ is a morphism of varieties.

3.1. Semi-invariant functions.

Lemma 18. Let X be an irreducible normal variety, and let $f \in \mathcal{O}(X)$ be a non-constant function such that the zero set $Z := \mathcal{V}_X(f) \subset X$ is an irreducible hypersurface. Let $G \subseteq \operatorname{Aut}(X)$ be a connected subgroup which stabilizes Z. Then we have the following.

- (1) The function f is a G-semi-invariant with character $\chi: G \to \mathbb{C}^*$, i.e. $f(gx) = \chi(g)^{-1} \cdot f(x)$ for all $x \in X$ and $g \in G$.
- (2) The character $\chi \colon G \to \mathbb{C}^*$ is a regular map.

For the proof we need the following description of the invertible functions on a product variety due to Rosenlicht [Ros61, Theorem 2]. We denote for any variety X the group of invertible functions on X by $\mathcal{O}(X)^*$.

Lemma 19. Let X_1 , X_2 be irreducible varieties. Then $\mathcal{O}(X_1 \times X_2)^* = \mathcal{O}(X_1)^* \cdot \mathcal{O}(X_2)^*$.

Proof of Lemma 18. (1) Since X is normal, the local ring $R = \mathcal{O}_{X,Z}$ is a discrete valuation ring. Let \mathfrak{m} be the maximal ideal of R. By assumption, $fR = \mathfrak{m}^k$ for some k > 0. Since \mathfrak{m} is stable under G, the same is true for \mathfrak{m}^k . Hence, for every $g \in G$, there exists a unit $r_g \in R^*$ such that $gf = r_g \cdot f$ in R. Since f and gf have no zeroes in $X \setminus Z$, it follows that r_g is regular and nonzero in $X \setminus Z$. Moreover, the open set where $r_g \in R$ is defined and nonzero meets Z, hence r_g is a regular invertible function on X. Consider the map

$$\chi \colon G \to \mathcal{O}(X)^*, \ g \mapsto r_g.$$

For all $x \in X \setminus Z$, $g \in G$ we get $f(gx) = \chi(g)(x)^{-1}f(x)$, and f(gx), f(x) are both nonzero. Since for each morphism $\nu : A \to \operatorname{Aut}(X)$ with image in G, the map $\tilde{\nu} : A \times X \to X$, $(a, x) \mapsto \nu(a)(x)$ is a morphism, we see that

$$A \times (X \setminus Z) \to \mathbb{C}^*, \ (a, x) \mapsto \chi(\nu((a))(x) = f(x) \cdot f(\tilde{\nu}(a, x))^{-1}$$

is a morphism. If A is irreducible, then by Lemma 19 there exist invertible functions $q \in \mathcal{O}(A)^*$ and $p \in \mathcal{O}(X \setminus Z)^*$ such that $\chi(\nu((a))(x) = q(a)p(x)$. If, moreover, $\nu(a_0) = e \in G$ for some $a_0 \in A$, then

$$1 = r_e(x) = \chi(\nu(a_0))(x) = q(a_0)p(x) \quad \text{for all } x \in X \setminus Z \,,$$

i.e. p is a constant invertible function. In this case, the composition $\chi \circ \nu : A \mapsto \mathcal{O}(X)^*$ has image in \mathbb{C}^* . Since G is connected, this implies that the whole image of χ lies in \mathbb{C}^* .

(2) Choose $x_0 \in X \setminus Z$. As before, for each morphism $\nu \colon A \to \operatorname{Aut}(X)$ with image in G, the map

$$A \to \mathbb{C}^*, \quad a \mapsto \chi(\nu(a)) = f(x_0) \cdot f(\nu(a)(x_0))^{-1}$$

is also a morphism.

Lemma 20. Let X be an irreducible normal variety, and let $G \subseteq Aut(X)$ be a connected subgroup. Assume that $f_1, \ldots, f_n \in O(X)$ have the following properties.

(1) $Z_i := \mathcal{V}_X(f_i), i = 1, ..., n$, are irreducible *G*-invariant hypersurfaces; (2) $\bigcap_i Z_i$ contains an isolated point.

If $\chi_i: G \to \mathbb{C}^*$ is the character of f_i (see Lemma 18), then

 $\chi := (\chi_1, \dots, \chi_n) \colon G \to (\mathbb{C}^*)^n$

is a regular homomorphism with finite kernel.

Proof. Let G act on \mathbb{A}^n by

$$g(a_1,\ldots,a_n) := (\chi_1(g)^{-1} \cdot a_1,\ldots,\chi_n(g)^{-1} \cdot a_n).$$

Then the map $f := (f_1, \ldots, f_n) \colon X \to \mathbb{A}^n$ is *G*-equivariant. Let $Y \subseteq \mathbb{A}^n$ be the closure of f(X). By assumption, $f^{-1}(0) = \bigcap_i Z_i$ contains an isolated point, hence $f \colon X \to Y$ has finite degree, i.e. the field extension $\mathbb{C}(X) \supset \mathbb{C}(Y)$ is finite. This implies that the kernel K of $\chi \colon G \to (\mathbb{C}^*)^n$ is finite, because K embeds into $\operatorname{Aut}_{\mathbb{C}(Y)}(\mathbb{C}(X))$. By Lemma 18, χ is regular. \Box

3.2. Another centralizer result. For an irreducible normal variety X, we denote by $CH^1(X)$ the first Chow group, i.e. the free group of integral Weil divisors modulo linear equivalence (see [Har77, §6, Chp. II]).

Proposition 21. Let X be an irreducible normal variety of dimension n such that $\operatorname{CH}^1(X)$ is finite. Assume that for a prime p the group $(\mathbb{Z}/p\mathbb{Z})^n$ acts faithfully on X with a (not necessarily unique) fixed point x_0 which is a smooth point of X. Then $G := \operatorname{Cent}_{\operatorname{Aut}(X)}((\mathbb{Z}/p\mathbb{Z})^n)$ is a closed subgroup of $\operatorname{Aut}(X)$ and the identity component G° is a closed torus of dimension $\leq n$.

Proof. By [KS13, Lemma 2.2] we get a faithful representation of $(\mathbb{Z}/p\mathbb{Z})^n$ on $T_{x_0}X$, and thus we can find generators $\sigma_1, \ldots, \sigma_n$ such that $(T_{x_0}X)^{\sigma_i} \subset T_{x_0}X$ is a hyperplane for each i, and that $(T_{x_0}X)^{(\mathbb{Z}/p\mathbb{Z})^n} = 0$. By Proposition 17, the hypersurface $X^{\sigma_i} \subset X$ is smooth at x_0 , with tangent space $T_{x_0}(X^{\sigma_i}) = (T_{x_0}X)^{\sigma_i}$. Hence there is a unique irreducible hypersurface $Z_i \subseteq X$ that is contained in X^{σ_i} and contains x_0 ; thus Z_i is G° -stable. Moreover, since $(T_{x_0}X)^{(\mathbb{Z}/p\mathbb{Z})^n} = 0$, it follows that x_0 is an isolated point of $\bigcap_i Z_i$. Since a multiple of Z_i is zero in $\mathrm{CH}^1(X)$, there exist G° -semi-invariant functions $f_i \in \mathcal{O}(X)$ such that $\mathcal{V}_X(f_i) = Z_i$ (Lemma 18), and the corresponding characters χ_i define a regular homomorphism

$$\chi = (\chi_1, \dots, \chi_n) \colon G^{\circ} \to (\mathbb{C}^*)^n$$

with a finite kernel (Lemma 20). It follows that $\dim G^{\circ} \leq n$. Indeed, if $\nu: A \to \operatorname{Aut}(X)$ is an injective morphism with image in G° , then $\chi \circ \nu: A \to (\mathbb{C}^*)^n$ is a morphism with finite fibers, and so $\dim A \leq n$. This implies, by Theorem 9, that $G^{\circ} \subseteq \operatorname{Aut}(X)$ is an algebraic subgroup and that χ is a homomorphism of algebraic groups with a finite kernel. Hence G° is a torus. Since G is closed in $\operatorname{Aut}(X)$ the same holds for G° , see Proposition 8. \Box

3.3. **Proof of Theorem 2.** Now we can prove Theorem 2 which has the same conclusion as the proposition above, but under different assumptions. We have to show that the assumptions of Proposition 21 are satisfied. Since X is smooth, it follows that $\operatorname{CH}^1(X) \simeq \operatorname{Pic}(X)$ is finite. Proposition 16 implies that the fixed point variety $X^{(\mathbb{Z}/p\mathbb{Z})^n} \subseteq X$ is non-empty. Now the claims follow from Proposition 21.

3.4. Images of maximal tori under group isomorphisms.

Proposition 22. Let X and Y be irreducible quasi-projective varieties such that dim $Y \leq \dim X =: n$. Assume that the following conditions are satisfied:

- (1) X is quasi-affine and toric;
- (2) Y is smooth, $\chi(Y) \neq 0$, and $\operatorname{Pic}(Y)$ is finite.

If θ : Aut $(X) \xrightarrow{\sim}$ Aut(Y) is an isomorphism, then dim Y = n, and $\theta(T)^{\circ}$ is a closed torus of dimension n in Aut(Y) for each n-dimensional torus $T \subseteq \text{Aut}(X)$.

Proof. Let θ : Aut $(X) \to$ Aut(Y) be an isomorphism. Since $\chi(Y) \neq 0$ it follows that there is a prime p that does not divide $\chi(Y)$.

Let $n = \dim X$ and denote by $T \subset \operatorname{Aut}(X)$ a torus of dimension n. We have that $T = \operatorname{Cent}_{\operatorname{Aut}(X)}(T)$ (Lemma 10), and thus $\theta(T)$ is a closed subgroup of $\operatorname{Aut}(Y)$. Let $\mu_p \subset T$ be the subgroup generated by the elements of order p, and let $G := \operatorname{Cent}_{\operatorname{Aut}(Y)}(\theta(\mu_p))$ which is closed in $\operatorname{Aut}(X)$. Note that $\theta(T) \subseteq G$ and that dim Y = n by Remark 2. Now Theorem 2 implies that G° is a closed torus of dimension $\leq n$ in $\operatorname{Aut}(Y)$, and by Proposition 8 and Theorem 9, $\theta(T)^{\circ}$ is a closed connected algebraic subgroup of G° .

In order to prove that $\dim \theta(T)^{\circ} \ge n$ we construct closed subgroups $\{1\} = T_0 \subset T_1 \subset T_2 \subset \cdots \subset T_n = T$ with the following properties:

- (i) dim $T_i = i$ for all i;
- (*ii*) $\theta(T_i)$ is closed in $\theta(T)$ for all *i*.

It then follows that $\theta(T_i)^{\circ}$ is a connected algebraic subgroup of $\theta(T)^{\circ}$. Since the index of $\theta(T_i)^{\circ}$ in $\theta(T_i)$ is countable (Proposition 8), but the index of T_i in T_{i+1} is not countable, we see that $\dim \theta(T_{i+1})^{\circ} > \dim \theta(T_i)^{\circ}$, and so $\dim \theta(T)^{\circ} \ge n$.

(a) Assume first that X is a torus. Then $\operatorname{Aut}(X)$ contains a copy of the symmetric groups S_n , and we can find cyclic permutations $\tau_i \in \operatorname{Aut}(X)$ such that $T_i := \operatorname{Cent}_T(\tau_i)$ is a closed subtorus of dimension *i*, and $T_i \subset T_{i+1}$ for all 0 < i < n. It then follows that $\theta(T_i) = \operatorname{Cent}_{\theta(T)}(\theta(\tau_i))$ is closed in $\theta(T)$, and we are done.

(b) Now assume that X is not a torus. By Proposition 12 there exist one-dimensional unipotent subgroups U_1, \ldots, U_n of $\operatorname{Aut}(X)$ normalized by T such that the corresponding characters $\chi_1, \ldots, \chi_n \colon T \to \mathbb{C}^*$ are linearly independent. Since

$$\ker(\chi_i) = \{t \in T \mid t \circ u_i \circ t^{-1} = u_i \text{ for all } u_i \in U_i\} = \operatorname{Cent}_T(U_i)$$

it follows that

$$T_i := \bigcap_{k=1}^{n-i} \ker(\chi_k) = \operatorname{Cent}_T(U_1 \cup \dots \cup U_{n-i}) \subseteq T$$

is a closed algebraic subgroup of T of dimension i. It follows that the image $\theta(T_i) = \operatorname{Cent}_{\theta(T)}(\theta(U_1) \cup \cdots \cup \theta(U_n))$ is closed in $\theta(T)$, and the claim follows also in this case.

3.5. Proof of Theorem 1. Using Proposition 22, it is enough to show that a smooth toric variety Y with finite (and hence trivial) Picard group is quasi-affine.

For proving this, let $\Sigma \subset N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$ be the fan that describes Y where N is a lattice of rank n. Let $N' \subseteq N$ be the sublattice spanned by $\Sigma \cap N$ and let Y' be the toric variety corresponding to the fan Σ in $N'_{\mathbb{Q}} = N' \otimes_{\mathbb{Z}} \mathbb{Q}$. It follows from [Ful93, p. 29] that

$$Y \simeq Y' \times (\mathbb{C}^*)^k$$

where $k = \operatorname{rank} N/N'$. Thus Y' is a smooth toric variety with trivial Picard group. Hence it is enough to prove that Y' is quasi-affine and therefore we can assume k = 0, i.e. Σ spans $N_{\mathbb{Q}}$. By [Ful93, Proposition in §3.4] we get

$$0 = \operatorname{rank}\operatorname{Pic}(Y) = d - n$$

where d is the number of edges in Σ . Let $\sigma \subset N_{\mathbb{Q}}$ be the convex cone spanned by the edges of Σ and let σ^{\vee} denote the dual cone of σ in $M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q}$ where $M = \operatorname{Hom}(N, \mathbb{Z})$. Since d = n, the edges of Σ are linearly independent in $\mathbb{N}_{\mathbb{Q}}$ and thus σ is a simplex. From the inclusion of the cones of Σ in σ we get a morphism $f: Y \to \operatorname{Spec} \mathbb{C}[\sigma^{\vee} \cap M]$ by [Ful93, §1.4], and since each cone in Σ is a face of σ it is locally an open immersion (see [Ful93, Lemma in §1.3]). This implies that f is quasi-finite and birational and thus by Zariski's Main Theorem [Gro61, Corollaire 4.4.9] it is an open immersion. \Box

4. Proof of the Main Theorem

4.1. A first characterization.

Proposition 23. Let X be an irreducible quasi-affine variety. If $\operatorname{Aut}(\mathbb{A}^n) \xrightarrow{\sim} \operatorname{Aut}(X)$ is an isomorphism that maps an n-dimensional torus in $\operatorname{Aut}(\mathbb{A}^n)$ to an algebraic subgroup, then $X \simeq \mathbb{A}^n$ as a variety.

Proof. Since all *n*-dimensional tori in $\operatorname{Aut}(\mathbb{A}^n)$ are conjugate (see [BB66]), all *n*-dimensional tori are sent to algebraic subgroups of $\operatorname{Aut}(X)$ via θ . The standard maximal torus T in $\operatorname{Aut}(\mathbb{A}^n)$ acts via conjugation on the subgroup of standard translations $\operatorname{Tr} \subset \operatorname{Aut}(\mathbb{A}^n)$ with a dense orbit $O \subset T$ and thus we get $\operatorname{Tr} = O \circ O$.

This implies that $S := \theta(T)$ acts on $U := \theta(Tr)$ via conjugation and we get $U = \theta(O) \circ \theta(O)$. Hence, for fixed $u_0 \in \theta(O) \subset U$ the morphism

$$S \times S \to \operatorname{Aut}(X), \quad (s_1, s_2) \mapsto s_1 \circ u_0 \circ s_1^{-1} \circ s_2 \circ u_0 \circ s_2^{-1}$$

has image equal to U. Now it follows from Theorem 9 that U is a closed (commutative) algebraic subgroup of Aut(X) with no nontrivial element of finite order, hence a unipotent subgroup.

We claim that U has no non-constant invariants on X. Indeed, let $\rho: \mathbb{G}_a \times X \to X$ be the \mathbb{G}_a -action on X coming from a nontrivial element of U. If $f \in \mathcal{O}(X)^U$ is a U-invariant, then it is easy to see that

(*)
$$\rho_f(s,x) := \rho(f(x) \cdot s, x)$$

is a \mathbb{G}_a -action commuting with U. Since U is self-centralizing, we see that $\rho_f(s) \in U$ for all $s \in \mathbb{G}_a$. Moreover, formula (*) shows that for every finite dimensional subspace $V \subset \mathcal{O}(X)^U$ the map $V \to U$, $f \mapsto \rho_f(1)$, is a morphism which is injective. Indeed, $\rho_f(1) = \rho_h(1)$ implies that $\rho(f(x), x) = \rho(h(x), x)$ for all $x \in X$, hence f(x) = h(x) for all $x \in X \setminus X^{\rho}$. It follows that $\mathcal{O}(X)^U$ is finite-dimensional. Since X is irreducible, $\mathcal{O}(X)^U$ is an integral domain and hence equal to \mathbb{C} , as claimed.

Since X is irreducible and quasi-affine, the unipotent group U has a dense orbit which is closed, and so X is isomorphic to an affine space \mathbb{A}^m . Since m is the maximal number such that there exists a faithful action of $(\mathbb{Z}/2\mathbb{Z})^m$ on \mathbb{A}^m (see Remark 2), we finally get m = n.

If X is an affine variety, then X has the structure of a so-called affine indgroup, see e.g. [Kum02, Sta13, FK17] for more details. The following result is a special case of [Kra17, Theorem 1.1]. It is an immediate consequence of Proposition 23 above, because a homomorphism of affine ind-groups sends algebraic groups to algebraic groups.

Corollary 24. Let X be an irreducible affine variety. If there is an isomorphism $\operatorname{Aut}(X) \simeq \operatorname{Aut}(\mathbb{A}^n)$ of affine ind-groups, then $X \simeq \mathbb{A}^n$ as a variety.

Corollary 25. Let X be a smooth, irreducible quasi-projective variety such that $\chi(X) \neq 0$ and $\operatorname{Pic}(X)$ is finite. If there is an isomorphism $\operatorname{Aut}(\mathbb{A}^n) \simeq \operatorname{Aut}(X)$ of abstract groups and if dim $X \leq n$, then $X \simeq \mathbb{A}^n$ as a variety.

Proof. Theorem 1 shows that for an isomorphism θ : $\operatorname{Aut}(\mathbb{A}^n) \xrightarrow{\sim} \operatorname{Aut}(X)$ and any *n*-dimensional torus $T \subseteq \operatorname{Aut}(\mathbb{A}^n)$, the identity component of the image $S := \theta(T)^\circ$ is a closed torus of dimension *n* in $\operatorname{Aut}(X)$, dim X = n, and *X* is quasi-affine. Thus we can apply Theorem 1 to θ^{-1} : $\operatorname{Aut}(X) \xrightarrow{\sim} \operatorname{Aut}(\mathbb{A}^n)$ and get that $\theta^{-1}(S)^{\circ}$ is a closed torus of dimension n in Aut(\mathbb{A}^n). Since

$$\theta^{-1}(S)^{\circ} \subseteq \theta^{-1}(S) \subseteq T$$
,

it follows that $\theta^{-1}(S) = T$, i.e. $\theta(T) = S$ is a closed *n*-dimensional torus in Aut(X). The assumptions of Proposition 23 are now satisfied for the isomorphism θ : Aut(\mathbb{A}^n) $\xrightarrow{\sim}$ Aut(X), and the claim follows. \Box

4.2. Proof of the Main Theorem. If (1) holds, i.e. X is a smooth, irreducible, quasi-projective variety of dimension $\leq n$ such that $\chi(X) \neq 0$ and $\operatorname{Pic}(X)$ is finite, then the claim follows from Corollary 25.

Now assume that the assumptions (2) are satisfied, i.e., that X is quasiaffine and toric of dimension $\geq n$. Let $T \subseteq \operatorname{Aut}(X)$ be a torus of maximal dimension. We can apply Theorem 1 to an isomorphism θ : $\operatorname{Aut}(X) \xrightarrow{\sim}$ $\operatorname{Aut}(\mathbb{A}^n)$ and find that $S := \theta(T)^{\circ} \subset \operatorname{Aut}(\mathbb{A}^n)$ is a closed torus of dimension n. Since the index of the standard *n*-dimensional torus in its normalizer in $\operatorname{Aut}(\mathbb{A}^n)$ has finite index and since all *n*-dimensional tori in $\operatorname{Aut}(\mathbb{A}^n)$ are conjugate (see [BB66]), it follows that S has finite index in $\theta(T)$. Hence $\theta^{-1}(S)$ has finite index in T. Since T is a divisible group, $\theta^{-1}(S) = T$ is an algebraic group. Thus we can apply Proposition 23 to the isomorphism θ^{-1} : $\operatorname{Aut}(\mathbb{A}^n) \xrightarrow{\sim} \operatorname{Aut}(X)$ and find that $X \simeq \mathbb{A}^n$ as a variety. \Box

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