COMBINATORICS OF THE TAME AUTOMORPHISM GROUP

STÉPHANE LAMY

Abstract. We study the group $\mathrm{Tame}(\mathbb{A}^3)$ of tame automorphisms of the 3-dimensional affine space, over a field of characteristic zero. We recover, in a unified and (hopefully) simplified way, previous results of Kuroda, Shestakov, Umirbaev and Wright, about the theory of reduction and the relations in $\mathrm{Tame}(\mathbb{A}^3)$. The novelty in our presentation is the emphasis on a simply connected 2-dimensional simplicial complex on which $\mathrm{Tame}(\mathbb{A}^3)$ acts by isometries.

Contents

Introduction

Let $\mathbf{k}$ be a field, and let $\mathbb{A}^n = \mathbb{A}^n_{\mathbf{k}}$ be the affine space over $\mathbf{k}$. We are interested in the group $\mathrm{Aut}(\mathbb{A}^n)$ of algebraic automorphisms of the affine space. Concretely,

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we choose once and for all an origin and a coordinate system $(x_1, \dots, x_n)$ for $\mathbb{A}^n$. Then any element $f \in \mathrm{Aut}(\mathbb{A}^n)$ is a map of the form

$$f\colon (x_1, \dots, x_n) \mapsto (f_1(x_1, \dots, x_n), \dots, f_n(x_1, \dots, x_n)),$$

where the f_i are polynomials in n variables, such that there exists a map g of the same form satisfying $f \circ g = \mathrm{id}$. We shall abbreviate this situation by writing $f = (f_1, \dots, f_n)$, and $g = f^{-1}$.

The group $\mathrm{Aut}(\mathbb{A}^n)$ contains the following natural subgroups. First we have the affine group $A_n = \mathrm{GL}_n(\mathbf{k}) \ltimes \mathbf{k}^n$. Secondly we have the group E_n of elementary automorphisms, which have the form

$$f\colon (x_1, \dots, x_n) \mapsto (x_1 + P(x_2, \dots, x_n), x_2, \dots, x_n),$$

for some choice of polynomial P in $n-1$ variables. The subgroup

$$\mathrm{Tame}(\mathbb{A}^n) = \langle A_n, E_n \rangle$$

generated by the affine and elementary automorphisms is called the subgroup of **tame automorphisms**.

A natural question is whether the inclusion $\mathrm{Tame}(\mathbb{A}^n) \subseteq \mathrm{Aut}(\mathbb{A}^n)$ is in fact an equality. It is a well-known result, which goes back to Jung (see e.g. [Lam02] for a review of some of the many proofs available in the literature), that the answer is *yes* for $n = 2$ (over any base field), and it is a result by Shestakov and Umirbaev [SU04b] that the answer is *no* for $n = 3$, at least when $\mathbf{k}$ is a field of characteristic zero.

The main purpose of the present paper is to give a self-contained reworked proof of this last result: see Theorem 4.1 and Corollary 4.2. We follow closely the line of argument by Kuroda [Kur10]. However, the novelty in our approach is the emphasis on a 2-dimensional simplicial complex $\mathcal{C}$ on which $\mathrm{Tame}(\mathbb{A}^3)$ acts by isometries.

In the work of Kuroda [Kur10], as in the original work of Shestakov and Umirbaev [SU04b], elementary reductions are defined with respect to one of the three coordinates of a fixed coordinate system. In contrast, we always work up to an affine change of coordinates. Indeed, our simplicial complex $\mathcal{C}$ is designed so that two tame automorphisms correspond to two vertices at distance 2 in the complex if and only if they differ by the composition of an automorphism of the form aea^{-1}, where a is affine and e is elementary. This allows to absorb the so-called "type I" and "type II" reductions of Shestakov and Umirbaev in the class of elementary reductions: In our terminology they become "elementary K-reductions" (see §3.D). On the other hand, the "type III" reductions, which are technically difficult to handle, are still lurking around. One can suspect that such reductions don't exist (as the most intricate "type IV" reductions which were excluded by Kuroda [Kur10]), and an ideal proof would settle this issue. Unfortunately we were not able to do so, and these hypothetical reductions still appear in our text under the name of "normalized proper K-reduction". See Example 6.5 for more comments on this issue.

One could say that the theory of Shestakov, Umirbaev and Kuroda consists in understanding the relations inside the tame group $\mathrm{Tame}(\mathbb{A}^3)$. This was made explicit by Umirbaev [Umi06], and then it was noticed by Wright [Wri15] that this can be rephrased in terms of an amalgamated product structure over three subgroups. In turn, it is known that such a structure is equivalent to the action of the group on a 2-dimensional simply connected simplicial complex, with fundamental domain a simplex. Our approach allows to recover a more transparent description of the relations in $\mathrm{Tame}(\mathbb{A}^3)$: we directly show that the natural complex on which $\mathrm{Tame}(\mathbb{A}^3)$ acts is simply connected (see Proposition 5.6), by observing that the reduction process of [Kur10, SU04b] corresponds to local homotopies.

We should stress once more that this paper contains no original result, and consists only in a new presentation of previous works by the above cited authors. In fact, for the sake of completeness we also include in Section 2 some preliminary results where we only slightly differ from the original articles [Kur08, SU04a].

Our motivation for reworking this material is to prepare the way for new results about $\mathrm{Tame}(\mathbb{A}^3)$, such as the linearizability of finite subgroups, the Tits alternative or the non simplicity. From our experience in related settings (see [BFL14, CL13]), such results should follow from some non-positive curvature properties of the simplicial complex. We plan to explore these questions in a follow-up paper.

1. Simplicial complex

We construct a $(n-1)$-dimensional simplicial complex on which the tame automorphism group of $\mathbb{A}^n$ acts. This construction makes sense over any base field $\mathbf{k}$.

1.A. **General construction.** For any $1 \le r \le n$, we call r**-tuple of components** a morphism

$$f\colon \mathbb{A}^n \to \mathbb{A}^r$$
$$x = (x_1, \dots, x_n) \mapsto (f_1(x), \dots, f_r(x))$$

that can be extended as a tame automorphism $f = (f_1, \dots, f_n)$ of $\mathbb{A}^n$. One defines n distinct types of vertices, by considering r-tuple of components modulo composition by an affine automorphism on the range, $r = 1, \dots, n$:

$$[f_1, \dots, f_r] := A_r(f_1, \dots, f_r) = \{a \circ (f_1, \dots, f_r); a \in A_r\}$$

where $A_r = \mathrm{GL}_r(\mathbf{k}) \ltimes \mathbf{k}^r$ is the r-dimensional affine group. We say that $v_r = [f_1, \dots, f_r]$ is a vertex of **type** $\boldsymbol{r}$, and that $(f_1, \dots, f_r)$ is a representative of v_r. We shall always stick to the convention that the index corresponds to the type of a vertex: for instance v_r, v_r', u_r, w_r, m_r will all be possible notation for a vertex of type r.

Now for any tame automorphism $(f_1, \dots, f_n) \in \mathrm{Tame}(\mathbb{A}^n)$ we attach a $(n-1)$-simplex on the vertices $[f_1]$, $[f_1, f_2]$, $\dots$, $[f_1, \dots, f_n]$. This definition is independent of a choice of representatives and produces a $(n-1)$-dimensional simplicial complex $\mathcal{C}_n$ on which the tame group acts by isometries, by the formulas

$$g \cdot [f_1, \dots, f_r] := [f_1 \circ g^{-1}, \dots, f_r \circ g^{-1}].$$

Lemma 1.1. *The group* $\mathrm{Tame}(\mathbb{A}^n)$ *acts on* $\mathcal{C}_n$ *with fundamental domain the simplex*

$$[x_1],\ [x_1, x_2],\ \dots,\ [x_1, \dots, x_n].$$

In particular the action is transitive on vertices of a given type.

Proof. Let $v_1, \dots, v_n$ be the vertices of a simplex (recall that the index corresponds to the type). By definition there exists $f = (f_1, \dots f_n) \in \mathrm{Tame}(\mathbb{A}^n)$ such that $v_i = [f_1, \dots, f_i]$ for each i. Then

$$[x_1, \dots, x_i] = [(f_1, \dots, f_i) \circ f^{-1}] = f \cdot v_i. \qquad \square$$

Remark 1.2. (1) One could make a similar construction by working with the full automorphism group $\mathrm{Aut}(\mathbb{A}^n)$ instead of the tame group. The complex $\mathcal{C}_n$ we consider is the gallery connected component of the vertex [id] in this bigger complex. See [BFL14, §6.2.1] for more details.

(2) When $n = 2$, the previous construction yields a graph $\mathcal{C}_2$. It is not difficult to show (see [BFL14, §2.5.2]) that $\mathcal{C}_2$ is isomorphic to the classical Bass-Serre tree of $\mathrm{Aut}(\mathbb{A}^2) = \mathrm{Tame}(\mathbb{A}^2)$.

1.B. **Degrees.** We shall compare polynomials in $\mathbf{k}[x_1, \dots, x_n]$ by using the graded lexicographic order on monomials. We find it more convenient to work with an additive notation, so we introduce the **degree** function, with value in $\mathbb{N}^n \cup \{-\infty\}$, by taking

$$\deg x_1^{a_1} x_2^{a_2} \dots x_n^{a_n} = (a_1, a_2, \dots, a_n)$$

and by convention $\deg 0 = -\infty$. We extend this order to $\mathbb{Q}^n \cup \{-\infty\}$, since sometimes it is convenient to consider difference of degrees, or degrees multiplied by a rational number. The **top component** of $g \in \mathbf{k}[x_1, \dots, x_n]$ is the uniquely defined $\bar{g} = c x_1^{d_1} \dots x_n^{d_n}$ such that

$$(d_1, \dots, d_n) = \deg g > \deg(g - \bar{g}).$$

If $f = (f_1, \dots, f_r)$ is a r-tuple of components, we call **top degree** of f the maximum of the degree of the f_i:

$$\operatorname{topdeg} f = \max \deg f_i \in \mathbb{N}^n.$$

Lemma 1.3. *Let $f = (f_1, \dots, f_r)$ be a r-tuple of components, and consider $V \subset \mathbf{k}[x_1, \dots, x_n]$ the vector space generated by the f_i. Then*

(1) *The set H of elements $g \in V$ satisfying* $\operatorname{topdeg} f > \deg g$ *is a hyperplane in V;*
(2) *There exist a sequence of degrees $\delta_r > \cdots > \delta_1$ and a flag of subspaces $V_1 \subset \cdots \subset V_r = V$ such that* $\dim V_i = i$ *and* $\deg g = \delta_i$ *for any* $g \in V_i \smallsetminus V_{i-1}$.

Proof. (1) Up to permuting the f_i we can assume $\operatorname{topdeg} f = \deg f_r$. Then for each $i = 1, \dots, r-1$ there exists a unique $c_i \in \mathbf{k}$ such that $\deg f_i > \deg(f_i + c_i f_r)$. The conclusion follows from the observation that an element of V is in H if and only if it is a linear combination of the $f_i + c_i f_r$, $i = 1, \dots, r-1$.

(2) Immediate, by induction on dimension. □

Using the notation of the lemma, we call $r\text{-}\deg f = (\delta_1, \dots, \delta_r)$ the ***r*-degree** of f, and $\deg f = \sum_{i=1}^r \delta_i$ the **degree** of f. Observe that for any affine automorphism $a \in A_r$ we have $r\text{-}\deg f = r\text{-}\deg(a \circ f)$, so we get a well-defined notion of r-degree and degree for any vertices of type r.

If $v_r = [f_1, \dots, f_r] \in \mathcal{C}_n$ with the f_i without constant term and the $\deg f_i$ pairwise distinct, we say that f is a **good representative** of v_r (we do not ask $\deg f_r > \cdots > \deg f_2 > \deg f_1$). We use a double bracket notation such as $v_1 = [\![f_1]\!]$, $v_2 = [\![f_1, f_2]\!]$ or $v_3 = [\![f_1, f_2, f_3]\!]$, to indicate that we are using a good representative.

1.C. **The complex in dimension 3.** Now we specialize the general construction to the dimension $n = 3$, which is our main interest in this paper. We drop the index and simply denote by $\mathcal{C}$ the 2-dimensional simplicial complex associated to $\mathrm{Tame}(\mathbb{A}^3)$. To get a first feeling of the complex one can draw pictures such as Figure 1, where we use the following convention for vertices: A ∘, • or ▪ corresponds respectively to a vertex of type 1, 2 and 3. However one should keep in mind, as the following formal discussion makes it clear, that the complex is not locally finite. A first step in understanding the geometry of the complex $\mathcal{C}$ is to understand the link of each type of vertices. In fact, we will see now that if the base field $\mathbf{k}$ is uncountable, then the link of any vertex or any edge also is uncountable.

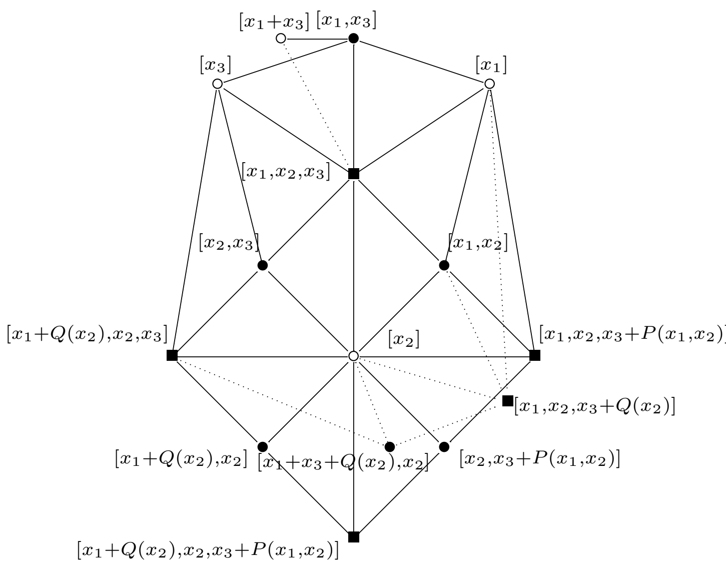


FIGURE 1. A few simplexes of the complex $\mathcal{C}$.

Consider first the link $\mathcal{L}(v_3)$ of a vertex of type 3. By transitivity of the action of $\mathrm{Tame}(\mathbb{A}^3)$, it is sufficient to describe the link $\mathcal{L}([\mathrm{id}])$. A vertex of type 1 at distance 1 from [id] has the form $[a_1x_1 + a_2x_2 + a_3x_3]$ where the a_i are uniquely defined up to a common multiplicative constant. In other words, vertices of type 1 in $\mathcal{L}([\mathrm{id}])$ are parametrized by $\mathbb{P}^2$. We denote by $\mathbb{P}^2(v_3)$ this projective plane of vertices of type 1 in the link of v_3. Similarly, vertices of type 2 in $\mathcal{L}(v_3)$ correspond to lines in $\mathbb{P}^2(v_3)$, and edges in $\mathcal{L}(v_3)$ correspond to incidence relations ("a point belongs to a line"). We denote by $\hat{\mathbb{P}}^2(v_3)$ the dual space of vertices of type 2. We will often refer to a vertex of type 2 as a "line in $\mathbb{P}^2(v_3)$". In the same vein, we will sometimes refer to a vertex of type 1 as being "the intersection of two lines in $\mathbb{P}^2(v_3)$", or we will express the fact that v_1 and v_2 are joined by an edge in $\mathcal{C}$ by saying "the line v_2 passes through v_1".

Now we turn to the description of the link of a vertex of type 2. We can assume $v_2 = [x_1, x_2]$, and one checks that vertices of type 1 in $\mathcal{L}(v_2)$ are parametrized by $\mathbb{P}^1$ and are of the form

$$[\alpha x_1 + \beta x_2],\ (\alpha : \beta) \in \mathbb{P}^1.$$

On the other hand vertices of type 3 in $\mathcal{L}(v_2)$ are of the form

$$[x_1, x_2, x_3 + P(x_1, x_2)],\ P \text{ without constant or linear part.}$$

Using the transitivity of the action of $\mathrm{Tame}(\mathbb{A}^3)$ on vertices of type 2, the following lemma and its corollary are then immediate:

Lemma 1.4. *The link $\mathcal{L}(v_2)$ of a vertex of type 2 is the complete bipartite graph between vertices of type 1 and 3 in the link.*

Corollary 1.5. *Let $v_2 = [f_1, f_2]$ and $v_3 = [f_1, f_2, f_3]$ be vertices of type 2 and 3. Then any vertex u_3 such that $v_2 \in \hat{\mathbb{P}}^2(u_3)$ has the form*

$$u_3 = [f_1, f_2, f_3 + P(f_1, f_2)]$$

where P is a polynomial in two variables.

In the situation of Corollary 1.5, we say that v_3 and u_3 are **neighbors**, with center v_2, and we denote this situation by $v_3 \between u_3$ (or $v_3 \between_{v_2} u_3$ if we want to insist on the center). We say that v_3 and u_3 are **simple neighbors** if up to a change of representatives the polynomial P depends on one variable only, that is, if we can write $v_3 = [f_1, f_2, f_3]$ and $u_3 = [f_1, f_2, f_3 + P(f_2)]$. In this situation we say that $v_2 = [f_1, f_2], v_1 = [f_2]$ is the **simple center** of the two neighbors, and when drawing pictures we represent this relation by adding an arrow on the edge from v_2 to v_1 (see Figure 2).

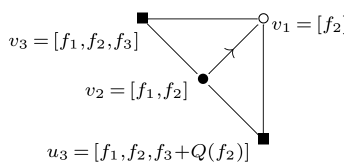


FIGURE 2. Simple neighbors with simple center v_2, v_1.

The link of a vertex of type 1 is more complicated. Let us simply mention without proof, since we won't need it in this paper (but see Lemma 5.5 for a partial result), that in contrast with the case of vertices of type 2 or 3, the link of a vertex of type 1 is a connected unbounded graph, which admits a projection to an unbounded tree.

2. Parachute Inequality and Principle of Two Maxima

I recall here two results from [Kur08] (in turn they were adaptations from [SU04b]). The Parachute Inequality is the most important; we also recall some direct consequences. From now on $\mathbf{k}$ denotes a field of characteristic zero.

2.A. **Degree of polynomials and forms.** Recall that we define a **degree** function on $\mathbf{k}[x_1, x_2, x_3]$ with value in $\mathbb{N}^3 \cup \{-\infty\}$ by taking $\deg x_1^{a_1} x_2^{a_2} x_3^{a_3} = (a_1, a_2, a_3)$ and by convention $\deg 0 = -\infty$. We compare degrees using the graded lexicographic order.

We introduce the notion of **virtual degree** in two distinct situations, which should be clear by context.

If $\varphi = \sum P_i y^i \in \mathbf{k}[x_1, x_2, x_3][y]$, and $g \in \mathbf{k}[x_1, x_2, x_3]$, we define the virtual degree of φ with respect to g as

$$\deg_{\text{virt}} \varphi(g) := \max_i (\deg P_i g^i) = \max_i (\deg P_i + i \deg g).$$

Denoting by I the set of indexes i that realize the maximum, we also define the **top component** of φ with respect to g as

$$\bar{\varphi}_g := \sum_{i \in I} \bar{P}_i y^i.$$

Similarly if $\varphi = \sum c_{i,j} y^i z^j \in \mathbf{k}[y,z]$, and $f, g \in \mathbf{k}[x_1,x_2,x_3]$, we define the virtual degree of φ with respect to f and g as

$$\deg_{\text{virt}} \varphi(f,g) := \max_{i,j} \deg f^i g^j = \max_{i,j} (i \deg f + j \deg g).$$

Observe that $\varphi(f,g)$ can be seen either as an element coming from $\mathbf{k}[f][y]$ or from $\mathbf{k}[y,z]$, and that the two possible notions of virtual degree coincide:

$$\deg_{\text{virt}} \varphi(g) = \deg_{\text{virt}} \varphi(f,g).$$

Example 2.1.

(1) Let $\varphi = x_3^2 y - x_3 y^2$, and $g = x_3$. Then $\varphi(g) = 0$, but
$$\deg_{\text{virt}} \varphi(g) = \deg x_3^3 = (0,0,3).$$

(2) Let $\varphi = y^2 - z^3$, and $g = x_1^3$, $h = x_1^2$. Then $\varphi(g,h) = 0$, but
$$\deg_{\text{virt}} \varphi(g,h) = \deg x_1^6 = (6,0,0).$$

We extend the notion of degree to algebraic differential forms. Given

$$\omega = \sum f_{i_1,\cdots,i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

where $k = 1, 2$ or 3 and $f_{i_1,\cdots,i_k} \in \mathbf{k}[x_1,x_2,x_3]$, we define

$$\deg \omega := \max\{\deg f_{i_1,\cdots,i_l} x_{i_1} \cdots x_{i_k}\}.$$

We gather some immediate remarks for future reference (observe that here we use the assumption $\operatorname{char} \mathbf{k} = 0$).

Lemma 2.2. *If ω, ω' are forms, and f is a non constant polynomial, we have*

$$\deg \omega + \deg \omega' \geq \deg \omega \wedge \omega';$$
$$\deg f = \deg df;$$
$$\deg f\omega = \deg f + \deg \omega.$$

2.B. **Parachute Inequality.** If $\varphi \in \mathbf{k}[x_1,x_2,x_3][y]$, we denote by $\varphi^{(i)} \in \mathbf{k}[x_1,x_2,x_3][y]$ the ith derivative of φ with respect to y. We simply write φ' instead of $\varphi^{(1)}$.

Lemma 2.3. *Let $\varphi \in \mathbf{k}[x_1,x_2,x_3][y]$ and $g \in \mathbf{k}[x_1,x_2,x_3]$. Then, for $m \geq 0$, the following two assertions are equivalent:*

(1) *For $i = 0, \dots, m-1$ we have*
$$\deg_{\text{virt}} \varphi^{(i)}(g) > \deg \varphi^{(i)}(g),$$
but
$$\deg_{\text{virt}} \varphi^{(m)}(g) = \deg \varphi^{(m)}(g).$$

(2) *There exists $\psi \in \mathbf{k}[x_1,x_2,x_3][y]$ such that $\psi(\bar{g}) \neq 0$ and*
$$\bar{\varphi}_g = (y - \bar{g})^m \cdot \psi.$$

Proof. Observe that we have the equivalences

$$\deg_{\text{virt}} \varphi(g) > \deg \varphi(g) \Leftrightarrow \bar{\varphi}_g(\bar{g}) = 0 \Leftrightarrow y - \bar{g} \text{ divides } \bar{\varphi}_g.$$

The implication (2) $\Rightarrow$ (1) is then direct.

To prove (1) $\Rightarrow$ (2), it is sufficient to show that if $\deg_{\text{virt}} \varphi^{(i)}(g) > \deg \varphi^{(i)}(g)$ for $i = 0, \dots, k-1$, then $\bar{\varphi}_g = (y - \bar{g})^k \cdot \psi_k$ for some $\psi_k \in \mathbf{k}[x_1,x_2,x_3][y]$. The remark above gives it for $k = 0$. Moreover, one checks that if $\bar{\varphi}_g$ depends on y then $\overline{\varphi'}_g = (\bar{\varphi}_g)'$, hence the result by induction. □

In the situation of Lemma 2.3, we call the integer m the **multiplicity** of φ with respect to g, and we denote it by $m(\varphi, g)$. In other words, the top component $\bar{g}$ is a multiple root of $\bar{\varphi}_g$ of order $m(\varphi, g)$.

Following Vénéreau [Vén11], where a similar inequality is proved, we call the following result a Parachute Inequality. Indeed its significance is that the real degree cannot drop too much with respect to the virtual degree. However we follow Kuroda for the proof.

Proposition 2.4 (Parachute Inequality, see [Kur08, Theorem 2.1])**.** *Let $f_0, \cdots, f_r \in \mathbf{k}[x_1, x_2, x_3]$ be algebraically independent, where $r = 1$ or 2. Let $\varphi \in \mathbf{k}[f_1, \cdots, f_r][y] \smallsetminus \{0\}$. Then*

$$\deg \varphi(f_0) \geq \deg_{\text{virt}} \varphi(f_0) - m(\varphi, f_0)(\deg \omega + \deg f_0 - \deg \omega \wedge df_0).$$

where $\omega = df_1$ if $r = 1$, or $\omega = df_1 \wedge df_2$ if $r = 2$.

Proof. Denoting as before φ' the derivative of φ with respect to y, we have

$$d\varphi(f_0) = \varphi'(f_0)df_0 + \text{other terms involving } df_1 \text{ or } df_2.$$

So we obtain $\omega \wedge d\varphi(f_0) = \varphi'(f_0)\omega \wedge df_0$. Using Lemma 2.2 this yields

$$\begin{aligned}\deg \omega + \deg \varphi(f_0) = \deg \omega + \deg d\varphi(f_0) &\geq \deg \omega \wedge d\varphi(f_0)\\ &= \deg \varphi'(f_0)\omega \wedge df_0 = \deg \varphi'(f_0) + \deg \omega \wedge df_0,\end{aligned}$$

which we can write as

$$-\deg \omega \wedge df_0 + \deg \omega + \deg \varphi(f_0) \geq \deg \varphi'(f_0). \tag{2.5}$$

Now we are ready to prove the inequality of the statement, by induction on $m(\varphi, f_0)$.

If $m(\varphi, f_0) = 0$, that is, if $\deg \varphi(f_0) = \deg_{\text{virt}} \varphi(f_0)$, there is nothing to do.

If $m(\varphi, f_0) \geq 1$, we have $m(\varphi', f_0) = m(\varphi, f_0) - 1$. Observe also that

$$\deg_{\text{virt}} \varphi'(f_0) \geq \deg_{\text{virt}} \varphi(f_0) - \deg f_0.$$

(In fact it is an equality except if the top component $\bar{\varphi}_{f_0}$ does not depend on y). By induction hypothesis, we have

$$\begin{aligned}\deg \varphi'(f_0) &\geq \deg_{\text{virt}} \varphi'(f_0) - m(\varphi', f_0)(\deg \omega + \deg f_0 - \deg \omega \wedge df_0)\\ &\geq \deg_{\text{virt}} \varphi(f_0) - \deg f_0 - (m(\varphi, f_0) - 1)(\deg \omega + \deg f_0 - \deg \omega \wedge df_0)\\ &= \deg_{\text{virt}} \varphi(f_0) - m(\varphi, f_0)(\deg \omega + \deg f_0 - \deg \omega \wedge df_0)\\ &\quad - \deg \omega \wedge df_0 + \deg \omega.\end{aligned}$$

Combining with (2.5), and canceling the terms $-\deg \omega \wedge df_0 + \deg \omega$ on each side, one obtains the expected inequality. □

2.C. **Consequences.** We shall use the Parachute Inequality 2.4 particularly when $r = 1$, and when we have a strict inequality $\deg_{\text{virt}} \varphi(f, g) > \deg \varphi(f, g)$. In this context the following easy lemma is crucial. One could say that this is here that we really use dimension 3.

Lemma 2.6. *Let $f, g \in \mathbf{k}[x_1, x_2, x_3]$ be algebraically independent, and $\varphi \in \mathbf{k}[y, z]$ such that $\deg_{\text{virt}} \varphi(f, g) > \deg \varphi(f, g)$. Then:*

(1) *There exist coprime* $p, q \in \mathbb{N}^*$ *such that*

$$p \deg g = q \deg f.$$

In particular, there exists $\delta \in \mathbb{N}^3$ *such that* $\deg f = p\delta$ *and* $\deg g = q\delta$.

(2) *Considering* $\varphi(f, g)$ *as coming from* $\varphi(f, y) \in \mathbf{k}[x_1, x_2, x_3][y]$*, we have*

$$\bar{\varphi}_g = (y^p - \bar{g}^p)^{m(\varphi,g)} \cdot \psi = (y^p - c\bar{f}^q)^{m(\varphi,g)} \cdot \psi$$

for some $c \in \mathbf{k}$*,* $\psi \in \mathbf{k}[x_1, x_2, x_3][y]$.

Proof. (1) We have $\varphi(f, g) = \sum c_{i,j} f^i g^j$. Since $\deg_{\text{virt}} \varphi(f, g) > \deg \varphi(f, g)$, there exist distinct (a, b) and (a', b') such that

$$\deg f^a g^b = \deg f^{a'} g^{b'} = \deg_{\text{virt}} \varphi(f, g).$$

Moreover we can assume that a is maximal for this property, and that a' is minimal. We obtain

$$(a - a') \deg f = (b' - b) \deg g.$$

Dividing by the GCD of $a - a'$ and $b' - b$ be get the expected relation.

(2) With the same notation, we have $b' = b + pm$ for some $m \geq 1$, and in particular $\deg_y \varphi(f, y) \geq p$. So if $p > \deg_y P(f, y)$ for some $P \in \mathbf{k}[f][y]$, we have $\deg_{\text{virt}} P(f, g) = \deg P(f, g)$. By the first assertion, there exists $c \in \mathbf{k}$ such that $\deg g^p > \deg(g^p - cf^q)$. By successive Euclidean divisions in $\mathbf{k}[f][y]$ we can write:

$$\varphi(f, y) = \sum R_i(y)(y^p - cf^q)^i$$

with $p > \deg_y R_i$ for all i. Denote by I the subset of indexes such that

$$\bar{\varphi}_g = \sum_{i \in I} \bar{R}_{i,g}(y^p - c\bar{f}^q)^i. \tag{2.7}$$

Let i_0 be the minimal index in I. We want to prove that $i_0 \geq m(\varphi, g)$. By contradiction, assume that $m(\varphi, g) > i_0$. Since $y - \bar{g}$ is a simple factor of $(y^p - c\bar{f}^q) = (y^p - \bar{g}^p)$, and is not a factor of any $\bar{R}_{i,g}$, we obtain that $(y - \bar{g})^{i_0+1}$ divide all sumands of (2.7) except $\bar{R}_{i_0,g}(y^p - c\bar{f}^q)^{i_0}$. In particular, $(y - \bar{g})^{i_0+1}$ doest not divide $\bar{\varphi}_g$: This is a contradiction with Lemma 2.3. □

We list now some consequences of the Parachute Inequality 2.4.

Corollary 2.8. *Let* $g, f \in \mathbf{k}[x_1, x_2, x_3]$ *be algebraically independent with* $\deg g \geq \deg f$*, and* $\varphi \in \mathbf{k}[y, z]$ *such that* $\deg_{\text{virt}} \varphi(g, f) > \deg \varphi(g, f)$*. Following Lemma 2.6, we write* $p \deg g = q \deg f$ *where* $p, q \in \mathbb{N}^*$ *are coprime. Then:*

(i) $\deg \varphi(g, f) \geq p \deg g - \deg g - \deg f + \deg df \wedge dg$*;*

(ii) If $\deg g \notin \mathbb{N} \deg f$*, then* $\deg \varphi(g, f) > \deg df \wedge dg$*;*

(iii) Assume $\deg g > \deg f$*,* $\deg g \geq \deg \varphi(g, f)$*, and* $\deg g \notin \mathbb{N} \deg f$*. Then* $p = 2$*, and* $q \geq 3$ *is odd. Moreover*

$$\deg \varphi(g, f) \geq \deg g - \deg f + \deg df \wedge dg = \Delta(g, f).$$

In particular if $\deg f \geq \deg \varphi(g, f)$*, then* $q = 3$.

(iv) Under the same assumptions as in the previous point, we have

$$\deg df \wedge d\varphi(g, f) \geq \deg g + \deg df \wedge dg.$$

Proof. (i) By Lemma 2.6, we have

$$\deg_{\text{virt}} \varphi(f,g) \geq qm(\varphi,g) \deg f. \tag{2.9}$$

On the other hand the Parachute Inequality 2.4 applied to $\varphi(f,y) \in \mathbf{k}[f][y]$ yields

$$\deg \varphi(f,g) \geq \deg_{\text{virt}} \varphi(f,g) - m(\varphi,g)(\deg f + \deg g - \deg df \wedge dg).$$

Finally dividing the right-hand side by $m(\varphi,g) \geq 1$, we obtain

$$\deg \varphi(f,g) \geq q \deg f + \deg df \wedge dg - \deg f - \deg g.$$

(ii) From Lemma 2.6 we have $\deg f = p\delta$ and $\deg g = q\delta$. The inequality (i) gives

$$\deg \varphi(f,g) \geq q \deg f - \deg f - \deg g + \deg df \wedge dg = (pq - p - q)\delta + \deg df \wedge dg. \tag{2.10}$$

The assumption $\deg g \notin \mathbb{N} \deg f$ implies $q > p \geq 2$. Thus $pq - p - q > 0$, and finally

$$\deg \varphi(f,g) > \deg df \wedge dg.$$

(iii) The assumptions imply $q > p \geq 2$. Since $q\delta = \deg g \geq \deg \varphi(f,g)$, we get from (2.10) that $q > pq - p - q$. This is possible only if $p = 2$, and so $q \geq 3$ is odd. Replacing p by 2 in (2.10), we get the inequality.

If $\deg f \geq \deg \varphi(f,g)$, we obtain $2\delta > (q-2)\delta$, hence $q = 3$.

(iv) Denoting $\varphi(f,g) = \sum c_{i,j} f^i g^j$, and

$$\varphi'(f) = \sum i c_{i,j} f^{i-1} g^j;$$
$$\varphi'(g) = \sum j c_{i,j} f^i g^{j-1}.$$

we have $d\varphi(g,f) = \varphi'(f)df + \varphi'(g)dg$. In particular $d\varphi(g,f) \wedge df = \varphi'(g) dg \wedge df$, and

$$\deg d\varphi(g,f) \wedge df = \deg \varphi'(g) + \deg df \wedge dg.$$

We want to show $\deg \varphi'(g) \geq \deg g$. Recall that by (2.9), $\deg_{\text{virt}} \varphi(g) \geq 2m(\varphi,g) \deg g$, and so

$$\deg_{\text{virt}} \varphi'(g) = \deg_{\text{virt}} \varphi(g) - \deg g \geq 2(m(\varphi,g) - 1) \deg g + \deg g.$$

The Parachute Inequality 2.4 then gives

$$\begin{aligned}
\deg \varphi'(g) &\geq \deg_{\text{virt}} \varphi'(g) - m(\varphi',g)(\deg g + \deg f - \deg df \wedge dg) \\
&\geq 2(m(\varphi,g)-1)\deg g + \deg g - (m(\varphi,g)-1)(\deg g + \deg f - \deg df \wedge dg) \\
&= (m(\varphi,g)-1)(\deg g - \deg f + \deg df \wedge dg) + \deg g \\
&\geq \deg g.
\end{aligned}$$

□

Corollary 2.11. *Let $f,g,h \in \mathbf{k}[x_1,x_2,x_3]$ be algebraically independent, and $\varphi \in \mathbf{k}[y,z]$ such that $\deg_{\text{virt}} \varphi(f,g) > \deg \varphi(f,g)$. Following Lemma 2.6, we write $p \deg g = q \deg f$ where $p,q \in \mathbb{N}^*$ are coprime. Assume that $\deg h > \deg(h + \varphi(g,f))$. Then*

$$\deg(h + \varphi(g,f)) > p \deg g - \deg df \wedge dh - \deg g$$
$$(= q \deg f - \deg df \wedge dh - \deg g).$$

Proof. The Parachute Inequality 2.4 applied to $\psi = h + \varphi(f, y) \in \mathbf{k}[f, h][y]$ gives

$$\deg h + \varphi(f, g) \geq \deg_{\text{virt}} \psi(g) - m(\psi, g)(\deg df \wedge dh + \deg g - \deg df \wedge dg \wedge dh). \quad (2.12)$$

By assumption $\deg h > \deg(h + \varphi(f, g))$, hence $\deg_{\text{virt}} \varphi(f, g) > \deg \varphi(f, g) = \deg h$. Thus not only $\deg_{\text{virt}} \psi(g) = \deg_{\text{virt}} \varphi(g)$, but also $\bar{\psi}_g = \bar{\varphi}_g$, hence $m(\psi, g) = m(\varphi, g) \geq 1$. By Lemma 2.6, we obtain

$$\deg_{\text{virt}} \psi(g) \geq pm(\varphi, g) \deg g = pm(\psi, g) \deg g$$

Replacing in (2.12), and dividing by $m(\psi, g)$, we get the result. □

2.D. **Principle of Two Maxima.** The proof of the following result, which we call the Principle of Two Maxima, is one of the few places where the formalism of Poisson brackets used by Shestakov and Umirbaev seems to be more transparent (at least for us) than the formalism of differential forms used by Kuroda. In this section we propose a definition that encompasses the two points of view, and then we recall the proof following [SU04a, Lemma 5].

Proposition 2.13 (Principle of Two Maxima, [Kur08, Theorem 5.2] and [SU04a, Lemma 5])**.** *Let (f_1, f_2, f_3) be an automorphism. Then the maximum between the following three degrees is realized at least twice:*

$$\deg f_1 + \deg df_2 \wedge df_3, \quad \deg f_2 + \deg df_1 \wedge df_3, \quad \deg f_3 + \deg df_1 \wedge df_2.$$

Let Ω be the space of algebraic 1-forms $\sum f_i dg_i$ where $f_i, g_i \in \mathbf{k}[x_1, x_2, x_3]$. We consider Ω as a free module of rank three over $\mathbf{k}[x_1, x_2, x_3]$, with basis dx_1, dx_2, dx_3, and we denote by

$$\mathbb{T} = \bigoplus_{p=0}^{\infty} \Omega^{\otimes p}$$

the associative algebra of tensorial powers of Ω, where as usual $\Omega^{\otimes 0} = \mathbf{k}[x_1, x_2, x_3]$. The degree function on Ω extends naturally to a degree function on $\mathbb{T}$. Recall that $\mathbb{T}$ has a natural structure of Lie algebra: For any $\omega, \mu \in \mathbb{T}$, we define their bracket as

$$[\omega, \mu] = \omega \otimes \mu - \mu \otimes \omega.$$

In particular, if $df, dg \in \Omega$ are 1-forms, we have

$$[df, dg] = df \otimes dg - dg \otimes df = df \wedge dg.$$

It is easy to check that the bracket satisfies the Jacobi identity: For any $\alpha, \beta, \gamma \in \mathbb{T}$, we have

$$\begin{aligned}
&[[\alpha, \beta], \gamma] + [[\beta, \gamma], \alpha] + [[\gamma, \alpha], \beta] \\
&\quad = \alpha \otimes \beta \otimes \gamma - \beta \otimes \alpha \otimes \gamma - \gamma \otimes \alpha \otimes \beta + \gamma \otimes \beta \otimes \alpha \\
&\quad + \beta \otimes \gamma \otimes \alpha - \gamma \otimes \beta \otimes \alpha - \alpha \otimes \beta \otimes \gamma + \alpha \otimes \gamma \otimes \beta \\
&\quad + \gamma \otimes \alpha \otimes \beta - \alpha \otimes \gamma \otimes \beta - \beta \otimes \gamma \otimes \alpha + \beta \otimes \alpha \otimes \gamma \\
&\quad = 0
\end{aligned}$$

since each one of the six possible permutations appears twice, with different signs. The proof of the Principle of Two Maxima 2.13 now follows from the observation:

Lemma 2.14. *Let $f, g, h \in \mathbf{k}[x_1, x_2, x_3]$. Then*

$$\deg\,[[df, dg]\,, dh] = \deg h + \deg df \wedge dg.$$

Proof. We have
$$dh = \sum_{1\le k\le 3} \tfrac{\partial h}{\partial x_k} dx_k$$
and
$$[df, dg] = df \wedge dg = \sum_{1\le i<j\le 3} \left(\tfrac{\partial f}{\partial x_i}\tfrac{\partial g}{\partial x_j} - \tfrac{\partial f}{\partial x_j}\tfrac{\partial g}{\partial x_i}\right) dx_i \wedge dx_j.$$
Thus
$$[[df, dg], dh] = \sum_{1\le k\le 3}\sum_{1\le i<j\le 3} \tfrac{\partial h}{\partial x_k}\left(\tfrac{\partial f}{\partial x_i}\tfrac{\partial g}{\partial x_j} - \tfrac{\partial f}{\partial x_j}\tfrac{\partial g}{\partial x_i}\right) [dx_i \wedge dx_j, dx_k]\,;$$
and the result follows, since $\deg\,[dx_i \wedge dx_j, dx_k] = \deg x_i x_j x_k$. □

Proof of the Principle of Two Maxima 2.13. Since by the Jacobi identity
$$[[df_1, df_2], df_3] + [[df_2, df_3], df_1] + [[df_3, df_1], df_2] = 0,$$
the dominant terms must cancel each other. In particular the maximum of the degrees, which are computed in Lemma 2.14, is realized at least twice: This is the five lines proof of the Principle of Two Maxima by Shestakov and Umirbaev! □

3. Geometric theory of reduction

In this section we mostly follow Kuroda [Kur10], but we reinterpret his theory of reduction in a combinatorial way in the complex $\mathcal{C}$.

3.A. **Degree of automorphisms and vertices.** Recall that in §1.B we defined a notion of degree for an automorphism $f = (f_1, f_2, f_3) \in \mathrm{Tame}(\mathbb{A}^3)$. The point is that we want a degree that is adapted to the theory of reduction of Kuroda, so for instance taking the maximal degree of the three components of an automorphism is not good, because we would not detect a reduction of the degree on one of the two lower components (such reductions do exist, see §6). We also want a definition that is adapted to working on the complex $\mathcal{C}$, so directly taking the sum of the degree of the three components is no good either, since it would not give a degree function on vertices of $\mathcal{C}$.

Recall that the **3-degree** of $f \in \mathrm{Tame}(\mathbb{A}^3)$, or of the vertex $v_3 = [f]$, is the triple $(\delta_1, \delta_2, \delta_3)$ given by Lemma 1.3, in particular $\delta_3 > \delta_2 > \delta_1$, and the **degree** of f is the sum
$$\deg f = \delta_1 + \delta_2 + \delta_3 \in \mathbb{N}^3.$$
We call δ_3 the **top degree** of f. Similarly we have a 2-degree (ν_1, ν_2) associated with any vertex v_2 of type 2, and a degree $\deg v_2 = \nu_1 + \nu_2$. Finally for a vertex of type 1 the notions of 1-degree and degree coincide.

Lemma 3.1. *Let v_3 be a vertex of type 3. Then*
$$\deg v_3 \ge (1, 1, 1)$$
with equality if and only if $v_3 = [\mathrm{id}]$.

Proof. If $v_3 = [f]$ with $\deg v_3 = (1, 1, 1)$, then the **3-degree** $(\delta_1, \delta_2, \delta_3)$ of v_3 must be equal to $(0, 0, 1), (0, 1, 0), (0, 0, 1)$, hence the result. □

Let v_3 be a vertex with 3-degree $(\delta_1, \delta_2, \delta_3)$. The unique $m_1 \in \mathbb{P}^2(v_3)$ such that $\deg m_1 = \delta_1$ is called the **minimal vertex** in $\mathbb{P}^2(v_3)$, and the unique $m_2 \in \hat{\mathbb{P}}^2(v_3)$ such that $\deg m_2 = (\delta_1, \delta_2)$ is called the **minimal line** in $\mathbb{P}^2(v_3)$. If $v_2 \in \hat{\mathbb{P}}^2(v_3)$ has 2-degree (ν_1, ν_2), there is a unique degree δ such that v_3 has degree $\nu_1 + \nu_2 + \delta$. We denote this situation by $\delta := \deg(v_3 \smallsetminus v_2)$. There is also a unique v_1 such that v_2 passes through v_1 and $\deg v_1 = \nu_1$: we call v_1 the minimal vertex of v_2. Observe that if $v_2 = m_2 \in \hat{\mathbb{P}}^2(v_3)$, then the minimal vertex of v_2 coincides with the minimal vertex of v_3, and if $v_2 \neq m_2$, then v_1 is the intersection of v_2 with m_2.

A **good triangle** T in $\mathbb{P}^2(v_3)$ is the data of three distinct lines m_2, v_2, u_2, such that m_2 is the minimal line, v_2 passes through the minimal vertex m_1, and u_2 does not pass through m_1. Equivalently, a good triangle corresponds to a good representative $v_3 = \llbracket f_1, f_2, f_3 \rrbracket$ with $\deg f_1 > \deg f_2 > \deg f_3$, by putting $m_2 = \llbracket f_2, f_3 \rrbracket$, $v_2 = \llbracket f_1, f_3 \rrbracket$, $u_2 = \llbracket f_1, f_2 \rrbracket$.

Let $v_2 \in \hat{\mathbb{P}}^2(v_3)$ be a vertex with 2-degree (δ_1, δ_2). We say that v_2 has **inner resonance** if $\delta_2 \in \mathbb{N}\delta_1$. We say that v_2 has **outer resonance** in v_3 if $\deg(v_3 \smallsetminus v_2) \in \mathbb{N}\delta_1 + \mathbb{N}\delta_2$.

3.B. **Elementary reductions.** Let v_3, v_3' be vertices of type 3.

Recall that v_3' is a **neighbor** of v_3 if $v_3' \neq v_3$ and there exists a vertex v_2 of type 2 such that $d(v_3, v_2) = d(v_3', v_2) = 1$. We denote this situation by $v_3' \between v_3$, or if we want to make v_2 explicit, by $v_3' \between_{v_2} v_3$ We also say that v_2 is the **center** of $v_3 \between v_3'$.

We say that v_3' is an **elementary reduction** (resp. a **weak elementary reduction**) of v_3 with center v_2, if $\deg v_3 > \deg v_3'$ (resp. $\deg v_3 \geq \deg v_3'$) and $v_3' \between_{v_2} v_3$. Let v_1 be the minimal vertex in the line v_2. We say that v_1, v_2, v_3 is the **pivotal simplex** of the reduction. Moreover we say that the reduction is **optimal** if v_3' has minimal degree among all neighbors of v_3 with center v_2, and that the reduction is **simple** if v_3 and v_3' are simple neighbors, as defined in §1.C.

Lemma 3.2. *Let v_3' be a neighbor of $v_3 = [f_1, f_2, f_3]$ with center $v_2' = \llbracket f_1, f_2 \rrbracket$. Then there exists a polynomial $P(f_1, f_2)$ such that $v_3' = \llbracket f_1, f_2, f_3 + P(f_1, f_2) \rrbracket$.*

Moreover:

- *If v_3' is a weak elementary reduction of v_3, then $\deg f_3 \geq \deg P(f_1, f_2)$;*
- *If v_3' is an elementary reduction of v_3, then $\deg(v_3 \smallsetminus v_2') = \deg P(f_1, f_2)$ (which is also equal to $\deg f_3$ if $v_3 = \llbracket f_1, f_2, f_3 \rrbracket$).*

Proof. From Corollary 1.5 we know that v_3' has the form $v_3' = [f_1, f_2, f_3 + P(f_1, f_2)]$. Since by assumption $\deg f_1 \neq \deg f_2$, there exist $a, b \in \mathbf{k}$ such that $(f_1, f_2, f_3 + P(f_1, f_2) + af_1 + bf_2)$ is a good representative for v_3'. So up to changing P we can assume $v_3' = \llbracket f_1, f_2, f_3 + P(f_1, f_2) \rrbracket$.

If v_3' is a weak elementary reduction of v_3, then we have

$$\deg f_1 + \deg f_2 + \deg f_3 \geq \deg v_3 \geq \deg v_3' = \deg f_1 + \deg f_2 + \deg(f_3 + P(f_1, f_2)).$$

So $\deg f_3 \geq \deg(f_3 + P(f_1, f_2))$, which implies $\deg f_3 \geq \deg P(f_1, f_2)$.

Finally if v_3' is an elementary reduction of v_3, that is, $\deg v_3 > \deg v_3'$, then the same computation gives $\deg(v_3 \smallsetminus v_2') > \deg(f_3 + P(f_1, f_2))$, which implies $\deg(v_3 \smallsetminus v_2') = \deg P(f_1, f_2)$. □

Remark 3.3. Under the assumptions of Lemma 3.2, if v_3' is a simple weak reduction of v_3 with center $v_2 = \llbracket f_1, f_2 \rrbracket, v_1 = \llbracket f_2 \rrbracket$, and moreover:

(1) v_2 is not the minimal line of v_3;

(2) v_2 has no inner resonance;

then $v_3' = [\![f_1, f_2, f_3 + P(f_2)]\!]$.

Indeed a priori we have $P(f_1, f_2) = af_1 + Q(f_2)$, and we would need to change the representative for v_3 in order to insure $a = 0$. But our assumptions imply that $\deg f_1 \notin \mathbb{N} \deg f_2$, and $\deg f_1 = \operatorname{topdeg} v_3$, so that

$$\deg f_1 > \deg f_3 \geq \deg(af_1 + Q(f_2)) = \max\{\deg af_1, \deg Q(f_2)\},$$

which gives $a = 0$.

Lemma 3.4 (Square Lemma)**.** *Let v_3, v_3', v_3'' be three vertices such that:*

- *$v_3' \between_{v_2'} v_3$ and $v_3'' \between_{v_2''} v_3$ for some v_2', v_2'' that are part of a good triangle of v_3 (this is automatic if $v_2' \neq v_2''$, and v_2' or v_2'' is the minimal line of v_3);*
- *Denoting v_1 the common vertex of v_2' and v_2'', v_3'' is a simple neighbor of v_3 with center v_2'', v_1;*
- *$\deg v_3 \geq \deg v_3'$, $\deg v_3 \geq \deg v_3''$, with at least one of the inequality being strict.*

Then there exists u_3 such that:

- *$u_3 \between v_3'$, $u_3 \between v_3''$;*
- *$\deg v_3 > \deg u_3$.*

Proof. We take $v_1 = [\![f_2]\!]$, $v_2' = [\![f_1, f_2]\!]$, $v_2'' = [\![f_2, f_3]\!]$. Since v_2' and v_2'' are part of a good triangle, we have $v_3 = [\![f_1, f_2, f_3]\!]$. By Lemma 3.2, and since v_3'' is a simple neighbor of v_3, there exist $a \in \mathbf{k}$, $Q \in \mathbf{k}[f_2]$ and $P \in \mathbf{k}[f_1, f_2]$ such that

$$v_3' = [\![f_1, f_2, f_3 + P(f_1, f_2)]\!];$$
$$v_3'' = [\![f_1 + af_3 + Q(f_2), f_2, f_3]\!].$$

We have

$$\deg f_3 \geq \deg(f_3 + P(f_1, f_2)), \quad \deg f_1 \geq \deg(f_1 + af_3 + Q(f_2)),$$

with one of the inequality being strict. We will show that one of the following choices for u_3 is good (see Figure 3):

(1) $u_3 = [f_1 + Q(f_2), f_2, f_3 + P(f_1, f_2)]$;
(2) $u_3 = [f_1 + af_3 + Q(f_2), f_2, af_3 + Q(f_2)]$.

In any case u_3 is a neighbor of both v_3' and v_3'', so the problem is only to check that for one of the choice we have $\deg v_3 > \deg u_3$.

If $a = 0$, then the following inequalities show that choice (1) is good:

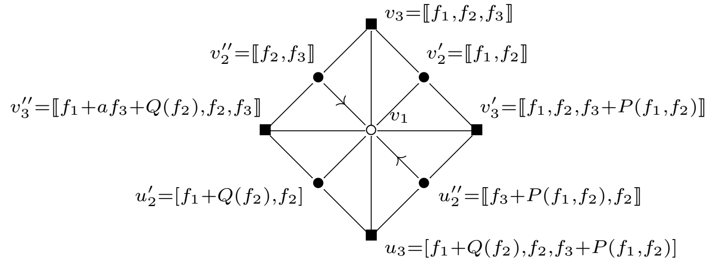
$$\begin{aligned}\deg v_3 &= \deg f_3 + \deg f_2 + \deg f_1 \\ &> \deg(f_3 + P(f_1, f_2)) + \deg f_2 + \deg(f_1 + Q(f_2)) \\ &\geq \deg u_3.\end{aligned}$$

Now we assume $a \neq 0$.

First consider the case $\deg v_3 = \deg v_3''$, that is, $\deg f_1 = \deg(f_1 + af_3 + Q(f_2))$. If $\deg f_3 > \deg(af_3 + Q(f_2))$ then choice (2) is good. If $\deg(af_3 + Q(f_2)) \geq \deg f_3$ then the equality $\deg f_1 = \deg(f_1 + af_3 + Q(f_2))$ implies $\deg f_1 \geq \deg f_3$. Moreover by assumption $\deg f_1 \neq \deg f_3$, so we have $\deg f_1 > \deg f_3$, which implies $\deg f_1 + Q(f_2) = \deg(f_1 + af_3 + Q(f_2))$, and so choice (1) is good.

Now consider the case $\deg v_3 > \deg v_3''$, that is, $\deg f_1 > \deg(f_1 + af_3 + Q(f_2))$. If $\deg f_3 \geq \deg(af_3 + Q(f_2))$ then choice (2) is good. If $\deg(af_3 + Q(f_2)) > \deg f_3$

Choice (1)



Choice (2)

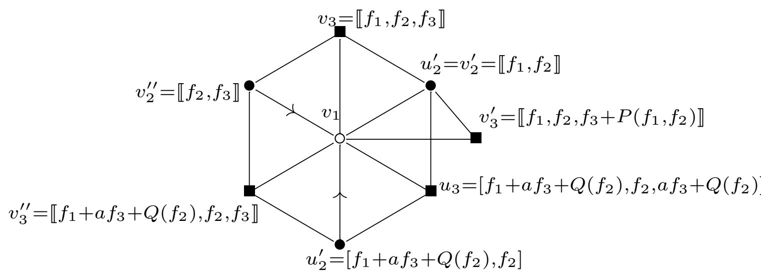


FIGURE 3. Square Lemma 3.4.

then as before $\deg f_1 > \deg f_3$. Then $\deg f_1 > \deg(f_1 + af_3 + Q(f_2))$ implies that $\deg f_1 > \deg(f_1 + Q(f_2))$, and so choice (1) is good. □

3.C. **Differential degree and delta degree.** If $f_1, f_2 \in \mathbf{k}[x_1, x_2, x_3]$ are two algebraically independent polynomials with $\deg f_1 > \deg f_2$, we introduce the degree

$$\Delta(f_1, f_2) := \deg f_1 - \deg f_2 + \deg df_1 \wedge df_2 \in \mathbb{Z}^3.$$

Observe that for any $a \in \mathbf{k}$

$$d(f_1 + af_2) \wedge df_2 = df_1 \wedge df_2,$$

so the following definitions do not depend on a choice of representative. If $v_2 = [\![f_1, f_2]\!]$ is a vertex of type 2, we denote

$$d(v_2) = \deg df_1 \wedge df_2 \quad \text{and} \quad \Delta(v_2) = \Delta(f_1, f_2).$$

We call $d(v_2)$ and $\Delta(v_2)$ respectively the **differential degree** and the **delta degree** of v_2. Observe that for any vertex v_2 we have

$$\Delta(v_2) > d(v_2).$$

The Principle of Two Maxima 2.13 translates as follows.

Lemma 3.5. *Let v_3 be a vertex of type 3, and $v_1 \in m_2$ a vertex in the minimal line of v_3.*

(1) *The function $v_2 \in \mathbb{P}^2(v_3) \mapsto \deg(v_3 \smallsetminus v_2) + d(v_2)$ is constant, except possibly at a unique line u_2 where it attains its minimum.*

(2) *The function $v_2 \in \mathbb{P}^2(v_3) \mapsto \Delta(v_2)$ is constant on the set of lines distinct from m_2 and passing through v_1, except possibly at a unique line u_2' where it attains its minimum. In this case, u_2' is equal to the vertex u_2 of the previous assertion.*

Proof. (1) let v_2, v_2', v_2'' be three lines in $\mathbb{P}^2(v_3)$, not passing through the same point. There exists (f_1, f_2, f_3) a (in general not good) representative of v_3, such that $v_2 = [f_1, f_2], v_2' = [f_1, f_3], v_2'' = [f_2, f_3]$. The Principle of Two Maxima 2.13 says that at least two of the following three quantities are equal and realizes the maximum:

$$\deg(v_3 \smallsetminus v_2) + d(v_2),\ \deg(v_3 \smallsetminus v_2') + d(v_2'),\ \deg(v_3 \smallsetminus v_2'') + d(v_2'').$$

The result follows.

(2) Let $v_2 \neq m_2$ a line passing through v_1. Now we can choose a good representative $[\![f_1, f_2, f_3]\!]$ of v_3 such that $v_1 = [\![f_2]\!]$, $m_2 = [\![f_2, f_3]\!]$, $v_2 = [\![f_1, f_2]\!]$. We have

$$\Delta(v_2) = \deg f_1 - \deg f_2 + \deg df_1 \wedge df_2 = \operatorname{topdeg} v_3 - \deg v_1 + d(v_2).$$

So for such lines passing through v_1 and distinct from m_2, the equality $\Delta(v_2) = \Delta(v_2')$ is equivalent to $d(v_2) = d(v_2')$. Since moreover $\deg(v_3 \smallsetminus v_2) = \deg(v_3 \smallsetminus v_2') = \deg f_3$, the result follows from the first assertion. □

3.D. ***K*-reductions.** We introduce now the key concept of K-reduction, where we let the reader decide for himself whether the K should stand for "Kuroda" or for "Kazakh". Let v_3 and u_3 be vertices of type 3.

We say that u_3 is an **elementary *K*-reduction** of v_3 if $u_3 \between v_3$ with center v_2 and

(K1) the vertex v_2 has no inner resonance;
(K2) the vertex v_2 has no outer resonance in v_3;
(K3) v_2 is not the minimal line in $\mathbb{P}^2(v_3)$;
(K4) $\Delta(v_2) > \deg(u_3 \smallsetminus v_2)$;
(K5) $\deg v_3 > \deg u_3$;

If v_1 is the minimal point in v_2, as before we call v_1, v_2, v_3 the **pivotal simplex** of the elementary K-reduction (denoted by $\circlearrowleft$ on Figure 4).

We say that u_3 is a **proper *K*-reduction** of v_3 if there exists w_3 such that $w_3 \between v_3$ (with center m_2) and $w_3 \between u_3$ (with center $w_2 \neq m_2$), such that

(K6) $\deg w_3 \geq \deg v_3$;
(K7) m_2 is the minimal line in $\mathbb{P}^2(w_3)$;
(K8) u_3 is an elementary K-reduction of w_3.

Observe that the pivot v_1 of the reduction is the common vertex of the lines m_2 and w_2 in $\mathbb{P}^2(w_3)$. The simplex v_1, w_2, w_3 is still called the **pivotal simplex** of the reduction.

Let v_1, v_2, v_3 be a simplex, and let $s \geq 3$ be an odd integer. We say that the simplex v_1, v_2, v_3 has **Strong Pivotal Form $\circlearrowleft(s)$** if

($\circlearrowleft$1) $\deg v_1 = 2\delta$ and $\deg v_2 = (2\delta, s\delta)$ for some $\delta \in \mathbb{N}^3$;
($\circlearrowleft$2) the vertex v_2 has no outer resonance in v_3;
($\circlearrowleft$3) v_2 is not the minimal line in $\mathbb{P}^2(v_3)$;
($\circlearrowleft$4) $\deg(v_3 \smallsetminus v_2) \geq \Delta(v_2)$;

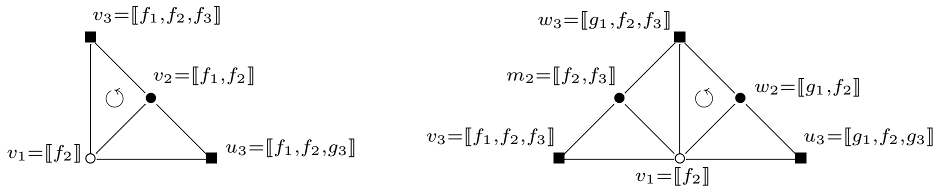


FIGURE 4. Elementary and proper K-reductions.

In all the previous definitions we took care of working with vertices, and not with particular representatives. However when proving things it will often be useful to choose representatives.

Set-Up 3.6.

(1) Let u_3 be an elementary K-reduction of v_3, with pivotal simplex v_1, v_2, v_3. Then there exist representatives

$$v_1 = [\![f_2]\!] \qquad v_3 = [\![f_1, f_2, f_3]\!]$$
$$v_2 = [\![f_1, f_2]\!] \qquad u_3 = [\![f_1, f_2, g_3]\!]$$

such that $g_3 = f_3 + \varphi_3(f_1, f_2)$, where $\varphi_3 \in \mathbf{k}[x, y]$.

(2) Let u_3 be a proper K-reduction of v_3, with pivotal simplex v_1, w_2, w_3. Then there exist representatives

$$v_1 = [\![f_2]\!] \qquad w_3 = [\![g_1, f_2, f_3]\!]$$
$$m_2 = [\![f_2, f_3]\!] \qquad v_3 = [\![f_1, f_2, f_3]\!]$$
$$w_2 = [\![g_1, f_2]\!] \qquad u_3 = [\![g_1, f_2, g_3]\!]$$

such that $g_1 = f_1 + \varphi_1(f_2, f_3)$ and $g_3 = f_3 + \varphi_3(g_1, f_2)$, where $\varphi_1, \varphi_3 \in \mathbf{k}[x, y]$.

Proof. (1) Pick any good representatives $v_1 = [\![f_2]\!]$, $v_2 = [\![f_1, f_2]\!]$, $v_3 = [\![f_1, f_2, f_3]\!]$, and apply Lemma 3.2 to get g_3.

(2) Pick any good representative $v_1 = [\![f_2]\!]$ (which is unique up to a multiplicative constant), and then pick f_3, g_1 such that $v_2 = [\![f_2, f_3]\!]$ and $w_2 = [\![g_1, f_2]\!]$. Since v_2 is the minimal line in $\hat{\mathbb{P}}^2(w_3)$, we have $\deg g_1 > \deg f_2$ and $\deg g_1 > \deg f_3$, hence (g_1, f_2, f_3) is a good representative for w_3. Now apply Lemma 3.2 twice to get f_1 and g_3. □

We can rephrase results from Corollary 2.8 with the previous definitions (see also Example 6.3 for some complements):

Proposition 3.7. *Let v_3 be a vertex that admits an elementary reduction with pivotal simplex v_1, v_2, v_3.*

(1) *Assume v_2 has no inner resonance, and no outer resonance in v_3. Then*

$$\deg(v_3 \smallsetminus v_2) > d(v_2).$$

(2) *If moreover v_2 is not the minimal line in $\mathbb{P}^2(v_3)$, then*

$$\deg(v_3 \smallsetminus v_2) \geq \Delta(v_2),$$

so that v_1, v_2, v_3 has Strong Pivotal Form $\circlearrowleft(s)$ for some odd $s \geq 3$.

(3) *In particular, the pivotal simplex of a K-reduction has Strong Pivotal Form $\circlearrowleft(s)$ for some odd $s \geq 3$.*

Proof. (1) We pick good representatives $v_1 = [\![f_2]\!]$, $v_2 = [\![f_1, f_2]\!]$, $v_3 = [\![f_1, f_2, f_3]\!]$. By Lemma 3.2, the elementary reduction has the form $u_3 = [\![f_1, f_2, f_3 + P(f_1, f_2)]\!]$ with $\deg f_3 = \deg P(f_1, f_2)$. Since v_2 has no outer resonance in v_3, we have $\deg f_3 \notin \mathbb{N} \deg f_1 + \mathbb{N} \deg f_2$, hence

$$\deg_{\text{virt}} P(f_1, f_2) > \deg P(f_1, f_2).$$

Since moreover v_2 has no inner resonance, we can apply Corollary 2.8(ii) to get the inequality $\deg(v_3 \smallsetminus v_2) > d(v_2)$.

(2) The condition that v_2 is not the minimal line in $\mathbb{P}^2(v_3)$ is equivalent to

$$\max\{\deg f_1, \deg f_2\} > \deg f_3 = \deg P(f_1, f_2),$$

hence the second assertion follows from Corollary 2.8(iii).

(3) The last assertion is an immediate corollary. □

Lemma 3.8. *Let u_3 be an elementary K-reduction of v_3 with center v_2, and let m_2 be the minimal line in $\mathbb{P}^2(v_3)$. Then*

(1) *$d(m_2) \geq \deg(v_3 \smallsetminus m_2) + d(v_2)$;*

(2) *For any line u_2 in $\mathbb{P}^2(v_3)$ distinct from v_2, we have*

$$\deg(v_3 \smallsetminus u_2) + d(u_2) > \deg(v_3 \smallsetminus v_2) + d(v_2)$$
$$\deg(v_3 \smallsetminus u_2) + d(u_2) \geq 2 \deg(v_3 \smallsetminus m_2) + d(v_2).$$

(3) *For any line u_2 in $\mathbb{P}^2(v_3)$ distinct from m_2 and v_2, we have*

$$d(u_2) > d(m_2) > d(v_2).$$

Proof. We use the notation from Set-Up 3.6.

(1) One the one hand:

$$df_2 \wedge df_3 = df_2 \wedge dg_3 - df_2 \wedge d\varphi_3(f_1, f_2).$$

On the other hand, combining Corollary 2.8(iv) for the first inequality and (K4) for the strict inequality:

$$\deg df_2 \wedge d\varphi_3(f_1, f_2) \geq \deg f_1 + \deg df_1 \wedge df_2 > \deg f_2 + \deg g_3 \geq \deg df_2 \wedge dg_3.$$

So $\deg df_2 \wedge df_3 = \deg df_2 \wedge d\varphi_3(f_1, f_2)$, and again by Corollary 2.8(iv) we obtain the expected inequality:

$$d(m_2) = \deg df_2 \wedge df_3 \geq \deg f_1 + \deg df_1 \wedge df_2 = \deg(v_3 \smallsetminus m_2) + d(v_2).$$

(2) From the previous inequality we get

$$\deg(v_3 \smallsetminus m_2) + d(m_2) > \deg(v_3 \smallsetminus v_2) + d(v_2).$$

By the Principle of Two Maxima, as formulated in Lemma 3.5(1), we get

$$\deg(v_3 \smallsetminus u_2) + d(u_2) = \deg(v_3 \smallsetminus m_2) + d(m_2),$$

which together with assertion (1) gives the two expected inequalities.

(3) Since by definition of the minimal line $\deg(v_3 \smallsetminus m_2) > \deg(v_3 \smallsetminus u_2)$, the previous equality gives $d(u_2) > d(m_2)$, and $d(m_2) > d(v_2)$ follows directly from (1). □

Corollary 3.9. *Assume that v_3 admits an elementary K-reduction, and that one of the following holds:*

(1) *u_2, v_2, w_2 is a good triangle in $\mathbb{P}^2(v_3)$, and $d(u_2) > d(w_2) > d(v_2)$;*
(2) *m_2 is the minimal line in $\mathbb{P}^2(v_3)$, v_2 is another line in $\mathbb{P}^2(v_3)$ and $d(m_2) > d(v_2)$.*

Then v_2 is the center of the K-reduction.

Proof. (1) The assumption implies that v_2 is the unique minimum of the function

$$u_2 \in \hat{\mathbb{P}}^2(v_3) \mapsto d(u_2).$$

By Lemma 3.8(3) the center of a K-reduction must be such a minimum.

(2) Pick u_2 such that u_2, m_2, v_2 is a good triangle of $\mathbb{P}^2(v_3)$: There is at least a one parameter space of such lines, so we can assume that u_2 is not the center of the K-reduction. By 3.8(3) we have $d(u_2) > d(m_2)$, and then we can apply the previous point. □

3.E. Proper K-reductions.

Proposition 3.10. *Let u_3 be a proper K-reduction of v_3 (via w_3). Then (using notation from Set-Up 3.6):*

(1) *$g_1 = f_1 + \varphi_1(f_2, f_3)$ with $\deg_{\text{virt}} \varphi_1(f_2, f_3) = \deg \varphi_1(f_2, f_3)$.*
(2) *If the pivotal simplex has Strong Pivotal Form $\circlearrowleft(s)$ with $s \geq 5$, then v_3 is a simple neighbor of w_3, with center m_2, v_1, and $\deg v_3 = \deg w_3$.*

Proof. (1) Assume by contradiction that $\deg_{\text{virt}} \varphi_1(f_2, f_3) > \deg \varphi_1(f_2, f_3)$. By non resonance (K1) we can apply Corollary 2.8(ii) and Lemma 3.8(1) to get the contradiction

$$\deg \varphi_1(f_2, f_3) > \deg df_2 \wedge df_3 = d(m_2) > \deg g_1.$$

(2) We have

$$\deg g_1 \geq \deg \varphi_1(f_2, f_3) = \deg_{\text{virt}} \varphi_1(f_2, f_3).$$

By Proposition 3.7, we know that v_1, w_2, w_3 has Strong Pivotal Form $\circlearrowleft(s)$, in particular $\deg g_1 = s\delta$, $\deg f_2 = 2\delta$ and by ($\circlearrowleft$4) we have $\deg f_3 > (s-2)\delta$. As soon as $s \geq 5$ we have $s - 2 > \frac{s}{2}$, and in this case φ_1 has the form $\varphi_1(f_2, f_3) = af_3 + Q(f_2)$, as expected. Moreover since $\deg Q(f_2) = 2r\delta$ for some $r \geq 2$, we have $\deg g_1 > \deg \varphi(f_2, f_3)$ hence $\deg f_1 = \deg g_1$ and $\deg v_3 = \deg w_3$. □

We say that a K-reduction is **normalized** if

- either it is an elementary K-reduction;
- or, if u_3 is a proper K-reduction of v_3 via w_3, the vertex v_3 is not a simple neighbor of w_3 with center m_2, v_1.

By Proposition 3.10(2), in the case of a normalized proper K-reduction the pivotal simplex has Strong Pivotal Form $\circlearrowleft(3)$. We now prove the converse, with some estimation of the degrees involved.

Lemma 3.11. *Assume that u_3 is a proper K-reduction of v_3, via w_3, and that the pivotal simplex has Strong Pivotal Form $\circlearrowleft(3)$. Then the reduction is normalized, and using representatives as from Set-Up 3.6, we have:*

$$\deg g_1 = 3\delta, \quad \deg f_2 = 2\delta, \quad \tfrac{3}{2}\delta \geq \deg f_3 > \delta,$$
$$\deg df_1 \wedge df_3 = \delta + \deg df_2 \wedge df_3 \geq 4\delta + \deg dg_1 \wedge df_2,$$
$$\deg df_1 \wedge df_2 = \deg f_3 + \deg df_2 \wedge df_3.$$

Moreover we have the implications:

$$\deg w_3 > \deg v_3 \Longrightarrow \deg f_3 = \tfrac{3}{2}\delta, \quad \deg f_1 > \tfrac{5}{2}\delta.$$
$$\deg w_3 = \deg v_3 \Longrightarrow \deg f_1 = \deg g_1 = s\delta.$$

In any case we have

$$\deg df_1 \wedge df_2 > \deg df_1 \wedge df_3 > \deg df_2 \wedge df_3,$$

m_2 is the minimal line of v_3, and for any line $\ell_2 \in \hat{\mathbb{P}}^2(v_3)$, we have

$$d(\ell_2) > d(m_2).$$

Proof. The equalities $\deg g_1 = 3\delta$ and $\deg f_2 = 2\delta$ come from the fact that the pivotal simplex has Strong Pivotal Form $\circlearrowleft(3)$. Property $(\circlearrowleft 4)$ gives $\deg f_3 > \delta$, and Proposition 3.10(1) says that $\deg_{\text{virt}} \varphi_1(f_2, f_3) = \deg \varphi_1(f_2, f_3)$. So there exists $a, b, c \in \mathbf{k}$, $a \neq 0$, such that

$$w_3 = [\![g_1 = f_1 + af_3^2 + bf_2 + cf_3, f_2, f_3]\!],$$

and $3\delta = \deg g_1 \geq 2\deg f_3$, with an equality in the case $\deg g_1 > \deg f_1$. In particular the K-reduction is normalized. By Lemma 3.8(1) we have

$$\deg df_2 \wedge df_3 = d(m_2) \geq \deg(w_3 \smallsetminus m_2) + d(w_2) = \deg g_1 + \deg dg_1 \wedge df_2. \tag{3.12}$$

This implies

$$\deg g_1 + \deg df_2 \wedge df_3 > \deg f_3 + \deg dg_1 \wedge df_2.$$

By the Principle of Two Maxima 2.13, we get

$$\deg g_1 + \deg df_2 \wedge df_3 = \deg f_2 + \deg dg_1 \wedge df_3. \tag{3.13}$$

Now $dg_1 \wedge df_3 = df_1 \wedge df_3 + b df_2 \wedge df_3$, and the previous equality implies $\deg dg_1 \wedge df_3 > \deg df_2 \wedge df_3$, so that

$$\deg dg_1 \wedge df_3 = \deg df_1 \wedge df_3.$$

Now combining (3.12) and (3.13) we get the expected inequality

$$\deg df_1 \wedge df_3 = \delta + \deg df_2 \wedge df_3 \geq 4\delta + \deg dg_1 \wedge df_2.$$

Finally, since $\deg f_1 + \deg f_3 \geq \deg df_1 \wedge df_3$, when $\deg f_3 = \frac{3}{2}\delta$ we also get $\deg f_1 > \frac{5}{2}\delta$.

For the last equality we start again from $g_1 = f_1 + af_3^2 + bf_2 + cf_3$, which gives

$$dg_1 \wedge df_2 = df_1 \wedge df_2 + af_3 df_2 \wedge df_3 + c df_2 \wedge df_3.$$

Since (3.12) implies $\deg(f_3 df_2 \wedge df_3) > \deg dg_1 \wedge df_2$, the first two terms on the right-hand side must have the same degree, which is the expected equality. □

Now we can justify the terminology of "reduction", which was by no mean obvious in the proper case:

Proposition 3.14. *Let u_3 be a proper K-reduction of a vertex v_3. Then*

$$\deg v_3 > \deg u_3.$$

Proof. We use the notation from Set-Up 3.6. Observe that if $\deg w_3 = \deg v_3$, then the proposition is obvious from (K5) and (K8).

Assume first that the reduction is not normalized, that is, $g_1 = f_1 + af_3 + Q(f_2)$. Since we know that $\deg g_1 = s\delta$, $\deg f_2 = 2\delta$ and $s\delta > \deg f_3 > (s-2)\delta$, we obtain that $\deg g_1 = \deg f_1 > \deg(af_3 + Q(f_2))$, hence $\deg w_3 = \deg v_3$ and we are done.

Now assume we have a normalized proper K-reduction, and that $\deg w_3 > \deg v_3$. By condition (K4) we have

$$\delta + \deg dg_1 \wedge df_2 > \deg g_3.$$

Adding $3\delta = \deg g_1$, and using Lemma 3.11, we get

$$\deg df_1 \wedge df_3 \geq 4\delta + \deg dg_1 \wedge df_2 > \deg g_1 + \deg g_3.$$

Finally adding $\deg f_2$ we get

$$\begin{aligned} \deg v_3 &= \deg f_1 + \deg f_2 + \deg f_3 \\ &\geq \deg df_1 \wedge df_3 + \deg f_2 \\ &> \deg g_1 + \deg g_3 + \deg f_2 \\ &= \deg u_3. \end{aligned}$$

□

Lemma 3.15 (Normalization of a K-reduction)**.** *Let u_3 be a non-normalized proper K-reduction of v_3, via w_3. Then there exists u_3' such that*

(1) *$u_3' \between v_3$ and $u_3' \between u_3$;*
(2) *u_3' is an elementary K-reduction of v_3.*

Proof. By Lemma 3.11 the pivotal simplex of the reduction has Strong Pivotal Form $\circlearrowleft(s)$ for some odd $s \geq 5$. Hence by Proposition 3.10 we have $\deg v_3 = \deg w_3$. By the Square Lemma 3.4, we get the existence of u_3' with $u_3' \between v_3$, $u_3' \between u_3$ and $\deg w_3 > \deg u_3'$. Hence $\deg v_3 > \deg u_3'$, which is (K5).

Looking at the proof of the Square Lemma 3.4, we see that we are in Choice (1): indeed $\deg v_3 = \deg w_3$ and by non-resonance $\deg(f_3 + Q(f_2)) \geq \deg f_3$.

More precisely (see Figure 5):

- v_3 and w_3 have same 3-degrees and v_2 and w_2 have same 2-degrees, which gives (K1) and (K2);
- m_2 is the minimal line of v_3, and is distinct from v_2, which gives (K3);
- $v_2 = [\![g_1 - Q(f_2), f_2]\!]$ so $d(v_2) = d(w_2)$, which gives (K4). □

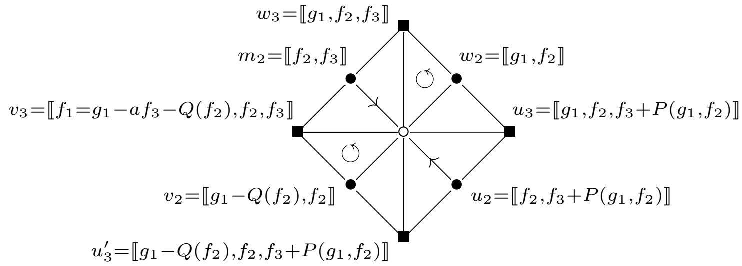


FIGURE 5. Normalization of a K-reduction.

Corollary 3.16. *Let v_3 admitting a K-reduction u_3, and write $v_3 = [\![f_1, f_2, f_3]\!]$ as in Set-Up 3.6. Then one of the following holds:*

(1) *Any line u_2 in $\mathbb{P}^2(v_3)$ has no inner resonance;*
(2) *$\deg f_1 = 2\deg f_3$ and $2\deg f_1 = 3\deg f_2$.*

Proof. If the K-reduction is elementary, we have $\deg f_1 = s\delta$, $\deg f_2 = 2\delta$ and $\deg f_3 > (s-2)\delta$ for some odd s. Moreover when $s \geq 5$, we cannot have $\deg f_3 = \frac{s-1}{2} \deg f_2$ by property (K2). This leaves the two stated possibilities.

If the K-reduction is proper, either $\deg g_1 = \deg f_1$ and we can apply the previous case; or $\deg g_1 > \deg f_1$ and we can assume the K-reduction is normalized. Then by Lemma 3.11 we have $3\delta > \deg f_1 > \frac{5}{2}\delta$, $\deg f_2 = 2\delta$, $\deg f_3 = \frac{3}{2}\delta$, hence $\deg f_1 \notin \mathbb{N} \deg f_2 + \mathbb{N} \deg f_3$, $\deg f_2 \notin \mathbb{N} \deg f_3$ and we are in case (1). □

3.F. **Stability of K-reductions.** Consider v_3 a vertex that admits a normalized K-reduction. In this section we want to show that most elementary reductions of v_3 still admit a K-reduction. First we prove two lemmas that give some constraint on the (weak) elementary reductions that such a vertex v_3 can admit.

Lemma 3.17. *Let u_3 be a normalized K-reduction of v_3, with pivot v_1. Let u_2 be any line in $\mathbb{P}^2(v_3)$ not passing through v_1. Then v_3 does not admit a weak elementary reduction with center u_2.*

Proof. We start with the notation from Set-Up 3.6. Replacing f_1, f_3 by $f_1 + af_2$, $f_3 + bf_2$ for some $a, b \in \mathbf{k}$, we can assume that $u_2 = [\![f_1, f_3]\!]$. However, it is possible with such a choice that $[f_2, f_3]$ is not a good representative anymore of m_2. There are two possibilities:

(1) either (f_1, f_2, f_3) is still a good representative for v_3, and $\deg f_2 = \deg(v_3 \smallsetminus u_2)$;
(2) or $\deg f_2 = \deg f_3 > \deg m_1$ where m_1 is the minimal point of v_3, and $\deg f_2 > \deg(v_3 \smallsetminus u_2)$.

Assume by contradiction that v_3 admits a weak elementary reduction with center u_2: there exists a non-linear polynomial $P \in \mathbf{k}[x, y]$ such that

$$\deg(v_3 \smallsetminus u_2) \geq \deg P(f_1, f_3).$$

On the other hand we know from Proposition 3.7 that the pivotal simplex of the K-reduction has Strong Pivotal Form $\circlearrowleft(s)$, hence $\deg f_3 > (s-2)\delta \geq \delta$ and $2 \deg f_3 > \deg f_2 = 2\delta$. In consequence we have $\deg_{\text{virt}} P(f_1, f_3) > \deg f_2 \geq \deg(v_3 \smallsetminus u_2)$, so that

$$\deg_{\text{virt}} P(f_1, f_3) > \deg P(f_1, f_3).$$

By Lemma 3.8(2) we have

$$\deg(v_3 \smallsetminus u_2) + d(u_2) \geq 2 \deg(v_3 \smallsetminus m_2) + d(v_2). \tag{3.18}$$

In the first alternative of Corollary 3.16, we can apply Corollary 2.8(ii) to get c

$$\deg(v_3 \smallsetminus m_2) > \deg(v_3 \smallsetminus u_2) > d(u_2).$$

This is contradictory with (3.18).

Now consider the second alternative from Corollary 3.16, which implies

$$\begin{aligned} \tfrac{1}{2} \deg(v_3 \smallsetminus m_2) &= \tfrac{1}{2} \deg f_1 = \deg f_3 \\ \tfrac{2}{3} \deg(v_3 \smallsetminus m_2) &= \tfrac{2}{3} \deg f_1 = \deg f_2 \geq \deg(v_3 \smallsetminus u_2) \end{aligned} \tag{3.19}$$

We apply Corollary 2.8(i) which gives

$$\deg(v_3 \smallsetminus u_2) \geq \deg P(f_1, f_3) \geq d(u_2) - \deg f_3,$$

which we rewrite as

$$\deg f_3 + 2 \deg(v_3 \smallsetminus u_2) \geq \deg(v_3 \smallsetminus u_2) + d(u_2).$$

Combining with (3.18) and (3.19) we get the contradiction

$$(\tfrac{1}{2} + \tfrac{4}{3}) \deg(v_3 \smallsetminus m_2) > 2 \deg(v_3 \smallsetminus m_2). \qquad \square$$

Lemma 3.20. *Let u_3 be a normalized proper K-reduction of v_3, with pivot v_1. Let $v_2' \neq m_2$ be a line in $\mathbb{P}^2(v_3)$ passing through v_1. If v_3' is a weak elementary reduction of v_3 with center v_2', then this reduction is simple with center v_2', v_1.*

Proof. We use the notation from Set-Up 3.6, and set $v_2' = \llbracket h_1, f_2 \rrbracket$ with $h_1 = f_1 + af_3$. Then $v_3 = \llbracket f_1, f_2, f_3 \rrbracket = \llbracket h_1, f_2, f_3 \rrbracket$ and $v_3' = \llbracket h_1, f_2, f_3 + P(h_1, f_2) \rrbracket$ for some polynomial P. We want to prove that $P(h_1, f_2) \in \mathbf{k}[f_2]$. It is sufficient to prove $\deg h_1 > \deg_{\text{virt}} P(h_1, f_2)$. Assume the contrary. Then

$$\deg_{\text{virt}} P(h_1, f_2) \geq \deg h_1 > \deg f_3 \geq \deg P(h_1, f_2).$$

By Corollary 3.16 we have $\deg h_1 = \deg f_1 \notin \mathbb{N} \deg f_2$, so we can apply Corollary 2.8(ii) to get

$$\deg f_1 > \deg P(h_1, f_2) > \deg dh_1 \wedge df_2 = \deg(df_1 \wedge df_2 - a df_2 \wedge df_3).$$

Then Lemmas 3.11 and 3.8(1) give

$$\deg df_1 \wedge df_2 = \deg f_3 + \deg df_2 \wedge df_3 > \deg f_3 + \deg f_1,$$

hence a contradiction. $\square$

Proposition 3.21 (Stability of a K-reduction)**.** *Let u_3 be a normalized K-reduction of v_3, and v_3' a weak elementary reduction of v_3. If u_3 is an elementary K-reduction, assume moreover that the centers of $v_3' \between v_3$ and $u_3 \between v_3$ are distinct. Then u_3 is a proper K-reduction of v_3', and moreover:*

1. *If u_3 is an elementary K-reduction of v_3, then u_3 is a (possibly non-normalized) proper K-reduction of v_3' via w_3' that satisfies $\deg w_3' = \deg v_3$;*
2. *If u_3 is a normalized proper K-reduction of v_3 via w_3, then u_3 also is a normalized proper K-reduction of v_3' via w_3.*

Proof. We denote by v_2' the center of $v_3' \between v_3$, and by $v_1 = \llbracket f_2 \rrbracket$, $v_2 = \llbracket f_1, f_2 \rrbracket$, $v_3 = \llbracket f_1, f_2, f_3 \rrbracket$ the pivotal simplex of the K-reduction u_3. By Lemma 3.17, the line v_2' passes through v_1.

First assume that u_3 is an elementary K-reduction of v_3, which is (K8). If v_2' is the minimal line of v_3, since by assumption $\deg v_3 \geq \deg v_3'$, we directly get (K6) and (K7), so that u_3 is a proper K-reduction of v_3' (via v_3). Now assume that v_2' is not the minimal line of v_3, so $v_2' = \llbracket f_1 + af_3, f_2 \rrbracket$, and $a \neq 0$ since we assume $v_2' \neq v_2$. Then by Lemma 3.2 we can write $v_3' = \llbracket f_1 + af_3, f_2, f_3 + P(f_1 + af_3, f_2) \rrbracket$ with $\deg f_3 \geq \deg P(f_1 + af_3, f_2)$. If we can show that P depends only on f_2 we are done: indeed then u_3 is a proper K-reduction of v_3' via $w_3' = \llbracket f_1, f_2, f_3 + P(f_2) \rrbracket$, where $m_2' = \llbracket f_2, f_3 + P(f_2) \rrbracket$ is the minimal line of w_3' (see Figure 6). To show that P depends only on f_2 it is sufficient to show that $\deg f_3 \geq \deg_{\text{virt}} P(f_1 + af_3, f_2)$. By contradiction, assume that this is not the case. Then

$$\deg_{\text{virt}} P(f_1 + af_3, f_2) > \deg f_3 \geq \deg P(f_1 + af_3, f_2).$$

Since v_2' has the same 2-degree as v_2, it has no inner resonance, and by Corollary 2.8(ii) we get

$$\deg P(f_1 + af_3, f_2) > \deg(df_1 \wedge df_2 + a df_3 \wedge df_2).$$

By Lemma 3.8 we have $\deg df_3 \wedge df_2 > df_1 \wedge df_2$ and $\deg df_3 \wedge df_2 > \deg f_3$, so finally we obtain the contradiction

$$\deg P(f_1 + af_3, f_2) > \deg df_2 \wedge df_3 > \deg f_3.$$

Now assume that u_3 is a normalized proper K-reduction of v_3. If $v_2' = m_2$ the minimal line of v_3, which is also the minimal line of the intermediate vertex w_3, then the conclusion is direct. Otherwise by Lemma 3.20 the reduction from v_3 to v_3' is simple with center v_2', v_1. But then by lemma 3.2 we should have $v_2' = [\![f_1 + bf_3, f_2]\!]$ and $v_3' = [\![f_1 + bf_3, f_2, f_3 + P(f_2)]\!]$. By Lemma 3.11 we have

$$\deg f_2 = 2\delta > \tfrac{3}{2}\delta \geq \deg f_3,$$

so $\deg P(f_2) > \deg f_3$ and we get a contradiction with $\deg v_3 \geq \deg v_3'$. □

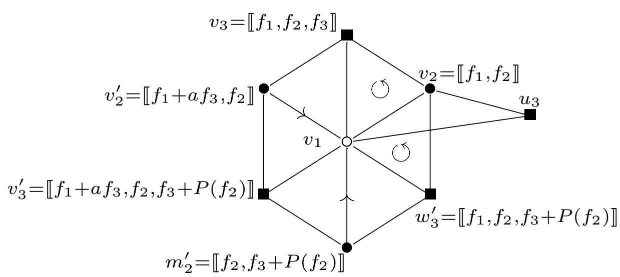


FIGURE 6. Stability of an elementary K-reduction.

4. Reducibility Theorem

4.A. **Reduction paths.** Let v_3, v_3' be vertices of type 3. A **reduction path** of length $n \geq 0$ from v_3 to v_3' is a sequence of type 3 vertices $v_3(0), v_3(1), \dots, v_3(n)$ such that:

- $v_3(0) = v_3$ and $v_3(n) = v_3'$;
- For all $i = 0, \dots, n-1$, $v_3(i+1)$ is either an elementary reduction, or a K-reduction, of $v_3(i)$.

Furthermore we say that a reduction path is **optimal** if each elementary reduction in the path is optimal, and **normalized** if each K-reduction in the path is normalized.

Given a vertex v_3 with a choice of good triangle T, we call elementary T-reduction any elementary reduction with center one of the three lines of T. We say that a vertex v_3 is **reducible** is for any good triangle T, v_3 admits either an elementary T-reduction or a (proper or elementary) K-reduction. We say that v_3 is **totally reducible** if v_3 admits a reduction path to the vertex [id], such that each vertex in the path is reducible.

In the following sections we want to prove the main result:

Theorem 4.1 (Reducibility Theorem)**.** *Any vertex of type 3 in the complex $\mathcal{C}$ is totally reducible.*

We remark that this result immediately implies that $\mathrm{Tame}(\mathbb{A}^3)$ is a proper subgroup of $\mathrm{Aut}(\mathbb{A}^3)$:

Corollary 4.2. *The Nagata's automorphism*

$$f = (x_1 + 2x_2(x_2^2 - x_1x_3) + x_3(x_2^2 - x_1x_3)^2,\ x_2 + x_3(x_2^2 - x_1x_3),\ x_3)$$

is not tame.

Proof. Denote $f = (f_1, f_2, f_3)$ the components of f. Assume that f is tame, let $v_3 = [\![f_1, f_2, f_3]\!]$ be the associated vertex in $\mathcal{C}$, and let T be the good triangle associated with this representative. We have $\deg f_1 = (2, 0, 3)$, $\deg f_2 = (1, 0, 2)$ and $\deg f_3 = (0, 0, 1)$.

On the one hand, if f admits a K-reduction, by Proposition 3.7 one of the f_i (the pivot of the reduction) should have a degree of the form 2δ: this is not the case.

On the other hand, the degrees of the f_i are pairwise $\mathbb{N}$-independent, so for any distinct $i, j \in \{1, 2, 3\}$ and any polynomial P we have $\deg_{\text{virt}} P(f_i, f_j) = \deg P(f_i, f_j)$. This implies that if f admits an elementary T-reduction, then one of the $\deg f_i$ should be a $\mathbb{N}$-combination of the other two. Again this is not the case.

Thus v_3 is not reducible, a contradiction. □

4.B. **Reduction of a strongly pivotal simplex.** First we describe the set-up that we shall use in this section.

Set-Up 4.3. Let v_1, v_2, v_3 be a simplex in $\mathcal{C}$ with Strong Pivotal Form $\circlearrowleft(s)$ for some odd $s \geq 3$. We choose some good representatives $v_1 = [\![f_2]\!]$, $v_2 = [\![f_1, f_2]\!]$ and $v_3 = [\![f_1, f_2, f_3]\!]$. Condition ($\circlearrowleft$1) means that

$$\deg f_1 = s\delta, \quad \deg f_2 = 2\delta.$$

By ($\circlearrowleft$3) we have $\deg f_1 > \deg f_3$. By ($\circlearrowleft$4) we have

$$\deg f_3 \geq (s-2)\delta + \deg df_1 \wedge df_2.$$

Condition ($\circlearrowleft$2) is equivalent to the condition

$$\deg f_3 \notin \mathbb{N} \deg f_2.$$

Since $\deg f_3 > (s-2)\delta \geq \delta$, we also obtain

$$\deg f_3^2 > \deg f_2 \ \text{ and } \ \deg f_2 \notin \mathbb{N} \deg f_3. \tag{4.4}$$

In particular $m_2 = [\![f_2, f_3]\!]$, which is the minimal line of v_3, has no inner resonance. Observe also that

$$\deg f_1 \notin \mathbb{N} \deg f_3 \text{ except if } s = 3 \text{ and } \deg f_1 = 2 \deg f_3. \tag{4.5}$$

Lemma 4.6. *Assume Set-Up 4.3. Then v_3 does not admit a normalized proper K-reduction.*

Proof. Assume v_3 admits a normalized proper K-reduction, via w_3. Then we get a contradiction as follows:

$$\begin{aligned} d(m_2) &> d(w_3 \smallsetminus m_2) && \text{by Lemma 3.8(1)} \\ &\geq d(v_3 \smallsetminus m_2) && \text{by (K6)} \\ &> d(v_3 \smallsetminus v_2) \geq \Delta(v_2) && \text{by } (\circlearrowleft 3) \text{ and } (\circlearrowleft 4) \\ &> d(v_2) > d(m_2) && \text{by Lemma 3.11.} \end{aligned}$$

□

Lemma 4.7. *Assume Set-Up 4.3. Assume that v_3 admits a weak elementary reduction v_3' with center v_2'. Then v_2' passes through v_1.*

Proof. By contradiction, assume that v_2' does not pass through v_1. We take $v_1 = [\![f_2]\!]$, and we can choose f_1 such that $v_2 = [\![f_1, f_2]\!]$ and $v_2 \cap v_2' = [\![f_1]\!]$. Then let $[\![h_3]\!]$ be the intersection of v_2' with the minimal line of $\mathbb{P}^2(v_3)$. Then we have $v_2' = [\![f_1, h_3]\!]$, $v_3 = [f_1, f_2, h_3]$, and by Lemma 3.2 $v_3' = [\![f_1, f_2 + \varphi_2(f_1, h_3), h_3]\!]$ for some polynomial φ_2. We have $\deg f_1 > \deg f_2 \geq \deg \varphi_2(f_1, h_3)$, and either $\deg f_2 = \deg h_3$ or $\deg f_3 = \deg h_3$, hence in any case by (4.4)

$$\deg_{\text{virt}} \varphi_2(f_1, h_3) > \deg \varphi_2(f_1, h_3).$$

By Lemma 2.6 there exist coprime $q > p \geq 2 \in \mathbb{N}^*$ such that

$$q \deg h_3 = p \deg f_1 = ps\delta.$$

Observe that even if (f_2, h_3) is not a good representative of the minimal line in $\mathbb{P}^2(v_3)$, in any case we have $\deg h_3 \geq \deg(v_3 \smallsetminus v_2) = \deg f_3$, and Property (⟲4) gives

$$\deg h_3 \geq (s-2)\delta + \deg df_1 \wedge df_2.$$

Then Corollary 2.11 yields:

$$\begin{aligned} 2\delta = \deg f_2 \geq \deg(f_2 + \varphi_2(f_1, h_3)) &\geq p \deg f_1 - \deg df_1 \wedge df_2 - \deg h_3 \\ &\geq p \deg f_1 - \deg h_3 + (s-2)\delta - \deg h_3 \end{aligned}$$

Multiplying by q and replacing $\deg f_1 = s\delta$ and $q \deg h_3 = ps\delta$ we get:

$$0 \geq (pqs - 2ps + sq - 4q)\delta,$$

hence

$$0 \geq ps(q-2) + q(s-4).$$

This implies $s = 3$, and we get the contradiction:

$$0 \geq 3pq - 6p - q = (3p-1)(q-2) - 2 \geq 5 - 2. \qquad \square$$

Lemma 4.8. *Assume Set-Up 4.3. Assume that v_3 admits an elementary reduction v_3' with center m_2, the minimal line of v_3. Assume moreover that v_3' is reducible. Then v_3' also admits an elementary reduction with center m_2.*

Proof. By Lemma 3.2 we can write $v_3' = [\![f_1', f_2, f_3]\!]$, where f_1' has the form

$$f_1' = f_1 + \varphi_1(f_2, f_3)$$

for some polynomial φ_1. Working with the good triangle associated with the representative (f_1', f_2, f_3), we want to prove that v_3' does not admit a K-reduction, nor an elementary reduction with center $[\![f_1', f_2]\!]$ or $[\![f_1', f_3]\!]$: Indeed since v_3' is reducible by assumption, the only remaining possibility will be that v_3' admits an elementary reduction with center $m_2 = [\![f_2, f_3]\!]$, as expected. The proof is quite long, so we prove several facts along the way. The first one is:

Fact 4.9. *If $\deg_{\text{virt}} \varphi_1(f_2, f_3) > \deg \varphi_1(f_2, f_3)$ then Lemma 4.8 holds.*

Proof. If $\deg_{\text{virt}} \varphi_1(f_2, f_3) > \deg \varphi_1(f_2, f_3)$ then by Lemma 2.6, there exist p, q coprime such that $q \deg f_2 = p \deg f_3$. Observe that (4.4) implies $p, q \neq 1$. We have

$$\begin{aligned} s\delta = \deg f_1 > \deg f_1' &= \deg f_1 + \varphi_1(f_2, f_3) && \text{by Lemma 3.2} \\ &> p \deg f_3 - \deg df_1 \wedge df_2 - \deg f_3 && \text{by Corollary 2.11} \\ &\geq p \deg f_3 - (\deg f_3 - (s-2)\delta)) - \deg f_3 && \text{by (⟲4)} \\ &= (p-2)\deg f_3 - 2\delta + s\delta. \end{aligned}$$

Multiplying by p, recalling that $\deg f_2 = 2\delta$, $p \deg f_3 = q \deg f_2$ and putting δ in factor we get:

$$0 > 2q(p-2) - 2p = (2p-4)(q-1) - 4 \geq 2p - 8. \tag{4.10}$$

It follows that $3 \geq p$. Now we deduce $p = 3$. If $p = 2$, then $\deg f_3 = q\delta$. Condition (↺4) gives

$$s\delta > \deg f_3 > (s-2)\delta,$$

so $q = s - 1$, which contradicts q coprime with 2.

Replacing $p = 3$ in the first inequality of (4.10) we get $6 > 2q$, hence $q = 2$. We obtain $\deg f_3 = \frac{4}{3}\delta$, and the condition $\deg f_3 > (s-2)\delta$ yields $s = 3$. Finally

$$\deg f_1 = 3\delta, \quad \deg f_2 = 2\delta, \quad \deg f_3 = \tfrac{4}{3}\delta, \quad 3\delta > \deg f_1' > \tfrac{7}{3}\delta.$$

First these values are not compatible with v_3' admitting a K-reduction. Indeed, an elementary K-reduction would imply $2 \deg f_1' = s' \deg f_3$ for some odd integer s', and since $3\delta > \deg f_1'$, by Lemma 3.11 a normalized proper reduction would imply $\deg f_3 = \frac{3}{2}\delta$. On the other hand, if v_3' admit an elementary reduction with center $[\![f_1', f_j]\!]$ with $j = 2$ or 3, by Corollary 2.8(iii) there exists $q' \geq 3$ odd such that $2 \deg f_1' = q' \deg f_j$: contradiction. □

From now on we assume $\deg_{\text{virt}} \varphi_1(f_2, f_3) = \deg \varphi_1(f_2, f_3)$.

Fact 4.11. $\deg f_1 = 2 \deg f_3$.

Proof. By contradiction, assume $\deg f_1 \neq 2 \deg f_3$. Then $\deg f_1 \notin \mathbb{N} \deg f_3$ by (4.5). Moreover we know that $\deg f_1 \notin \mathbb{N} \deg f_2$ and $\deg f_2 + \deg f_3 > \deg f_1$. This is not compatible with the equalities

$$\deg f_1 = \deg \varphi_1(f_2, f_3) = \deg_{\text{virt}} \varphi_1(f_2, f_3). \qquad \square$$

We deduce from (4.5) and Fact 4.11 that $s = 3$, so that

$$\deg f_1 = 3\delta, \quad \deg f_2 = 2\delta, \quad \deg f_3 = \tfrac{3}{2}\delta,$$

and there exist $a, c, e \in \mathbf{k}$ such that

$$\varphi_1(f_2, f_3) = a f_3^2 + c f_3 + e f_2 \text{ with } a \neq 0. \tag{4.12}$$

Now come some technical facts.

Fact 4.13. $\deg df_1 \wedge df_3 = \deg df_1' \wedge df_3 = \delta + \deg df_2 \wedge df_3$.

Proof. We have $\frac{3}{2}\delta = \deg f_3 > \deg df_1 \wedge df_2$, so

$$3\delta > \deg f_3 + \deg df_1 \wedge df_2.$$

Since $\deg f_1 = 3\delta$ we get

$$\deg f_1 + \deg df_2 \wedge df_3 > \deg f_3 + \deg df_1 \wedge df_2.$$

By the Principle of Two Maxima 2.13 we have

$$\deg f_2 + \deg df_1 \wedge df_3 = \deg f_1 + \deg df_2 \wedge df_3.$$

Passing $\deg f_2$ to the right-hand side we get the first expected equality

$$\deg df_1 \wedge df_3 = \deg f_1 - \deg f_2 + \deg df_2 \wedge df_3 = \delta + \deg df_2 \wedge df_3.$$

From (4.12) we get

$$df_1' \wedge df_3 = df_1 \wedge df_3 + e\, df_2 \wedge df_3.$$

By the previous equality we obtain $\deg df_1 \wedge df_3 = \deg df_1' \wedge df_3$. □

Fact 4.14. $\deg df_1' \wedge df_2 = \frac{3}{2}\delta + \deg df_2 \wedge df_3$.

Proof. From (4.12) we get

$$df_1' \wedge df_2 = df_1 \wedge df_2 + 2a f_3\, df_3 \wedge df_2 + c\, df_3 \wedge df_2.$$

By (↻4) we have $\deg f_3 > \deg df_1 \wedge df_2$, so $2a f_3\, df_3 \wedge df_2$ has strictly larger degree than the two other terms of the right-hand side. Finally,

$$\deg df_1' \wedge df_2 = \deg f_3\, df_3 \wedge df_2 = \tfrac{3}{2}\delta + \deg df_2 \wedge df_3. \qquad \square$$

Fact 4.15. $\deg f_1' > \delta$.

Proof. Consider $P = f_1 + ay^2 + cy + e f_2 \in \mathbf{k}[f_1, f_2][y]$. We have

$$\deg_{\text{virt}} P(f_3) = \deg f_1 > \deg f_1' = \deg P(f_3).$$

On the other hand $P' = 2ay + c$, so that $\deg_{\text{virt}} P'(f_3) = \deg P'(f_3) = \deg f_3$. Thus $m(P, f_3) = 1$, and the Parachute Inequality 2.4 yields

$$\begin{aligned} \deg f_1' = \deg P(f_3) &\geq \deg_{\text{virt}} P(f_3) + \deg df_1 \wedge df_2 \wedge df_3 - \deg df_1 \wedge df_2 - \deg f_3 \\ &> \deg f_1 - \deg df_1 \wedge df_2 - \deg f_3 \\ &= \deg f_3 - \deg df_1 \wedge df_2. \end{aligned}$$

Recall that by (↻4) we have

$$\deg f_3 \geq \deg f_1 - \deg f_2 + \deg df_1 \wedge df_2 = \delta + \deg df_1 \wedge df_2.$$

Replacing in the previous inequality we get the result. $\square$

Fact 4.16. *The vertices* $[\![f_1', f_2]\!]$ *and* $[\![f_1', f_3]\!]$ *do not have outer resonance in* $v_3' = [\![f_1', f_2, f_3]\!]$.

Proof. Fact 4.15 implies

$$2\deg f_1' > \deg f_2 > \deg f_3,$$

so that $\deg f_3$ is not a $\mathbb{N}$-combination of $\deg f_1'$ and $\deg f_2$. We also have

$$2\deg f_3 > \deg f_2,$$

which implies that $\deg f_2$ is not a $\mathbb{N}$-combination of $\deg f_1'$ and $\deg f_3$. $\square$

Now we are ready to finish the proof of Lemma 4.8.

Fact 4.17. v_3' *does not admit an elementary* K*-reduction.*

Proof. By contradiction, assume that v_3' admits an elementary K-reduction. From Facts 4.13 and 4.14 we have

$$\deg f_1' \wedge f_2 > \deg f_1' \wedge f_3 > \deg f_2 \wedge f_3.$$

By Corollary 3.9(1), it follows that the center of the reduction is $[\![f_2, f_3]\!]$. But then by Proposition 3.7(2) we should have $2\deg f_2 = s' \deg f_3$ for some odd integer $s' \geq 3$, and this is not compatible with $\deg f_2 = 2\delta$ and $\deg f_3 = \frac{3}{2}\delta$. $\square$

Fact 4.18. v_3' *does not admit a normalized proper* K*-reduction.*

Proof. By contradiction, assume that v_3' admits a normalized proper K-reduction u_3 via w_3. Since v_2' is the minimal line of both v_3' and v_3, we get that u_3 is also a proper K-reduction of v_3 via w_3. This is a contradiction with Lemma 4.6. $\square$

Fact 4.19. v_3' *does not admit an elementary reduction with center* $[\![f_1', f_2]\!]$.

Proof. By contradiction, assume there exists $\varphi_3(f_1', f_2)$ such that

$$\deg \varphi_3(f_1', f_2) = \deg f_3.$$

By Fact 4.16 we have $\deg_{\text{virt}} \varphi_3(f_1', f_2) > \deg \varphi_3(f_1', f_2)$, and on the other hand Proposition 3.7 gives

$$\tfrac{3}{2}\delta = \deg f_3 = \deg \varphi_3(f_1', f_2) > \deg df_1' \wedge df_2.$$

This is a contradiction with Fact 4.14. □

Fact 4.20. *v_3' does not admit an elementary reduction with center $[\![f_1', f_3]\!]$.*

Proof. By contradiction, assume there exists $\varphi_2(f_1', f_3)$ such that

$$\deg \varphi_2(f_1', f_3) = \deg f_2.$$

By Fact 4.16 we have $\deg_{\text{virt}} \varphi_2(f_1', f_3) > \deg \varphi_2(f_1', f_3)$. By Lemma 2.6 there exist $p, q \in \mathbb{N}^*$ with $p \wedge q = 1$, and $\gamma \in \mathbb{N}^3$ such that

$$\deg f_1' = p\gamma, \qquad \deg f_3 = q\gamma.$$

Corollary 2.8(i) then yields

$$2\delta = \deg f_2 = \deg \varphi_2(f_1', f_3) \geq pq\gamma + \deg df_1' \wedge df_3 - p\gamma - q\gamma.$$

By Fact 4.13 we know that $\deg df_1' \wedge df_3 = \delta + \deg df_2 \wedge df_3$, so that

$$\delta \geq \deg df_2 \wedge df_3 + (pq - p - q)\gamma. \tag{4.21}$$

We know that $q\gamma = \deg f_3 = \frac{3}{2}\delta > \delta$, and by Fact 4.15 we have $p\gamma = \deg f_1' > \delta$, so

$$\min\{p, q\} > pq - p - q.$$

By Fact 4.16 we know that $p \neq 1$ and $q \neq 1$. The only possibilities for the pair (p, q) are then $(2, 3)$ or $(3, 2)$.

If $(p, q) = (2, 3)$ then Fact 4.15 gives the contradiction

$$3\delta = 2\deg f_3 = 3\deg f_1' > 3\delta.$$

If $(p, q) = (3, 2)$, one checks that

$$\deg f_1' = \tfrac{9}{4}\delta, \quad \gamma = \tfrac{3}{4}\delta, \quad pq - p - q = 1.$$

The inequality (4.21) becomes

$$\tfrac{\delta}{4} > \deg df_2 \wedge df_3.$$

Corollary 2.11 then yields

$$\begin{aligned} \deg(f_2 + \varphi_2(f_1', f_3)) &> 2\deg f_1' - \deg f_2 \wedge f_3 - \deg f_1' \\ &\geq \left(\tfrac{9}{4} - \tfrac{1}{4}\right)\delta \\ &= 2\delta. \end{aligned}$$

This is a contradiction with $2\delta = \deg f_2 > \deg(f_2 + \varphi_2(f_1', f_3))$. □

This finishes the proof of Lemma 4.8. □

We introduce now an induction hypothesis that will be the corner-stone for the proof of the Reducibility Theorem 4.1.

Induction Hypothesis 4.22 (for degrees ν, μ in $\mathbb{N}^3$)**.** *Let v_3 be a totally reducible vertex such that $\nu \geq \deg v_3$. Then any neighbor u_3 of v_3 with $\mu \geq \deg u_3$ also is totally reducible.*

Lemma 4.23. *Let $\mu > \nu \in \mathbb{N}^3$ be two consecutive degrees, and assume Induction Hypothesis 4.22 for degrees ν, ν. Let v_3 be a totally reducible vertex with $\mu \geq \deg v_3$. Then v_3 admits an optimal and normalized reduction path to* [id]*, such that each vertex in the path is reducible.*

Proof. The lemma is obvious when $\deg v_3 = (1,1,1)$, that is, when $v_3 = [\mathrm{id}]$. Now we proceed by induction on the degree of v_3, and assume (Secondary Induction Hypothesis) that the lemma is true for any reducible vertex of degree less than $\deg v_3$.

First assume that u_3 is non-normalized proper K-reduction of v_3, appearing as the first step of a reduction path from v_3. By Lemma 3.15, there exists u_3' an elementary K-reduction of v_3, such that $u_3' \between u_3$. Since $\nu \geq \deg v_3 > \deg u_3$, by the Induction Hypothesis 4.22 we get that u_3' is totally reducible. We conclude by the Secondary Induction Hypothesis.

Finally assume that u_3 is a non-optimal elementary reduction of v_3, with center v_2, such that u_3 is totally reducible. Let u_3' be an optimal elementary reduction with the same center v_2. By the Induction Hypothesis 4.22 we get that u_3' is totally reducible, and again, we conclude by the Secondary Induction Hypothesis. □

Proposition 4.24. *Let $\mu > \nu \in \mathbb{N}^3$ be two consecutive degrees, and assume Induction Hypothesis 4.22 for degrees ν, ν. Let v_3 be a vertex that is part of a simplex v_1, v_2, v_3 with Strong Pivotal Form $\circlearrowleft(s)$ for some odd $s \geq 3$. Assume that v_3 is totally reducible, and that $\mu \geq \deg v_3$. Then*

1. *v_3 admits an elementary reduction;*
2. *Any such elementary reduction admits v_2 as a center;*
3. *Any such optimal elementary reduction is an elementary K-reduction.*

In particular there exists an elementary K-reduction u_3 of v_3 with center v_2, such that u_3 is totally reducible.

Proof. We write $v_3 = [\![f_1, f_2, f_3]\!]$ as in Set-Up 4.3. By Lemma 4.6 there is no normalized proper K-reduction for v_3. By Lemma 4.7 there is no elementary reduction of v_3 with center $[\![f_1, f_3]\!]$. So by reducibility of v_3, there exists a reduction path from v_3 that starts with one of the following elementary reductions, which gives (1):

(i) An elementary reduction v_3' with center $[\![f_2, f_3]\!]$;
(ii) An elementary K-reduction;
(iii) An elementary reduction with center $v_2 = [\![f_1, f_2]\!]$.

In case (i), by Lemma 4.23 we can moreover assume that the elementary reduction v_3' is optimal. Then Lemma 4.8 gives a contradiction.

In case (ii), by Lemma 3.8(1) we have

$$d(m_2) > \operatorname{topdeg} v_3.$$

Moreover by ($\circlearrowleft$4) we have

$$\operatorname{topdeg} v_3 > \deg(v_3 \setminus v_2) \geq \Delta(v_2) > d(v_2).$$

By Corollary 3.9(2) we conclude that v_2 is the center of the K-reduction, hence we are in case (iii), which gives (2).

Finally assume that u_3 is an optimal reduction of v_3 with center v_2. We want to prove that $\Delta(v_2) > d(u_3 \setminus v_2)$, that is, Property (K4), which will imply that u_3 is a K-reduction of v_3. By contradiction, assume $d(u_3 \setminus v_2) \geq \Delta(v_2)$, which is condition $(\circlearrowleft 4)$. Then v_1, v_2, u_3 has Strong Pivotal Form, and from assertion (2) we conclude that u_3 admits an elementary reduction with center v_2. This contradicts the optimality of the reduction from v_3 to u_3. □

Corollary 4.25. *Let $\mu > \nu \in \mathbb{N}^3$ be two consecutive degrees, and assume Induction Hypothesis 4.22 for degrees ν, ν. Let v_3 be a totally reducible vertex with $\mu \geq \deg v_3$, and assume that u_3 is an optimal elementary reduction of v_3 with pivotal simplex v_1, v_2, v_3. If v_2 has no inner resonance, and no outer resonance in v_3, then either v_2 is the minimal line of v_3, or u_3 is an elementary K-reduction of v_3.*

Proof. Assume v_2 is not the minimal line of v_3. By Proposition 3.7, the simplex v_1, v_2, v_3 has Strong Pivotal Form, so we can apply Proposition 4.24(3). □

4.C. **Vertex with two low degree neighbors.**

Set-Up 4.26. Let $\mu > \nu \in \mathbb{N}^3$ be two consecutive degrees, and assume the Induction Hypothesis 4.22 for degrees ν, μ. Let v_3, v_3', v_3'' be vertices such that

- $\mu \geq \deg v_3$, $\deg v_3 > \deg v_3'$, $\deg v_3 \geq \deg v_3''$;
- $v_3' \between_{v_2'} v_3$ and $v_3'' \between_{v_2''} v_3$ with $v_2' \neq v_2''$;
- v_3' is totally reducible (hence v_3 also is by the Induction Hypothesis);
- v_2' is minimal, in the sense that if u_3 is an elementary reduction of v_3 with center u_2, which is the first step of a reduction path, then $\deg u_2 \geq \deg v_2'$.

We denote by $v_1 = [\![f_2]\!]$ the intersection point of the lines v_2' and v_2''. We fix choices of f_1, f_3 such that $v_2' = [\![f_1, f_2]\!]$ and $v_2'' = [\![f_2, f_3]\!]$. Observe that it is possible that $\deg f_1 = \deg f_3$, and in this case (f_1, f_2, f_3) is not a good representative of v_3. In any case there exist some non linear polynomials in two variables P_1, P_3 without constant terms such that

(1) $v_3 = [f_1, f_2, f_3]$;
(2) $v_3' = [\![f_1, f_2, f_3 + P_3(f_1, f_2)]\!]$;
(3) $v_3'' = [\![f_1 + P_1(f_2, f_3), f_2, f_3]\!]$.

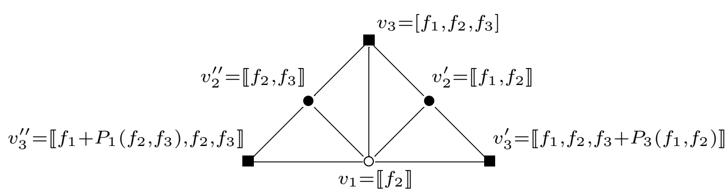


Figure 7. Set-Up 4.26.

In this section we shall prove:

Proposition 4.27. *Assume Set-Up 4.26. Then v_3'' is totally reducible.*

We divide the proof in several lemmas. The proposition will be a direct consequence of Lemmas 4.29, 4.31, 4.32 and 4.33.

We start with a consequence from the minimality of v_2' and v_3'.

Lemma 4.28. *Assume Set-Up 4.26. If v_2' has inner resonance, then v_2' is the minimal line of v_3.*

Proof. Assume first that there exists $a \in \mathbf{k}$ and $r \geq 2$ such that $\deg f_2 > \deg(f_2 + af_1^r)$. Then consider the vertex $w_3 = [f_1, f_2 + af_1^r, f_3]$, which satisfies $\deg v_3 > \deg w_3$. Assume by contradiction that v_2' is not the minimal line of v_3. Then we have $\deg f_2 > \deg f_3$, hence $m_2 = [f_1, f_3]$ is the minimal line of v_3. By the Square Lemma 3.4, we find u_3 such that $u_3 \between w_3$, $u_3 \between v_3'$ and $\deg v_3 > \deg u_3$. By applying the Induction Hypothesis 4.22 successively to $v_3' \between u_3$ and $u_3 \between w_3$, we find that w_3 is totally reducible. So we can take w_3 as the first step of a reduction path from v_3, which contradicts the minimality of v_2'.

Now assume that there exist $a \in \mathbf{k}$ and $r \geq 2$ such that $\deg f_1 > \deg(f_1 + af_2^r)$. If v_2' is not the minimal line, we have $\deg f_1 \geq \deg f_3$.

If $\deg f_1 = \deg f_3$, there exists $b \in \mathbf{k}$ such that $\deg f_1 > \deg(f_1 + bf_3)$. Then $[f_2, f_1 + bf_3]$ is the minimal line of v_3, and we consider $w_3' = [f_1 + af_2^r, f_2, f_1 + bf_3]$, which satisfies $\deg v_3 > \deg w_3'$. By the Square Lemma 3.4 and the Induction Hypothesis 4.22, we obtain that w_3' can be chosen to be the first step of a reduction path from v_3. This contradicts the minimality of v_2'.

If $\deg f_1 > \deg f_3$, $v_2'' = \llbracket f_2, f_3 \rrbracket$ is the minimal line of v_3. We consider $w_3'' = [f_1 + af_2^r, f_2, f_3]$ which satisfies $\deg v_3 > \deg w_3''$. As before w_3'' can be chosen to be the first step of a reduction path from v_3, with center v_2'': again this contradicts the minimality of v_2'. □

Lemma 4.29. *Assume Set-Up 4.26. If v_2' is not the minimal line in v_3, and has outer resonance in v_3, then v_3'' is totally reducible.*

Proof. First consider the case $\deg f_2 = \operatorname{topdeg} v_3$. The assumption on outer resonance means that $\deg f_3 = \deg f_1^r$ for some $r \geq 2$. The existence of a weak reduction via v_2'' gives the existence of $P(f_2, f_3)$ with $\deg f_1 \geq P(f_2, f_3)$. We have $\deg f_2 \notin \mathbb{N} \deg f_3$, otherwise we would have $\deg f_2 \in \mathbb{N} \deg f_1$, that is, v_2' would have inner resonance, and by Lemma 4.28 this would contradict $v_2' \neq m_2$. Corollary 2.8(iii) then gives $2 \deg f_2 = 3 \deg f_3$, and $\deg(v_3 \smallsetminus v_2'') \geq \Delta(v_2)$, that is, ($\circlearrowleft$1) and ($\circlearrowleft$4). In other words, ($\circlearrowleft$2), ($\circlearrowleft$3) and the existence of such a P implies that $[f_3], v_2'', v_3$ has Strong Pivotal Form $\circlearrowleft(3)$. But then by Lemma 4.7 the center v_2' of the elementary reduction from v_3 to v_3' should pass through $[f_3]$: contradiction.

Now consider the case $\deg f_1 = \operatorname{topdeg} v_3$. In particular $\deg f_1 > \deg f_2$ and $v_1 = \llbracket f_2 \rrbracket$ is on the minimal line of v_3. Now we distinguish two subcases (see Figure 8):

- Case $\deg f_1 > \deg f_3$. Then there exists $r \geq 2$ such that $\deg f_3 = \deg f_2^r$, and v_2'' is the minimal line of v_3. We apply the Square Lemma 3.4 to get u_3 common neighbor of v_3'' and $w_3' = [f_1, f_2, f_3 + af_2^r]$, and then we conclude by the Induction Hypothesis 4.22.
- Case $\deg f_1 = \deg f_3$. Then there exists $b \neq 0$ and $r \geq 2$ such that $\deg(bf_1 + f_3) = \deg f_2^r$. Then there exists $a \in \mathbf{k}$ such that $u_3' = [f_1, f_2, bf_1 + f_3 + af_2^r]$ and $u_3'' = [bf_1 + f_3 + af_2^r, f_2, f_3]$ are (simple) elementary reductions of v_3 with respective centers v_2' and v_2''. Moreover u_3' and u_3'' are neighbors, with center $[bf_1 + f_3 + af_2^r, f_2]$. Again we conclude by applying the Induction Hypothesis 4.22 to $v_3' \between u_3'$, $u_3' \between u_3''$ and $u_3'' \between v_3''$ successively. □

Lemma 4.30. *Assume Set-Up 4.26. Assume v_2' has no outer resonance in v_3. Then the vertex v_1 is on the minimal line m_2 of v_3.*

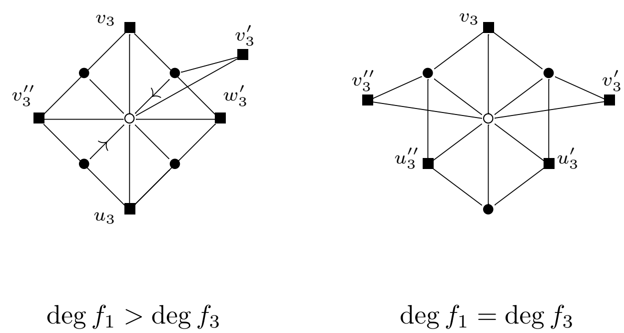


FIGURE 8. Lemma 4.29, case $\deg f_1 = \operatorname{topdeg} v_3$.

Proof. If v_2' is the minimal line of v_3, there is nothing to prove. So we can assume that v_2' is not the minimal line of v_3, and by Lemma 4.28 we can assume that v_2' has no inner resonance.

By contradiction, assume that v_1 is not on m_2. In particular, we have

$$\deg f_2 > \max\{\deg f_1, \deg f_3\}.$$

We are going to prove that v_3' is a K-reduction of v_3 with pivot $[f_1]$. Since v_3'' is a weak elementary reduction of v_3 with center v_2'', and $v_2'' = [\![f_2, f_3]\!]$ does not pass through $[f_1]$, Lemma 3.17 will give the expected contradiction.

We show that v_3' is an elementary K-reduction of $v_3 = [f_1, f_2, f_3]$, with pivot $v_1' = [f_1]$. Conditions (K1) and (K2) come from our assumptions of non resonance. Condition (K3) follows from our assumption $v_2' \neq m_2$. Condition (K5) is immediate. Finally, since $\deg f_2 > \deg f_3 \geq P(f_1, f_2)$, Corollary 2.8(iii) gives

$$P(f_1, f_2) \geq \Delta(f_2, f_1),$$

hence (K4). □

We conclude the case where v_2' is not equal to the minimal line m_2 of v_3 with the following:

Lemma 4.31. *Assume Set-Up 4.26, $v_2' \neq m_2$ is not the minimal line in v_3, and has no outer resonance in v_3. Then there exists u_3 an elementary K-reduction of v_3 with center v_2'. In particular, v_3'' is totally reducible, and there exists a reduction path starting with a proper K-reduction from v_3'' to u_3.*

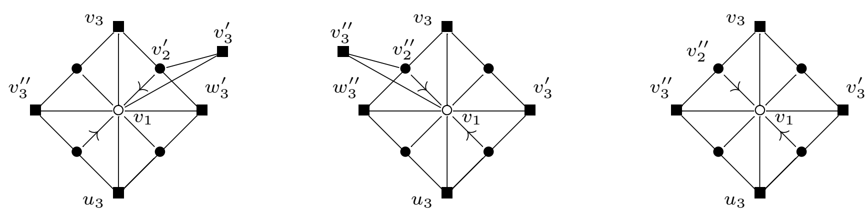
Proof. By Lemma 4.28, we know that v_2' has no inner resonance. Then Corollary 4.25 says that any optimal elementary reduction of v_3 with center v_2' is an elementary K-reduction. The last assertion follows by Stability of K-reductions 3.21. □

Now we treat the situation where $v_2' = m_2$ is the minimal line of v_3, and first we identify some cases that we can handle with the Square Lemma 3.4.

Lemma 4.32. *Assume Set-Up 4.26, and $v_2' = m_2$. In the following cases, v_3'' is totally reducible:*

(1) *v_3 admits a simple elementary reduction with center v_2', v_1;*
(2) *v_3 admits a simple elementary reduction with center v_2'', v_1;*

(3) *v_3'' is a simple weak elementary reduction of v_3.*



Case (1) Case (2) Case (3)

FIGURE 9. Lemma 4.32.

Proof. Since $m_2 = [\![f_1, f_2]\!]$, we have $v_3 = [\![f_1, f_2, f_3]\!]$ with $\deg f_3 = \operatorname{topdeg} v_3$.

(1) Denote by w_3' a simple elementary reduction of v_3 with center v_2', v_1. By the Square Lemma 3.4, there exists u_3 a neighbor of both w_3' and v_3'' such that $\deg v_3 > \deg u_3$ (see Figure 9). Then the Induction Hypothesis 4.22 applied to $v_3' \between w_3'$, to $w_3' \between u_3$, and then to $u_3 \between v_3''$ yields that all these vertices are totally reducible.

(2) Denote by w_3'' a simple elementary reduction of v_3 with center v_2'', v_1. By the Square Lemma 3.4, there exists u_3 a neighbor of both w_3'' and v_3' such that $\deg v_3 > \deg u_3$. Then the Induction Hypothesis 4.22 applied to $v_3' \between u_3$, to $u_3 \between w_3''$ and finally to $w_3'' \between v_3''$ yields that all these vertices are totally reducible.

(3) By the Square Lemma 3.4, there exists u_3 a neighbor of both v_3'' and v_3' such that $\deg v_3 > \deg u_3$. Then the Induction Hypothesis 4.22 applied to $v_3' \between u_3$, and to $u_3 \between v_3''$ yields that all these vertices are totally reducible. □

Lemma 4.33. *Assume Set-Up 4.26, $v_2' = m_2$, and that we are not in one of the cases covered by Lemma 4.32. Then there exists u_3'' an elementary K-reduction of v_3 with center v_2'' such that u_3'' is totally reducible. In particular, by the Induction Hypothesis 4.22, v_3'' is totally reducible.*

Proof. It is sufficient to check that the simplex v_1, v_2'', v_3 has Strong Pivotal Form $\circlearrowleft(s)$ for some odd $s \geq 3$: indeed then one can apply Proposition 4.24 to get the result.

On the one hand $\deg f_3 > \deg f_1 \geq \deg P_1(f_2, f_3)$, and on the other hand since we are not in the situation of Lemma 4.32(3) we have $P_1(f_2, f_3) \notin \mathbf{k}[f_2]$, so that

$$\deg_{\text{virt}} P_1(f_2, f_3) > \deg P_1(f_2, f_3).$$

We also have $\deg f_3 \notin \mathbb{N} \deg f_2$ since otherwise we could apply Lemma 4.32(1). So we are in the hypotheses of Corollary 2.8(iii), and there exist a degree δ and an odd integer $s \geq 3$ such that $\deg f_2 = 2\delta$, $\deg f_3 = s\delta$ and

$$s\delta > \deg P_1(f_2, f_3) \geq \Delta(f_3, f_2).$$

It remains to check ($\circlearrowleft$2): if $v_2'' = [\![f_2, f_3]\!]$ had outer resonance in v_3 then we would have $\deg f_1 \in \mathbb{N} \deg f_2$, and we could apply Lemma 4.32(2), contrary to our assumption. □

4.D. **Proof of the Reducibility Theorem.** Clearly Theorem 4.1 is a corollary of

Proposition 4.34. *If a vertex v_3 of type 3 is totally reducible, then any neighbor of v_3 also is totally reducible.*

Proof. We plan to prove Proposition 4.34 by induction on degree: we need to prove that for any $\nu \in \mathbb{N}^3$, the Induction Hypothesis 4.22 holds for degrees ν, ν. Clearly when $\nu = (1,1,1)$ this is true (because empty!).

Let $\mu > \nu$ be two consecutive degrees in $\mathbb{N}^3$. It is sufficient to prove the two following facts.

Fact 4.35. *Assume the Induction Hypothesis 4.22 for degrees ν, ν. Then it also holds for degrees ν, μ.*

Fact 4.36. *Assume the Induction Hypothesis 4.22 for degrees ν, μ. Then it also holds for degrees μ, μ.*

To prove Fact 4.35, consider v_3' a totally reducible vertex with $\nu \geq \deg v_3'$, and let v_3 be a neighbor of v_3', with center v_2', and with $\deg v_3 = \mu$ (otherwise there is nothing to prove).

If v_2' is the minimal line of v_3, then v_3' is a T-reduction of v_3 for any good triangle T and we are done.

If v_2' is not the minimal line of v_3, and has no inner or outer resonance in v_3, then by Corollary 4.25 any optimal reduction u_3 of v_3 with respect to the center v_2' is an elementary K-reduction of v_3. Since u_3 is a neighbor of v_3' and $\nu \geq \deg u_3$, we conclude by the Induction Hypothesis 4.22.

Finally assume that v_2' has resonance, and that v_2' is not the minimal line of v_3. By Lemma 3.2 we can write

$$v_3 = [\![f_1, f_2, f_3]\!], \quad v_3' = [\![f_1, f_2, g_3]\!], \quad v_2' = [\![f_1, f_2]\!],$$

with $\deg f_1 > \max\{\deg f_3, \deg f_2\}$ and $g_3 = f_3 + P(f_1, f_2)$ for some polynomial P.

If v_2' has inner resonance, then $\deg f_1 = r \deg f_2$ for some $r \geq 2$. There exists $a \in \mathbf{k}$ such that $v_3'' = [f_1 - af_2^r, f_2, f_3]$ is an elementary reduction of v_3 with center $[\![f_2, f_3]\!]$, which is the minimal line of v_3. Then we can apply the Square Lemma 3.4 to get u_3 with $\nu \geq \deg u_3$ and $u_3 \between v_3'$, $u_3 \between v_3''$. Again we conclude by the Induction Hypothesis 4.22.

If v_2' has no inner resonance, but has outer resonance in v_3, then $\deg f_1 > \deg f_3 > \deg f_2$ and $\deg f_3 = r \deg f_2$ for some $r \geq 2$. There exists $Q(f_2)$ such that $\deg f_3 > \deg(f_3 + Q(f_2))$ and $\deg(f_3 + Q(f_2)) \notin \mathbb{N} \deg f_2$. Let T be a good triangle of v_3, one of the line has the form $u_2 = [\![f_1 + af_3, f_2]\!]$. Set $u_3 = [\![f_1 + af_3, f_2, f_3 + Q(f_2)]\!]$, which is an elementary reduction of v_3 with center u_2, and a neighbor of $w_3 = [\![f_1, f_2, f_3 + Q(f_2)]\!]$. By the Induction Hypothesis 4.22 applied successively to $v_3' \between w_3$ and $w_3 \between u_3$, we obtain that u_3 is totally reducible.

To prove Fact 4.36, consider v_3 a totally reducible vertex with $\deg v_3 = \mu$, and let v_3'' be a neighbor of v_3, with center v_2'', such that

$$\deg v_3 \geq \deg v_3''.$$

First assume that v_3 admits a reduction path such that the first step is a proper K-reduction v_3'. Then by Stability of K-reduction 3.21, we obtain that v_3' is also a K-reduction of v_3'', and we are done.

Now we assume that v_3 admits a reduction path such that the first step v_3' is an elementary reduction, with center v_2'. Moreover we assume that v_2' has minimal degree between all possible such first center of the path.

If $v_2' = v_2''$, then v_3'' is a neighbor of v_3' and by the Induction Hypothesis 4.22 we are done. If on the contrary v_2' and v_2'' are two different lines in $\mathbb{P}^2(v_3)$, we are in Set-Up 4.26. Then we conclude by Proposition 4.27. □

5. Simple connectedness

Now that the Reducibility Theorem 4.1 is proved, all previous results that were dependent of a reducibility assumption become stronger. This is the case in particular for:

- The Induction Hypothesis 4.22, which is always true;
- Lemma 4.8;
- Proposition 4.24 and Corollary 4.25;
- Set-Up 4.26, hence also all results in §4.C.

In particular we single out the following striking consequences of the Reducibility Theorem 4.1.

Proposition 5.1. *Let u_3 be an elementary K-reduction of v_3, with center v_2. Then v_3 does not admit any elementary reduction with center distinct from v_2.*

Proof. By contradiction, assume that v_3' is an elementary reduction of v_3, with center v_2' distinct from v_2. Then by Stability of K-reduction 3.21(1), there exists w_3 with $\deg w_3 = \deg v_3$ such that u_3 is a proper K-reduction of v_3' via w_3. In particular, v_3' is an elementary reduction of w_3 with center m_2, the minimal line of w_3. Now consider v_3'' an optimal elementary reduction of w_3 with center m_2: Lemma 4.8 gives a contradiction. □

Corollary 5.2. *Let u_3 be a proper K-reduction of v_3, via w_3. Then $\deg w_3 = \deg v_3$.*

Proof. By Proposition 5.1 we cannot have $\deg w_3 > \deg v_3$. □

We also obtain that the situation of Lemma 4.33 never happens:

Corollary 5.3. *Assume Set-Up 4.26, and $v_2' = m_2$. Then we are in one of the cases covered by Lemma 4.32.*

Proof. Otherwise by Lemma 4.33 there would exist an elementary K-reduction of v_3 with center v_2'', in addition to the elementary reduction with center v_2', in contradiction with Proposition 5.1. □

We call **locally geodesic loop** of length $n \geq 2$, with base point the vertex [id], a sequence of vertices $v_3(i)$, $i = 0, \dots, n$, such that

1. $v_3(0) = v_3(n) = [\mathrm{id}]$;
2. For all $i = 0, \dots, n-1$, $v_3(i) \between v_3(i+1)$ with center $v_2(i)$;
3. For all $i = 0, \dots, n-1$, $v_2(i) \neq v_2(i+1)$, where by convention $v_2(n) = v_2(0)$.

The **maximal vertex** of such a loop is defined as the vertex $v_3(i_0)$ that realizes the maximum $\max \deg v_3(i)$, with i_0 maximal. In particular, we have

$$\deg v_3(i_0) > \deg v_3(i_0 + 1) \text{ and } \deg v_3(i_0) \geq \deg v_3(i_0 - 1).$$

Lemma 5.4. *Let $v_3(i_0)$ be the maximal vertex of a locally geodesic loop. Then the path $v_3(i_0-1), v_2(i_0-1), v_3(i_0), v_2(i_0), v_3(i_0+1)$ is homotopic in $\mathcal{C}$ to a path from $v_3(i_0-1)$ to $v_3(i_0+1)$ where all type 3 intermediate vertices have degree strictly less than* $\deg v_3(i_0)$.

Proof. Let v_2' be a vertex of minimal degree such that $v_3 = v_3(i_0)$ admits an elementary reduction v_3' with center v_2'. Then we apply Set-Up 4.26 twice, taking v_3'' to be successively $v_3(i_0-1)$ and $v_3(i_0+1)$ (the case where v_3' and v_3'' share the same center being trivial).

If $v_2' = m_2$ then by Corollary 5.3 we are in one of the cases covered by Lemma 4.32, and the homotopy is clear in all cases (see Figure 9).

If $v_2' \neq m_2$, and v_2' has outer resonance in v_3, then we are in one of the two cases covered by Lemma 4.29, and again the homotopy is clear (see Figure 8).

If $v_2' \neq m_2$, and v_2' has no outer resonance in v_3, then by Lemma 4.31 there exists u_3 a K-reduction of v_3 with center v_2'. By Proposition 5.1, this can only happen in the case $v_3'' = v_3(i_0-1)$. By Stability of K-reduction 3.21, u_3 is a proper K-reduction of $v_3(i_0-1)$, and up to a local homotopy (see Figure 6) we can assume that the intermediate vertex is $v_3(i_0)$. If this proper K-reduction is not normalized, we obtain the expected homotopy from the normalization process (see Figure 5). Otherwise, by Proposition 5.1 again, we get $\deg v_3(i_0-1) = \deg v_3(i_0)$, and by Stability of K-reduction 3.21 we get that u_3 is a normalized proper K-reduction of $v_3(i_0-2)$. Iterating this process, we obtain the contradiction that u_3 is a proper K-reduction of $[\mathrm{id}] = v_3(0)$. □

We need one last ingredient before proving the simple connectedness of the complex $\mathcal{C}$.

Lemma 5.5. *The link of a vertex of type 1 is a connected graph.*

Proof. By transitivity of the action of $\mathrm{Tame}(\mathbb{A}^3)$ on vertices of type 1, it is sufficient to work with $\mathcal{L}([x_3])$. Let $v_3 = [\![f_1, f_2, x_3]\!]$ be a vertex of type 3 in $\mathcal{L}([x_3])$. First observe that v_3 does not admit an elementary K-reduction: if $s \geq 5$ we should have $\deg x_3 = 2\delta$ for some $\delta \in \mathbb{N}^3$, and if $s = 3$ we should have $\deg f_2 = 2\delta$ for some $\delta \in \mathbb{N}^3$, and $(0,0,2) = \deg x_3^2 > \deg f_2$: impossible. It follows that v_3 also does not admit a proper K-reduction: such a reduction would be via $w_3 = [\![g_1, f_2, x_3]\!]$, but we just proved that such a w_3 cannot admit an elementary K-reduction. By Theorem 4.1, we conclude that v_3 admits an elementary reduction v_3', which clearly must admits $[x_3]$ as pivot (since there is no polynomial $P \in \mathbf{k}[x_1, x_2, x_3]$ with $\deg x_3 > \deg P$). In particular, v_3' also is in $\mathcal{L}([x_3])$, and by induction on degree, we obtain the existence of reduction path from v_3 to $[\mathrm{id}]$ that stays in $\mathcal{L}([x_3])$. □

Now we recover a result of [Umi06] and [Wri15], about relations in $\mathrm{Tame}(\mathbb{A}^3)$. Precisely, Umirbaev gives an algebraic description of the relations, and Wright shows that this result can be rephrased in terms of an amalgamated product structure over three subgroups. Our proof follows the same strategy as in [BFL14, Proposition 3.10].

Proposition 5.6. *The complex $\mathcal{C}$ is simply connected.*

Proof. Let γ be a loop in $\mathcal{C}$. We want to show that it is homotopic to a trivial loop. Without loss in generality, we can assume that the image of γ is contained in the 1-skeleton of the square complex, and that $\gamma(0) = [\![x_1, x_2, x_3]\!]$ is the vertex of type 3 associated with the identity.

A priori (the image of) γ is a sequence of arbitrary edges. By Lemma 5.5, we can perform a homotopy to avoid each vertex of type 1. So now we assume that vertices in γ are alternatively of type 2 and 3: Precisely for each i, $\gamma(2i)$ has type 3 and $\gamma(2i+1)$ has type 2. We can also assume that γ is locally injective.

Let $v_3(i) = \gamma(2i)$, this is a locally geodesic loop. Let $v_3(i_0)$ be the maximal vertex of the loop, and δ its degree. Then by Lemma 5.4 we can conclude by induction on the couple (δ, i_0), ordered with lexicographic order. □

Since $\mathrm{Tame}(\mathbb{A}^3)$ acts on a simply connected 2-dimensional simplicial complex with fundamental domain a simplex, we can recover the group from the data of the stabilizers of each type of vertex. This is a simple instance of the theory of developable complexes of groups in the sense of Haefliger (see [BH99, III.$\mathcal{C}$]). Following Wright we can phrase this remark as follows:

Corollary 5.7 ([Wri15, Theorem 2])**.** *The group* $\mathrm{Tame}(\mathbb{A}^3)$ *is the amalgamated product of the three following subgroups along their pairwise intersections:*

$$\begin{aligned}\mathrm{Stab}([x_1,x_2,x_3]) &= A_3;\\ \mathrm{Stab}([x_1,x_2]) &= \{(ax_1+bx_2+c, a'x_1+b'x_2+c', \alpha x_3 + P(x_1,x_2))\};\\ \mathrm{Stab}([x_1]) &= \{(ax_1+b, f_2(x_1,x_2,x_3), f_3(x_1,x_2,x_3))\}.\end{aligned}$$

6. Examples

We gather in this last section a few examples of interesting reductions.

Example 6.1 (Elementary K-reduction with $s=3$)**.** Let

$$\begin{aligned}g &= (x_1,\, x_2,\, x_3 + x_1^2 - x_2^3),\\ t_1 &= (x_1 + \alpha x_2 x_3 + x_3^3,\, x_2 + x_3^2,\, x_3).\end{aligned}$$

Clearly in the composition $g\circ t_1$ the terms of degree 6 cancel each other. Moreover, if we choose $\alpha = \frac{3}{2}$ this is also the case for terms of degree 5:

$$g\circ t_1 = \left(x_1 + \tfrac{3}{2}x_2x_3 + x_3^3,\, x_2 + x_3^2,\, x_3 + x_1^2 - x_2^3 + 3x_1x_2x_3 + \tfrac{x_3^2}{4}(8x_1x_3 - 3x_2^2)\right).$$

Consider now a triangular automorphism preserving the quadratic form $8x_1x_3 - 3x_2^2$ that appears as a factor:

$$t_2 = (x_1,\, x_2 + x_1^2,\, x_3 + \tfrac{3}{4}x_1x_2 + \tfrac{3}{8}x_1^3).$$

A direct computation shows that the automorphism $f = g\circ t_1\circ t_2 = (f_1, f_2, f_3)$ admits the following 3-degree:

$$(9,0,0),\ (6,0,0),\ (7,0,1).$$

Finally, $u_3 = [t_1\circ t_2]$, whose 3-degree is

$$(9,0,0),\ (6,0,0),\ (3,0,0),$$

is an elementary K-reduction of $v_3 = [f_1, f_2, f_3]$. Following notation from Proposition 3.7, we have $s = 3$ and $\delta = (3,0,0)$. Moreover

$$\begin{aligned}df_1\wedge df_2 = -\tfrac{3}{2}(x_1^2 - x_2)dx_2\wedge dx_3 + (\tfrac{27}{16}x_1^2x_2 + \tfrac{9}{8}x_2^2 + \tfrac{3}{2}x_1x_3 + 1)dx_1\wedge dx_2\\ + (-\tfrac{9}{4}x_1^3 - \tfrac{3}{2}x_1x_2 + 2x_3)dx_1\wedge dx_3.\end{aligned}$$

so that $\deg df_1\wedge df_2 = (4,0,1)$, from the contribution of the factor $x_1^3 dx_1\wedge dx_3$.

Example 6.2 (Elementary K-reduction with $s = 5$)**.** One can apply the same strategy to produce examples of K-reduction with $s \geq 3$ an arbitrary odd number. I give the construction for $s = 5$, and leave the generalization to the reader. Let

$$\begin{aligned} g &= (x_1,\, x_2,\, x_3 + x_1^2 - x_2^5), \\ t_1 &= (x_1 + \alpha x_2^2 x_3 + \beta x_2 x_3^3 + x_3^5,\, x_2 + x_3^2,\, x_3). \end{aligned}$$

Observe that $\alpha x_2^2 x_3 + \beta x_2 x_3^3 + x_3^5$ is homogeneous of degree 5, by putting weight 1 on x_3 and weight 2 on x_2. By choosing $\alpha = \frac{15}{8}$, $\beta = \frac{5}{2}$, we minimize the degree of the composition:

$$g \circ t_1 = \left(x_1 + \tfrac{15}{8} x_2^2 x_3 + \tfrac{5}{2} x_2 x_3^3 + x_3^5,\, x_2 + x_3^2,\, x_3 + \tfrac{1}{8} x_3^4 (16 x_1 x_3 - 5 x_2^3) + \cdots\right).$$

Now take the following triangular automorphism, which preserves the polynomial $16 x_1 x_3 - 5 x_2^3$:

$$t_2 = \left(x_1,\, x_2 + x_1^2,\, x_3 + \tfrac{5}{16}(3 x_1 x_2^2 + 3 x_1^3 x_2 + x_1^5)\right).$$

We compute the 3-degree of $f = g \circ t_1 \circ t_2 = (f_1, f_2, f_3)$:

$$(25, 0, 0),\ (10, 0, 0),\ (20, 3, 0).$$

Finally, $u_3 = [t_1 \circ t_2]$, whose 3-degree is

$$(25, 0, 0),\ (10, 0, 0),\ (5, 0, 0),$$

is an elementary K reduction of $v_3 = [f_1, f_2, f_3]$. Here we have $s = 5$ and $\delta = (5, 0, 0)$. Moreover

$$\begin{aligned} df_1 \wedge df_2 = &-\tfrac{15}{8}\left(x_1^2 + x_2\right)^2 dx_2 \wedge dx_3 + \left(2x_3 - \tfrac{5}{8}(5x_1^5 - 9x_1^3 x_2 - 3x_1 x_2^2)\right) dx_1 \wedge dx_3 \\ &+ \left(\tfrac{75}{128}(5x_1^4 x_2^2 + 9x_1^2 x_2^3 + 3x_2^4) + \tfrac{15}{8}(x_1^3 x_3 + 2x_1 x_2 x_3) + 1\right) dx_1 \wedge dx_2, \end{aligned}$$

so that $\deg df_1 \wedge df_2 = (5, 3, 0)$, from the contribution of the factor $x_1^4 x_2^2 dx_1 \wedge dx_2$.

Example 6.3 (Elementary reduction without Strong Pivotal Form)**.** We give examples of elementary reduction that show that the three assumptions in Proposition 3.7 are necessary to get Strong Pivotal Form.

(1) Let $f = (f_1, f_2, f_3) \in \mathrm{Tame}(\mathbb{A}^3)$ and $r \geq 2$ such that

$$\deg f_1 > \deg f_3 = r \deg f_2.$$

In particular there exists $a \in \mathbf{k}$ such that $\deg f_3 > \deg f_3 + a f_2^r$. For instance $f = (x_1 + x_3^3, x_2, x_3 + x_2^2)$ is such an automorphism, for $r = 2$ and $a = -1$. Then $u_3 = [\![f_1, f_2, f_3 + a f_2^r]\!]$ is an elementary reduction of $v_3 = [\![f_1, f_2, f_3]\!]$, and the pivotal simplex does not have Strong Pivotal Form. Observe that $v_2 = [\![f_1, f_2]\!]$ has outer resonance in v_3.

(2) Let $u_3 = [\![f_1, f_2, f_3 + P(f_1, f_2)$ be an elementary K-reduction of $w_3 = [\![f_1, f_2, f_3]\!]$, with $2 \deg f_1 = s \deg f_2$ for some odd $s \geq 3$. For instance we can start with one of the examples 6.1 or 6.2. Pick any integer $r \geq \frac{s+1}{2}$. Then $v_3' = [\![f_1 + f_2^r, f_2, f_3 + P(f_1 - f_2^r, f_2)]\!]$ is an elementary reduction of $v_3 = [\![f_1 + f_2^r, f_2, f_3]\!]$, and the pivotal simplex does not have Strong Pivotal Form. Observe that the center $v_2 = [\![f_1 + f_2^r, f_2]\!]$ has inner resonance.

(3) With the notation of Example 6.1 or 6.2, the elementary reduction from $[\![g \circ t_1]\!]$ to $[\![t_1]\!]$ gives an example of an elementary reduction where the center m_2, which is the minimal line, does not have inner or outer resonnance, and again the pivotal simplex does not have Strong Pivotal Form.

Example 6.4 (Non-normalized proper K-reduction)**.** Pick the elementary K-reduction from Example 6.2, and set $v_3' = [f_1 + f_2^2, f_2, f_3]$, which is a weak elementary reduction of v_3. Then u_3 is a non-normalized proper K-reduction of v_3', via v_3. This corresponds to case (1) of Stability of a K-reduction 3.21. We mention again that it is an open question whether there exists any normalized proper K-reduction.

Non Example 6.5 (Hypothetical type II and type III reductions)**.** From Proposition 3.7 we know that if v_3 admits an elementary K-reduction with coefficient $s = 3$, then $v_3 = [\![f_1, f_2, f_3]\!]$ with $\deg f_1 = 3\delta$, $\deg f_2 = 2\delta$ and $\deg f_3 > \delta$. It is not clear if there exists an example of such a reduction with $\frac{3}{2}\delta > \deg f_3$, or even $2\delta > \deg f_3$. Observe that such an example would be the key for the existence of the following reductions (for the definition of a reduction of type II or III, see the original paper [SU04b], or [Kur10, §7]):

(1) If $\frac{3}{2}\delta > \deg f_3$, then $v_3' = [\![f_1 + f_3^2, f_2, f_3]\!]$ would admit a normalized proper K-reduction, via v_3: This would correspond to a type III reduction.

(2) If $2\delta > \deg f_3 > \frac{3}{2}\delta$, then v_3 would admit an elementary K-reduction such that the pivot $[f_2]$ is distinct from the minimal vertex $[f_3]$: This would correspond to a type II reduction.

(3) If $\frac{3}{2}\delta > \deg f_3$, then $v_3'' = [\![f_1, f_2 + f_3^2, f_3]\!]$ would be an example of a vertex that admits a reduction along a center with outer resonance in v_3'', but that does not admit a reduction with center the minimal line of v_3'' (see Lemma 4.29).

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Institut de Mathématiques de Toulouse, Université Paul Sabatier, 118 route de Narbonne, 31062 Toulouse Cedex 9, France

E-mail address: slamy@math.univ-toulouse.fr