

CHARACTERIZATION OF AFFINE TORIC VARIETIES BY THEIR AUTOMORPHISM GROUPS

ALVARO LIENDO, ANDRIY REGETA, AND CHRISTIAN URECH

ABSTRACT. We show that complex affine toric surfaces are determined by the abstract group structure of their regular automorphism groups in the category of complex normal affine surfaces using properties of the Cremona group. As a generalization to arbitrary dimensions, we show that complex affine toric varieties, with the exception of the algebraic torus, are uniquely determined in the category of complex affine normal varieties by their automorphism groups seen as ind-groups.

1. INTRODUCTION

In the whole paper we work over the field \mathbb{C} of complex numbers and varieties are considered to be irreducible. Let T be the complex algebraic torus, i.e. $T \cong \mathbb{G}_m^n$, where $\mathbb{G}_m = (\mathbb{C}^*, \cdot)$ is the multiplicative group of the base field \mathbb{C} . A toric variety is a normal algebraic variety endowed with a T -action having a Zariski dense open orbit. In particular, affine spaces, projective spaces and algebraic tori are toric varieties. The automorphism groups of affine toric varieties of dimension greater than one are never algebraic groups and most of the time they are infinite dimensional.

In this paper we are interested in the following question: is a toric variety uniquely determined by its automorphism group? Our work can be seen in the context of the Erlangen program of Felix Klein, in which he suggested to understand geometrical objects through their groups of symmetries ([Kle93]). Note that in general it is impossible to characterize affine algebraic varieties by their groups of regular automorphisms since most of them have a trivial automorphism group. In this paper, we show that in the toric case the automorphism group is in most of the cases sufficiently rich to uniquely determine the underlying variety. Our first main result is:

Theorem 1.1. *Let S_1 be an affine toric surface and let S_2 be a normal affine surface. If $\text{Aut}(S_1)$ and $\text{Aut}(S_2)$ are isomorphic as groups, then S_1 and S_2 are isomorphic.*

This theorem can be seen to be in the spirit of the previous papers [Des06a] and [Ca14] in which the authors consider abstract embeddings of continuous groups into $\text{Bir}(\mathbb{P}^n)$.

An important class of elements in $\text{Aut}(S)$, where S is an affine surface, are *algebraic* elements, i.e. elements that are contained in an algebraic group acting regularly on S (see Section 2.3). The main idea of the proof of Theorem 1.1 is to consider an automorphism of an affine surface S as an element of $\text{Bir}(S)$, the group of birational transformations of S . We show that an element in $\text{Bir}(S)$ is algebraic if and only if an iterate of it is divisible (Theorem 3.1). From this purely group theoretical characterization we obtain that algebraic elements are preserved under group homomorphisms and we will be able to reconstruct the corresponding surfaces.

Much less is known about the group structure of $\text{Bir}(X)$ if X is a variety of dimension greater than two. Hence, we are not able to generalize Theorem 1.1 to arbitrary dimensions. However, the automorphism group of an affine variety comes with the additional structure of an ind-group (see Section 2.1). We show that the automorphism group as an ind-group determines a toric variety in most of the cases:

Theorem 1.2. *Let X be an affine toric variety different from the algebraic torus and let Y be a normal affine variety. If $\text{Aut}(X)$ and $\text{Aut}(Y)$ are isomorphic as ind-groups, then X and Y are isomorphic.*

2010 Mathematics Subject Classification: Primary 14M25, secondary 14E07, 14R10.

Key words: Affine toric varieties, automorphism groups, Cremona groups.

The first author was supported by Fondecyt project number 1160864. The second author was partially supported by SNF, project number P2BSP2_165390 and the third author was supported by the SNF, project number P2BSP2_175008.

In fact, we show that toric varieties and their automorphism groups are uniquely determined by the weights of their root subgroups. In the case of finite dimensional algebraic groups this is a property of reductive groups, see [Sp98, Section 4.4].

Theorem 1.2 can be seen as a generalisation of the results from [Kr15] (see also [Reg17]) showing that the complex affine space is characterized by its automorphism group seen as an ind-group. The assumption that X is not an algebraic torus in Theorem 1.2 is necessary, as the following result shows:

Theorem 1.3. *Let T be an algebraic torus and let C be a smooth affine curve. If C has trivial automorphism group and no invertible global functions, then $\text{Aut}(T)$ and $\text{Aut}(C \times T)$ are isomorphic as ind-groups.*

Note that there exist many curves C with the properties required in Theorem 1.3. For instance, take any smooth complete curve with a trivial automorphism group and remove one point.

The normality condition in Theorem 1.1 and Theorem 1.2 cannot be removed. Indeed, let S_d be the quotient of \mathbb{A}^2 by the cyclic group $\mu_d = \{\xi \in \mathbb{C} \mid \xi^d = 1\}$, $|\mu_d| > 1$, where μ_d acts on \mathbb{A}^2 by scalar multiplication. Then S_d is a toric surface and the ring of regular functions of S_d is $\mathcal{O}(S_d) = \bigoplus_{k \geq 0} \mathbb{C}[x, y]_{dk}$, where $\mathbb{C}[x, y]_{dk}$ is the vector space of homogeneous polynomials of degree dk . Denote by S_d^l the variety with the ring of regular functions $\mathbb{C} \oplus \bigoplus_{k \geq l} \mathbb{C}[x, y]_{dk}$ for some $l > 0$. Then S_d^l is not normal and its normalization equals S_d . One can show that $\text{Aut}(S_d)$ and $\text{Aut}(S_d^l)$ are isomorphic as ind-groups for any $d, l \in \mathbb{Z}_{>0}$, $d > 1$ (see [Reg17] for details).

However, in the particular case of the two-dimensional affine space, one can remove the normality hypothesis:

Theorem 1.4. *Let S be a complex affine surface such that $\text{Aut}(S)$ and $\text{Aut}(\mathbb{A}^2)$ are isomorphic as groups. Then S is isomorphic to \mathbb{A}^2 .*

In [Des06b] it is shown that all group automorphisms of $\text{Aut}(\mathbb{A}^2)$ are inner up to automorphisms of the base-field \mathbb{C} . Together with Theorem 1.4 this implies directly:

Corollary 1.5. *Let S be a complex affine surface and $\varphi: \text{Aut}(S) \rightarrow \text{Aut}(\mathbb{A}^2)$ a group isomorphism. Then S is isomorphic to \mathbb{A}^2 and φ is given, up to a field automorphism of the base-field \mathbb{C} , by conjugation with an isomorphism between S and \mathbb{A}^2 .*

This corollary can be seen as an algebraic analogue to a result of Filipkiewicz ([Fil82]) about isomorphisms between groups of diffeomorphisms of manifolds without boundary.

Acknowledgements. The authors would like to thank J. Blanc, M. Brion, S. Cantat, H. Kraft, A. Perechko and M. Zaidenberg for useful discussions.

2. AUTOMORPHISMS AND BIRATIONAL TRANSFORMATIONS OF AFFINE VARIETIES

In this section we recall some results that we will need about automorphisms and birational transformations of normal affine varieties.

2.1. Ind-groups. The notion of an ind-group was introduced by Shafarevich who called these objects infinite dimensional groups, see [Sh66]. We refer to [Ku02] and the paper in preparation [FK18] for basic notations in this context.

Definition 2.1. An *ind-variety* is a set V together with an ascending filtration $V_0 \subset V_1 \subset V_2 \subset \dots \subset V$ such that the following conditions are satisfied:

- (1) $V = \bigcup_{k \geq 0} V_k$;
- (2) each V_k has a structure of an algebraic variety;
- (3) for every $k \in \mathbb{Z}_{\geq 0}$, the embedding $V_k \subset V_{k+1}$ is closed in the Zariski-topology.

A *morphism* between ind-varieties $V = \bigcup_k V_k$ and $W = \bigcup_m W_m$ is a map $\varphi: V \rightarrow W$ such that for any k there is an $m \in \mathbb{Z}_{\geq 0}$ such that $\varphi(V_k) \subset W_m$ and such that the induced map $V_k \rightarrow W_m$ is a

morphism of algebraic varieties. An *isomorphism* of ind-varieties is defined in the usual way. An ind-variety V can be equipped with a topology: a subset $S \subset V$ is open if $S_k := S \cap V_k \subset V_k$ is open for all k . A closed subset $S \subset V$ has the natural structure of an ind-variety and is called an *ind-subvariety*. An ind-variety V is called *affine* if all the V_k are affine varieties.

Definition 2.2. An affine ind-variety G is called an *ind-group* if the underlying set G is a group such that the map $G \times G \rightarrow G$, defined by $(g, h) \mapsto gh^{-1}$, is a morphism of ind-varieties.

A closed subgroup H of G is again an ind-group under the closed ind-subvariety structure on G . A closed subgroup H of an ind-group G is an algebraic subgroup if and only if H is an algebraic subset of G . A proof of the next proposition can be found in [St13]:

Proposition 2.3. *Let X be an affine variety. Then $\text{Aut}(X)$ has the structure of an ind-group such that for any algebraic group G , a regular G -action on X induces an ind-group homomorphism $G \rightarrow \text{Aut}(X)$.*

We need the following definition:

Definition 2.4. An ind-group G is *connected* if for every element $g \in G$ there exists an irreducible curve D and a morphism $D \rightarrow G$ whose image contains the identity element e and g . The *connected component* G^0 of an ind-group G is the maximal connected subgroup of G which contains e .

The following observation will turn out to be useful:

Lemma 2.5. *Let $U \subset \text{Aut}(X)$ be a commutative subgroup which coincides with its centraliser. Then U is a closed subgroup of $\text{Aut}(X)$*

Proof. Let $u \in U$ and define $G_u = \{g \in \text{Aut}(X) \mid gu = ug\}$. Since $ug = gu$ is a closed condition on each filter set, we obtain that $G_u \subset \text{Aut}(X)$ is a closed subgroup. Hence, $\bigcap_{u \in U} G_u = U$ is closed in $\text{Aut}(X)$. \square

2.2. The Zariski topology on $\text{Bir}(X)$. Let X be a variety and denote by $\text{Bir}(X)$ its group of birational transformations. Blanc and Furter show in [BF13] that $\text{Bir}(\mathbb{P}^n)$ is not an ind-group. However, it still comes with the so-called *Zariski topology*, which has been introduced by Demazure ([De70]). Let A be a variety and let

$$f: A \times X \rightarrow A \times X$$

be an A -birational map, i.e. f is the identity on the first factor, that induces an isomorphism between open subsets U and V of $A \times X$ such that the projections from U and from V to A are both surjective. From this definition it follows that each $a \in A$ defines an element in $\text{Bir}(X)$ and we obtain a map $A \rightarrow \text{Bir}(X)$. A map of this form is called a *morphism*. The *Zariski topology* is now defined to be the finest topology on $\text{Bir}(X)$ such that all the morphisms $A \rightarrow \text{Bir}(X)$ for all varieties A are continuous with respect to the Zariski topology on A . For all $g \in \text{Bir}(X)$ the maps $\text{Bir}(X) \rightarrow \text{Bir}(X)$ given by $x \mapsto x^{-1}$, $x \mapsto g \circ x$ and $x \mapsto x \circ g$ are continuous.

Assume that X is the projective n -space \mathbb{P}^n . With respect to homogeneous coordinates $[x_0 : \dots : x_n]$ an element $f \in \text{Bir}(\mathbb{P}^2)$ is given by $[x_0 : \dots : x_n] \mapsto [f_0 : \dots : f_n]$, where the $f_i \in \mathbb{C}[x_0, \dots, x_n]$ are homogeneous polynomials of the same degree d without a non-constant common factor. We call d the *degree* of f . Denote by $\text{Bir}(\mathbb{P}^n)_{\leq d}$ the elements of $\text{Bir}(\mathbb{P}^n)$ of degree $\leq d$. In [BF13], it is shown that $\text{Bir}(\mathbb{P}^n)_{\leq d}$ is a closed subset for all d .

2.3. Algebraic subgroups of $\text{Bir}(X)$ and $\text{Aut}(X)$. An *algebraic subgroup* of $\text{Bir}(X)$ is the image of an algebraic group G by a morphism $G \rightarrow \text{Bir}(X)$ that is also an injective group homomorphism. In the case of $\text{Bir}(\mathbb{P}^n)$ it can be shown that algebraic groups are closed and of bounded degree. On the other hand, closed subgroups of bounded degree of $\text{Bir}(\mathbb{P}^2)$ are algebraic subgroups with a unique algebraic group structure that is compatible with the Zariski topology (see [BF13]). An element $g \in \text{Bir}(\mathbb{P}^2)$ is called *algebraic* if the closure of $\langle g \rangle$ in $\text{Bir}(X)$ is an algebraic group; this is equivalent to $\{\deg(g^n)\}_{n \in \mathbb{Z}}$ being bounded. A group $G \subset \text{Aut}(X)$ is called an *algebraic subgroup* if the induced action $G \times X \rightarrow X$ is a regular action. An element $g \in \text{Aut}(X)$ is *algebraic* if it is contained in an algebraic subgroup. An

algebraic element $g \in \text{Aut}(X)$ is also an algebraic element in $\text{Bir}(X)$. But a priori it is not clear whether an automorphism $g \in \text{Aut}(X)$ that is an algebraic element in $\text{Bir}(S)$, is an algebraic element in $\text{Aut}(X)$. However, in Proposition 3.12 we will see that for a normal affine surface S the two notions coincide, i.e. an element $g \in \text{Aut}(S)$ is algebraic in $\text{Aut}(S)$ if and only if g is algebraic in $\text{Bir}(S)$.

3. CHARACTERIZATION OF ALGEBRAIC ELEMENTS ON SURFACES

3.1. Divisibility in the Cremona group. Recall that an element f in a group G is called *divisible by n* if there exists an element $g \in G$ such that $g^n = f$. An element is called *divisible* if it is divisible by all $n \in \mathbb{Z}^+$. We use divisibility in order to characterize algebraic elements in $\text{Bir}(S)$ for surfaces S :

Theorem 3.1. *Let S be a surface and $f \in \text{Bir}(S)$. Then the following two conditions are equivalent:*

- (a) *there exists a k such that f^k is divisible;*
- (b) *f is algebraic.*

In order to prove Theorem 3.1, we need some results from dynamics of birational transformations of surfaces. Let H be an ample divisor class on S and denote by f^*H the total transform of H under $f \in \text{Bir}(S)$. The *degree of f with respect to H* is defined as

$$\deg_H(f) = f^*H \cdot H.$$

If f is an element in $\text{Bir}(\mathbb{P}^2)$ and H the class of a hypersurface then $\deg_H(f) = \deg(f)$, the degree we have defined in Section 2.2. Let H_1 and H_2 be two different ample divisors. Then there exists a constant $C > 0$ such that

$$\frac{1}{C} \deg_{H_1}(f) \leq \deg_{H_2}(f) \leq C \deg_{H_1}(f)$$

for all $f \in \text{Bir}(S)$ (see for example [Da17, Theorem 2]).

If we fix an ample divisor H on S , we can associate to each $f \in \text{Bir}(X)$ its degree sequence $\{\deg_H(f^n)\}_{n \in \mathbb{Z}_+}$. The growth of the degree sequence of a birational transformation carries information about its dynamical behavior. The following theorem is crucial for the understanding of groups of birational transformations in dimension two:

Theorem 3.2 (Gizatullin; Diller, Favre; Cantat ([Gi80, DF01, Ca15])). *Let S be a complex projective surface, H an ample divisor on S and $f \in \text{Bir}(S)$ a birational transformation. Then we are in exactly one of the following cases:*

- (a) *the sequence $\{\deg_H(f^n)\}_{n \in \mathbb{Z}_+}$ is bounded, which is equivalent to f being algebraic;*
- (b) *$\deg_H(f^n) \sim cn$ for some constant $c > 0$ and f preserves a rational fibration;*
- (c) *$\deg_H(f^n) \sim cn^2$ for some constant $c > 0$ and f preserves an elliptic fibration;*
- (d) *$\deg_H(f^n) \sim c\lambda^n$ for some constant $c > 0$, where λ is a Pisot or Salem number.*

Blanc and Déserti gave lower bounds for the constant c appearing in the cases (b) and (c) of Theorem 3.2 if the surface S is rational:

Theorem 3.3 ([BD15]). *Let $f \in \text{Bir}(\mathbb{P}^2)$ and let H be the divisor class of a line. Assume that $\deg_H(f^n) \sim cn$, then $c \geq 1/2$. If $f \in \text{Bir}(\mathbb{P}^2)$ such that $\deg_H(f^n) \sim cn^2$, then $c \geq 1/3$.*

We also need the following:

Theorem 3.4 ([Ca11]). *Let S be a projective surface with an ample divisor H and $f \in \text{Bir}(S)$ such that $\deg(f^n)$ grows exponentially with n . Then the centralizer of f equals $\langle f \rangle$ up to finite index.*

Theorem 3.3 can be generalized to non-rational surfaces of negative Kodaira dimension:

Lemma 3.5. *Let $S = C \times \mathbb{P}^1$, where C is a smooth projective, non-rational curve. Then there exists an ample divisor class H on S such that for all $f \in \text{Bir}(S)$ we are in one of the following cases:*

- (a) *the sequence $\{\deg_H(f^n)\}_{n \in \mathbb{Z}_+}$ is bounded and f is algebraic;*
- (b) *$\deg_H(f^n) \sim cn$ for some constant $c > 1/2$.*

To prove Lemma 3.5, we need some birational geometry. Let f be a birational transformation of a projective surface X . Whenever we speak of base-points, we consider both, proper and infinitely near base-points. A base-point p of f is called *persistent* if there exists an integer k such that p is a base-point of f^k for all $k \geq N$ but p is not a base-point of f^{-k} for any $k \geq N$. In [BD15], the authors show that if f has no persistent base-points then $\varphi f \varphi^{-1}$ has no persistent base-points, where $\varphi: S \dashrightarrow S'$ is any birational transformation to some smooth projective surface S' . In the same reference it is proven that f has no persistent base-points if and only if f is conjugate to an automorphism of a smooth, projective surface. Another important fact from [BD15] is the following:

Theorem 3.6 ([BD15]). *Let S be a smooth projective complex surface and $f \in \text{Bir}(S)$. Denote by $\mathfrak{b}(f^n)$ the number of base-points of f^n . Then there exists a non-negative integer ν such that the set $\{\mathfrak{b}(f^n) - n\nu \mid n \geq 0\} \subset \mathbb{Z}$ is bounded.*

Proof of Lemma 3.5. Since C is not rational, f preserves the \mathbb{P}^1 -fibration given by the first projection. We have $\text{Pic}(S) \simeq \text{Pic}(C) \oplus \text{Pic}(\mathbb{P}^1)$, where the embedding of $\text{Pic}(C)$ into $\text{Pic}(S)$ is given by the pullback of the first projection $\pi_1: S \rightarrow C$ and the embedding of $\text{Pic}(\mathbb{P}^1)$ into $\text{Pic}(S)$ by the pullback of the second projection $\pi_2: S \rightarrow \mathbb{P}^1$. Let $P \in \text{Pic}(C)$ and $Q \in \text{Pic}(\mathbb{P}^1)$ be the divisor class of a single point in C and \mathbb{P}^1 , respectively and let $F_P := \pi_1^*P$ and $S := \pi_2^*Q$ in $\text{Pic}(S)$. Define the ample divisor $H := F_P + S$.

If $\{\deg_H(f^n)\}_{n \in \mathbb{Z}_+}$ is bounded, then f is algebraic, by Theorem 3.2. Assume now that the degree-sequence $\{\deg_H(f^n)\}_{n \in \mathbb{Z}_+}$ is unbounded. Since f preserves the fibration given by the first projection, we have that $f^*F_P = F_{P'}$, where $P' \in \text{Pic}(C)$ is the divisor class of another point in C . Moreover, we have $(f^n)^*S = aS + D$, for some $D \in \text{Pic}(C)$ and $a \in \mathbb{Z}$. Since the pullback of any member of the linear system S by f intersects each fiber of π_1 exactly once, we have that $a = 1$ and hence $(f^n)^*(F_P + S) = S + F_{P'} + D$. It follows that $\deg_H(f^n) = (f^n)^*(F_P + S) \cdot (F_P + S) = \deg(D) + 2$ and therefore $\deg(D) = \deg_H(f^n) - 2$. We obtain that the total transform $(f^n)^*S$ has self-intersection $((f^n)^*S) \cdot ((f^n)^*S) = 2 \deg(D) = 2(\deg_H(f^n) - 2)$. Since f preserves the \mathbb{P}^1 -fibration, all base-points have order one. The divisor class S has self-intersection 0, hence we obtain that f^n must have $2(\deg_H(f^n) - 2)$ base-points. By Theorem 3.6, the number of base-points of f^n grows asymptotically like Kn for some integer K , hence $\deg_H(f^n)$ grows asymptotically like $(K/2)n$. \square

Lemma 3.7. *Let $n > 0$ and $A \in \text{GL}_n(\mathbb{Z})$ an element such that A^k is divisible for some $k \neq 0$. Then A is of finite order.*

Proof. It is enough to show that there exists no divisible element of infinite order in $\text{GL}_n(\mathbb{Z})$. Let $B \in \text{GL}_n(\mathbb{Z})$ be of infinite order. For a prime p let $\varphi_p: \text{GL}_n(\mathbb{Z}) \rightarrow \text{GL}_n(\mathbb{F}_p)$ be the homomorphism given by reduction modulo p . We may choose p such that B is not contained in the kernel of φ_p . The image $\varphi_p(B)$ is then not divisible by $k := |\text{GL}_n(\mathbb{F}_p)|$. Hence, B is not divisible by k . \square

Lemma 3.8. *Let S be a complex projective surface of non-negative Kodaira dimension and $f \in \text{Bir}(S)$. If f^k is divisible for some $k \neq 0$, then f is algebraic.*

Proof. Since the Kodaira dimension of S is non-negative, there exists a unique minimal model S' in the birational equivalence class of S and we have $\text{Bir}(S) = \text{Bir}(S') = \text{Aut}(S')$ (see [Ba01, Corollary 10.22]). The action of $\text{Aut}(S')$ on the cohomology groups $H^k(S'; \mathbb{Z})$ yields a homomorphism $\rho: \text{Aut}(S') \rightarrow \text{GL}(H^*(S'; \mathbb{Z}))$, where $H^*(S'; \mathbb{Z})$ denotes the direct sum over all cohomology groups. The kernel of ρ is known to be an algebraic group ([Li78]). Let $f^k \in \text{Bir}(S)$ be a divisible element. Lemma 3.7 shows that $\rho(f)$ has finite order, and so f^k is contained in the kernel of ρ for some k . Therefore, f is algebraic. \square

Lemma 3.9. *Let S be a complex projective surface and $f \in \text{Bir}(S)$ an element such that f^k is divisible for some $k \neq 0$. Then f is algebraic in $\text{Bir}(S)$.*

Proof. First we consider the case, where S is rational. Let $f \in \text{Bir}(S)$ and H an ample divisor on S . If f is of finite order, there exists a k such that $f^k = \text{id}$, which is a divisible element. So we may assume that f is of infinite order. We consider the four cases, given by Theorem 3.2. If $\{\deg_H(f^n)\}$ is bounded,

Theorem 3.2 implies that f is algebraic. If $\deg(f^n) \sim cn$, assume that there is a $g \in \text{Bir}(\mathbb{P}^2)$ and a $k \geq 0$ such that $g^k = f$. It follows that $\deg_H(g^n) \sim \frac{c}{k}n$. By Theorem 3.3, the constant c has to be at least $1/2$, so $k \leq 2c$. If $\deg_H(f^n) \sim cn^2$ we obtain similarly that if $g^k = f$ then $k \leq 3c$. In both cases it follows that f is only divisible by finitely many elements, hence no power of f is divisible. Finally, if $\deg(f^n) \sim c\lambda^n$, we observe that every element that divides f , centralizes f . So, by Theorem 3.4, there are only finitely many elements $g \in \text{Bir}(\mathbb{P}^2)$ that divide f and again, no power of f is divisible.

If S is non-rational and of Kodaira dimension $-\infty$, we use Lemma 3.5 and proceed with a similar argument as in the rational case.

If S is of Kodaira dimension ≥ 0 , the result follows from Lemma 3.7. \square

An algebraic group H is called *anti-affine* if $\mathcal{O}(H) = \mathbb{C}$. If G is an arbitrary connected algebraic group, there exists a central anti-affine group $G_{ant} \subset G$ such that G/G_{ant} is linear (see [Br09]). We thank Brion for pointing out the reference [Br09] to us in the context of divisibility. Denote by \mathbb{G}_a the additive group $(\mathbb{C}, +)$ of the field of complex numbers.

Lemma 3.10. *Let G be an algebraic group and $g \in G$. Then there exists a $k > 0$ such that g^k is divisible.*

Proof. We may assume that G is connected. If G is linear, consider the Zariski-closure $A := \overline{\langle g \rangle}$, which is a commutative subgroup of G . Hence, $A \simeq \mathbb{G}_m^{n_1} \times \mathbb{G}_a^{n_2} \times H$, for some $n_1 \geq 0$, $n_2 \in \{0, 1\}$ and a finite group H . Let k be the order of H . Then g^k is contained in $U \simeq \mathbb{G}_m^{n_1} \times \mathbb{G}_a^{n_2} \times \{\text{id}\} \subset A$, which is a group in which every element is divisible.

Let $G_{ant} \subset G$ be a central anti-affine group such that G/G_{ant} is linear. By [Br09, Lemma 1.6], every element in G_{ant} is divisible. Let now $g \in G$ be arbitrary. As G/G_{ant} is linear there exists a k such that the class of $[g^k]$ is divisible in G/G_{ant} , i.e. for every $n \geq 0$ there exists an element $f \in G$ such that $f^n h = g^k$ for some $h \in G_{ant}$. Since G_{ant} is divisible, there is a $h' \in G_{ant}$ satisfying $h'^n = h$ and hence $(fh')^n = g^k$, i.e. g^k is divisible. \square

Proof of Theorem 3.1. By Lemma 3.9, every divisible element is algebraic. On the other hand, let $f \in \text{Bir}(S)$ be an algebraic element. Then f is contained in an algebraic subgroup $G \subset \text{Bir}(S)$. By Lemma 3.10, we obtain that f is divisible. \square

3.2. Algebraic elements. A well-known theorem of Weil (see [We55]) implies that an algebraic element $f \in \text{Bir}(X)$, where X is a smooth projective variety, can always be regularized, i.e. there exists a smooth projective surface Y and a birational transformation $\varphi: X \dashrightarrow Y$ such that $\varphi f \varphi^{-1}$ is an algebraic automorphism of Y . We need a slightly finer version of this result:

Lemma 3.11. *Let X be a smooth projective surface and $U \subset X$ an open dense subset. Let $f \in \text{Bir}(X)$ be an algebraic birational transformation such that the restriction of f to U is an automorphism. Then there exists a smooth projective surface Y and a birational transformation $\varphi: X \dashrightarrow Y$ such that the restriction of φ to U induces an isomorphism to its image and such that $\varphi f \varphi^{-1}$ is an automorphism of Y .*

Proof. Since f is algebraic, it is, by the theorem of Weil, conjugate to an automorphism and has therefore no persistent base-points. Note that for any integer k , the base-points of f^k are contained in the closed subset $X \setminus U$ or are infinitely near to it. We follow now closely the proof of Proposition 3.5 in [BD15] in order to construct a smooth projective surface Y and a birational transformation $\varphi: X \dashrightarrow Y$ with the desired properties.

Let K be the set of all points of S (proper or infinitely near) that are base-points of f^i and f^{-j} for some $i, j > 0$. Note that all the points of K are contained in $X \setminus U$ or are infinitely near to it. One can show that K is finite and that one can blow up all the points of K to obtain an *algebraically stable* model of X ; i.e. let $\alpha_1: Z \rightarrow X$ be the blow up of the points K and $\tilde{f} := \alpha_1^{-1} f \alpha_1$, then no point in Z is a base-point of \tilde{f}^i and \tilde{f}^{-j} for any $i, j > 0$ (see proof of Theorem 0.1 in [DF01] or proof of Proposition 3.5 in [BD15]).

If Z contains a (-1) -curve C that is contracted by \tilde{f} we contract it by a birational morphism $\beta_1: Z \rightarrow Y_1$. Note that C is contained in $Y \setminus \alpha^{-1}(U)$, since the restriction of \tilde{f} to $\alpha_1^{-1}(U)$ is an automorphism. We obtain a birational transformation $\beta_1 \tilde{f} \beta_1^{-1}$ of Y_1 , which is again algebraically stable. We repeat the process finitely many times until we obtain a birational morphism

$$\alpha_2 = \beta_n \beta_{n-1} \cdots \beta_1: Z \rightarrow Y_n =: Y$$

such that $\alpha_2 \tilde{f} \alpha_2^{-1}$ contracts no (-1) -curve. Again, all the curves contracted by α_2 are contained in $Y \setminus \alpha_1^{-1}(U)$. Let $\varphi := \alpha_2 \alpha_1^{-1}$. We claim that the birational transformation $\hat{f} := \varphi f \varphi^{-1}$ is an automorphism of Y . Since we only blew up points and contracted lines that are contained in $X \setminus U$ or infinitely near to it, this claim will finish the proof.

In order to prove the claim, assume that the birational transformation \hat{f} is not an automorphism. We will deduce a contradiction. Let $\tau_1: V \rightarrow Y$ and $\tau_2: V \rightarrow Y$ be a resolution of \hat{f} , i.e. τ_1 is a birational morphism that blows up the base-points of \hat{f} and τ_2 a birational morphism that blows up the base-points of \hat{f}^{-1} , in particular $\hat{f} = \tau_2 \tau_1^{-1}$. Let $C \subset V$ be a curve that is contracted by τ_2 to a base-point $p \in Y$ of \hat{f}^{-1} . Note that such a curve always exists and that $\tau_1(C) \subset Y$ has to be a curve. Since \hat{f} is algebraically stable, there is no $k > 0$ such that p is a base-point of \hat{f}^k . Let $l > 0$ be such that p is no base-point of \hat{f}^{-l} . Note that such an l exists since otherwise p would be persistent. Let W be a resolution of $\hat{f}^{1-l} \tau_1^{-1}: Y \dashrightarrow V$ and $\pi_1: W \rightarrow Y$ the birational morphism blowing up all the base-points of $\hat{f}^{1-l} \tau_1^{-1}$ and $\pi_2: W \rightarrow V$ the birational morphism blowing up all the base-points of $(\hat{f}^{1-l} \tau_1^{-1})^{-1}$. Since p is not a base-point of \hat{f}^{-l} , the morphism π_1 contracts the curve $\pi_2^{-1}(C)$. Moreover, there are no base-points of \hat{f}^{-l} on C or infinitely near to it, since otherwise p would be a base-point of \hat{f}^{-l} . Thus the curve $\pi_2^{-1}(C) \subset W$ has still self-intersection (-1) . Since $\tau_1(C)$ is not a (-1) -curve, but is contracted by \hat{f}^{-1+l} , it has to contain a base-point q of \hat{f}^{-1+l} . Using again that \hat{f} is algebraically stable, we obtain that q is not a base-point of \hat{f} , in particular, q is not blown up by τ_1 . So π_2 has to blow up q . But this contradicts $\pi_2^{-1}(C)$ having self-intersection (-1) . \square

Proposition 3.12. *Let S be a normal affine surface and let $g \in \text{Aut}(S)$ be an automorphism. If g is an algebraic element in $\text{Bir}(S)$, then g is an algebraic element in $\text{Aut}(S)$.*

Proof. Let \tilde{X} be a projective completion of S . Let $\pi: X \rightarrow \tilde{X}$ be a smooth resolution. The birational morphism π is given by blowing up the singular points of \tilde{X} . Let $f := \pi^{-1} g \pi$ and denote by E the exceptional divisor of the singular points in S , or infinitely near to S , that are blown up. The restriction of f to $\pi^{-1}(U)$ is an automorphism. By Lemma 3.11, there exists a birational transformation $\varphi: X \rightarrow Y$, such that the restriction of φ to $\varphi^{-1}(S)$ is an isomorphism. Let $\tilde{f} := \varphi f \varphi^{-1}$. By construction, \tilde{f} is an algebraic automorphism of Y and it preserves the closed set $Y \setminus \varphi(\pi^{-1}(U))$ as well as the closed set $\varphi(E)$. Therefore, $G := \overline{\langle \tilde{f} \rangle}$, the closure of the group generated by \tilde{f} , is an algebraic group and preserves $Y \setminus \varphi(\pi^{-1}(U))$ as well as the closed set $\varphi(E)$. We therefore obtain an algebraic action of G on $\varphi(\pi^{-1}(U))$, which we can pull back to $\pi^{-1}(U)$, and, since the action of G preserves E , it induces an algebraic action on S . This implies that g is algebraic in $\text{Aut}(S)$. \square

Proposition 3.13. *Let S_1 and S_2 be normal affine surfaces, $\varphi: \text{Aut}(S_1) \rightarrow \text{Aut}(S_2)$ an abstract group isomorphism and $g \in \text{Aut}(S_1)$ an algebraic element. Then $\varphi(g)$ is an algebraic element in $\text{Aut}(S_2)$.*

Proof. The element g is algebraic in $\text{Aut}(S_1)$ and therefore divisible. Since divisibility is preserved by group homomorphisms, we obtain that $\varphi(g)$ is divisible in $\text{Aut}(S_2)$. Therefore, by Theorem 3.1, $\varphi(g)$ is algebraic in $\text{Bir}(Y)$. Proposition 3.12 implies that $\varphi(g)$ is algebraic in $\text{Aut}(S_2)$. \square

4. TORIC VARIETIES

4.1. Root subgroups. In this section we describe so-called *root* subgroups of $\text{Aut}(X)$ for a given affine variety X . We always consider $\mathbb{G}_a \cong \text{Spec } \mathbb{C}[s]$ as an algebraic variety so that s is an affine coordinate of \mathbb{G}_a at the identity.

Definition 4.1. Let $T \subset \text{Aut}(X)$ be a torus in $\text{Aut}(X)$. A closed subgroup $U \subset \text{Aut}(X)$ isomorphic to \mathbb{G}_a is called a *root subgroup* with respect to T if the normalizer of U in $\text{Aut}(X)$ contains T . Since \mathbb{G}_a contains no non-trivial closed normal subgroups, such a subgroup U is equivalent to a normalized \mathbb{G}_a -action on X , i.e. a \mathbb{G}_a -action on X whose image in $\text{Aut}(X)$ is normalized by T .

Let $U \subset \text{Aut}(X)$ be a root subgroup with respect to T . Since T is in the normalizer, we can define an action $\varphi: T \rightarrow \text{Aut}(U)$ of T on U given by $t \cdot s = t \circ s \circ t^{-1}$ for all $t \in T$ and $s \in U$. Furthermore, since $\text{Aut}(U) \cong \mathbb{G}_m$, such an action corresponds to a character of the torus $\chi: T \rightarrow \mathbb{G}_m$, which does not depend on the choice of automorphism between $\text{Aut}(U)$ and \mathbb{G}_m . Hence, T and U span an algebraic subgroup in $\text{Aut}(X)$ isomorphic to $\mathbb{G}_a \rtimes_{\chi} T$.

Definition 4.2. The *weight* of the root subgroup U is the unique character $\chi: T \rightarrow \mathbb{G}_m$ of the torus T such that $t \circ s \circ t^{-1}(x) = \chi(t) \cdot s(x)$ for every $t \in T$, $s \in U$ and $x \in X$.

Assume that the algebraic torus T acts linearly and regularly on a vector space A of countable dimension. We say that A is *multiplicity-free* if the weight spaces A_{χ} are all of dimension less or equal than one for every character $\chi: T \rightarrow \mathbb{G}_m$ of the torus T . In our proof of Theorem 1.2, we will use the following lemma that is due to Kraft:

Lemma 4.3 ([Kr15, Lemma 5.2]). *Let X be a normal affine variety and let $T \subset \text{Aut}(X)$ be a torus. If there exists a root subgroup $U \subset \text{Aut}(X)$ with respect to T such that $\mathcal{O}(X)^U$ is multiplicity-free, then $\dim T \leq \dim X \leq \dim T + 1$.*

Additive group actions on affine varieties can be described by a certain kind of derivations. We recall some of the basics here (see [Fr06] for details). Let $\lambda: \mathbb{G}_a \rightarrow \text{Aut}(X)$ be a \mathbb{G}_a -action on an affine variety X . This action induces a derivation on the level of regular functions by

$$\delta_{\lambda}: \mathcal{O}(X) \rightarrow \mathcal{O}(X), \quad f \mapsto \left[\frac{d}{ds} \lambda^*(f) \right]_{s=0}$$

which has the property that for every $f \in \mathcal{O}(X)$ there exists an $\ell \in \mathbb{N}$ with $\delta_{\lambda}^{\ell}(f) = 0$. Derivations having this property are called *locally nilpotent*. Furthermore, every \mathbb{G}_a -action on X arises from such a locally nilpotent derivation δ and the \mathbb{G}_a -action $\alpha_{\delta}: \mathbb{G}_a \rightarrow \text{Aut}(X)$ is recovered from δ via

$$(\alpha_{\delta}(s))^*: \mathcal{O}(X) \rightarrow \mathcal{O}(X), \quad f \mapsto \exp(s\delta)(f) := \sum_{i=0}^{\infty} \frac{s^i \delta^i(f)}{i!}.$$

Let $T \subseteq \text{Aut}(X)$ be an algebraic torus. The choice of such a T is equivalent to fixing an M -grading on the ring $\mathcal{O}(X)$ of regular functions, where M is the character lattice of the torus. We follow the standard convention to consider M as an abstract additive lattice and to denote the character corresponding to $m \in M$ by χ^m .

Recall that a linear map $\delta: A \rightarrow B$ between M -graded \mathbb{C} -vector spaces is called *homogeneous* if there exists an $e \in M$ such that for every homogeneous element f of degree m , the image $\delta(f)$ is homogeneous of degree $m + e$. We call the element $e \in M$ the *degree* of δ and denote it by $\deg \delta$.

The next lemma states that normalized \mathbb{G}_a -actions are in one to one correspondence with locally nilpotent derivations that are homogeneous with respect to the M -grading of $\mathcal{O}(X)$. A proof can be found in [Li11, Lemma 2].

Proposition 4.4. *Let X be an affine variety and fix a torus $T \subseteq \text{Aut}(X)$. A locally nilpotent derivation δ on $\mathcal{O}(X)$ is normalized by T if and only if it is homogeneous with respect to the M -grading on $\mathcal{O}(X)$ given by T . The weight of the corresponding root subgroup is $\chi^{\deg \delta}$.*

4.2. Root subgroups of toric varieties. An affine toric variety is a normal affine variety endowed with a faithful action of an algebraic torus T that acts with an open orbit. An affine toric variety X is called *non-degenerate* if it has no torus factor, i.e. if it is not isomorphic to $Y \times \mathbb{A}_*^1$ for some variety Y , where $\mathbb{A}_*^1 = \mathbb{A}^1 \setminus \{0\}$.

In this section we first recall the well known description of affine toric varieties by means of strongly convex rational polyhedral cones, details can be found in reference texts about toric geometry such as [Od88, Ful93, CLS11]. Then we provide a description of root subgroups of affine toric varieties.

Let M and N be dual lattices of rank n and consider the duality pairing $M \times N \rightarrow \mathbb{Z}$, defined by $(m, p) \mapsto \langle m, p \rangle = p(m)$. Let $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ be the corresponding real vector spaces and let T be the algebraic torus $T = \text{Spec } \mathbb{C}[M] = N \otimes_{\mathbb{Z}} \mathbb{C}^* \cong \mathbb{G}_{\text{m}}^n$. With this choice M is the character lattice of T and N the lattice of 1-parameter subgroups of T .

By a well known construction, affine toric varieties can be described via strongly convex rational polyhedral cones in the vector space $N_{\mathbb{R}}$. Let σ be a strongly convex rational polyhedral cone in $N_{\mathbb{R}}$ and let $\mathbb{C}[\sigma^{\vee} \cap M]$ be the semigroup algebra $\mathbb{C}[\sigma^{\vee} \cap M] = \bigoplus_{m \in \sigma^{\vee} \cap M} \mathbb{C}\chi^m$, where the multiplication rule is given by $\chi^m \cdot \chi^{m'} = \chi^{m+m'}$ and $\chi^0 = 1$. In the following, we denote $\sigma^{\vee} \cap M$ by σ_M^{\vee} .

The main result about affine toric varieties is that $X_{\sigma} := \text{Spec } \mathbb{C}[\sigma_M^{\vee}]$ is an affine toric variety, where the comorphism $\alpha^*: \mathbb{C}[\sigma_M^{\vee}] \rightarrow \mathbb{C}[M] \otimes_{\mathbb{C}} \mathbb{C}[\sigma_M^{\vee}]$ of the T -action is given by $\chi^m \mapsto \chi^m \otimes \chi^m$. Furthermore, every affine toric variety arises via this construction.

We now describe root subgroups of the automorphism group of a toric variety. Let $\sigma \subseteq N_{\mathbb{R}}$ be a strongly convex rational polyhedral cone. Following the usual convention, we identify a ray $\rho \subseteq \sigma$ with its shortest non-trivial vector, called its primitive vector. The set of all the rays of σ is denoted by $\sigma(1)$.

Definition 4.5. We say that a lattice vector $\alpha \in M$ is a *root* of σ if there exists $\rho_{\alpha} \in \sigma(1)$ such that $\langle \alpha, \rho_{\alpha} \rangle = -1$ and $\langle \alpha, \rho \rangle \geq 0$, for every $\rho \in \sigma(1)$ different from ρ_{α} . We call the ray ρ_{α} the *distinguished ray* of the root α . We denote by $\mathcal{R}(\sigma)$ the set of all roots of σ and by $\mathcal{R}_{\rho}(\sigma)$ the set of all roots of σ with distinguished ray ρ .

Let $\alpha \in \mathcal{R}(\sigma)$. One checks that the linear map given by

$$\delta_{\alpha}: \mathbb{C}[\sigma_M^{\vee}] \rightarrow \mathbb{C}[\sigma_M^{\vee}], \quad \chi^m \mapsto \langle m, \rho_{\alpha} \rangle \cdot \chi^{m+\alpha}$$

is a homogeneous locally nilpotent derivation of the algebra $\mathbb{C}[\sigma_M^{\vee}]$. Furthermore, it was proven implicitly in [De70] and explicitly in [Li10, Theorem 2.7] that every homogeneous locally nilpotent derivation of the algebra $\mathbb{C}[\sigma_M^{\vee}]$ arises this way. We summarize these results in the following proposition.

Proposition 4.6. *Let X_{σ} be the toric variety given by a strongly convex rational polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$. The root subgroups of $\text{Aut}(X_{\sigma})$ with respect to T are in one to one correspondence with the roots of the cone σ . The correspondence is given by assigning to every $\alpha \in \mathcal{R}(\sigma)$ the root subgroup whose homogeneous locally nilpotent derivation is δ_{α} with weight character χ^{α} .*

It is straightforward to verify from Proposition 4.6 that the root subgroups corresponding to δ_{α} and $\delta_{\alpha'}$ commute if they have the same distinguished ray. The following corollary follows directly from Proposition 4.6 since all actions of tori of dimension $\dim X_{\sigma}$ on a toric variety X_{σ} are conjugate (see [De82]).

Corollary 4.7. *Let X_{σ} be an affine toric variety and let $T \subset \text{Aut}(X_{\sigma})$ be a maximal subtorus. Then all the root subgroups of $\text{Aut}(X_{\sigma})$ with respect to T have different weights.*

4.3. Toric surfaces. In this section we prove our first main result stated in Theorem 1.1. The next lemma is known and can, for example, be found in [KRS17, Lemma 10].

Lemma 4.8. *Let X be an affine toric variety and let T be a torus of dimension $\dim X$ which acts faithfully on X . Then the centralizer of T in $\text{Aut}(X)$ equals T . In particular, T is closed in $\text{Aut}(X)$.*

Lemma 4.9. *Let X be an affine variety and $G, H \subset \text{Aut}(X)$ be commuting algebraic subgroups. Then the closure of the group $\langle G, H \rangle$ generated by G and H is an algebraic group.*

Proof. Let $\text{Aut}(X) = \cup W_i$ be a filtration of an ind-group. Since G and H are algebraic subgroups of $\text{Aut}(X)$, $G \subset W_i, H \subset W_j$. Then $G \times H \subset W_i \times W_j \subset W_k$, because $\text{Aut}(X)$ is an ind-group. The claim follows. \square

Proposition 4.10. *If $\mathbb{G}_{\text{a}} \times \mathbb{G}_{\text{m}}$ acts on a normal affine surface X , then X is toric.*

Proof. Let X be an affine surface and let $\lambda: \mathbb{G}_a \times \mathbb{G}_m \rightarrow \text{Aut}(X)$ be a $\mathbb{G}_a \times \mathbb{G}_m$ action on X . The \mathbb{G}_m -action $\lambda|_{\mathbb{G}_m}$ corresponds to a \mathbb{Z} -grading on the ring of regular functions $\mathcal{O}(X) = \bigoplus_{i \in \mathbb{Z}} A_i$ and in this context the action $\lambda|_{\mathbb{G}_a}$ is determined by a root subgroup of weight zero, or equivalently, by a homogeneous locally nilpotent $\delta: \mathcal{O}(X) \rightarrow \mathcal{O}(X)$ of degree zero. In [Li10] a classification of homogeneous locally nilpotent derivations on normal affine T -varieties was given. Since δ has degree zero, there must be elements of the kernel of δ in all graded pieces A_i and so, in the notation of [Li10, 3.17 and Lemma 3.23], we have $\omega := \omega(\delta) = \{0\}$ and $A_\omega = \mathcal{O}(X)$. Now, [Li10, Remark 3.24 (1) and Lemma 3.23] yields that X is toric. \square

Recall that every connected one-dimensional affine algebraic group is isomorphic to \mathbb{G}_m or to \mathbb{G}_a . The proof of the next lemma is straightforward and we leave it to the reader.

Lemma 4.11. *Let G be a connected one-dimensional affine algebraic group that acts non-trivially on a connected one-dimensional affine algebraic group H by conjugation. Then $G \cong \mathbb{G}_m$ and $H \cong \mathbb{G}_a$.*

Let $G_{d,e} = \langle g \rangle$ be the cyclic group of order d of $\text{Aut}(\mathbb{A}^2)$ that is given by $g: (x, y) \mapsto (\xi^e x, \xi y)$ where ξ is a d -th primitive root of unity, $0 \leq e < d$ and $(e, d) = 1$. Every affine toric surface is either isomorphic to $\mathbb{A}_*^1 \times \mathbb{A}_*^1$, to $\mathbb{A}^1 \times \mathbb{A}_*^1$, or to some $X_{d,e} = \mathbb{A}^2/G_{d,e}$. Furthermore, the toric surface $X_{d,e}$ is described in standard correspondence between toric varieties and convex polyhedral rational cones by the cone σ spanned by $\rho_1 = \beta_2^*$ and $\rho_2 = d\beta_1^* - e\beta_2^*$ in $N_{\mathbb{R}}$, where $\{\beta_1^*, \beta_2^*\}$ is a \mathbb{Z} -basis of the 1-parameter subgroup lattice N of the 2-dimensional torus ([CLS11, Proposition 10.1.3]). Let $\{\beta_1, \beta_2\}$ be the corresponding dual \mathbb{Z} -basis of the character lattice M of the 2-dimensional torus. By [CLS11, Proposition 10.1.3] we have the following lemma:

Lemma 4.12. *The toric surface $X_{d,e}$ is isomorphic to $X_{d',e'}$ if and only if $d' = d$, and $e = e'$ or $ee' = 1 \pmod{d}$.*

Every automorphism of the toric surface $X_{d,e}$ is induced by a unique automorphism of \mathbb{A}^2 and the description of the automorphism groups in [AZ12] implies in particular, that $\mathbb{A}^1 \times \mathbb{A}_*^1$ is the only affine toric surface whose automorphism group consists only of algebraic elements.

For the proof of Theorem 1.1 we need to describe the characters of root subgroups of affine toric surfaces $X_{d,e}$. By Proposition 4.6, root subgroups in a toric variety are uniquely determined by their weight characters. Let e' and a be the unique integers with $0 \leq e' < d$ such that $ee' = 1 + ad$. We have the following lemma.

Lemma 4.13. *The weight characters of the root subgroups of $\text{Aut}(X_{d,e})$ are:*

- with distinguished ray ρ_1 the characters χ^α with $\alpha = -\beta_2 + l \cdot \beta_1$, for all $l \in \mathbb{Z}_{\geq 0}$; and
- with distinguished ray ρ_2 the characters χ^α with $\alpha = (a\beta_1 + e'\beta_2) + k \cdot (e\beta_1 + d\beta_2)$, for all $k \in \mathbb{Z}_{\geq 0}$.

Proof. By Proposition 4.6, weight characters χ^α correspond to vectors $\alpha \in M$ with $\langle \alpha, \rho_\alpha \rangle = -1$ for some ray ρ_α in $\sigma(1)$ and $\langle \alpha, \rho \rangle \geq 0$ for all the other rays. The ray ρ_α is called the distinguished ray. In this case we have only two rays: $\rho_1 = \beta_2^*$ and $\rho_2 = d\beta_1^* - e\beta_2^*$. Assume that α is of the form $\alpha = c_1\beta_1 + c_2\beta_2$ and let ρ_1 be distinguished ray. Then we have $c_2 = -1$ and $c_1d - c_2e \geq 0$. This yields the first family in the lemma. Let now ρ_2 be the distinguished ray. Then we have $c_1d - c_2e = -1$ and $c_2 \geq 0$. A straightforward computation yields the second family in the lemma. \square

Proposition 4.14. *Let S_1 be an affine toric surface that is not isomorphic to $\mathbb{A}_*^1 \times \mathbb{A}_*^1$ and let S_2 be an affine surface. Let $T \subset \text{Aut}(S_1)$ be a maximal torus. Assume that there is an isomorphism of groups $\varphi: \text{Aut}(S_1) \rightarrow \text{Aut}(S_2)$. Then we have the following:*

- the image $\varphi(T)$ is a maximal torus in $\text{Aut}(S_2)$;
- the normalization of S_2 is an affine toric surface that is not isomorphic to $\mathbb{A}_*^1 \times \mathbb{A}_*^1$;
- the normalization of S_2 is isomorphic to $\mathbb{A}_*^1 \times \mathbb{A}^1$ if and only if S_1 is isomorphic to $\mathbb{A}_*^1 \times \mathbb{A}^1$.

Proof. Since S_1 is not isomorphic to $\mathbb{A}_*^1 \times \mathbb{A}_*^1$, $\text{Aut}(S_1)$ contains a root subgroup with respect to the maximal torus T with a non-trivial character. Choose $t_1, t_2 \in T$ of infinite order and a unipotent element

$u \in \text{Aut}(S_1)$ in such a way that t_1 commutes with u and $t_2 \circ u \circ t_2^{-1} = u^2$. By Proposition 3.13, the elements $\varphi(t_1), \varphi(t_2), \varphi(u) \in \text{Aut}(S_2)$ are algebraic. We observe that the two algebraic groups $\overline{\langle \varphi(t_1) \rangle}$ and $\overline{\langle \varphi(t_2) \rangle}$ commute with each other and do not coincide. If one of the groups $\overline{\langle \varphi(t_1) \rangle}$ or $\overline{\langle \varphi(t_2) \rangle}$ has dimension strictly bigger than 1, Proposition 4.10 implies that the normalization of S_2 is toric. Hence, we may assume that the groups $\overline{\langle \varphi(t_1) \rangle}$ and $\overline{\langle \varphi(t_2) \rangle}$ are one-dimensional. In case one of the groups $\overline{\langle \varphi(t_1) \rangle}^\circ$ or $\overline{\langle \varphi(t_2) \rangle}^\circ$ is isomorphic to \mathbb{G}_m , the normalization of S_2 is again toric by Proposition 4.10. The only case we have to consider is when both groups $\overline{\langle \varphi(t_1) \rangle}$ and $\overline{\langle \varphi(t_2) \rangle}$ are isomorphic to \mathbb{G}_a . But then $\overline{\langle \varphi(t_2) \rangle}$ acts nontrivially by conjugation on $\overline{\langle \varphi(u) \rangle}$. This is not possible by Lemma 4.11. Therefore, the groups $\overline{\langle \varphi(t_1) \rangle}$ and $\overline{\langle \varphi(t_2) \rangle}$ are isomorphic to \mathbb{G}_m . Hence, from Lemma 4.9 it follows that the algebraic group generated by $\varphi(t_1)$ and $\varphi(t_2)$ contains a two-dimensional algebraic torus which coincides with its centralizer by Lemma 4.8. We obtain that $\varphi(T)$ is a maximal algebraic torus which proves the first claim of the statement. Since every algebraic group action on S_2 lifts to an action on the normalization of S_2 it follows that the normalization of S_2 is toric. Moreover, $\mathbb{A}_1^* \times \mathbb{A}_1^*$ is the only toric surface such that the automorphism group contains an infinite set of elements that normalize the maximal torus but do not centralize it. Since S_1 is not isomorphic to $\mathbb{A}_1^* \times \mathbb{A}_1^*$ by assumption, the normalization of S_2 is therefore not isomorphic to $\mathbb{A}_1^* \times \mathbb{A}_1^*$.

As $\mathbb{A}_*^1 \times \mathbb{A}^1$ is the only affine toric surface such that all elements from its automorphism group are algebraic, the third claim of the statement holds. \square

In the proof of the next lemma we will need the notion of an *irreducible derivation* of a \mathbb{C} -algebra A which is a derivation that can not be written as a multiple of a non-constant element of A and another derivation. Recall that $\text{Aut}(\mathbb{A}_*^1 \times \mathbb{A}_*^1) \simeq \text{GL}_2(\mathbb{Z}) \times \mathbb{G}_m^2$, where $\text{GL}_2(\mathbb{Z})$ is the group of monomial transformations.

Lemma 4.15. *Let S be an affine normal surface. If $\text{Aut}(S)$ is isomorphic to $\text{Aut}(\mathbb{A}_*^1 \times \mathbb{A}_*^1)$ as an abstract group, then S is isomorphic to $\mathbb{A}_*^1 \times \mathbb{A}_*^1$ as a variety.*

Proof. Let $\varphi: \text{Aut}(\mathbb{A}_*^1 \times \mathbb{A}_*^1) \xrightarrow{\sim} \text{Aut}(S)$ be an isomorphism of groups and let $T \subset \text{Aut}(\mathbb{A}_*^1 \times \mathbb{A}_*^1)$ be a maximal subtorus. Since T coincides with its centralizer (see Lemma 4.8), $\varphi(T) \subset \text{Aut}(S)$ is a closed subgroup by Lemma 2.5. Let $d \in T$ be an element of infinite order. Then, by Proposition 3.13, $\varphi(d)$ is an algebraic element of $\text{Aut}(S)$. Hence $\overline{\langle \varphi(d) \rangle} \subset \varphi(T)$ is a commutative algebraic subgroup of positive dimension.

If S is a toric surface, the claim of the statement follows from Proposition 4.14. So assume that S is not toric. In this case, Proposition 4.10 implies that S does not admit a faithful action of \mathbb{G}_m^2 or of $\mathbb{G}_a \times \mathbb{G}_m$. Hence, for all algebraic elements $h \in \text{Aut}(\mathbb{A}_*^1 \times \mathbb{A}_*^1)$ of infinite order the algebraic subgroup $\overline{\langle \varphi(h) \rangle} \subset \text{Aut}(S)$ is one-dimensional. Assume that there are $t_1, t_2 \in T$ of infinite order such that the commutative groups $\overline{\langle \varphi(t_1) \rangle}^\circ$ and $\overline{\langle \varphi(t_2) \rangle}^\circ$ do not coincide. Since, by assumption, S is not toric, $\overline{\langle \varphi(t_1) \rangle}^\circ$ and $\overline{\langle \varphi(t_2) \rangle}^\circ$ are both isomorphic to \mathbb{G}_a . If $\overline{\langle \varphi(t_1) \rangle}^\circ$ and $\overline{\langle \varphi(t_2) \rangle}^\circ$ have different orbits, the algebraic group generated by $\overline{\langle \varphi(t_1) \rangle}^\circ$ and $\overline{\langle \varphi(t_2) \rangle}^\circ$ is unipotent and acts with an open orbit on S , which implies that $S \cong \mathbb{A}^2$ - a contradiction to our assumption that S is not toric. Hence, all the one-dimensional unipotent algebraic subgroups of $\varphi(T)^\circ$ have the same orbits and therefore the same ring of invariants, which we denote by $\mathcal{O}(S)^{\varphi(T)^\circ}$. In fact, there exists an irreducible locally nilpotent derivation δ of $\mathcal{O}(S)$ such that

$$\varphi(T)^\circ = \{\exp(cf\delta) \mid c \in \mathbb{C}, f \in \mathcal{O}(S)^{\varphi(T)^\circ}\}.$$

Denote by U the \mathbb{G}_a -action on S that corresponds to δ . Then $\mathcal{O}(S)^{\varphi(T)^\circ} = \mathcal{O}(S)^U$ and $\mathcal{O}(S)^U$ is finitely generated.

The image of the monomial transformations $\varphi(\text{GL}_2(\mathbb{Z}))$ normalizes $\varphi(T)$ and hence it also normalizes $\varphi(T)^\circ$. This induces an action of $\varphi(\text{GL}_2(\mathbb{Z}))$ on $\mathcal{O}(S)^U$. Let us observe that the kernel K of this action acts on U and is a non-solvable group. Since the automorphism group of $U \simeq \mathbb{G}_a$ is solvable, K acts on \mathbb{G}_a with a non-trivial kernel \tilde{K} . Hence, by the structure of $\varphi(T)^\circ$, the kernel \tilde{K} acts trivially on $\varphi(T)^\circ$.

But this means that $\mathrm{GL}_2(\mathbb{Z})$ acts on a subgroup of $\varphi(T)$ of countable index with a non-trivial kernel, which implies that the action of $\mathrm{GL}_2(\mathbb{Z})$ on a countable index subgroup of T has a nontrivial kernel. Since such a subgroup is dense in T it follows that $\mathrm{GL}_2(\mathbb{Z})$ acts on T with a nontrivial kernel which is not the case. Therefore, S is toric. \square

Lemma 4.16. *Let S_1 and S_2 be two affine surfaces endowed with positive dimensional maximal tori T_1 and T_2 respectively. Assume that there is an isomorphism $\varphi: \mathrm{Aut}(S_1) \rightarrow \mathrm{Aut}(S_2)$ such that $\varphi(T_1) = T_2$. Then any root subgroup of $\mathrm{Aut}(S_1)$ with respect to T_1 is sent by φ to a root subgroup of $\mathrm{Aut}(S_2)$ with respect to T_2 .*

Proof. If $\mathrm{Aut}(S_1)$ contains no root subgroup, the lemma is trivially true. Otherwise, let U be a root subgroup of $\mathrm{Aut}(S_1)$ with respect to T_1 , then T_1 acts on U with two orbits. Hence, $\varphi(U)$ is a group normalized by $T_2 = \varphi(T_1)$ which also acts on $\varphi(U)$ with two orbits. Since any orbit of the algebraic group action is open in its closure, we obtain, by considering the kernel of the actions, that $\overline{\varphi(U)}$ is a quasi-affine curve. This implies that $\overline{\varphi(U)}$ is a one-dimensional algebraic group. Moreover, $\overline{\varphi(U)}$ is a union of $\varphi(U)$ and at most one point. Then $\overline{\varphi(U)}$ contains at most one root of unity. Hence, $\overline{\varphi(U)}$ is isomorphic to \mathbb{G}_a and normalized by $\varphi(T_1) = T_2$. Moreover, $\overline{\varphi(U)} = \varphi(U)$ and the claim follows. \square

Proof of Theorem 1.4. Let $\mathrm{Tr} = \{(x + c, y + d) \mid c, d \in \mathbb{C}\} \subset \mathrm{Aut}(\mathbb{A}^2)$ be the subgroup of translations. The group Tr coincides with its centralizer in $\mathrm{Aut}(\mathbb{A}^2)$. Hence, $\varphi(\mathrm{Tr}) \subset \mathrm{Aut}(S)$ is a closed subgroup by Lemma 2.5. The maximal torus $T \subset \mathrm{Aut}(\mathbb{A}^2)$ given by the group of diagonal automorphisms acts on Tr by conjugation with finitely many orbits. By Lemma 4.14, $\varphi(T) \subset \mathrm{Aut}(S)$ is a maximal subtorus. Therefore, the closed subgroup $\varphi(T) \subset \mathrm{Aut}(S)$ also acts on $\varphi(\mathrm{Tr})$ faithfully and with finitely many orbits and hence $\varphi(\mathrm{Tr}) \subset \mathrm{Aut}(S)$ is an algebraic subgroup of dimension 2. Since Tr does not contain elements of finite order, the group $\varphi(\mathrm{Tr})$ is unipotent. If two different \mathbb{G}_a -actions in $\varphi(\mathrm{Tr})$ have different orbits, then $\varphi(\mathrm{Tr})$ acts with an open orbit and because $\varphi(\mathrm{Tr})$ is unipotent it follows that $S \cong \mathbb{A}^2$. Now assume that all the \mathbb{G}_a -actions from $\varphi(\mathrm{Tr})$ have the same orbits. Then the ring of invariants $\mathcal{O}(S)^{\varphi(\mathrm{Tr})}$ contains non-constant functions and there exists a locally nilpotent derivation δ such that every \mathbb{G}_a -action in $\varphi(\mathrm{Tr})$ is of the form $\{\exp(cg\delta) \mid c \in \mathbb{C}\}$ for some $g \in \mathcal{O}(S)^{\varphi(\mathrm{Tr})}$. But in this case for any $k \geq 0$ and any $f \in \mathcal{O}(S)^{\varphi(\mathrm{Tr})}$, the \mathbb{G}_a -action $\{\exp(cf^k\delta) \mid c \in \mathbb{C}\}$ commutes with all the \mathbb{G}_a -actions in $\varphi(\mathrm{Tr})$ which implies that the centralizer of $\varphi(\mathrm{Tr})$ is infinite-dimensional. This contradicts the fact that $\dim \varphi(\mathrm{Tr})$ has dimension two and the claim follows. \square

Proof of Theorem 1.1. Let $\varphi: \mathrm{Aut}(S_1) \xrightarrow{\sim} \mathrm{Aut}(S_2)$ be an isomorphism of groups and fix a maximal torus $T_1 \subset \mathrm{Aut}(S_1)$. If S_1 or S_2 are isomorphic to $\mathbb{A}_*^1 \times \mathbb{A}_*^1$, to $\mathbb{A}_*^1 \times \mathbb{A}^1$, or to \mathbb{A}^2 then the claim follows from Lemma 4.15, Proposition 4.14, or Theorem 1.4 respectively. Now let S_1 be isomorphic to some $X_{d,e}$ different from \mathbb{A}^2 . By Lemma 4.14, S_2 is a toric surface $X_{\bar{d},\bar{e}}$. Moreover, $T_2 = \varphi(T_1)$ is a 2-dimensional torus.

By Lemma 4.16, all the root subgroups of $\mathrm{Aut}(X_{d,e})$ with respect to T_1 are mapped by φ to root subgroups of $\mathrm{Aut}(X_{\bar{d},\bar{e}})$ with respect to T_2 . Hence, to conclude the proof, it is enough to show that we can recover $X_{d,e}$ from the abstract group structure of its root subgroups and their relationship with the torus. Assume that the torus T_1 acts on a root subgroup \mathbb{G}_a with character χ . The center of the semidirect product $\mathbb{G}_a \rtimes_{\chi} T_1$ is exactly $\{0\} \rtimes_{\chi} \ker \chi$. The image $\varphi(\mathbb{G}_a)$ is a root subgroup of $\mathrm{Aut}(S_2)$ with respect to the torus $\varphi(T_1) = T_2$ with some character χ_2 . Hence the kernel of χ is mapped under φ to the kernel of χ_2 . We consider now the kernel of the characters of two root subgroups with different distinguished rays and look at their intersection. More precisely, let χ^{α_1} be a character with distinguished ray ρ_1 and χ^{α_2} a character with distinguished ray ρ_2 . By Lemma 4.13 we have

$$\alpha_1 = -\beta_2 + l \cdot \beta_1 \text{ and } \alpha_2 = (a\beta_1 + e'\beta_2) + k \cdot (e\beta_1 + d\beta_2).$$

Recall that e' and a are the only integers with $0 \leq e' < d$ such that $ee' = 1 + ad$. Define

$$K_{l,k} := \ker \chi^{\alpha_1} \cap \ker \chi^{\alpha_2}.$$

From $\chi^{\alpha_1} = 1$ we obtain $\chi^{\beta_2} = (\chi^{\beta_1})^l$ and replacing this into $\chi^{\alpha_2} = 1$ we obtain $(\chi^{\beta_1})^{a+l \cdot e' + k \cdot e + lk \cdot d} = 1$. This yields that the order of $K_{l,k}$ is

$$|K_{l,k}| = |a + l \cdot e' + k \cdot e + lk \cdot d|.$$

We fix a character χ^{α_1} of $\text{Aut}(S_1)$ with distinguished ray ρ_1 . Now we consider all the characters χ^{α_2} with distinguished ray ρ_2 . These are exactly the characters corresponding to the root subgroups that do not commute with the root subgroup corresponding to χ^{α_1} . By considering the intersections of the kernels, we obtain a sequence of integers $\{|K_{l,k}|\}$, where l is fixed and k varies. We observe that this sequence, after possibly some finite number of terms, forms an arithmetic progression. By varying l , we obtain a set of such sequences. The smallest common difference of these arithmetic progressions is d and the second smallest common difference is $d + e$.

Analogously, for every fixed $k \in \mathbb{Z}_{\geq 0}$ the sequence of integers $|K_{l,k}|$, for all $l \in \mathbb{Z}_{\geq 0}$, after possibly some finite number of terms, forms an arithmetic progressions. The smallest common difference of these arithmetic progressions is d and the second smallest common difference is $d + e'$.

Since $\varphi(K_{l,k})$ is again the intersection of the kernels of two non-commuting characters, the sequences of integers $|K_{l,k}|$ for l or k fixed are the same for $\text{Aut}(S_1)$ and $\text{Aut}(S_2)$.

This yields that the isomorphism $\varphi: \text{Aut}(S_1) \xrightarrow{\sim} \text{Aut}(S_2)$ can only exist if $S_1 = X_{d,e}$ and $S_1 = X_{\tilde{d},\tilde{e}}$ with $\tilde{d} = d$, and $\tilde{e} = e$ or $\tilde{e} = e'$. This is, S_1 is isomorphic to S_2 by Lemma 4.12. \square

4.4. Higher dimensional toric varieties. The following lemma is needed for the proof of Theorem 1.2.

Lemma 4.17. *Let X_σ be an affine toric variety of dimension n and let $T \subset \text{Aut}(X_\sigma)$ be a maximal subtorus. Then there exists an $(n-1)$ -dimensional torus $H \subset T$ such that all root subgroups of $\text{Aut}(X)$ with respect to H have different weights.*

Proof. Take any $(n-1)$ -dimensional subtorus $H \subset T$ such that $N_H \cap \sigma = \{0\}$, where N_H is the sublattice of N of 1-parameter subgroups of T that are contained in H and recall that $\sigma \subset N_{\mathbb{R}}$. It is clear that every T -root subgroup is also a H -root subgroup. Now, [Ko14, Proposition 1] shows that every H -root subgroup is also T -root subgroup and so Corollary 4.7 implies that the weights of H -root subgroups are also different. \square

We now show that affine toric varieties are determined by their set of roots.

Lemma 4.18. *Let σ and σ' be strongly convex rational polyhedral cones in $N_{\mathbb{R}}$. If $\mathcal{R}(\sigma) = \mathcal{R}(\sigma')$ then $\sigma = \sigma'$. In particular, a toric variety X_σ is completely determined by the set of its roots with respect to any fixed maximal torus in $\text{Aut}(X_\sigma)$.*

Proof. To prove the lemma, it is enough to show that the cone σ of a toric variety X_σ can be recovered from the set of roots $\mathcal{R}(\sigma)$. Since any strongly convex rational polyhedral cone is the convex hull of its rays, it is enough to show that every ray $\rho \in \sigma(1)$ can be recovered from the set of roots. By [Li10, Remark 2.5] the set $\mathcal{R}_\rho(\sigma)$ of roots with $\rho \in \sigma(1)$ as distinguished ray is not empty. Hence, to recover σ from $\mathcal{R}(\sigma)$ it is enough to recover for every $e \in \mathcal{R}(\sigma)$ its distinguished ray.

By [Li10, Remark 2.5], the lattice vector $m+e \in \mathcal{R}_\rho(\sigma)$ for every $e \in \mathcal{R}_\rho(\sigma)$ and every $m \in \rho^\perp \cap \sigma_M^\vee$. Let us fix now a root $e \in \mathcal{R}(\sigma)$. By the preceding consideration, there exists a hyperplane $H \subset M_{\mathbb{R}}$ such that the linear span of $H \cap (\mathcal{R}(\sigma) - e)$ equals H . Take now $L = H^\perp \subset N_{\mathbb{R}}$ the line orthogonal to H . The line L is composed of two rays and has only two primitive vectors $\pm p \in L$. The distinguished ray of e is given by $\rho_e = -\langle e, p \rangle \cdot p$ since this way $\langle e, \rho_e \rangle = -\langle e, p \rangle^2 = -1$. \square

Proposition 4.19. *Let X and Y be affine varieties of dimension n . Assume that both varieties X and Y admit a structure of toric variety. If $\text{Aut}(X)$ and $\text{Aut}(Y)$ are isomorphic as ind-groups, then X and Y are isomorphic.*

Proof. Fix an isomorphism $\varphi: \text{Aut}(X) \rightarrow \text{Aut}(Y)$ of ind-groups. Let $T \subset \text{Aut}(X)$ be a maximal subtorus of dimension n . Since the groups $\text{Aut}(X)$ and $\text{Aut}(Y)$ are isomorphic, the root subgroups

of $\text{Aut}(X)$ with respect to T are sent to root subgroups of $\text{Aut}(Y)$ with respect to $\varphi(T)$. Moreover, weights are preserved under this isomorphism. Now, Lemma 4.18 implies that $X \cong Y$ as a toric variety. In particular, $X \cong Y$ as a variety. \square

We now proceed to prove the last two remaining main results of this paper.

Proof of Theorem 1.2. Let $\varphi: \text{Aut}(X) \rightarrow \text{Aut}(Y)$ be an isomorphism of ind-groups and let n be the dimension of X . By Lemma 4.17 there is a subtorus $H \subset \text{Aut}(X)$ of dimension $n - 1$ such that all the root subgroups of $\text{Aut}(X)$ have different weights with respect to H . Since φ is an isomorphism of ind-groups, all the root subgroups of $\text{Aut}(Y)$ have different weights with respect to the algebraic torus $\varphi(H) \subset \text{Aut}(Y)$. Hence, by Lemma 4.3 we have $\dim Y \leq n$. Since X admits an n -dimensional faithful torus action, the same holds for Y and we conclude that Y is also a toric variety of dimension n . Now the theorem follows from Proposition 4.19. \square

Proof of Theorem 1.3. Let $\varphi: C \times T \rightarrow C \times T$ be an automorphism of $C \times T$. For every $z \in C$, we define $\varphi_z: T \rightarrow C$ to be the map given by $\varphi_z(t) = \text{pr}_1 \circ \varphi(z, t)$ for all $t \in T$, where $\text{pr}_1: C \times T \rightarrow C$ is the first projection. The map φ_z is constant. Indeed, if φ_z is not constant then it is dominant and since T is a rational variety, this implies that C is unirational, which in dimension one implies that C is rational. But the only affine rational curve without invertible functions is \mathbb{A}^1 , whereas, by assumption, C has a trivial automorphism group.

This yields that the first projection pr_1 is preserved by φ and so φ descends to an automorphism $\tilde{\varphi}: C \rightarrow C$. But C has no nontrivial automorphism, so $\tilde{\varphi} = \text{id}_C$ is the identity. Hence, we obtain that $\varphi(z, t) = (z, \psi(z, t))$ for all $z \in C, t \in T$.

For every $t \in T$ we let $\psi_t: C \rightarrow T$ be the map given by $\psi_t(z) = \psi(z, t)$. The map φ_t is also constant. Indeed, the comorphism $\psi_t^*(z): \mathbb{C}[M] \rightarrow \mathcal{O}(C)$ sends invertible elements to invertible elements, but $\mathbb{C}[M]$ is generated by invertible elements while C admits no invertible global function other than the constants so the image of ψ_t^* is the base field. We obtain that $\psi(z, t) = \tilde{\psi}(t)$ for some automorphism $\tilde{\psi}: T \rightarrow T$ of the torus T and for all $z \in C, t \in T$. This yields that the automorphism φ is a product $\varphi = \text{id}_C \times \tilde{\psi}$, which proves the theorem. \square

REFERENCES

- [AZ12] Ivan Arzhantsev and Mikhail Zaidenberg, *Acyclic curves and group actions on affine toric surfaces*, In: Affine Algebraic Geometry, Osaka, March 3-6, 2011, World Scientific, Singapore, 2013, 1-41.
- [Ba01] Badescu, Lucian, *Algebraic surfaces*, Springer-Verlag, New York, 2001.
- [BD15] Jérémy Blanc, Julie Déserti, *Degree growth of birational maps of the plane*. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), Vol. 14, No. 2. (2015), 507–533.
- [BF13] Jérémy Blanc, Jean-Philippe Furter, *Topologies and structures of the Cremona groups*, Ann. of Math., Volume 178, Issue 3, (2013) 1173-1198.
- [Br09] Michel Brion, *Anti-affine algebraic groups*, Journal of Algebra, Volume 321, Issue 3, (2009) 934-952.
- [Ca11] Serge Cantat, *Sur les groupes de transformations birationnelles des surfaces*. Ann. of Math. (2), Vol. 174, No. 1, 299–340, (2011).
- [Ca14] Serge Cantat, *Morphisms between Cremona groups, and characterization of rational varieties*. Compositio Mathematica, Vol. 150, No. 7, 1107-1124, (2014).
- [Ca15] Serge Cantat, *The Cremona group*. Preprint: <https://perso.univ-rennes1.fr/serge.cantat/Articles/Survey-Cremona-SLC.pdf>.
- [CLS11] David A. Cox, John B. Little, and Henry K. Schenck. *Toric varieties*, volume 124 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2011.
- [Da17] Nguyen-Bac Dang. *Degrees of iterates of rational maps on normal projective varieties*, arXiv:1701.0776, (2017)
- [De82] A. S. Demushkin *Combinatoric Invariance of Toric Singularities*, Mosc. Univ. Math. Bull. 37, no. 2, 104-111 (1982)
- [Des06a] Julie Déserti, *Sur les automorphismes du groupe de Cremona*, Compositio Mathematica 146 (2006), no. 6, 1459-1478.
- [Des06b] Julie Déserti, *Sur le groupe des automorphismes polynomiaux du plan affine*, J. Algebra 297 (2006), no. 2, 584-599.
- [De70] Michel Demazure. *Sous-groupes algébriques de rang maximum du groupe de Cremona*. Ann. Sci. École Norm. Sup. (4), 3:507–588, 1970.

- [DF01] Jeffrey Diller, Charles Favre, *Dynamics of bimeromorphic maps of surfaces*. Amer. J. Math., Vol. 123, No. 6, 1135–1169, (2001)
- [Fil82] Richard P. Filipkiewicz. *Isomorphisms between diffeomorphism groups*. Ergodic Theory Dynamical Systems, 2(2):159171, 1982.
- [Fr06] G. Freudenburg, *Algebraic theory of locally nilpotent derivations*. Encyclopaedia of Mathematical Sciences 136, Invariant Theory and Algebraic Transformation Groups 7, Springer, 2006.
- [FK18] Jean-Philippe Furter and Hanspeter Kraft, *On the geometry of the automorphism group of affine n -space*, in preparation, 2017.
- [Ful93] William Fulton. *Introduction to toric varieties*, volume 131 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1993. The William H. Roever Lectures in Geometry.
- [Gi80] Marat H. Gizatullin. *Rational G -surfaces*. Izv. Akad. Nauk SSSR Ser. Mat. Vol. 44, No. 1, 110–144, 239 (1980).
- [Kle93] Felix Klein, *Vergleichende Betrachtungen über neuere geometrische Forschungen*, Mathematische Annalen, 43, 1, 63–100 (1893).
- [Ku02] Shrawan Kumar, *Kac-Moody groups, their flag varieties and representation theory*, Progress in Mathematics, vol. 204, Birkhäuser Boston Inc., Boston, MA, 2002.
- [Kr15] Hanspeter Kraft *Automorphism Groups of Affine Varieties and a Characterization of Affine n -Space*, arXiv:1501.06362 .
- [KRS17] Hanspeter Kraft, Andriy Regeta and Immanuel van Santen (born Stampfli), *Is the affine space determined by its automorphism group?*, arXiv:1707.06883
- [Ko14] Kotenkova, P.Y., *On restriction of roots on affine T -varieties*, Beitr. Algebra Geom., October 2014, Volume 55, Issue 2, pp 621-634.
- [Li78] Lieberman, D. I., *Compactness of the Chow scheme: applications to automorphisms and deformations of Kähler manifolds*, in *Fonctions de plusieurs variables complexes, III (Sém. François Norguet, 1975–1977)*, Springer Verlag, Berlin, 1978.
- [Li10] Alvaro Liendo. *Affine \mathbb{T} -varieties of complexity one and locally nilpotent derivations*. *Transform. Groups*, 15(2):389–425, 2010.
- [Li11] Alvaro Liendo. *Roots of the affine Cremona group*. *Transform. Groups* **16** (2011), no. 4, 1137–1142.
- [Od88] Tadao Oda. *Convex bodies and algebraic geometry. An introduction to the theory of toric varieties*, volume 15 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)*. Springer-Verlag, Berlin, 1988.
- [Reg17] Andriy Regeta, *Characterization of n -dimensional normal affine SL_n -varieties*, arXiv:1702.01173
- [Sh66] I. R. Shafarevich, *On some infinite-dimensional groups*, Rend. Mat. e Appl. (5) **25** (1966), no. 1-2, 208–212.
- [Sp98] T. A. Springer, *Linear algebraic groups*. Progress in Mathematics, 9. Birkhuser Boston, Inc., Boston, MA, 1998.
- [St13] Immanuel Stampfli, *Contributions to automorphisms of affine spaces*, PhD thesis, University of Basel, 2013.
- [We55] André Weil, *On algebraic groups of transformations*. American Journal of Mathematics, Vol. 77, 355–391, (1955).

INSTITUTO DE MATEMÁTICA Y FÍSICA, UNIVERSIDAD DE TALCA, CASILLA 721, TALCA, CHILE.

E-mail address: aliendo@inst-mat.utalca.cl

UNIVERSITÄT ZU KÖLN, MATHEMATISCHES INSTITUT, WEYERTAL 86-90, 50931 KÖLN, GERMANY.

E-mail address: andriyregeta@gmail.com

IMPERIAL COLLEGE LONDON, MATHEMATICS DEPARTMENT, 180 QUEEN'S GATE, LONDON SW7 2AZ, UK.

E-mail address: christian.urech@gmail.com