

Algebraic stability and degree growth of monomial maps

Jan-Li Lin

Received: 2 August 2010 / Accepted: 9 February 2011 / Published online: 30 March 2011
© Springer-Verlag 2011

Abstract Given a rational monomial map, we consider the question of finding a toric variety on which it is algebraically stable. We give conditions for when such variety does or does not exist. We also obtain several precise estimates of the degree sequences of monomial maps.

1 Introduction

Given an $n \times n$ integer matrix $A = (a_{i,j})$, there is an associated monomial map $f_A : (\mathbb{C}^*)^n \rightarrow (\mathbb{C}^*)^n$ defined by

$$f_A(x_1, \dots, x_n) = \left(\prod_j x_j^{a_{1,j}}, \dots, \prod_j x_j^{a_{n,j}} \right).$$

Monomial maps fit nicely into the framework of toric varieties and equivariant maps on them. In this paper, we try to make extensive use of the toric method to study the dynamics of monomial maps.

The idea of applying the theory of toric varieties to monomial maps is in fact not new. For example, Favre [3] used the orbit-cone correspondence of the torus action to translate a criterion of algebraic stability to a condition about cones in a fan, and uses it to classify monomial maps in the case of toric surfaces. In order to generalize his result to higher dimension, one needs a good understanding on pulling back cohomology classes under rational maps. So we start from a formula of pulling back divisors in toric varieties (Proposition 3.1).

We then prove a criterion for algebraic stability (Theorem 4.2). Results about stability are proven using the criterion. For example, we proved that every monomial polynomial map is algebraically stable on $(\mathbb{P}^1)^n$. Also, we generalize some results of [3] to higher dimension.

J.-L. Lin (✉)

Department of Mathematics, Indiana University, Bloomington, IN 47405, USA
e-mail: janlin@indiana.edu

Theorem 4.7 Suppose that $A \in \mathbf{M}_n(\mathbb{Z})$ is an integer matrix.

- (1) If there is a unique eigenvalue λ of A of maximal modulus, with algebraic multiplicity one; then $\lambda \in \mathbb{R}$, and there exists a simplicial toric birational model X (maybe singular) and a $k \in \mathbb{N}$ such that f_A^k is strongly algebraically stable on X .
- (2) If $\lambda, \bar{\lambda}$ are the only eigenvalues of A of maximal modulus, also with algebraic multiplicity one, and if $\lambda = |\lambda| \cdot e^{2\pi i\theta}$, with $\theta \notin \mathbb{Q}$; then there is no toric birational model which makes f_A strongly algebraically stable.

For the definition of (strongly) algebraically stable, see Sect. 4. We note that many of the results concerning stabilization of monomial maps in this paper have been obtained independently by Jonsson and Wulcan [6]. For more detail, see the remark after Theorem 4.7.

Next, we focus on the projective space \mathbb{P}^n in Sect. 5. For \mathbb{P}^n , the pull back f^* of a rational map $f: \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ on $H^{1,1}(\mathbb{P}^n; \mathbb{R})$ is given by the degree of f . Thus we consider the degree sequence $\{\deg(f^k)\}_{k=1}^\infty$. Results about the degree sequence of monomial maps can be found in [1, 5].

In particular, one can define the *asymptotic degree growth*

$$\delta_1(f_A) = \lim_{k \rightarrow \infty} (\deg(f_A^k))^{\frac{1}{k}}.$$

Hasselblatt and Propp [5, Theorem 6.2] proved that $\delta_1(f_A) = \rho(A)$, the spectral radius of the matrix A . We refine the above result and obtain the following description of the asymptotic behavior of the degree sequence for a general monomial map.

Theorem 5.1 Given an $n \times n$ integer matrix A with nonzero determinant, assume that $\rho(A)$ is the spectral radius of A . Then there exist two positive constants $C_1 \geq C_0 > 0$ and a unique integer ℓ with $0 \leq \ell \leq n - 1$, such that

$$C_0 \cdot k^\ell \cdot \rho(A)^k \leq \deg(f_A^k) \leq C_1 \cdot k^\ell \cdot \rho(A)^k$$

for all $k \in \mathbb{N}$.

In fact, $(\ell + 1)$ is the size of the largest Jordan block of A among the ones corresponding to eigenvalues of maximal modulus.

If the matrix A has some better property, then we can describe the degree sequence more precisely. This is the content of Theorems 5.5, 5.6, and the following theorem.

Theorem 5.7 Assume that the matrix A is diagonalizable, and assume for each eigenvalue λ of A of maximum modulus, $\lambda/\bar{\lambda}$ is a root of unity. Then there is a positive integer p , and p constants $C_0, C_1, \dots, C_{p-1} \geq 1$, such that

$$\deg(f_A^{pk+l}) = C_l \cdot |\lambda_1|^{pk+l} + O(|\lambda_2|^{pk+l}),$$

where $l = 0, 1, \dots, p - 1$.

The above theorems about the degree sequences of monomial maps can be generalized to the case of weighted projective spaces. On weighted projective spaces, we have the notion of *weighted degree* of a toric map, and their growth under iterations follows the same pattern as the degree growth of monomial maps in projective spaces. This generalization is suggested to us by Mattias Jonsson. We introduce weighted projective space briefly, and explain the generalization in Sect. 5.3.

2 Toric varieties

In this section, we give a brief survey of basic definitions and properties of toric varieties. For more detail, we refer the readers to [2] or [4].

2.1 Cones and affine toric varieties

Let $N \cong \mathbb{Z}^n$ be a lattice of rank n , and $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$. A *polyhedral cone* in $N_{\mathbb{R}}$ is of the form $\sigma = \left\{ \sum_{i=1}^k r_i v_i \mid r_i \in \mathbb{R}_{\geq 0}, v_i \in N_{\mathbb{R}} \right\}$ for some finite set of vectors v_1, \dots, v_k .

The *dimension* of σ is the dimension of the \mathbb{R} -span of σ . A cone is *strongly convex* if it does not contain any line. A cone is *rational* if we can choose the generators v_1, \dots, v_k from the lattice N . In what follows, by a cone we always mean “a strongly convex, rational polyhedral cone”.

From the lattice N we can form the *dual lattice* $M := \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$, with dual pairing denoted by $\langle \cdot, \cdot \rangle$. The *dual cone* σ^{\vee} of σ is defined by

$$\sigma^{\vee} = \{u \in M_{\mathbb{R}} \mid \langle u, v \rangle \geq 0 \text{ for all } v \in \sigma\}.$$

Let $S_{\sigma} := \sigma^{\vee} \cap M$, the variety $U_{\sigma} := \text{Spec}(\mathbb{C}[S_{\sigma}])$ is called the *affine toric variety* associated to the cone σ . More concretely, a closed point in U_{σ} corresponds to a semigroup morphism $(S_{\sigma}, +) \rightarrow (\mathbb{C}, \cdot)$ which sends $0 \in S_{\sigma}$ to $1 \in \mathbb{C}$.

For example, let $N = \mathbb{Z}^n$, and let σ be the cone in $N_{\mathbb{R}} \cong \mathbb{R}^n$ generated by the standard basis e_1, \dots, e_n . Then the affine toric variety $U_{\sigma} \cong \mathbb{C}^n$.

We will need the following definitions. One dimensional cones are also called *rays*. On a ray, the nonzero integral point of the smallest norm is called *the ray generator*. A cone is *simplicial* if it is generated by linearly independent vectors. A cone is *smooth* if it is generated by part of a basis for the lattice N . A cone σ is smooth if and only if the corresponding affine variety U_{σ} is smooth.

2.2 Fans and general toric varieties

A fan Σ in $N_{\mathbb{R}}$ is a finite collection of cones in $N_{\mathbb{R}}$ such that each face of a cone in Σ is again in Σ , and the intersection of two cones in Σ is a face of each.

From a fan Σ , we can construct the *toric variety* $X(\Sigma)$ corresponding to Σ . For cones $\sigma, \tau \in \Sigma$, we glue U_{σ} and U_{τ} along the open subvariety $U_{\sigma \cap \tau}$. The resulting variety is the toric variety $X(\Sigma)$.

We use the notion $\Sigma(k)$ to denote the set of all k -dimensional cones of Σ . A fan Σ is *complete* if $|\Sigma| = N_{\mathbb{R}}$, where $|\Sigma| := \bigcup_{\sigma \in \Sigma} \sigma$ is the *support* of Σ .

Example 2.1 Let $N = \mathbb{Z}^n$, e_1, \dots, e_n be the standard basis of N , and $e_0 = -(e_1 + \dots + e_n)$. For any proper subset I of the set $\mathbf{n} = \{0, 1, \dots, n\}$, let σ_I be the cone generated by $\{e_i \mid i \in I\}$. The set $\Sigma = \{\sigma_I \mid I \subsetneq \mathbf{n}\}$ forms a fan. The toric variety associated to the fan is the projective space $X(\Sigma) \cong \mathbb{P}^n$.

Example 2.2 We will construct a fan Σ that corresponds to the product of the projective line $(\mathbb{P}^1)^n$. The rays are generated by the standard basis vectors e_1, \dots, e_n and their negatives $-e_1, \dots, -e_n$. The maximal cones are generated by vectors of the form $\{(s_1 e_1), \dots, (s_n e_n)\}$, where $s_i \in \{+1, -1\}$ are the signs. All other cones in Σ are faces of some maximal cone.

2.3 The orbits of the torus action

Every toric variety $X(\Sigma)$ is equipped with a torus action, thus $X(\Sigma)$ can be written as the disjoint union of the orbits. The orbits are in 1–1 correspondence with cones in the fan Σ as follows. For each cone $\tau \in \Sigma$, let $x_\tau \in U_\tau$ be the closed point corresponding to the semigroup morphism $S_\sigma \rightarrow \mathbb{C}$.

$$u \in S_\sigma \mapsto \begin{cases} 1 & \text{if } u \in \sigma^\perp, \\ 0 & \text{otherwise.} \end{cases}$$

We define O_τ to be the orbit of x_τ under the torus action. The closure of an orbit O_τ in $X(\Sigma)$ is denoted by $V(\tau)$.

2.4 Toric maps

Suppose $A : N \rightarrow N'$ is a homomorphism of lattices, Σ is a fan in N , and Σ' is a fan in N' .

Definition Given a cone $\sigma \in \Sigma$, we say that σ *maps regularly* to Σ' by A if there is a cone $\sigma' \in \Sigma'$ such that $A(\sigma) \subseteq \sigma'$. In this case, we call the smallest such cone in Σ' the *cone closure* of the image of σ , and denote it by $\overline{A(\sigma)}$.

If $A : N \rightarrow N'$ is a homomorphism such that every cone of Σ maps regularly to Σ' , then A induces a morphism of varieties $f_A : X(\Sigma) \rightarrow X(\Sigma')$ which is *equivariant* under the torus action. Conversely, every equivariant morphism $X(\Sigma) \rightarrow X(\Sigma')$ is induced by a homomorphism of lattices satisfying the above property. Equivariant morphisms will map orbits to orbits. If $\sigma \in \Sigma$, then f_A maps $O_\sigma \subset X(\Sigma)$ to $O_{\overline{A(\sigma)}} \subset X(\Sigma')$.

More generally, any homomorphism of lattices $A : N \rightarrow N'$ induces an *equivariant rational map* $f_A : X(\Sigma) \dashrightarrow X(\Sigma')$. On a complete toric variety, f_A is *dominant* if and only if $A_{\mathbb{R}} = (A \otimes \mathbb{R}) : N_{\mathbb{R}} \rightarrow N'_{\mathbb{R}}$ is surjective.

3 Divisors on toric varieties

We will recall basic definitions and properties of divisors in a toric variety, then prove a formula about pulling back divisors.

3.1 Weil divisors and divisor class groups

In a toric variety $X(\Sigma)$, let $\Sigma(1) = \{\tau_1, \dots, \tau_d\}$ be the set of rays in Σ . A T -invariant Weil divisor, T -Weil divisor for short, is of the form $\sum_{i=1}^d a_i V(\tau_i)$ where $a_i \in \mathbb{Z}$. The group of T -Weil divisors is denoted by $\text{WDiv}_T(X(\Sigma))$. The principal divisors in $\text{WDiv}_T(X(\Sigma))$ are in 1-1 correspondence to elements of M . The quotient

$$A_{n-1}(X(\Sigma)) := \text{WDiv}_T(X(\Sigma))/M$$

is the *divisor class group* of $X(\Sigma)$.

3.2 Cartier divisors and Picard groups

In a complete toric variety $X(\Sigma)$ of dimension n , the torus invariant Cartier divisors, or T -Cartier divisors, is given by the following data. For each cone $\sigma \in \Sigma(n)$, we specify an element $u(\sigma) \in M$. The datum $\{u(\sigma) | \sigma \in \Sigma(n)\}$ are required to satisfy the compatibility

condition that $[u(\sigma)] = [u(\sigma')]$ in $M/M(\sigma \cap \sigma')$, where $M(\sigma \cap \sigma') = (\sigma \cap \sigma')^\perp \cap M$. We write $D = \{u(\sigma)\}$ and call it the Cartier divisor defined by the data $\{u(\sigma)\}$. We denote the group of all T -Cartier divisor by $\text{CDiv}_T(X(\Sigma))$.

Every T -Cartier divisor $D = \{u(\sigma)\}$ gives rise to a unique T -Weil divisor

$$[D] = \sum_{\tau_i \in \Sigma(1)} -\langle u(\sigma), v_i \rangle \cdot V(\tau_i),$$

with σ any maximal cone such that $v_i \in \sigma$.

The data $\{u(\sigma) | \sigma \in \Sigma(n)\}$ also defines a continuous piecewise linear function ψ_D on $N_{\mathbb{R}}$. The restriction of ψ_D to the maximal cone σ is given by $u(\sigma)$, i.e., $\psi_D(v) = \langle u(\sigma), v \rangle$ for $v \in \sigma$. Conversely, a continuous piecewise linear, integral (i.e., given by an element of M on each cone) function ψ on $N_{\mathbb{R}}$ determines a unique T -Cartier divisor D , with $[D] = \sum -\psi(v_i) \cdot V(\tau_i)$. The function ψ is called the *support function* of the Cartier divisor D . On a complete toric variety, a T -Cartier divisor is *ample* if and only if its support function is strictly convex [4, p.70].

One can identify M as a subgroup of $\text{CDiv}_T(X(\Sigma))$. Each $u \in M$ is identified with the Cartier divisor such that $u(\sigma) = u$ for all $\sigma \in \Sigma(n)$. The quotient $\text{CDiv}_T(X(\Sigma))/M$ is the *Picard group* of $X(\Sigma)$, and is denoted by $\text{Pic}(X(\Sigma))$.

We conclude this section by mentioning relations between Picard groups and cohomology groups. For a complete toric variety X , we have $\text{Pic}(X) = H^2(X; \mathbb{Z})$. If X is also simplicial, then

$$H^{1,1}(X) := H^1(X, \Omega_X) = H^2(X; \mathbb{C}) = \text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{C}.$$

3.3 Pulling back divisors

The main result of this section is the following.

Proposition 3.1 *Let Σ, Σ' be complete fans, and $f_A : X(\Sigma) \dashrightarrow X(\Sigma')$ be a dominant toric rational map induced by $A : N \rightarrow N'$. The pull back of a Cartier divisor D via f_A is the Weil divisor*

$$f_A^* D = \sum_{\tau_i \in \Sigma(1)} -\psi_D(Av_i) \cdot V(\tau_i). \quad (3.1)$$

Here ψ_D is the support function of D , and v_i is the ray generator of τ_i .

Proof We can refine the fan Σ to get a fan $\tilde{\Sigma}$ such that A induces a toric morphism from $X(\tilde{\Sigma})$ to $X(\Sigma)$. In order to distinguish from f_A , we call this morphism \tilde{f}_A . The morphism $\pi : X(\tilde{\Sigma}) \rightarrow X(\Sigma)$ is induced by the identity map on N . It is proper and birational. So we have the following diagram.

$$\begin{array}{ccc} & X(\tilde{\Sigma}) & \\ \pi \swarrow & & \searrow \tilde{f}_A \\ X(\Sigma) & \xrightarrow{\tilde{f}_A \circ \pi^{-1}} & X(\Sigma') \end{array}$$

To pull back the divisor D , we use $f_A^* D = \pi_*(\tilde{f}_A^* D)$. Once we show that $\tilde{f}_A^* D$ is given by (3.1), it is independent of refinement.

If ψ_D is the support function of D , then $\tilde{f}_A^* D$ will have $\psi_D \circ A$ as its support function. Thus, as a Weil divisor, we have

$$\tilde{f}_A^* D = \sum_{\tau_i \in \tilde{\Sigma}(1)} -\psi_D(Av_i) \cdot V(\tau_i).$$

The fan $\tilde{\Sigma}$ is a subdivision of Σ , and π is induced by identity. The push forward map π_* is given by

$$\pi_* V(\tau) = \begin{cases} V(\tau) & \text{if } \tau \in \Sigma(1), \\ 0 & \text{if } \tau \notin \Sigma(1). \end{cases}$$

Therefore, combining the two steps, we obtain

$$f_A^* D = \pi_*(\tilde{f}_A^* D) = \sum_{\tau_i \in \Sigma(1)} -\psi_D(Av_i) \cdot V(\tau_i).$$

□

Notice that, if D is a principal divisor, i.e., ψ_D is a linear function, then the pull back $f_A^* D$ will again be principal, given by the linear function $\psi_D \circ A$. Thus it induces a map, also denoted by f_A^* , from the Picard group to the divisor class group.

$$f_A^* : \text{Pic}(X(\Sigma')) \rightarrow A_{n-1}(X(\Sigma)).$$

If the fan Σ is smooth, then $\text{Pic}(X(\Sigma)) \cong A_{n-1}(X(\Sigma))$, and the pull back map induces a map on Picard groups.

$$f_A^* : \text{Pic}(X(\Sigma')) \rightarrow \text{Pic}(X(\Sigma)).$$

If the fan Σ is simplicial, then every Weil divisor D is \mathbb{Q} -Cartier. Thus if we denote $G_{\mathbb{Q}} = G \otimes_{\mathbb{Z}} \mathbb{Q}$ for an abelian group G , then we have maps

$$\begin{aligned} f_A^* : \text{CDiv}_T(X(\Sigma'))_{\mathbb{Q}} &\rightarrow \text{CDiv}_T(X(\Sigma))_{\mathbb{Q}}, \\ f_A^* : \text{Pic}(X(\Sigma'))_{\mathbb{Q}} &\rightarrow \text{Pic}(X(\Sigma))_{\mathbb{Q}}. \end{aligned}$$

We use the same symbol f_A^* here to avoid inventing too many notations, and we will state clearly whether we talk about divisors or divisor classes.

What we do for pulling back \mathbb{Q} -Cartier divisors in the simplicial case is as follows. An element $D \in \text{CDiv}_T(X(\Sigma'))_{\mathbb{Q}}$ can be identified with a *rational* support function ψ_D . The composition $(\psi_D \circ A)$ is piecewise linear on $\tilde{\Sigma}$, not on Σ . We make an interpolation and obtain a piecewise linear function on Σ . If we denote the modifying (interpolation) function by $\mu = \mu_{\tilde{\Sigma}, \Sigma}$, we can conclude the following:

Corollary *For complete, simplicial toric varieties $X(\Sigma)$, $X(\Sigma')$, and a dominant toric rational map $f_A : X(\Sigma) \dashrightarrow X(\Sigma')$, we can write the procedure of pulling back divisors as $f_A^* D = \mu_{\tilde{\Sigma}, \Sigma}(\psi_D \circ A)$.*

4 Algebraic stability

For the rest of this paper, all toric varieties are assumed to be complete and simplicial.

4.1 Definition and a geometric criterion

Here we define algebraic stability in the case of toric maps. For a general discussion on algebraic stable maps, see [8].

Definition A toric rational map $f_A : X(\Sigma) \dashrightarrow X(\Sigma)$ is *strongly algebraically stable* if $(f_A^k)^* = (f_A^*)^k$ as maps of $\text{CDiv}_T(X(\Sigma))_{\mathbb{Q}}$ for all $k \in \mathbb{N}$. It is *algebraically stable* if $(f_A^k)^* = (f_A^*)^k$ as maps of $\text{Pic}(X(\Sigma))_{\mathbb{Q}}$, for all k .

Notice that $(f_A^k)^* = (f_A^*)^k$ on $\text{CDiv}_T(X(\Sigma))_{\mathbb{Q}}$ implies $(f_A^k)^* = (f_A^*)^k$ on $\text{Pic}(X(\Sigma))_{\mathbb{Q}}$, so the condition for strongly algebraic stability is indeed stronger. It is not clear to us whether the two conditions are equivalent or not in general. However, if we assume that the toric variety $X = X(\Sigma)$ is projective, then the two conditions are equivalent. We will prove that later in this section.

Our next goal is to prove a geometric characterization of strongly algebraically stable maps. We need to prove a lemma first. Given two homomorphisms of lattices $A : N \rightarrow N'$ and $B : N'' \rightarrow N$, they induce two toric rational maps $f_A : X(\Sigma) \rightarrow f(\Sigma')$ and $f_B : X(\Sigma'') \rightarrow f(\Sigma)$.

Lemma 4.1 $(f_A \circ f_B)^* = f_B^* \circ f_A^*$ as maps

$$\text{CDiv}_T(X(\Sigma''))_{\mathbb{Q}} \rightarrow \text{CDiv}_T(X(\Sigma'))_{\mathbb{Q}}$$

if and only if for each ray in Σ'' , the cone closure of its image maps regularly to Σ' . That is, for each $\tau \in \Sigma''(1)$, there exists a $\sigma' \in \Sigma'$ such that $A(\overline{B(\tau)}) \subset \sigma'$.

Proof First, suppose that the geometric condition is satisfied, we want to show $(f_A \circ f_B)^* = f_B^* \circ f_A^*$. Remember that $(f_A \circ f_B)^* D = \mu(\psi_D \circ (A \circ B))$ and $(f_B^* \circ f_A^*) D = f_B^*(f_A^* D) = \mu(\mu(\psi_D \circ A) \circ B)$, where μ is the modifying function. So it is enough to show that, for all $\tau_i \in \Sigma(1)$ and v_i the ray generator of τ_i ,

$$(\psi_D \circ (A \circ B))(v_i) = (\mu(\psi_D \circ A) \circ B)(v_i),$$

that is, $\psi_D(A(\overline{Bv_i})) = \mu(\psi_D \circ A)(Bv_i)$.

Since $A(\overline{B(\tau_i)}) \subset \sigma'$ for some $\sigma' \in \Sigma'$ and ψ_D is linear on σ' , hence $(\psi_D \circ A)$ is linear on $\overline{B(\tau_i)}$. The interpolation μ therefore does not do anything on $\overline{B(\tau_i)}$, and we have $\mu(\psi_D \circ A)(Bv_i) = (\psi_D \circ A)(Bv_i)$.

Conversely, if for some ray $\tau \in \Sigma(1)$, $\overline{B(\tau)}$ does not map regularly by A . This means that $A(\overline{B(\tau)})$ is not contained in any cone of Σ' . We will construct a divisor $D \in \text{CDiv}_T(X(\Sigma''))$ such that $(f_A \circ f_B)^* D \neq (f_B^* \circ f_A^*) D$. Let $\gamma_1, \dots, \gamma_m$ be the one-dimensional faces of Σ' , and for $i = 1, \dots, m$, let

$$a_i = \begin{cases} 0 & \text{if } \gamma_i \text{ is a face of } \overline{(A \circ B)(\tau)}, \\ 1 & \text{otherwise.} \end{cases}$$

Define $D = \sum_{i=1}^m a_i \cdot V(\tau'_i)$, and let ψ_D be the support function of D . First, observe that $\psi_D(v) = 0$ if and only if $w \in \overline{(A \circ B)(\tau)}$. Thus for the divisor $(f_A \circ f_B)^* D$, the coefficient of $V(\tau)$ is 0.

On the other hand, since $A(\overline{B(\tau)})$ is not contained in $\overline{(A \circ B)(\tau)}$, there is some one dimensional face τ_0 of $\overline{B(\tau)}$ such that $A(\tau_0) \notin \overline{(A \circ B)(\tau)}$. Let v_0 be the ray generator of τ_0 , then $\psi_D(Av_0) > 0$. Thus $\mu(\psi_D \circ A)$ is strictly positive in the relative interior of $\overline{B(\tau)}$, which contains Bv . This implies that the coefficient of $V(\tau)$ for the divisor $(f_B^* \circ f_A^*) D$ is strictly positive. Therefore we have $(f_A \circ f_B)^* D \neq (f_B^* \circ f_A^*) D$. \square

Theorem 4.2 *A toric rational map $f_A : X(\Sigma) \dashrightarrow X(\Sigma)$ is strongly algebraically stable if and only if for all ray $\tau \in \Sigma(1)$ and for all $n \in \mathbb{N}$, $\overline{A^n(\tau)}$ maps regularly to Σ by A .*

Proof First assume that $f = f_A$ is strongly algebraically stable. Thus for all $n \in \mathbb{N}$, we have $(f^n)^* = (f^*)^n$ and $(f^{n+1})^* = (f^*)^{n+1}$. This gives us

$$(f \circ f^n)^* = (f^{n+1})^* = (f^*)^{n+1} = (f^*)^n \circ f^* = (f^n)^* \circ f^*.$$

By the above lemma, the equality $(f \circ f^n)^* = (f^n)^* \circ f^*$ implies that $\overline{A^n(\tau)}$ is mapped regularly to Σ by f .

Conversely, assume that $\overline{A^n(\tau)}$ is mapped regularly to Σ by f for all $n \in \mathbb{N}$. This tells us that $(f \circ f^n)^* = (f^n)^* \circ f^*$ for all $n \in \mathbb{N}$. Thus we have $(f^n)^* = (f^*)^n$ for all $n \in \mathbb{N}$ by an induction argument. \square

In fact, let $\sigma_n = \overline{A^n(\tau)}$, the next lemma implies that, not only σ_n maps to Σ regularly, also the cone closure $\overline{A(\sigma_n)}$ is equal to $\sigma_{n+1} = \overline{A^n(\tau)}$.

Lemma 4.3 *Assume further that $A_{\mathbb{R}}$ is surjective, and $\overline{B(\tau)}$ maps regularly to Σ' , then for all $\tau \in \Sigma''$, we have $\overline{A(B(\tau))} = \overline{(A \circ B)(\tau)}$.*

That is, if σ is the smallest cone in Σ that contains $B(\tau)$, and σ' is the smallest cone in Σ' that contains $A(\sigma)$, then σ' will be the smallest cone in Σ' that contains $A(B(\tau))$.

Proof Obviously, $\overline{(A \circ B)(\tau)}$ is a face of $\overline{A(B(\tau))}$. Thus there is a supporting hyperplane H' of $\overline{A(B(\tau))}$ in $N'_{\mathbb{R}}$ such that

$$\overline{A(B(\tau))} \cap H' = \overline{(A \circ B)(\tau)}.$$

The preimage $H = A^{-1}(H')$ will then be a supporting hyperplane of $\overline{B(\tau)}$ in $N_{\mathbb{R}}$, so $\overline{B(\tau)} \cap H$ is a face of $\overline{B(\tau)}$ that contains $B(\tau)$. By the minimality of $\overline{B(\tau)}$, we must have $\overline{B(\tau)} \cap H = \overline{B(\tau)}$, i.e., $\overline{B(\tau)} \subset H$. Thus, $\overline{A(B(\tau))} \subset H'$, and by the minimality of $\overline{A(B(\tau))}$, we know $\overline{A(B(\tau))} \subset H'$. Therefore,

$$\overline{A(B(\tau))} = \overline{A(B(\tau))} \cap H' = \overline{(A \circ B)(\tau)}.$$

\square

With Theorem 4.2 and Lemma 4.3, we can describe the behavior, under iterations, of an strongly algebraically stable toric rational map f_A very concretely, as follows. For each ray $\tau \in \Sigma(1)$, let $\sigma_1 = \overline{A(\tau)}$ be the smallest cone containing $A(\tau)$, then σ_1 will map regularly to some cone in N . Assume $\sigma_2 = \overline{A(\sigma_1)} = \overline{A^2(\tau)}$ is the smallest such cone. Here the second equality is due to the lemma. Then σ_2 will map regularly again to some smallest $\sigma_3 = \overline{A(\sigma_2)} = \overline{A^2(\sigma_1)} = \overline{A^3(\tau)}$, and so on.

4.2 Algebraic stable versus strongly algebraic stable

Now we can prove the equivalence of algebraic stable and strongly algebraic stable in the projective case. The equivalence of the two conditions, and a proof in the general case is mentioned to us by Charles Favre. We adapted his proof to a proof for toric varieties.

Given two integer matrices $A, B \in \mathbf{M}_n(\mathbb{Z})$ with nonzero determinants, which induce two dominant toric rational maps $f_A, f_B : X \dashrightarrow X$.

Lemma 4.4 *Let D be an ample, T -invariant divisor on X , then the difference $(f_B^* \circ f_A^*)D - (f_A \circ f_B)^*D$ is an effective \mathbb{Q} -Cartier divisor.*

Proof Write

$$(f_B^* \circ f_A^*)D = \sum_{\tau \in \Sigma(1)} a_\tau V(\tau), \quad (f_A \circ f_B)^*D = \sum_{\tau \in \Sigma(1)} b_\tau V(\tau).$$

We will show that $a_\tau \geq b_\tau$ for every $\tau \in \Sigma(1)$, which is equivalent to the lemma.

Let $\psi = \psi_D$ be the support function of D . For some $\tau \in \Sigma(1)$, let $v \in \tau$ be the ray generator. Let $\sigma = \overline{B\tau}$ be the smallest cone which contains $B\tau$, and assume that u_1, \dots, u_d are the generators of the cone σ . Then there are positive numbers r_1, \dots, r_d such that $B(v) = r_1 u_1 + \dots + r_d u_d$.

By the formula for pulling back divisors, to compute a_τ , we need to apply the *interpolation* process, and obtain

$$a_\tau = -[r_1 \psi(Au_1) + \dots + r_d \psi(Au_d)].$$

We can also see that

$$b_\tau = -\psi((A \circ B)(v)) = -\psi(r_1 Au_1 + \dots + r_d Au_d).$$

Now the fact $a_\tau \geq b_\tau$ comes from the fact that ψ is (strictly) convex since D is ample. \square

Proposition 4.5 *For a projective, complete, simplicial toric variety $X = X(\Sigma)$, a toric rational map f_A is strongly algebraically stable if and only if it is algebraically stable.*

Proof Since strongly AS implies AS, it suffices to show the other direction. Assume that f_A is not strongly AS, then there is a ray τ and a positive integer k such that $A^k(\overline{A\tau})$ is not contained in any cone of Σ .

Let D be any ample divisor, using the same notation as in the proof of the above lemma, with $B = A^k$, we can see that $a_\tau > b_\tau$, since the $A(u_i)$'s are not in a same cone, and ψ is strictly convex.

Thus the difference between the support functions of $(f_A^{k+1})^*D$ and that of $(f^*)^{k+1}D$ is a nonnegative function which is strictly positive on τ , hence cannot be linear. This means $(f^{k+1})^*D \neq (f^*)^{k+1}D$ in $\text{Pic}(X)$. \square

4.3 Applications of the criterion

We will apply the above criterion (Theorem 4.2) to give some results about stabilization in certain cases.

First, suppose all entries of A are non-negative, i.e., f_A is a polynomial monomial map. There is a nice nonsingular toric model on which f_A is algebraically stable, namely $(\mathbb{P}^1)^n$.

Proposition 4.6 *Every monomial polynomial map is strongly algebraically stable on $(\mathbb{P}^1)^n$, hence algebraically stable.*

Proof Let Σ be the fan such that $X(\Sigma) = (\mathbb{P}^1)^n$. The rays of Σ are given by $\tau_i = \mathbb{R}_{\geq 0} \cdot e_i$ and $-\tau_i$, for $i = 1, \dots, n$. The morphism A maps each of τ_i into the cone σ_+ generated by e_1, \dots, e_n , and maps each of $-\tau_i$ into the cone σ_- generated by $-e_1, \dots, -e_n$.

Observe that the compositions of polynomial maps are still polynomial maps. So A^k are all polynomial monomial maps for $k \geq 1$. Also notice that $A^k(\tau_i) \subset \sigma_+$, so $\overline{A^k(\tau_i)}$ is a face of σ_+ . Hence there is a subset of indexes $I \subset \{1, \dots, n\}$ such that $\overline{A^k(\tau_i)}$ is generated by

$\{e_i | i \in I\}$. Since each $A^k(e_i) \in \sigma_+$, we have that $A(\overline{A^k(\tau_i)}) \subset \sigma_+$. This means $\overline{A^k(\tau_i)}$ maps regularly for all k . By symmetry, we also know that $A(\overline{A^k(-\tau_i)}) \subset \sigma_-$. Therefore, the map f_A is strongly algebraically stable on $X(\Sigma) = (\mathbb{P}^1)^n$. \square

The above property is about maps on a fixed toric variety $(\mathbb{P}^1)^n$. Next, we will fix some map, and ask whether there exists a toric variety on which the map is strongly algebraically stable. We give partial answers for maps satisfying some conditions.

Theorem 4.7 *Suppose that $A \in \mathbf{M}_n(\mathbb{Z})$ is an integer matrix.*

- (1) *If there is a unique eigenvalue λ of A of maximal modulus, with algebraic multiplicity one; then $\lambda \in \mathbb{R}$, and there exists a simplicial toric birational model X (maybe singular) and a $k \in \mathbb{N}$ such that f_A^k is strongly algebraically stable on X .*
- (2) *If $\lambda, \bar{\lambda}$ are the only eigenvalues of A of maximal modulus, also with algebraic multiplicity one, and if $\lambda = |\lambda| \cdot e^{2\pi i\theta}$, with $\theta \notin \mathbb{Q}$; then there is no toric birational model which makes f_A strongly algebraically stable.*

Proof For (1), let $v \in \mathbb{R}^n$ be the eigenvector corresponding to the largest real eigenvalue λ , then the subspace $\mathbb{R}v$ is attracting. We can find integral vectors v_1, \dots, v_n , linearly independent over \mathbb{R} , such that

- v is in the interior of the cone generated by v_1, \dots, v_n .
- $A^n(\mathbb{R}v_i) \rightarrow \mathbb{R}v$ for all $i = 1, \dots, n$, as elements of $\mathbb{R}\mathbb{P}^n$.

The rays $\{\mathbb{R}_{\geq 0} \cdot v_i, \mathbb{R}_{\geq 0} \cdot (-v_i) \mid i = 1, \dots, n\}$ generates a fan Σ similar to the way we form $\mathbb{P}^1 \times \dots \times \mathbb{P}^1$. That is, the maximal cones of Σ are generated by the sets $\{s_1 v_1, \dots, s_n v_n\}$ where $s_i \in \{+1, -1\}$. All other cones are faces of some maximal cone. It is easy to see that for some k , f_A^k is strongly algebraically stable on $X(\Sigma)$.

To prove (2), let $\lambda, \bar{\lambda}$ be the largest eigenvalue pair, and $\Gamma \subset \mathbb{R}^n$ be the two dimensional invariant subspace corresponding to them. Since the fan Σ is complete, there is at least one ray $\tau \in \Sigma(1)$ such that under iterations, $A^k \tau$ will approach Γ . Moreover, since $A|_\Gamma$ is an irrational rotation on rays, we know that for all $v \in \Gamma$, there is a sequence k_i such that $A^{k_i} \tau \rightarrow \mathbb{R}_{\geq 0} \cdot v$.

Consider the set $\Sigma \cap \Gamma = \{\sigma \cap \Gamma \mid \sigma \in \Sigma\}$, it is a fan in Γ . Each cone in it is strictly convex, but not necessarily rational. Pick $v_0 \in \Gamma$ which lies in the interior of some two dimensional cone of $\Sigma \cap \Gamma$, and pick a sequence k_i such that $A^{k_i} \tau \rightarrow \tau_0 = \mathbb{R}_{\geq 0} \cdot v_0$.

Since $A^{k_i} \tau \rightarrow \tau_0$ and Σ consists of only finitely many cones, there must be some k such that $\tau_0 \in \overline{A^k \tau}$. But τ_0 is in the interior of some two dimensional cone of $\Sigma \cap \Gamma$, so we know that $A^k \tau \cap \Gamma$ is a two dimensional cone in Γ . Finally, we know that $\overline{A^k \tau} \cap \Gamma$ cannot map regularly under all A^k , so $\overline{A^k \tau}$ cannot either. Thus A can never be made strongly algebraically stable. \square

We do not know what the correct statement would be for the missing case $\lambda = |\lambda| \cdot e^{2\pi i\theta}$, with $\theta \in \mathbb{Q}$.

Remark Some of our results were obtained independently by Jonsson and Wulcan [6]. One of the main theorems in their paper [6, Theorem A'] deal with smooth stabilization of a monomial map by refining a given fan. This aspect of the stabilization is more delicate and is not discussed in our paper. Part (1) of Theorem 4.7 in the current paper coincides with Theorem B' in [6]. They also discuss the special case of monomial maps on toric surfaces (two dimensional toric varieties), which is not dealt in this paper.

5 Monomial maps on projective spaces

The motivation for studying toric rational maps comes from the study of monomial maps on projective spaces. So let us come back to monomial maps and try to understand more about them with the help of techniques from toric varieties.

5.1 Pulling back divisors and divisor classes

Given an $n \times n$ integer matrix $A = (a_{i,j})_{1 \leq i,j \leq n}$, the associated monomial map $\mathbb{C}^n \rightarrow \mathbb{C}^n$ is given by

$$(X_1, \dots, X_n) \mapsto \left(\prod_{j=1}^n X_j^{a_{1,j}}, \dots, \prod_{j=1}^n X_j^{a_{n,j}} \right).$$

We then use the embedding $\mathbb{C}^n \hookrightarrow \mathbb{P}^n$ defined by $(X_1, \dots, X_n) \mapsto [1; X_1; \dots; X_n]$ to identify \mathbb{C}^n with the open subset $U_0 = \{x_0 \neq 0\} \subset \mathbb{P}^n$. After homogenizing, there is another integer matrix, with size $(n+1) \times (n+1)$, denoted by $h(A) = (b_{i,j})_{0 \leq i,j \leq n}$, such that

$$f_A([x_0; \dots; x_n]) = \left[\prod_{j=0}^n x_j^{b_{0,j}}; \dots; \prod_{j=0}^n x_j^{b_{n,j}} \right].$$

Recall the structure of the fan associated to the projective space. The one dimensional cones are generated by the standard basis e_1, \dots, e_n and $e_0 = -(e_1 + \dots + e_n)$. Denote them by $\tau_i = \mathbb{R}_{\geq 0} \cdot e_i$ for $i = 0, \dots, n$. Consider the divisors $D_i = V(\tau_i) = \{x_i = 0\}$. Pulling back the defining equation $x_i = 0$ gives us the equation $\prod_{j=0}^n x_j^{b_{i,j}} = 0$, which means

$$f_A^*(D_i) = b_{i,0} \cdot D_0 + b_{i,1} \cdot D_1 + \dots + b_{i,n} \cdot D_n.$$

On the other hand, by Proposition 3.1, if ψ_i is the support function of the divisor D_i , then

$$f_A^*(D_i) = -\psi_i(Ae_0) \cdot D_0 - \psi_i(Ae_1) \cdot D_1 - \dots - \psi_i(Ae_n) \cdot D_n.$$

Thus we obtain the equality $b_{i,j} = -\psi_i(Ae_j)$. The formulae of ψ_i are as follows.

$$\begin{cases} \psi_0(a_1, \dots, a_n) = \min\{0, a_1, \dots, a_n\}, \\ \psi_i(a_1, \dots, a_n) = \min\{0, -a_i, a_j - a_i; j \neq i\} \quad \text{for } i = 1, \dots, n. \end{cases} \quad (5.1)$$

Next, we turn our attention to the pull back of divisor classes, i.e., elements of $\text{Pic}(\mathbb{P}^n)$. We know that $\text{Pic}(\mathbb{P}^n) \cong \mathbb{Z}$, and we have the map $\deg : \text{CDiv}_T(\mathbb{P}^n) \rightarrow \text{Pic}(\mathbb{P}^n)$ given by $\deg(\sum a_i V(\tau_i)) = \sum a_i$. For a monomial map f_A on the projective space, the degree of the map is given by $\deg(f_A^*D)$ for any divisor D of degree one. We also denote this number by $\deg(f_A)$. If ψ is the support function for D , then the degree of the monomial map f_A is given by

$$\deg(f_A) = \sum_{i=0}^n -\psi(Ae_i). \quad (5.2)$$

For instance, let ψ be any one of the ψ_i listed in (5.1), then we can get a concrete formula for $\deg(f_A)$. In particular, let $\psi = \psi_0$, we have

$$\begin{aligned} -\psi(a_1, \dots, a_n) &= -\psi_0(a_1, \dots, a_n) \\ &= -\min\{0, a_1, \dots, a_n\} \\ &= \max\{0, -a_1, \dots, -a_n\}. \end{aligned}$$

Then we rediscover the formula in [5, Proposition 2.14].

$$\deg(f_A) = \sum_{j=1}^n \max_{1 \leq i \leq n} \{0, -a_{ij}\} + \max_{1 \leq i \leq n} \left\{ 0, \sum_{j=1}^n a_{ij} \right\}.$$

The definition of algebraic stability for rational maps on \mathbb{P}^n states that f_A is algebraically stable if and only if $\deg(f_A^k) = \deg(f_A)^k$ for all k . Another property of the degree sequence is that it is *submultiplicative*, i.e., $\deg(f_A^{k+k'}) \leq \deg(f_A^k) \cdot \deg(f_A^{k'})$. We will use this property in the next section.

5.2 Estimates of the degree sequence

In this section, we are going to study the degree sequence $\{\deg(f_A^k)\}_{k=1}^\infty$. We are particularly interested in the asymptotic behavior of the degree sequence. An important numerical invariant is the *asymptotic degree growth*

$$\delta_1(f_A) = \lim_{k \rightarrow \infty} (\deg(f_A^k))^{\frac{1}{k}}.$$

It is known that for a monomial map f_A , $\delta_1(f_A) = \rho(A)$, the spectral radius of the matrix A [5, Theorem 6.2]. We will refine this result and give more precise estimates on the degree growth of a monomial map.

For two sequences $\{\alpha_k\}_{k=1}^\infty$ and $\{\beta_k\}_{k=1}^\infty$ of positive real numbers, we say that they are *asymptotically equivalent*, denoted by $\alpha_k \sim \beta_k$, if there exists two positive constants $c_1 \geq c_0 > 0$, independent of k , such that $c_0 \cdot \beta_k \leq \alpha_k \leq c_1 \cdot \beta_k$ for all k .

The main result for general monomial maps is the following theorem.

Theorem 5.1 *Given an $n \times n$ integer matrix A with nonzero determinant, assume that $\rho(A)$ is the spectral radius of A . Then there exist two positive constants $C_1 \geq C_0 > 0$ and a unique integer ℓ with $0 \leq \ell \leq n - 1$, such that*

$$C_0 \cdot k^\ell \cdot \rho(A)^k \leq \deg(f_A^k) \leq C_1 \cdot k^\ell \cdot \rho(A)^k \quad (5.3)$$

for all $k \in \mathbb{N}$. Or, equivalently, $\deg(f_A^k) \sim k^\ell \cdot \rho(A)^k$.

In fact, $(\ell + 1)$ is the size of the largest Jordan block of A among the ones corresponding to eigenvalues of maximal modulus.

In formula (5.2), notice that the right hand side can be defined over the real numbers because. Thus, we define a function $v : \mathbf{M}_n(\mathbb{R}) \rightarrow \mathbb{R}$ by

$$v(M) = \sum_{i=0}^n -\psi(Me_i). \quad (5.4)$$

Proposition 5.2 *The following properties hold for the function ν .*

- (i) *Any support function ψ of a T -divisor of degree one on \mathbb{P}^n will give the same ν , i.e., ν is independent of the choice of ψ .*
- (ii) *ν is a continuous function when we equip $\mathbf{M}_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ and \mathbb{R} with the usual topology of the Euclidean spaces.*
- (iii) *$\nu(M) \geq 0$, and $\nu(M) = 0$ if and only if $M = 0$. Thus, in fact, we have $\nu : \mathbf{M}_n(\mathbb{R}) \rightarrow \mathbb{R}_{\geq 0}$.*
- (iv) *$\nu(rM) = r \cdot \nu(M)$ for $r \geq 0$.*
- (v) *$\nu(M + M') \leq \nu(M) + \nu(M')$.*

Proof First, notice that (ii) is true because ψ is continuous, and (iv) is true because ψ is linear on each ray. Then (i) follows by (ii), (iv), and the fact that $\nu(A) = \deg(f_A)$ for $A \in \mathbf{M}_n(\mathbb{Z})$, which is independent of ψ .

Once we know that ν is independent of the choice of ψ , one can pick any ψ , e.g. $\psi = \psi_0$, and prove (iii) and (v) directly. However, we would like to offer a more intrinsic explanation for (iii) and (v).

Since ψ is the support function for a degree one divisor D on \mathbb{P}^n , we know that D is very ample, and hence ψ is strictly convex (see [4, p.70]). The first part of (iii), and (v), can be easily deduced from convexity. Strict convexity is needed to show that $\nu(M) = 0$ implies $M = 0$.

Suppose $M \neq 0$, then Me_0, Me_1, \dots, Me_n cannot be all zero. But since $Me_0 + \dots + Me_n = 0$, and the cones in the fan for \mathbb{P}^n are strongly convex (they do not contain any line through the origin), Me_0, \dots, Me_n cannot all lie in the same cone. Thus by strict convexity, we know

$$\nu(M) = - \sum_{i=0}^n \psi(Me_i) > -\psi\left(\sum_{i=0}^n Me_i\right) = \psi(0) = 0.$$

□

By properties (iii)–(v), we know that the only reason to prevent ν from being a norm is that we may have $\nu(M) \neq \nu(-M)$. Indeed, for the $n \times n$ identity matrix I_n , we have $\nu(I_n) = 1$, but $\nu(-I_n) = n$. So ν is not a norm. However, if we define $\bar{\nu}(M) = \nu(M) + \nu(-M)$, then $\bar{\nu}$ is a norm.

Before we prove Theorem 5.1, we recall the following elementary lemma from linear algebra.

Lemma 5.3 *For an $n \times n$ matrix $A \in \mathbf{M}_n(\mathbb{C})$ and any norm $\|\cdot\|$ defined on $\mathbf{M}_n(\mathbb{C})$, there exists two positive constants $c_1 \geq c_0 > 0$ and a unique integer ℓ with $0 \leq \ell \leq n - 1$, such that*

$$c_0 \cdot k^\ell \cdot \rho(A)^k \leq \|A^k\| \leq c_1 \cdot k^\ell \cdot \rho(A)^k \quad (5.5)$$

for all $k \in \mathbb{N}$. Here $\rho(A)$ is the spectral radius of A , and $(\ell + 1)$ is the size of the largest Jordan block among those blocks corresponding to eigenvalues of maximal modulus $\rho(A)$.

Proof of Theorem 5.1 For the matrix $A \in \mathbf{M}_n(\mathbb{Z})$, consider the set

$$\left\{ \frac{A^k}{k^\ell \rho(A)^k} \mid k \in \mathbb{N} \right\} \subset \mathbf{M}_n(\mathbb{R}).$$

By Lemma 5.3, it is a subset of a compact set $S = \{M \in \mathbf{M}_n(\mathbb{R}) \mid c_0 \leq \|M\| \leq c_1\}$ for some $c_1 \geq c_0 > 0$. Since v is continuous, we have $v(S) \subset [C_0, C_1]$ for some reals $C_1 \geq C_0 \geq 0$. Moreover, $0 \notin S$, thus $C_0 > 0$. This gives us

$$C_0 \leq v\left(\frac{A^k}{k^\ell \cdot \rho(A)^k}\right) \leq C_1.$$

for all $k \in \mathbb{N}$, with $C_1 \geq C_0 > 0$.

Finally, since $k^\ell \cdot \rho(A)^k > 0$, and $v(A^k) = \deg(f_A^k)$, we have

$$C_0 \cdot k^\ell \cdot \rho(A)^k \leq \deg(f_A^k) \leq C_1 \cdot k^\ell \cdot \rho(A)^k$$

This concludes the proof. \square

Corollary 5.4 *If A is diagonalizable, then we have*

$$C_0 \cdot \rho(A)^k \leq \deg(f_A^k) \leq C_1 \cdot \rho(A)^k \quad (5.6)$$

for some constants $C_1 \geq C_0 \geq 1$.

Proof In the diagonalizable case, $\ell = 0$, hence we have (5.6). Recall that the degree sequence is submultiplicative. Thus, if we have $\frac{\deg(f_A^k)}{\rho(A)^k} = r < 1$ for some k , then

$$\frac{\deg(f_A^{kj})}{\rho(A)^{kj}} \leq \frac{\deg(f_A^k)^j}{\rho(A)^{kj}} = r^j \rightarrow 0 \text{ as } j \rightarrow +\infty.$$

This contradicts the existence of $C_0 > 0$. Therefore, $\frac{\deg(f_A^k)}{\rho(A)^k} \geq 1$ for all k , and we can choose $C_0 \geq 1$. \square

If we impose more conditions on the matrix A , we can obtain more precise estimates on the degree sequence.

Theorem 5.5 *Assuming that the matrix A is diagonalizable, and there is a unique eigenvalue λ_1 of maximal modulus, which is real and positive. Also, assume that the eigenvalues of A are arranged as $\lambda_1 > |\lambda_2| \geq |\lambda_3| \geq \dots \geq |\lambda_m|$ for some m . Then there is a constant $C \geq 1$ such that*

$$\deg(f_A^k) = C \cdot \lambda_1^k + O(|\lambda_2|^k).$$

Proof First, given a vector $v \in \mathbb{R}^n$, since A is diagonalizable, we can represent v uniquely as

$$v = v_1 + v_2 + \dots + v_m,$$

where each $v_j \in \mathbb{C}^n$ is an eigenvector corresponding to λ_j . We have $v_1 \in \mathbb{R}^n$ since λ_1 is real. Thus

$$A^k v = \lambda_1^k v_1 + \lambda_2^k v_2 + \dots + \lambda_m^k v_m.$$

Let ψ be the support function of some degree one divisor D in \mathbb{P}^n . For each k , there is some maximal cone σ_k such that $A^k v \in \sigma_k$. Let L_k be the linear function such that $L_k|_{\sigma_k} = \psi|_{\sigma_k}$.

Notice that L_k can be defined on \mathbb{C}^n as a linear map, and we have

$$\begin{aligned}\psi(A^k v) &= L_k(A^k v) = L_k\left(\sum_{j=1}^m \lambda_j^k v_j\right) \\ &= \lambda_1^k \cdot L_k(v_1) + \sum_{j=2}^m \lambda_j^k \cdot L_k(v_j) \\ &= L_k(v_1) \cdot \lambda_1^k + O(|\lambda_2|^k).\end{aligned}$$

There are two cases here: $v_1 \neq 0$, or $v_1 = 0$.

First, if $v_1 \neq 0$, then for the rays $\tau = \mathbb{R}_{\geq 0} \cdot v$, we know $A^k \tau \rightarrow \mathbb{R}_{\geq 0} \cdot v_1$. Thus for large k , we can choose σ_k so that both $A^k v \in \sigma_k$ and $v_1 \in \sigma_k$. Since $L_k|_{\sigma_k} = \psi|_{\sigma_k}$ for the cone σ_k , we know that for large k , the value $L_k(v_1) = \psi(v_1)$ is independent of k , and $\psi(A^k v) = \psi(v_1) \cdot \lambda_1^k + O(|\lambda_2|^k)$. Second, if $v_1 = 0$, then it is obvious that $\psi(A^k v) = O(|\lambda_2|^k)$.

Now let's look at the fan structure of projective spaces. For the ray generators e_0, e_1, \dots, e_n of \mathbb{P}^n , if e_i is decomposed as

$$e_i = v_{i,1} + v_{i,2} + \dots + v_{i,m} \quad (5.7)$$

for $i = 0, 1, \dots, n$, where each $v_{i,j}$ is an eigenvector corresponding to the eigenvalue λ_j . Then

$$\psi(A^k e_i) = \psi(v_{i,1}) \cdot \lambda_1^k + O(|\lambda_2|^k).$$

If we set

$$C = \sum_{i=0}^n -\psi(v_{i,1}), \quad (5.8)$$

then we can compute the degree sequence $\deg(f_A^k)$ as

$$\begin{aligned}\deg(f_A^k) &= \deg((f_A^k)^* D) \\ &= \sum_{i=1}^n -\psi(A^k e_i) \\ &= C \cdot \lambda_1^k + O(|\lambda_2|^k).\end{aligned}$$

The fact that $C \geq 1$ is a consequence of Corollary 5.4. \square

Notice that, on our way to prove the theorem, we also derive a concrete formula for the constant C in (5.8).

Theorem 5.6 *Assuming that the matrix A is diagonalizable, and there is a unique eigenvalue λ_1 of maximal modulus, which is real and negative. Also assume that the eigenvalues of A are arranged as $(-\lambda_1) > |\lambda_2| \geq |\lambda_3| \geq \dots \geq |\lambda_m|$ for some m . Then there are two positive constants C_0, C_1 , not necessarily distinct, and satisfying $1 \leq C_0 \leq C_1^2$, such that*

$$\deg(f_A^{2k+l}) = C_l \cdot |\lambda_1|^{2k+l} + O(|\lambda_2|^{2k+l}),$$

where $l = 0, 1$.

Proof We consider the subsequences $\{\deg(f_A^{2k})\}$ and $\{\deg(f_A^{2k+1})\}$. Since A^2 satisfies the condition in Theorem 5.5, with the unique eigenvalue $|\lambda_1|^2$ with maximal modulus, thus

$$\deg(f_A^{2k}) = C_0 \cdot |\lambda_1|^{2k} + O(|\lambda_2|^{2k})$$

for some $C_0 \geq 1$. For the subsequence $\{\deg(f_A^{2k+1})\}$, we consider Ae_i instead of e_i in (5.7) in the proof of Theorem 5.5, and apply the map f_A^2 on these vectors. We then get

$$\deg(f_A^{2k+1}) = C_1 \cdot |\lambda_1|^{2k+1} + O(|\lambda_2|^{2k+1})$$

for some $C_1 \geq 1$.

Finally, for any k , we have,

$$\deg(f_A^{4k+2}) / |\lambda_1|^{4k+2} \leq \left(\deg(f_A^{2k+1}) / |\lambda_1|^{2k+1} \right)^2.$$

As $k \rightarrow \infty$, the left side converges to C_0 , while the right side converges to C_1^2 . So the relation $C_0 \leq C_1^2$ follows. This completes the proof. \square

The idea in the proof of Theorem 5.6 of considering subsequences can be pushed further to prove the following more general result.

Theorem 5.7 Assume that the matrix A is diagonalizable, and assume for each eigenvalue λ of A of maximum modulus, $\lambda/\bar{\lambda}$ is a root of unity. Then there is a positive integer p , and p constants $C_0, C_1, \dots, C_{p-1} \geq 1$, such that

$$\deg(f_A^{pk+l}) = C_l \cdot |\lambda_1|^{pk+l} + O(|\lambda_2|^{pk+l}),$$

where $l = 0, 1, \dots, p-1$.

Proof Notice that there is an integer p such that the eigenvalue of A^p of maximum modulus is unique and positive, so we can use the same argument as Theorem 5.6 to the subsequences

$$\left\{ \deg(f_A^{pk}) \right\}, \left\{ \deg(f_A^{pk+1}) \right\}, \dots, \left\{ \deg(f_A^{pk+p-1}) \right\}.$$

The theorem then follows. \square

Under the assumption of Theorem 5.7, the sequence $\{\deg(f_A^k)/|\lambda_1|^k\}_{k=1}^\infty$ has finitely many limit points, namely, C_0, \dots, C_{p-1} . The following proposition shows a different behavior of the sequence $\{\deg(f_A^k)/|\lambda_1|^k\}_{k=1}^\infty$ when we have a maximal eigenvalue λ such that $\lambda/\bar{\lambda}$ is not a root of unity. Therefore, we cannot expect Theorem 5.7 holds for general diagonalizable matrices.

Proposition 5.8 For a 2×2 integer matrix A , suppose it has a conjugate pair $\lambda, \bar{\lambda}$ of eigenvalues such that $\lambda/\bar{\lambda}$ is not a root of unity. Then the sequence $\{\deg(f_A^k)/|\lambda|^k\}_{k=1}^\infty$ is dense in some closed interval contained in $[1, \infty)$.

Proof First, notice that

$$\frac{\deg(f_A^k)}{|\lambda|^k} = \frac{v(A^k)}{|\lambda|^k} = v\left(\frac{A^k}{|\lambda|^k}\right) = v((A/|\lambda|)^k). \quad (5.9)$$

Since $\lambda/\bar{\lambda}$ is not a root of unity, we can conjugate $A/|\lambda|$ to some irrational rotation matrix, i.e., we can write

$$A/|\lambda| = P \cdot \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \cdot P^{-1},$$

for some $\theta \notin 2\pi\mathbb{Q}$. Thus the closure of the set $S = \{(A/|\lambda|)^k | k \in \mathbb{N}\}$ is

$$\bar{S} = \left\{ P \cdot \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \cdot P^{-1} \mid t \in [0, 2\pi] \right\}.$$

\bar{S} is, topologically, a circle inside $\mathbf{M}_2(\mathbb{R})$. Since v is continuous, $v(\bar{S}) = \overline{v(S)}$ is connected and compact. Thus it is either a point or a closed interval.

We claim that $v(\bar{S})$ cannot be a point. If $v(\bar{S}) = \{C\}$, then we will have $\deg(f_A^k) = C \cdot |\lambda|^k$ for all $k \in \mathbb{N}$. In this case, the degree sequence $d_k = \deg(f_A^k)$ satisfies a linear recurrence $d_{k+1} = |\lambda| \cdot d_k$. This contradicts a theorem of Bedford and Kim [1, Theorem 1.1], which asserts that if the matrix A has a complex eigenvalue λ of maximal modulus, and $\lambda/\bar{\lambda}$ is not a root of unity, then the degree sequence for f_A cannot satisfy any linear recurrence relation.

Hence, $v(\bar{S}) = \overline{v(S)}$ is a closed interval. By (5.9), $v(S)$ is exactly the set $\{\deg(f_A^k)/|\lambda|^k; k \in \mathbb{N}\}$. Finally, by Corollary 5.4, we further know that the interval $\overline{v(S)}$ is contained in $[1, +\infty)$. This concludes the proof. \square

5.3 Degree growth on weighted projective spaces

Weighted projective spaces are generalizations of the usual projective spaces. The results we obtained in the last subsection about the degree growth of monomial maps on projective spaces can be generalized to weighted projective spaces.

For arbitrary positive integers d_0, \dots, d_n , the associated *weighted projective space*, denoted by $\mathbb{P}(d_0, \dots, d_n)$, is defined as

$$\mathbb{P}(d_0, \dots, d_n) = (\mathbb{C}^{n+1} - \{0\}) / \sim$$

where the equivalent relation is given by $(x_0, \dots, x_n) \sim (\zeta^{d_0}x_0, \dots, \zeta^{d_n}x_n)$ for $\zeta \in \mathbb{C}^*$.

A weighted projective space $\mathbb{P}(d_0, \dots, d_n)$ such that no n of the d_0, d_1, \dots, d_n have a common factor is called *well formed*. A standard fact (see [7, Proposition 3.6]) shows that it is sufficient to only consider well formed weighted projective spaces. We will make that assumption from now on. Also, for simplicity of notation, we will denote $\mathbb{P}(d_0, \dots, d_n)$ simply by \mathbb{P} when there is no confusion. The usual projective space will still be denoted by \mathbb{P}^n .

To construct $\mathbb{P}(d_0, \dots, d_n)$ as a toric variety, one uses the same fan as in the construction of the projective spaces. That is, the cones are generated by proper subsets of $\{e_0, \dots, e_n\}$. The lattice N' is taken to be generated by the vectors $e'_i := e_i/d_i$, $i = 0, \dots, n$. Let $\tau_i = \mathbb{R}_{\geq 0} \cdot e_i$ be the rays for the fan of \mathbb{P} , the well formed-ness of \mathbb{P} implies that e'_i is the ray generator for τ_i for $i = 0, \dots, n$.

If we define the map $\theta : \mathbb{Z}^{n+1} \rightarrow N'$ by $\theta(a_0, \dots, a_n) = a_0e'_0 + \dots + a_ne'_n$, then θ is a surjective homomorphism, and $\ker(\theta)$ is the subgroup $\mathbb{Z} \cdot (d_0, \dots, d_n)$. Hence $N' \cong \mathbb{Z}^{n+1} / \mathbb{Z} \cdot (d_0, \dots, d_n)$, and its dual lattice is

$$M' = (N')^\vee \cong \{(a_0, \dots, a_n) \in \mathbb{Z}^{n+1} \mid a_0d_0 + \dots + a_nd_n = 0\}.$$

We have $\text{WDiv}_T(\mathbb{P}) \cong \bigoplus_{i=0}^n \mathbb{Z} \cdot V(\tau_i)$. Define the *weighted degree* $\deg' : \text{WDiv}_T(\mathbb{P}) \rightarrow \mathbb{Z}$ by $\deg'(a_i \cdot V(\tau_i)) = \sum_{i=0}^n a_id_i$. It is a surjective homomorphism with kernel canonically isomorphic to M' . Therefore, we have $A_{n-1}(\mathbb{P}) \cong \mathbb{Z}$, and the isomorphism is induced by the weighted degree.

Let $m = \text{lcm}(d_0, \dots, d_n)$, one can show that a T_N -invariant Weil divisor D is Cartier if and only if $m \mid \deg'(D)$. As a consequence, the image of the Picard group $\text{Pic}(\mathbb{P}) \subset A_{n-1}(\mathbb{P}) \cong \mathbb{Z}$ under the isomorphism is the subgroup $m\mathbb{Z}$. Therefore, we will look at \mathbb{Q} -Weil divisors and rational support function in this subsection.

Let ψ be a rational support function, then ψ induces a \mathbb{Q} -Cartier divisor on \mathbb{P} , whose associated \mathbb{Q} -Weil divisor is $D' = \sum_{i=0}^n -\psi(e'_i) \cdot V(\tau_i)$. Also, ψ induces a \mathbb{Q} -Cartier divisor on \mathbb{P}^n , with associated \mathbb{Q} -Weil divisor $D = \sum_{i=0}^n -\psi(e_i) \cdot V(\tau_i)$. A basic fact is the following.

Lemma 5.9 *Assume the above notations, then the weighted degree of D' is the same as the degree of D , i.e., $\deg'(D') = \deg(D)$.*

Proof This can be verified as follows:

$$\deg'(D') = \sum_{i=0}^n -d_i \cdot \psi(e'_i) = \sum_{i=0}^n -d_i \cdot \psi(e_i/d_i) = \sum_{i=0}^n -\psi(e_i) = \deg(D).$$

□

Let $A \in \text{End}(N')$, then A induces a toric rational map $f_A : \mathbb{P} \rightarrow \mathbb{P}$. Using the standard basis e_1, \dots, e_n of $N'_{\mathbb{R}} \cong N_{\mathbb{R}}$, we can represent A as an $n \times n$ matrix with *rational* entries. The following proposition tells us how to compute the weighted degree of f_A .

Proposition 5.10 *Assume the above notations, then the weighted degree of f_A is given by*

$$\deg'(f_A) = \nu(A),$$

where $\nu : \mathbf{M}_n(\mathbb{R}) \rightarrow \mathbb{R}$ is the function defined in (5.4).

Proof The weighted degree can be computed as $\deg'(f_A) = \deg(f_A^* D')$ for any \mathbb{Q} -divisor D' on \mathbb{P} of degree one. Thus, if D' is a \mathbb{Q} -divisor on \mathbb{P} of degree one, and $\psi = \psi_{D'}$ is the \mathbb{Q} -support function of D' , then

$$\deg'(f_A) = \sum_{i=0}^n -d_i \cdot \psi(Ae'_i) = \sum_{i=0}^n -\psi(Ae_i) = \nu(A).$$

The last equality holds because the degree of the \mathbb{Q} -divisor on \mathbb{P}^n associated to ψ also has degree one by Lemma 5.9. □

Since the weighted degree function is the same as the function ν , this tells us that the weighted degree growth of iterations of toric rational maps on \mathbb{P} follows the same results as the degree growth on \mathbb{P}^n .

Acknowledgements The author would like to thank Professor Eric Bedford for suggesting the problem and for many invaluable discussions. Without Professor Bedford's continuous support this work would have been impossible. He would also like to thank Jeff Diller, Mattias Jonsson, and Charles Favre for useful discussions and suggestions.

References

1. Bedford, E., Kyounghee, K.: Linear recurrences in the degree sequences of monomial mappings. *Ergod. Theory Dyn. Syst.* **28**(5), 1369–1375 (2008)
2. Danilov, V.I.: The geometry of toric varieties. *Russ. Math. Surv.* **33**, 97–154 (1978)
3. Favre, C.: Les applications monomiales en deux dimensions. *Mich. Math. J.* **51**(3), 467–475 (2003)
4. Fulton, W.: *Introduction to Toric Varieties*. In: *Annals of Math Studies*, vol. 131. Princeton University Press, Princeton (1993)
5. Hasselblatt, B., Propp, J.: Degree-growth of monomial maps. *Ergod. Theory Dyn. Syst.* **27**(5), 1375–1397 (2007)
6. Jonsson, M., Wulcan, E.: Stabilization of monomial maps. arXiv:1001.3938 [math.DS]

7. Reid, M.: Graded rings and varieties in weighted projective space. <http://www.warwick.ac.uk/~masda/surf/more/grad.pdf>
8. Sibony, N.: Dynamics of rational maps on \mathbb{P}^n . In: Complex dynamics and geometry. SMF/AMS Texts and Monographs, vol. 10, pp. 85–166. American Mathematical Society, Providence (2003)