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Notes on the Correspondence between Luigi Cremona and Max Noether

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This paper contains an analysis of the scientific climate in which the Italian school of algebraic geometry came into being and developed in a direction different from that of the German school, even though both shared a common initial matrix. Reference is made to a number of letters written by Max Noether to Luigi Cremona. © 1986 Academic Press, Inc.

Il presente lavoro contiene un'analisi del clima scientifico nel quale la Scuola Italiana di geometria algebrica nasceva e si sviluppava in una direzione diversa da quella tedesca, cui pure la univa una matrice iniziale comune, facendo riferimento ad alcune lettere inviate da Max Noether a Luigi Cremona. © 1986 Academic Press, Inc.

Diese Arbeit enthält eine Analyse des wissenschaftlichen Klimas, in dem die italienische Schule der algebraischen Geometrie geboren wurde und sich in einer Richtung entwickelte, die von derjenigen der deutschen Schule verschieden war, obwohl sie eine gemeinsame Herkunft zu Beginn verband. Zitiert sind einige Briefe, die Max Noether an Luigi Cremona richtete. © 1986 Academic Press, Inc.

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FOREWORD

Italian algebraic geometry started life in the second half of the 19th century, and coincided with the emergence of a new direction of geometric research connected with the work of the Italian geometer Luigi Cremona (1830–1903). It was around 1860, in fact, that Cremona abandoned the more specifically algebraic, demonstrative methods of his masters Bordoni and Brioschi to follow an autonomous approach in the direction of pure geometry. This new boost given to Cremona's work and activity stemmed from his earnest commitment to reviving Italian science [1]. As a result, scientific research in every sphere flourished, international contacts were restored, and new university chairs were established, including the Chair of Higher Geometry at Bologna University which, in 1860, Cremona was the first to occupy.

Foreign contacts also improved as a result of interest in synthetic geometry and the work of von Staudt, Poncelet, and others, which created common ground for

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research in algebraic geometry. However, this did not prevent Italian algebraic geometry from taking a path different from that in the rest of Europe, particularly the German school. Italian geometers placed greater store in the pure methods of synthetic and intuitive geometry, even though they did not make exclusive use of them. In Germany, on the other hand, the achievements in analytic algebra that provided geometry with a more rigorous scientific backing directed research toward today's axiomatic geometry.

The first section of this paper examines the scientific climate in Italy at the time Cremona was beginning his work. Although scientific studies had undergone a revival to a certain extent, the algebraic and analytical tools at the disposal of the Italian geometers were not comparable with those that existed in Germany. Consequently they did not foster the development of algebraic geometry except in the synthetic direction chosen by Cremona.

While Italian algebraic geometry originated in the work of Luigi Cremona, one of the fathers of German algebraic geometry was Max Noether (1844–1921). His wide-ranging research, partly conducted in conjunction with the algebraist Brill, is still one of the bulwarks of algebraic geometry. A number of letters written by Noether [2] to Cremona are discussed in the second section of this paper. These reveal some significant differences in approach between the two geometers. For example, Noether (with his "rigorous" view of the methods of algebraic geometry) could not accept a demonstration that Cremona had provided for one of his theorems using the intuitive methods that typified his work.

1. THE BIRTH OF THE ITALIAN SCHOOL OF ALGEBRAIC GEOMETRY

An analysis of the early writings of Luigi Cremona provides a fairly accurate idea of the new direction being taken by demonstration methods and geometric research, a direction which was to have a far-reaching effect on the development of algebraic geometry in Italy and elsewhere. This began around 1861, the year in which Cremona published his famous "Introduzione ad una teoria geometrica delle curve piane" [Cremona 1862], the only treatise in which he expounded his theories of plane algebraic curves using the synthetic method. (His first purely geometric work was published in 1860, however, on the subject of the twisted cubic [3].)

Before producing these purely geometric works, ranging from 1853 when he graduated in architectural engineering to 1860 when he was appointed to the Chair of Higher Geometry in Bologna, Cremona's papers drew broadly on the research of his masters, Bordoni and Brioschi, on the one hand, and the work of Chasles on the other. He subsequently moved on to the study of sphere-conjugate tangents, confocal spherical conics, and spatial curves. The latter study enabled him to demonstrate, and complete, Chasles' theorems of 1858–1859, using a simple analytical demonstration.

When Cremona read and reviewed Christian von Staudt's *Geometrie der Lage* in 1858, he was still a follower of Chasles, whose *Aperçu historique . . .* attracted his admiration—so much so that even though he greatly appreciated the work of von Staudt, he wrote that "the descriptive properties and the metrical properties of the figures are so closely connected, that it is inappropriate and disadvantageous to try to separate them so completely" [Cremona 1858, 125]. Later, however, Cremona came to consider von Staudt as "the father of pure geometry," to the extent that he even took his work as the basis for his own course on projective geometry:

> I therefore gave greater emphasis to graphic rather than metric properties; but I used the procedures of Staudt's *Geometrie der Lage* more often than Chasles' *Géometrie Supérieure*, although I never wholly excluded metric relations which would have had an adverse practical effect on teaching. (. . .) I could have copied Staudt by doing without any preparatory notions whatsoever, but in this case my work would have been too long, and I would not have been able to adapt it to the students in Technical Colleges, who are expected to have studied the usual fundamentals of mathematics in their first biennium. . . . [Cremona 1873, X]

These final remarks help to show the methodological conception that led Cremona to found "his" school of pure geometry. He was not primarily interested in rigor, although he appreciated rigorous, pure geometric reasoning. Even so, he did not frown on using other techniques (metric, algebraic, and analytic) when they helped to simplify his reasoning, and whenever the rigorously purer methods of von Staudt were unsuitable for developing the theory much further.

Although Italy helped only indirectly to rebuild the great edifice of projective geometry that had been started over fifty years earlier in Germany and France, Cremona and the Italian school that followed him were by no means less well endowed in knowledge and intuition than the great geometers who were working in other countries at the time. However, this was not the case for algebraic techniques, and above all analytic techniques.

The state of Italian mathematics around the middle of the 19th century certainly did not look very bright. Geometry dominated the Italian journals from 1850 to 1860. While problems of analytic geometry and elementary geometry still appeared, there were also developments in the direction of the study of surfaces of the second and third degrees, algebraic curves, etc. Very often, geometry was linked to astronomy and mechanics, and analysis was often related to physics. As far as the latter was concerned, there were many problems related to the integration of series, differential and linear equations, and integrals of higher order. But the algebraic problems were almost exclusively related to the solution of various types of equations. Furthermore, the influence of Brioschi's theory of determinants was also noticeable.

Unfortunately, the results in the fields of algebra and analysis were not generally very remarkable (Ulisse Dini's analytical work only began in 1864), even though Noether said in his commemoration of Brioschi that "if Brioschi did not seem an entirely original thinker who created new ideas and opened up new avenues, he has to be acknowledged as a mind endowed with an originality of his own . . ." [Noether 1898, 491].

Works by foreigners were rarely published in Italy, although occasionally articles by French algebraists appeared, including presentations and translations of works by Chasles and, above all, by the German mathematician Riemann. It was only in *Annali di Matematica* that articles began to appear after 1858 by Hermite, Hesse, Jonquières, Hirst, Cayley, Lebesgue, Roberts, and Clebsch. (It is interesting to note that more foreign works were published in 1867, as soon as Cremona became the editor of *Annali.*)

Generally speaking, there was not very much contact with foreign mathematicians; the main focus of attention was Germany. Steiner, Jacobi, Dirichlet, and Borchardt had all visited Italy in 1843–1844 [4], while Betti, Brioschi, and Casorati had been to Germany in 1858 [5]. Even Riemann had spent a long period in Italy, and his influence can be seen in Italian works of the period.

This, then, was the scientific climate in which Cremona had to work, but he had the advantage of the algebraic background he had obtained from Brioschi. Algebra, therefore, supplied many of the tools that Cremona would use in his geometrical works. If we add to this the influence that von Staudt had on Cremona, and his "natural inclination" toward the geometrical view of things, we have a fairly good idea of the features of the so-called Italian school of pure geometry, of which Cremona was certainly the founder. It was primarily his ideas that were to inspire Italian algebraic geometry throughout the first half of the 20th century, with such illustrious names as Corrado Segre, Francesco Severi, Federigo Enriques, and Guido Castelnuovo. Meanwhile, in Germany there was vigorous development of analysis and algebra [6], both of which were applied to geometry (not without a certain amount of Italian influence) leading to the foundation of algebraic geometry proper, thanks to the work of those like Plücker, Cayley, Sylvester, Clebsch, Zeuthen, Brill, and Noether.

Cremona's main field of research was birational transformations between two planes and between two spaces, which bear his name today. That is to say, he was concerned with research into the more general correspondences between two planes or two spaces that transform points into points and straight lines into algebraic curves. This research paved the way for the founding of a school of algebraic geometry by Cremona. The study of the invariants for these transformations underlies all the properties of algebraic geometry in the sense in which we understand it today, and forms part of a much broader program for the study of geometrical properties through their invariants by transformation, as presented by Felix Klein at Erlangen in 1872.

Cremona's program, and that of the distinguished school that grew up around him, began by separating geometrical from analytical properties, and operated in the domain of synthetic geometry using both the rigor of logic and intuition. However, viewed in terms of contemporary thought, which sees pure geometry as being wholly independent of analytical algebraic support and logically based on intuitive postulates, the work produced in Italy in the late 19th century was not strictly pure geometry. Even though Cremona did not use coordinates or algebraic developments, and only used pure reasoning to demonstrate all his propositions, he nevertheless employed several algebraic means. For instance, he described and "counted" the intersections of curves and algebraic surfaces instead of calculating them with systems of equations. He proceeded so, because, on the one hand, he did not want to "mask" the geometric phenomenon under examination. Yet, on the other hand, he was willing to use the results of algebraic systems, and denomination taken from the algebraic field (orders and kinds of algebraic forms and related meanings), if they could enhance the continuity of the geometric treatment he was giving [7]. It could therefore be said that the algebraic methods were subordinated to the synthetic method of proof, and no concessions made to the analytical approach.

This method of proceeding, in which intuition of results preceded deductive proofs, was certainly not likely to shield Italian geometers from criticism. Later, as the results obtained in Italy expanded, foreigners almost literally turned the work initiated by Cremona on its head. Instead of geometrically interpreting the results obtained using algebraic methods, as Cremona had done with Steiner's work [8], they felt the need to use algebra to redemonstrate the theorems which had been obtained in Italy using pure geometric reasoning. In a Commemoration of Cremona in 1904, Noether wrote about Cremona's *Introduzione ad una teoria geometrica delle curve piane:*

> . . . From today's point of view we note the algebraic foundation, which does not correspond to the purity of the method of synthetic geometry. And in the same way one misses the rigor, which we have now the right to pretend from the definitions and demonstrations, especially for the configuration theorems, both from the algebraic and the geometric point of view; but Cremona, who had had this intention for a long time, could not decide on a second draft of the book, for it would have meant exactly a completely new reorganization; and a further try for a substitution has not been made. Nevertheless, contemporary Italian science would especially be able to give to geometry an analogous work, which would take account of the great progress of the last 40 years on sure bases, which alone could guarantee it an everlasting utility. For Cremona's work has the historical merit of having established, with his methods and conceptions, the contact of pure geometry with the analytical-geometrical development which had emerged through the work of Plücker, Hesse and Clebsch, of Salmon and Cayley. [Noether 1904, 7]

Although this extract refers to a specific work, it was basically a criticism of the work done by Cremona and his followers in general. A lack of rigor deprived their results, said Noether, of "sound bases," to the extent that the validity of the work could not be guaranteed. Thus Cremona's merits were regarded as mainly "historical" in the sense that, thanks to his methods, he had established "the contact of the pure geometry with the analytical-geometrical development." As far as the achievements of his school were concerned, Noether believed that they could be of value only if they were reinterpreted using more rigorous schemata.

Max Noether [9] was closer to the Italian school than any other foreign mathematician, so far as the content of his own work, and sometimes his methods, was concerned. Noether had adopted Clebsch's program of algebraic geometry, but while Clebsch tended toward a geometric interpretation of the algebraic results, Noether's approach was closer to Plücker's. He was an algebraist, but he was not so much interested in the algorithm as in functional problems. He was also influenced by the school of synthetic geometry of Poncelet, Steiner, and Chasles, and even Cremona, so much so that he tried occasionally the path of geometric intuition. But he also influenced the Italian school, which drew on his results for new resources.

The correspondence between Luigi Cremona and Max Noether, discussed in the next section, shows how the two geometers exchanged some of their scientific views, and also reveals the similarities between their fields of research. However, differences emerge when one examines their methods: Noether asked Cremona to explain one of his works on the rational transformations of space and questioned Cremona's demonstration, which he found neither adequate nor convincing.

Even though we do not know Cremona's reply, he apparently did not feel obliged to give a more precise demonstration. But Noether himself set about doing so in order to appraise the full value of a result that he considered to be important.

2. CORRESPONDENCE BETWEEN LUIGI CREMONA AND MAX NOETHER

There are nine letters written in German by Max Noether to Luigi Cremona, mostly in 1871, when the ideas of both geometers were particularly close. Indeed, as both of them said, their articles were "virtually the same." This was mentioned by Cremona, in particular, at the end of his second note, "Sulle trasformazioni razionali dello spazio," in which he wrote:

> . . . I would like to take this opportunity to add another quotation to those given at the beginning of my first Note. It is from a new paper by Mr. Noether entitled "Ueber die eindeutigen Raumtransformationen" [*Mathematische Annalen* **3** (1871), 547], of which I received a copy on 7 May. In Mr. Noether's paper, and in my Note "Ueber die Abbildung algebraischer Flächen" [*Göttingische Nachrichten* (May 3, 1871)], which deal with the same subject (the application of third level transformations to representations of algebraic surfaces), which were published practically on the same day, the reader will find the most singular coincidences, even in the minutest details. This should not come as a surprise to anyone, and it is a source of great satisfaction to me: particularly since it was the excellent research work conducted by Mr. Noether, and set out in his previous works, that led me to take up these studies again, and which taken together with the results I had already obtained for plane figures, eventually led me to complete the general transformations in space, the purpose of this and the First Note (4 May) communicated to R. Istituto. [Cremona 1871a, 324]

Noether, in the earliest letter found at the Castelnuovo Institute of Mathematics (Rome) said:

> . . . Obviously I couldn't be less surprised than you at the remarkable coincidence of the most recent geometric works of both of us in relation not only to the subject, but also to the smaller details of the explanations. I can only say how happy I am that my ideas coincide with those of a recognized authority, whom I esteem much. [May 11, 1871]

As Noether wrote in his Commemoration of Cremona [Noether 1904], the two works coincided in the smallest details, "apart from the external form of the methods expressed more or less geometrically." Cremona's geometrical view undoubtedly enabled him to get around a number of algebraic procedures which not everyone was able, or willing, to do. Even Noether seemed to appreciate, above all, the fact that Cremona's methods were translatable into algebraic form, as he said in his Commemoration. On the subject of *Preliminari* [Cremona 1867], Noether said:

> . . . Here again Cremona's observations obviously are lacking in purity and foundational rigor; but they are also stimulating at a higher level because they can be translated into algebraic form, generally without any effort. Only a few deeper studies result in more difficult consequences through analytical means. . . .[Noether 1904, 8]

Noether's subsequent letters—apart from a short letter of thanks—all refer to the two notes just mentioned to "Sulle trasformazioni razionali dello spazio" [1871]. The issue raised is the "adequacy" of Cremona's demonstration of the assertions made in it.

In the note, Cremona sets out to obtain from a plane representation of a given surface any (rational) transformation whose omaloidical system contains the given surface, where an omaloidical system is a linear system of ∞^3 surfaces, such that three of them have one single variable intersection, and which therefore represents the set of surfaces in a birational transformation corresponding to the planes of the other space. To do this, he posits an omaloidical net of curves in the plane on which the given surface F of order $\mathbf{n}$ is represented, such that these curves, together with a fixed curve, constitute the images of the intersections of F with the rational surfaces of order $\mathbf{n}$, having the same multiple points and lines of F.

Using the methods of synthetic geometry, Cremona's demonstration is mainly based on the production of a great many examples, which took up sixteen of the twenty pages of the two notes. To all of this he offered the following conclusion:

> . . . The properties stated here can easily be demonstrated by considering that a rational curve of order n (in three-dimensional space) is determined by $4n$ conditions; that having to pass with r branches through a given point absorbs $2r$ conditions but that, if it must have an r-fold point as a point O which is simple for all ϕ surfaces of the homaloid system,* the number of absorbed conditions will be $r(r + 1)$. At point O there corresponds in space (x) a homaloid surface of order r (cutting all the ψ exclusively into fixed, fundamental curves, and forming part of each of the ψ of a network contained in the system); in the first case this curve fits twice into the Jacobian of the ψ, but $r + 1$ times in the second case. To a curve Λ of order i, which is r-fold for all the ϕ, there corresponds a surface (to be counted once in the Jacobian of the ψ), the order of which is equal to the number of nonfixed intersections of Λ with the rational curve corresponding to an arbitrary straight line of the space (x), and which cuts any ψ according to several fixed curves and i rational curves of order r.
>
> It may happen, as in cases 5, 6, and 9, that in addition to the fundamental curves required to identify the system, the ϕ still have a line R in common, determined by the former, which I shall assume to be of order i and r-fold for the ϕ. Therefore, to each point of R there will correspond lines coinciding with a single line R' of order r, which is i-fold for the ψ. The line R (and likewise line R') is not met by the rational curve corresponding to an arbitrary straight line in the space (x), and is $4r$ multiple for the Jacobian of the ϕ. Etc., etc.
>
> * Which will have a contact with it of order $r - 1$, so that by representing one of the two homaloid surfaces determining the curve, its image will have an r-fold point at o, the image of O. [Cremona 1871a, 324]

Noether's first objection, which he raised in his letter dated June 16, 1871, was that the list of examples was not as complete as it should have been. He added a further example:

. . . If this note pretends to give a complete list with respect to the omaloidical systems which originate from the general surfaces of the 3rd order, I miss a transformation that sometimes I met, but that probably you have noted yourself in the meantime.

The ϕ are surfaces of the 3rd order, which pass through 3 fixed lines that do not intersect each other, G_1, G_2, G_3, touch each other in one point, and intersect each other in a further fixed point. That means inversion of the 6th order. Here the K's are curves of the 3rd order, 1230^20, to the G's correspond on the plane conic sections 12456, 13456, 23456.

Cremona, however, had never intended to provide a compete list, judging from what he had written after giving the examples:

. . . I think that these examples are now sufficient to prove my assumption. . . . [Cremona 1871a, 324]

Noether's next two letters dealt with the approach used by Cremona for the demonstration which Cremona mentioned at the end of the second note. In his letter of July 18, 1871, Noether wondered how to interpret it, and whether there existed any precise basis for Cremona's assertion, or whether Cremona had directly determined the results which, according to Noether, would be complex:

. . . (You say) that the Jacobian surface of the ψ contains $(r + 1)$ times the surface of the rth order, which corresponds to a contact point of the $(r - 1)$st order of the ϕ, O. You seem to deduce this fact from the fact that the rational spatial curves of the mth order, which correspond to the lines of the space X, have the point O as an r-fold point, which makes out $r(r + 1)$ conditions for these curves. And in fact, as the two numbers, the reduction $4m - 4$ for the multiplicity of these curves and the degree $4m - 4$ of the Jacobian surface of the ψ, are coincident, so this conclusion is completely justified.

Nevertheless it doesn't seem to me completely satisfactory; for I cannot see any further reason for the coincidence of these two numbers, particularly for the equality of the reduction of the *conditions* for the curves in a *point O* with the number of the *intersection points* of a line with the corresponding piece of the Jacobian of the ψ.

The direct determination of how many times the surface corresponding to O is contained in the Jacobian one is difficult; it becomes easy only for the case $r = 2$, for then the ψ have the form

$$\psi_1 = \Lambda^2 B, \qquad \psi_2 = \Lambda C, \qquad \psi_3 = \Lambda D, \qquad \psi_4 = F$$

while for $r > 2$ the bundle of surfaces

$$K\Lambda B + \Lambda C + \mu D$$

gets complicated properties.

You would perhaps be so kind as to give me an explanation, if I correctly understood your conclusion, if you motivated more precisely this coincidence, or if you undertook a direct determination. Also the statement at the end of your note, that a line R lies $4r$ times in the Jacobian, I cannot motivate completely (if not in the way of the analogous results in §1 of may treatise on spatial transf.). . . .

We do not know Cremona's answer. According to the *Verzeichnis der schriftlichen Nachlässe in deutschen Archiven und Bibliotheken* [Mommsen 1971; Denecke & Brandis 1981], there are no surviving letters of Cremona to Noether. In Noether's next letter dated September 12, 1871, he thanked Cremona for his reply of August 20. However, one may assume that Cremona still had not sent any direct demonstration, since Noether—still dissatisfied—added:

. . . Perhaps it is only a personal fact that I am in this case not completely satisfied with the indirect way in which you deduce the facts about the behavior of the Jacobiana, a way which is actually completely sufficient for explaining the result and which can often be considered also as satisfying just as it is; for at the time I posed my question I hoped to find a direct insight into the behavior of the surface which corresponds to a higher contact point with respect to the Jacobiana, without reaching my aim.

The latter remark by Noether should not be taken, however, to imply that he had given up the attempt to provide a direct determination. He seemed to accept the line used by Cremona to prove the equality of the two numbers mentioned in his previous letter of July 18, 1871 (paragraph two), but he preferred to make this deduction by using an algebraic system which he knew. Believing that his calculations might be useful to someone, he finally asked Cremona to publish them in *Annali di Matematica Pura ed Applicata*, of which Cremona was the editor:

. . . Because that to an r-fold point of the ϕ, which absorbs k lin. conditions, must correspond in the Jacobiana of the ψ a locus of the kth order, is clearly deduced only from the fact that to the fund. Curves of the ϕ corresponds then only a locus of the $(4m - 4 - k)$th order, that means from an extension of one of the equations of the system of equations that I once handled:

$$4n' - 4 = \sum_i 2i\left\{(n-i)m_i - \frac{1}{2}ir_i\right\} - \sum_{ij} 2ij\,k_{ij} + \sum_{\rho} 2\rho^2 - \sum_{i,\rho} 2i\rho\lambda_{i,\rho} + \sum_{\sigma} \sigma(\sigma+1).$$

(n, n' orders of the ϕ and the ψ resp.; i-fold fund. curves of order m, rank r_i; ρ-fold fund. points; α_σ contact points of the $(\sigma - 1)$st order of the ϕ; the i-fold fund. curve $(m_i; r_i)$ goes with $\lambda_{i,\rho}$ branches through the ρ-fold point and is intersected by the j-fold fund. curve in $k_{i,j}$ points.)

I have now collected the observations which lead to these formulas, that is, an extension of the observations of Salmon and Cayley on the system of intersection points of three surfaces, about the conditions which are absorbed by a multiple curve, etc., and I tried particularly to research more precisely the limits of its validity. As it seems to me that to someone such a collection, which needs however many calculations, would be useful, so I dare to propose it to you as a little note for the annali (about one sheet).

I obviously want to present the whole thing as a collection and an extension, not as something new. [September 12, 1871]

The collection material to which Noether alludes in the last sentence was indeed published in *Annali di Matematica* under the heading "Sulle curve multiple di superficie algebraiche," in which the calculations given in the letter mentioned above constituted an appendix [Noether 1871, 163–177]. In a footnote, Noether added that he had reached the same conclusion using "a similar method"' to the one used by Mr. Cremona in the second of his notes [Noether 1871, 177].

CONCLUSION

The extracts from the correspondence between Luigi Cremona and Max Noether discussed herein, although referring to a specific work of Cremona, are highly significant in that they are a perfect reflection not only of Noether's opinion of Cremona and his school, as we have seen in the quotations contained in Section

1, but also more generally of what was to turn into a controversy concerning the subsequent Italian school of algebraic geometry.

However, more positively, these extracts represent evidence of the level of the international exchange of scientific information of the time, and of the respect and friendship associated with these exchanges.

NOTES

1. For a more detailed account of the life and work of Luigi Cremona see [Berzolari 1906; Loria 1904; Noether 1904].

2. These letters belong to a large collection of letters sent to Luigi Cremona by world-famous mathematicians, discovered in the library of the "G. Castelnuovo" Institute of mathematics by G. Israel and L. Nurzia. See [Israel & Nurzia 1983].

3. This refers to the last paragraph of the work "Solution des questions 494 et 499 . . ." [Cremona 1860].

4. Compare [Segre 1933].

5. Compare [Volterra 1902].

6. Compare, in particular, [Kline 1972].

7. Compare [Castelnuovo 1928; Israel 1981; Segre 1933; Severi 1928].

8. Cremona states this explicitly in the preface to [Cremona 1862, 305]: "Wishing to use the methods of pure geometry to demonstrate the extremely important theorems expressed by the distinguished *Steiner* in his short paper 'Allgemeine Eigenschaften der algebraischen Curven' [*Crelle's Journal* **47** (1854), 1–6], I was led to research a number of things. . . ."

9. For a more complete account of the life and work of Max Noether, see [Brill 1923; Castelnuovo, Enriques, & Severi 1925].

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