PISOT UNITS, SALEM NUMBERS AND HIGHER DIMENSIONAL PROJECTIVE MANIFOLDS WITH PRIMITIVE AUTOMORPHISMS OF POSITIVE ENTROPY

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Abstract. We show that, in any dimension greather than one, there is an abelian variety with primitive biregular automorphisms of positive entropy. We also show that there are smooth complex projective, hyperkähler fourfolds, Calabi-Yau fourfolds and rational fourfolds, with primitive biregular automrphisms of positive entropy. Besides geometry, Pisot units and Salem numbers play important roles in our proof.

1. Introduction

Thoughout this note, we work in the category of projective varieties defined over $\mathbb{C}$.

The aim of this note is to give an affirmative answer (Theorems 1.7, 1.8) to the following question asked by [Og15, Problem 1.1]:

Question 1.1. For each integer $\ell \geq 2$, is there a smooth projective variety of dimension ℓ with primitive biregular automorphisms of positive topological entropy?

Question 1.1 is related with both birational geometry, complex dynamics. and also number theory as we shall see. We recall the complex dynamical notion of topological entropy and closely related notions of dynamical degrees and relative dynamical degrees in Section 2 following [Bo73], [Gr03], [Yo87], [DS05], [DN11] and [Tr15]. The notion of primitivity of automorphism, introduced by De-Qi Zhang [Zh09], is purely algebro-geometric:

Definition 1.2. Let M be a smooth projective variety of dimension $\ell \geq 2$ and $f \in \mathrm{Bir}\,(M)$. f is called *imprimitive* if there are a dominant rational map $\pi : M \dashrightarrow B$ to a smooth projective variety B with $0 < \dim B < \dim M$ and with connected fibers, and a rational map $f_B : B \dashrightarrow B$, necessarily $f_B \in \mathrm{Bir}\,(B)$, such that

$$\pi \circ f = f_B \circ \pi \ .$$

Here smoothness assumption is harmless, as we work over $\mathbb{C}$. f is *primitive* if it is not imprimitive.

Note that if M is a curve, i.e., if $\ell = 1$, then any automorphism of M is primitive and of null topological entropy. So, from now, we assume that $\ell \geq 2$. A primitive birational automorphism is in some sense an *irreducible* automorphism in birational geometry. Compare with the following trivial:

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Remark 1.3. Let k be a field and V be a k-vector space of $\dim V = \ell \geq 1$. Let $f \in \mathrm{GL}\,(V)$. Then f is not irreducible in the usual sense if and only if there are a k-vector space W with $0 < \dim W < \dim V$, a surjective linear map $\pi : V \to W$ and $f_W \in \mathrm{GL}\,(W)$ such that $\pi \circ f = f_W \circ \pi$. The existence of irreducible (V, f) with $\dim V = \ell$ depends on k. For instance $\ell = 1$ if $k = \mathbb{C}$, $\ell = 1,\ 2$ if $k = \mathbb{R}$, while ℓ is arbitrary if $k = \mathbb{Q}$.

The existence of a primitive automorphism also depends on the classes of smooth projective varieties. Indeed, as again observed by De-Qi Zhang [Zh09] (see also [Og16, Lemma 3.2, Theorem 3.3] for the formulation here), we have the following rather strong constraint on projective varieties admitting primitive bitaional automorphisms:

Theorem 1.4. *Let M be a smooth projective variety of dimension $\ell \geq 2$. We denote the Kodaira dimension of M by $\kappa(M)$. Assume that the following two statements hold for M:*

(1) If $\kappa(M) = 0$ and $h^1(\mathcal{O}_M) = 0$, then M is birational to a minimal Calabi-Yau variety M', i.e., a normal projective varietry M' with only $\mathbb{Q}$-factorial terminal sinularities such that $\mathcal{O}_{M'}(mK_{M'}) \simeq \mathcal{O}_{M'}$ for some $m > 0$ and $h^1(\mathcal{O}_{M'}) = 0$; and

(2) If $\kappa(M) = -\infty$, then M is uniruled.

Then, if M has a primitive birational automorphism $f \in \mathrm{Bir}\,(M)$, then $\mathrm{ord}\, f = \infty$ and M falls into one of the following three exclusive classes:

(RC) M is a rationally connected manifold, i.e., a smooth projective variety whose two general closed points are connected by a rational curve on M;

(CY) M is birational to a minimal Calabi-Yau variety; or

(A) M is birational to an abelian variety.

Remark 1.5. The assumptions (1) and (2) in Theorem 1.4 hold if the minimal model problem and the abundance conjecture are affirmative in dimension ℓ, which are believed to be true. Theorem 1.4 is unconditional if $\ell \leq 3$ by the minimal model theory and abundance theorem for projective threefolds due to Kawamata, Miyaoka, Mori and Reid ([Mo88], [Ka92], see also [KMM87], [KM98]).

Remark 1.6. Smooth rational varieties are rationally connected. The most important classes of smooth projective varieties in (CY) are Calabi-Yau manifolds and projective hyperkähler manifolds. Recall that an ℓ-dimensional simply-connected smooth projective variety M is a Calabi-Yau manifold (resp. hyperkähler manifold) if $H^0(\Omega_M^j) = 0$ for $0 < j < \ell$ and $H^0(\Omega_M^\ell) = \mathbb{C}\omega_M$ for a nowhere vanishing regular ℓ-form ω_M (resp. $H^0(\Omega_M^2) = \mathbb{C}\eta_M$ for an everywhere non-degenerate regular 2-form η_M, and therefore ℓ is even).

When $\ell = 2,\ 3$, there are smooth projective, rational varieties, Calabi-Yau manifolds and abelian varieties, with primitive bireguler automorphisms of positive entropy (See eg. [Ca99], [BK09], [BK12], [Mc07], [Mc16], [Re12], [CO15], [Do16], [Og16] for surfaces in several classes and [OT14], [OT15] for threefolds).

Our main results are the following Theorems 1.7 and 1.8.

Theorem 1.7. *For each $\ell \geq 2$, there is an ℓ-dimensional abelian variety A with a primitive biregular automorphism $f \in \mathrm{Aut}\,(A)$ of positive topological entropy. There is also an ℓ-dimensional smooth projective variety M, birational to a minimal Calabi-Yau variety, with a primitive biregular automorphism of positive topological entropy.*

We construct A as the self product of an elliptic curve E and its desired automorphism by using Pisot units (Definition 3.1) in Theorem 3.6. M is then obtained by a standard resolution of the quotient variety $A/\langle -1_A \rangle$ (Corollary 3.7). Primitivity is checked by the so called product formula for the relative dynamical degrees ([DN11], [DNT12], [Tr15], see also Section 2 for summary and [Tr16]). Proof of Theorem 1.7 is extremely easy once one notices an avaiabilty of Pisot units, but I believe that it is important to guarantee the existence in any dimension $\ell \geq 2$. We also show the existence of rationally connected manifolds of dimension 4 and 5 with primitive biregular automorhisms of positive topological entropy as a byproduct (Corollary 3.9).

It is much more interesting to see the existence of more specific classes of manifolds, namely, projective hyperkähler manifolds, Calabi-Yau manifolds, smooth rational varieties, with primitive automorphisms of positive topological entropy. In this direction, we shall prove the following theorem in dimension 4 in a fairly explicit way:

Theorem 1.8. *There are* 4*-dimensional projective, hyperkähler manifold, Calabi-Yau manifold and smooth rational variety, with a primitive biregular automorphism of positive topological entropy.*

We construct such explicit four dimensional manifolds from K3 surfaces S with special automorphisms and their Hilbert schemes of two points $S^{[2]}$ (Theorems 5.2, 6.1, 7.1). Besides basic facts on toplogical entropy, dynamical degrees and relative dynamical degrees, Salem numbers (Definition 4.1) and hyperkähler geometry play crucial roles in checking primitivity of candidate automorphisms (Theorem 4.6). However, the following question is yet completely open, as our criterion Theorem 4.6 heavily relies on our assumption, dimension 4:

Question 1.9. Let $\ell \geq 3$. Are there 2ℓ-dimensional projective, hyperkähler manifolds, Calabi-Yau manifolds and smooth rational varieties, with primitive biregular (resp. birational) automorphisms of positive topological entropy (resp. of first dynamical degree > 1)?

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2. Entropy, Dynamical degrees and relative dynamical degrees

In this section we briefly recall definitions of entropy, dynamical degrees, relative dynamical degrees and their basic properties, used in this paper. No new result is included. Main references here are Dinh-Sibony [DS05], Dinh-Nguyen [DN11], Dinh-Nguyen-Troung [DNT12] and Troung [Tr15]. We state the results for birational automorphisms of smooth projective varieties, while all the results below are valid for any compact Kähler manifolds (if we use Kähler forms instead of Fubini-Study forms or hyperplane classes).

2.1. **Topological entropy and Gromv-Yomdin theorem.** Let $M = (M, d)$ be a compact metric space and $f : M \to M$ be a continuous surjective selfmap of M. The *topological entropy* of f is the fundamental invariant that measures *how fast two general points spread* *out under the action of the semi-group* $\{f^n | n \in \mathbf{Z}_{\geq 0}\}$. For the definition, we define the new distance $d_{f,n}$ on X, depending on f and n, by

$$d_{f,n}(x, y) = \max_{0 \leq j \leq n-1} d(f^j(x), f^j(y)) \text{ for } x, y \in X.$$

Let $\epsilon > 0$ be a positive real number. We call two points $x, y \in M$ (n, ϵ)-*separated* if $d_{f,n}(y, x) \geq \epsilon$, and a subset $F \subset M$ (n, ϵ)-*separated* if any two distinct points of F are (n, ϵ)-separated. Let

$$N_d(f, n, \epsilon) := \text{Max}\,\{|F| \mid F \subset M \text{ is } (n, \epsilon) - \text{separated }\}\ .$$

Note that $N_d(f, n, \epsilon)$ is a well-defined positive integer, as M is compact. The following definition was introduced by Bowen [Bo73]:

Definition 2.1. The *topological entropy* of f is defined by:

$$h_{\text{top}}(f) := \lim_{\epsilon \to +0} \text{limsup}_{n \to \infty} \frac{\log N_d(f, n, \epsilon)}{n}\ .$$

Note that $h_{\text{top}}(f)$ does not depend on the choice of the metric d of the topological space M.

Let M be a smooth projective variety and $f : M \to M$ be a surjective morphism. Then f is in particular continuous in the classical metric topology given by the Fubini-Study metric $d(*, **)$ under some embedding $M \subset \mathbb{P}^N$. So, one can speak of the topological entropy of f. One of the most fundamental properties of the topological entropy is the following cohomological characterization due to Gromov and Yomdin ([Gr03], [Yo87]):

Theorem 2.2. *Let M be a smooth projective variety of dimension ℓ and $f : M \to M$ be a surjective morphism. Then, $d_p(f) = r_p(f)$ and*

$$h_{\text{top}}(f) = \log \max_{0 \leq p \leq \ell} d_p(f) = \log \max_{0 \leq p \leq k} r_p\,(f) = \log r\,(f)\ .$$

Here $r_p\,(f)$ and $r\,(f)$ are the spectral radii of $f^*|H^{p,p}(X)$ and $f^*| \oplus_{p=0}^{2k} H^k(M, \mathbf{Z})$ respectively. The p-th dynamical degree $\lambda_p(f)$ is defined (and well-defined) by:

$$d_p(f) = \lim_{n \to \infty} (\int_X (f^n)^*(\omega^p) \wedge \omega^{k-p})^{\frac{1}{n}} = \lim_{n \to \infty} ((f^n)^*(H^p).H^{\ell-p})_M^{\frac{1}{n}} \geq 1\ .$$

Here H is the hyperplane class of M (under any embedding $M \subset \mathbb{P}^N$) and ω is the Fubini-Study form, i.e., the positive closed $(1, 1)$-form induced by the Fubini-Study form of $\mathbb{P}^N$. Note that the equality $d_p(f) = r_p(f)$ shows that one can replace $H^{p,p}(M)$ by $N^p(M)$, provided that M is projective (see eg. [Tr15], [Tr16]). Here $N^p(M)$ is the finitely generated free $\mathbb{Z}$-module consisting of the numerical equivalence classes of algebraic p-cocycles.

2.2. **Dynamical degrees.** One can define $r_p(f)$, $r(f)$ and $d_p(f)$ also for birational self map $f : M \dashrightarrow M$, i.e., for $f \in \text{Bir}\,(M)$. Precisely, for each $n \geq 1$, we define the pull back $(f^n)^*$ by

$$(f^n)^*(a) := (p_{1,n})_*((p_{2,n})^*(a))\ .$$

and $d_p(f)$ by the same formula above. Here $p_{1,n} : \tilde{M}_n \to M$ and $p_{2,n} : \tilde{M}_n \to M$ be any birational *morphisms* from a smooth projective variety $\tilde{M}_n$, depending on f and n, such that $f^n = p_{2,n} \circ (p_{1,n})^{-1}$. Note that $r_p(f)$ and $r(f)$ (we may just take $n = 1$) are not birational invarinat in general, as the standard Cremona involution of $\mathbb{P}^2$ shows. On the other hand, the dynamical degrees $d_p(f)$ are well-defined birational invariants, as Dinh-Sibony ([DS05]) proved:

Theorem 2.3. *Let M and M' be smooth projective varieties and $\mu : M \dashrightarrow M'$ be a birational map. Let $f \in \mathrm{Bir}\,(M)$ and $f' = \mu \circ f \circ \mu^{-1}$ the birational automorphism of M' induced by f and μ. Then, $d_p(f)$ is well defined, $d_p(f^m) = d_p(f)^m$ and $d_p(f) = d_p(f')$ for all p.*

2.3. **Relative dynamical degrees.**

Set-up 2.4. Let M and B be smooth projective varieties of dimension ℓ and b respectively, $f \in \mathrm{Bir}\,(M)$ and $\pi : M \to B$ be an f-equivariant surjective morphism so that there is $f_B \in \mathrm{Bir}\,(B)$ with $\pi \circ f = f_B \circ \pi$. Here we do not require that π has connected fibers. Let H_M and H_B be the hyperplane class of M and B (under any embedding to projective spaces) and ω_M and ω_B be the Fubini-Study forms induced by the embeddings.

Remark 2.5. If $f \in \mathrm{Bir}\,(M)$ is imprimitive and $\pi : M \dashrightarrow B$ be f-equivariant, then by replacing M by $\tilde{M}$, a resolution $\mu : \tilde{M} \to M$ of indeterminacy of π, f by $\tilde{f} = \mu^{-1} \circ f \circ \mu$ and π by the induced morphism $\mu \circ \pi$, we always obtain $\tilde{f}$-equivariant surjective morphism $\pi \circ \mu : \tilde{M} \to B$ as in Set-up 2.4. In this modification, we may loose biregularity of $\tilde{f}$ in general even if $f \in \mathrm{Aut}\,(M)$ but the values $d_p(f)$ remain the same, i.e., $d_p(\tilde{f}) = d_p(f)$ by Theorem 2.3. So, when we compute the dynamical degree $d_p(f)$, we may assume π is a surjective *morphism*.

The notion of the relative dynamical degrees $d_p(f|\pi)$ is defined by Dinh-Nguyen [DN11] as follows:

Definition 2.6. Under Set-up 2.4, the *relative p-th dynamical degree* $d_p(f|\pi)$ is defined as

$$\lim_{n\to\infty} (\int_M (f^n)^*(\omega_M^p) \wedge \omega_M^{\ell-b-p} \wedge \pi^*(\omega_B^b))^{\frac{1}{n}} = \lim_{n\to\infty} ((f^n)^*(H_M^p).H_M^{\ell-b-p}.\pi^*(H_B^b))_M^{\frac{1}{n}} \geq 1 \ .$$

$d_p(f|\pi)$ is well-defined by [DN11]. We may regard $\pi^*(H_B^b)$ as a general fiber class of π, up to positive constant multiple. So, the relative dynamical degree can be considered as the dynamical degree of f *restricted to a general fiber F*. However, to make the meaning of *restriction to F* rigorous, we need good fixed points of f_B^n on B (Compare with the argument in Theorem 4.6 in Section 4).

The following important result, called the *product formula*, was first proved by Dinh-Nguyen [DN11] (See also [Tr15] for purely algebro-geometric new proof):

Theorem 2.7. *Under Set-up 2.4, $d_p(f) = \max_{\max\{0,p-\ell+b\} \leq j \leq \min\{p,b\}} d_j(g) d_{p-j}(f|\pi)$.*

The following corollary was observed again by [DN11]:

Corollary 2.8. *Let M and M' be smooth projective varieties, $f \in \mathrm{Bir}\,(M)$ be a birational automorphism of M, $g : M \dashrightarrow M'$ be an f-equivariant generically finite dominant rational map and f' be the induced birational automorphism of M'. Then $d_p(f) = d_p(f')$ for every p.*

Proof. As $d_p(f)$ are birational invariant by Theorem 2.3, we may assume that g is a genericaly finite surjective morphism (cf. Remark 2.5). Then, by the product formula (Theorem

2.7), we have $d_p(f) = d_p(f')d_0(f|g)$ for all p. By definition of the relative dynamical degree, we have $d_0(f|g) = 1$. Hence $d_p(f) = d_p(f')$. □

See also for [OT15] for other algebro-geometric applications of the product formula.

3. Pisot units and Proof of Theorem 1.7.

Main results of this section are Theorems 3.3, 3.4 and thie corollaries Corollaries 3.7, 3.9. First, we recall the definition of Pisot unit:

Definition 3.1. Let $\overline{\mathbb{Z}} \subset \mathbb{C}$ be the ring of algebraic integers. A real algebraic integer $\alpha \in \overline{\mathbb{Z}} \cap \mathbb{R}$ is called a *Pisot number* if $|\alpha| > 1$ and $|\alpha'| < 1$ for all Galois conjugates $\alpha' \neq \alpha$ of α over $\mathbb{Q}$. Here, for a complex number a, we denote the real part of a, the imaginary part of a and the complex conjugate of a by $\mathrm{Re}\, a$, $\mathrm{Im}\, a$ and $\overline{a}$ respectively. We define the absolute value $|a|$ of a by

$$|a| := \sqrt{a\overline{a}} = \sqrt{(\mathrm{Re}\, a)^2 + (\mathrm{Im}\, a)^2} \in \mathbb{R}_{\geq 0}\ .$$

The degree of the Pisot number is the degree of the minimal polynomial of α over $\mathbb{Z}$ or equivalently over $\mathbb{Q}$. A Pisot number α of degree d is called a *Pisot unit* if α is invertible in $\overline{\mathbb{Z}}$, i.e. $\prod_{j=1}^{d} \alpha_j = \pm 1$, where α_i $(1 \leq j \leq d)$ are the Galois conjugates of α. We note that if α is a Pisot number or a Pisot unit of degree d then so is $-\alpha$. So, *from now on, we may and will assume that Pisot numbers are greater than* 1.

Example 3.2. (1) Let a be an integer such that $a \geq 3$. Then, by definition, the largest root of $X^2 - aX + 1 = 0$ is a Pisot unit of degree 2. Similarly, for a positive integer b, the largest root of $X^2 - bX - 1 = 0$ is also a Pisot unit of degree 2.

(2) The largest real roots

$$\alpha_3 = 1.324\ldots\ ,\ \alpha_4 = 1.380\ldots\ ,\ \alpha_5 = 1.443\ldots$$

of the equations

$$X^3 - X - 1 = 0\ ,\ X^4 - X^3 - 1 = 0\ ,\ X^5 - X^4 - X^3 + X^2 - 1 = 0$$

are Pisot units of degree 3, 4, 5 respectively. These three Pisot units $\alpha_3 < \alpha_4 < \alpha_5$ are the smallest three positive Pisot numbers ([BDGPS92, Theorem 7.2.1]).

There are plenty of Pisot units as the following theorem ([BDGPS92, Theorem 5.2.2]) shows:

Theorem 3.3. *Let $d \geq 1$ be any positive integer and K be any real field extension of $\mathbb{Q}$ of degree $d = [K : \mathbb{Q}]$ (for instance $K = \mathbb{Q}(\sqrt[d]{2})$). Then there is a Pisot unit $\alpha \in K$ of degree $d = [K : \mathbb{Q}]$.*

Pisot numbers naturally appear as the first dynamical degrees of birational automorphisms of rational surfaces, which are not conjugate to biregular automorphisms ([DF01], [BC13]). They are not necessarily Pisot units. Pisot numbers, or more precisely Pisot units, also play important roles in primitive automorphisms of abelian varieties:

Theorem 3.4. *Let d be an integer such that $d \geq 2$. Let A be a d-dimensional abelian variety and $f \in \mathrm{Aut}\,(A)$. Assume that $f^*|H^0(A, \Omega_A^1)$ has a Pisot number $\alpha > 1$ of degree d as its eigenvalue. Then α is a Pisot unit and f is a primitive automorphism of A of positive topological entropy. More precisely $h_{\mathrm{top}}(f) = 2 \log \alpha > 0$.*

Proof. By the Hodge decomposition theorem, it follows that

$$H^1(A,\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{C} = H^1(A,\mathbb{C}) = H^0(A,\Omega_A^1)\oplus\overline{H^0(A,\Omega_A^1)}$$

and the decomposition is compatible with the action of f^*. Thus the eigenvalues of $f^*|H^0(A,\mathbb{C})$ are α_j , $\overline{\alpha}_j$ $(1 \le j \le d)$, counted with multiplicities. Here $\alpha_1 := \alpha$ and α_j $(1 \le j \le d)$ are the Galois conjugates of α. As f is an automorphism, $f^*|H^0(A,\mathbb{Z})$ is invertible over $\mathbb{Z}$. Thus the determinant of $f^*|H^0(A,\mathbb{Z})$, which is $\pm\prod_j |\alpha_j|^2$, is ± 1. Hence $\prod_j \alpha_j$ is ± 1 as $\prod_j \alpha_j \in \mathbb{Z}$. This shows that α is a Pisot unit. As $\alpha = \alpha_1$ is a positive Pisot number, we may rename α_j so that

$$|\alpha_1| = \alpha > 1 > |\alpha_2| \ge \ldots \ge |\alpha_d| \ .$$

As A is an abelian variety, we have

$$H^{p,p}(A) = \wedge^p H^1(A,\Omega_A^1)\otimes_{\mathbb{C}} \wedge^p\overline{H^0(A,\Omega_A^1)} \ .$$

It follows that

$$d_k(f) = \prod_{j=1}^{k} |\alpha_j|^2 \ .$$

In particular,

$$d_1(f) = |\alpha_1|^2 \ , \ d_2(f) = |\alpha_1|^2|\alpha_2|^2 \ ,$$

and

$$h_{\mathrm{top}}(f) = 2\log\alpha > 0 \ ,$$

by $\alpha = |\alpha_1| > 1$ and $|\alpha_j| < 1$ for $j \ge 2$ and Theorem 2.2.

Lemma 3.5. *f is primitive.*

Proof. Assuming to the contrary that f is not primitive, we derive a contradiction.

Let $\pi : A \dashrightarrow B$ be an f-equivariant dominant rational map such that $0 < \dim B < \dim A$. We denote by d_j^B the j-th dynamical degree of $f_B \in \mathrm{Bir}\,(B)$ induced by f and d_j^F be the j-th relative dynamical degree of f with respect to π, or more precisely, the j-th relative dynamical degree of a resolution of indeterminacy of π (cf. Remark 2.5). Then, by the product formula (Theorem 2.7), we have

$$|\alpha_1|^2 = d_1(f) = \mathrm{Max}\,(d_0^B d_1^F, d_1^B d_0^F) = \mathrm{Max}(d_1^F, d_2^F) \ ,$$

$$|\alpha_1|^2|\alpha_2|^2 = d_2(f) = \mathrm{Max}_{i+j=2}(d_i^B d_j^F) \ge d_1^B d_1^F \ .$$

As $d_1^B \ge 1$ and $d_1^F \ge 1$, it follows that $d_1^B d_1^F \ge d_1^B, d_1^F$. Thus $d_2(f) \ge d_1(f)$, i.e.,

$$|\alpha_1|^2|\alpha_2|^2 \ge |\alpha_1|^2 \ ,$$

a contradiction to the fact that $|\alpha_2| < 1$ (and $|\alpha_1| \ne 0$). Hence f has to be primitive. □

This completes the proof of Theorem 3.4. □

Theorem 3.6. *Let d be an integer such that $d \ge 2$. Let $\alpha > 1$ be any Pisot unit of degree d, whose existence is guaranteed by Theorem 3.3. Let E be any elliptic curve. Then the d-dimensional abelian variety E^d admits a primitive automorphism f with*

$$h_{\mathrm{top}}(f) = 2\log\alpha > 0 \ .$$

Proof. Let
$$S_d(X) = X^d + a_d X^{d-1} + \ldots + a_2 X + a_1 \in \mathbb{Z}[X]$$
be the minimal polynomial of α. Note that $a_1 = \pm 1$ as α is a Pisot unit. Consider the matrix $M_d = (m_{ij}) \in M_d(\mathbb{Z})$ whose entries m_{ij} are 0 except
$$m_{dj} = -a_j \ , \ m_{i,i+1} = 1 \ (1 \le j \le d \ , \ 1 \le i \le d-1) \ ,$$
associated to the polynomial $S_d(X)$. For instance
$$M_2 = \begin{pmatrix} 0 & 1 \\ -a_1 & -a_2 \end{pmatrix} \ , \ M_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_1 & -a_2 & -a_3 \end{pmatrix} \ , \ M_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_1 & -a_2 & -a_3 & -a_4 \end{pmatrix} .$$
By definition of M_d, the characteristic polynomial of M_d is $S_d(X)$ and, as $a_1 = \pm 1$, we have $M_d \in \mathrm{GL}_d(\mathbb{Z})$. Let $(z_1, z_2, \ldots, z_d)$ be the standard complex coordinates of the universal cover $\mathbb{C}^d$ of E^d. Then, as $M_d \in \mathrm{GL}_d(\mathbb{Z})$, we can uniquely define the group automorphism $f \in \mathrm{Aut}_{\mathrm{group}}(E^d)$ by
$$f^*(z_1, z_2, \ldots, z_d)^t = M_d(z_1, z_2, \ldots, z_d)^t \ .$$
Here $(z_1, z_2, \ldots, z_d)^t$ is the transpose of $(z_1, z_2, \ldots, z_d)$. Then
$$f^*|H^0(E^d, \Omega^1_{E^d}) = M_d$$
under the basis $\langle dz_i \rangle_{i=1}^d$. Hence the characteristic polynomial of $f^*|H^0(E^d, \Omega^1_{E^d})$ is $S_d(X)$, the minimal polynomial of Pisot unit α of degree d. The result then follows from Theorem 3.4. □

The first part of Theorem 1.7 follows from Theorem 3.6.

Corollary 3.7. *Let d be an integer such that $d \ge 3$. Let $\alpha > 1$ be any Pisot unit of degree d, whose existence is guaranteed by Theorem 3.3. Then there is a d-dimensional smooth projective variety M, birational to a minimal Calabi-Yau variety, such that M admits a primitive automorphism f with $h_{\mathrm{top}}(f) = 2 \log \alpha > 0$.*

Proof. Let M be the blow up at the maximal ideals of the singular points of the quotient variety $\overline{M} := E^d/\langle -1_{E^d} \rangle$. Then $\overline{M}$ is a minimal Calabi-Yau variety as $d \ge 3$, and M is a smooth projective variety birational to $\overline{M}$. Moreover, the automorphism $f \in \mathrm{Aut}\,(E^d)$ in Theorem 3.4 descends to an automorphism $f_M \in \mathrm{Aut}\,(M)$ of M as $f \circ (-1_{E^d}) = (-1_{E^d}) \circ f$ and by the universality of the blow up. By Corollary 2.8, $d_p(f_M) = d_p(f)$ for all p. Thus, by Theorem 2.2, $h_{\mathrm{top}}(f_M) = h_{\mathrm{top}}(f)$, which is $2 \log \alpha$ by Theorem 3.6. f_M is also primitive by Lemma 3.8 below. □

Lemma 3.8. *Let U and V be smooth projective varieties of the same dimension d and $\mu : U \dashrightarrow V$ be a dominant rational map (necessarily generically finite). Let $f_U \in \mathrm{Bir}\,(U)$ and $f_V \in \mathrm{Bir}\,(V)$. Then, f_V is primitive if f_U is primitive and $\mu \circ f_U = f_V \circ \mu$.*

Proof. If $\pi : V \dashrightarrow B$ is an f_V-equivariant rational dominant map of connected fibers, then the Stein factorization of $\pi \circ \mu : U \dashrightarrow B$ is an f_U-equivariant rational dominant map of connected fibers. As f_U is primitive and $\dim V = \dim U = d$, it follows $\dim B = 0$ or $\dim B = d$. This means that f_V is primitive as well. □

Corollary 3.7 shows the second half part of Theorem 1.7.

Corollary 3.9. *Let α_d $(d = 3,\ 4,\ 5)$ be the first three smallest positive Pisot units of degree d respectively (Example 3.2). Then the logarithm $2\log\alpha_d$ $(d = 3,\ 4,\ 5)$ is realized as the topological entropy of a primitive automorphism of an abelian varieties of dimension $d = 3,\ 4,\ 5$ respectively. Moreover, $2\log\alpha_3$ is also realized as the topological entropy of a primitive automorphism of a 3-dimensional Calabi-Yau manifold and a 3-dimensional smooth rational variety, and $2\log\alpha_d$ $(d = 4,\ 5)$ is also realized as the topological entropy of a primitive automorphism of a d-dimensional smooth rationally connected variety R_d.*

Proof. In Theorem 3.4, we choose E to be $E_\omega = \mathbb{C}/\mathbb{Z} + \omega\mathbb{Z}$ $(\omega = (-1+\sqrt{-3})/2)$. Then $f_d := f \in \mathrm{Aut}\,(E_\omega^d)$ in Theorem 3.4, associated to α_d $(d = 3,\ 4,\ 5)$, satisfies the first requirement.

Let $\mu : R_d \to \overline{R}_d$ be the blow up at the maximal ideals of the singular points of the quotient variety $\overline{R}_d := E_\omega^d/\langle -\omega I_d\rangle$. Then R_d is a smooth projective variety. Moreover, the automorphism $f_d \in \mathrm{Aut}\,(E^d)$ descends to an automorphism $f_{R_d} \in \mathrm{Aut}\,(R_d)$ of M_d and f_{R_d} is primitive of positive topological entropy $2\log\alpha_d$, exactly for the same reason as in Corollary 3.7.

Let V_3 be the blow up at the maximal ideals of the singular points of the quotient variety $E_\omega^3/\langle \omega I_3\rangle$. Then V_3 is a smooth Calabi-Yau threefold and the automorphism f_{V_3} of V_3 induced by f_3 satisfies all the required properties.

R_3 is rational by [OT15], R_4 is unirational by [COV15] and R_5 is rationally connected by [KL09].

Here, we shall give an alternative uniform proof of rational connectedness of R_d $(d = 3,$ $4,\ 5)$, using the fact that $f_{R_d} \in \mathrm{Aut}\,(R_d)$ is primitive.

By construction, $\overline{R}_d$ $(d = 3,\ 4,\ 5)$ has numerically trivial canonical divisor and only isolated singular points which are the image of the fixed points of $\langle -\omega I_3\rangle$. Let $P \in \overline{R}_d$ be the image of the origin of E_ω^d. As $d \le 5$, $\overline{R}_d$ is klt but not canonical at P. Let $E \subset R_d$ be the exceptional divisor lying over P. Then we have

$$K_{R_d} \equiv -aE + E'\ ,\ a > 0\ ,$$

where E' is a divisor whose support lies over the singular points of $\overline{R}_d$ other than P.

Let $y \in R_d$ be a general point of R_d and set $x = \mu(y)$. As y is general, x is a smooth point of $\overline{R}_d$. Choose an ample divisor H of $\overline{R}_d$. Then, there is a positive integer m (may depends on y) and a complete intersection curves

$$\overline{C} = H_1 \cap H_2 \cap \ldots \cap H_{d-1}\ ,\ H_i \in |mH|$$

such that $\overline{C}$ is irreducible, $\overline{C} \ni x$, $\overline{C} \ni P$ and $\overline{C}$ contains no other singular points of $\overline{R}_d$. This is possible, as $\overline{R}_d$ has only finitely many singular points. Let $C \subset R_d$ be the strict transform of $\overline{C}$. Then

$$(K_{R_d}.C) = -a(E.C) + (E'.C) = -a(E.C) < 0\ .$$

Note that $y \in C$ as $x \in \overline{C}$ and x is a smooth point. Therefore, for a general point $y \in R_d$, there is an irreducible curve $C \subset R_d$ such that $y \in C$ and $(K_{R_d}.C) < 0$. Hence R_d is uniruled by the numerical criterion of the uniruledness due to Miyaoka-Mori ([MM86], see also [Ko96, Chap. IV, Theorem 1. 13]).

Now consider the maximal rationally connected fibration $\pi : R_d \dashrightarrow B$ of R_d. It is unique up to bitational equivalence ([KMM92], [Ko96, Chap. IV, Theorem 5.5]). In particular, π is f_{R_d}-equivariant. Here B is not uniruled by [GHS03]. Thus $\dim B < \dim R_d$ as R_d is uniruled. As f_{R_d} is primitive, it follows that $\dim B = 0$. Hence R_d is rationally connected. $\square$

The following question is yet open and is of its own interest (see [COV15] for some attempt):

Question 3.10. Are R_4 and R_5 constructed in our proof of Corollary 3.9 rational?

Note that R_1 and R_2 are obviously rational and R_3 is also rational ([OT15]) but if $d \geq 6$, then $\kappa(R_d) = 0$, hence R_d is never rational (even never uniruled), as $E_\omega^d/\langle -\omega I_d \rangle$ ($d \geq 6$) has only canonical singularities with numerically trivial canonical divisor.

4. Salem numbers and a criterion of the primitivity of automorphisms of a projective hyprkähler fourfold.

Our main result of this section is Theorem 4.6.

First, we recall the definition of Salem number (including quadratic units):

Definition 4.1. A polynomial $P(X) \in \mathbb{Z}[X]$ is called a *Salem polynomial* if it is irreducible over $\mathbb{Z}$, monic, of even degree $2d \geq 2$ and the complex zeroes of $P(x)$ are of the form $(1 \leq i \leq d-1)$:

$$a > 1 \ , \ 0 < a < 1 \ , \ \alpha_i \ , \ \alpha_{i+d-1} := \overline{\alpha}_i \in S^1 := \{z \in \mathbb{C} \,|\, |z| = 1\} \setminus \{\pm 1\} \ .$$

A *Salem number* is the largest real root $a > 1$ of a Salem polynomial $P(X)$ and we call the degree of $P(X)$ the degree of a. By definition, Salem numbers are always in $\overline{\mathbb{Z}}^{\times}$ and of even degree.

Example 4.2. Unlike Pisot numbers, it is unknown which is the smallest Salem number. The smallest known Salem number is the Lehmer number

$$\lambda_{\mathrm{Lehmer}} = 1.17628\ldots$$

Lehmer number is the real root > 1 of the following Salem polynomial of degree 10:

$$X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1 \ .$$

It is conjectured that the Lehmer number is the smallest Salem number and in fact it is the smallest one in degree ≤ 40 (see eg. the webpage [Mo03]). Recall that the first dynamical degree $d_1(f)$ of a smooth surface automorphism f is Salem number if $d_1(f) > 1$ (cf. Proposition 4.4). McMullen [Mc07, Theorem A.1] shows that the Lehmer number is the smallest Salem number among the first dynamical degrees $d_1(f) \neq 1$ of smooth surface automorohisms f ([Mc07, Theorem A.1]). He also shows that the Lehmer number is realized as $d_1(f)$ of a rational surface automorphism f and a projective K3 surface automorphism f ([Mc07], [Mc16]).

In this note, we frequently use the following elementary:

Lemma 4.3. *Let n be a positive integer and a is a Salem number of degree $2d$. Then so is a^n.*

Proof. Under the notation of the Galois conjugate of a in Definition 4.1, it is clear that a^n, $1/a^n$, α_i^n ($1 \le i \le 2d-2$) are Galois conjugate of a^n. As the elementary symmetric polynomials of X_i ($1 \le i \le m$) generate $\mathbb{Z}[X_1, \dots, Z_m]^{S_m}$ over $\mathbb{Z}$ as $\mathbb{Z}$-algebra, we have

$$P_n(X) := (X - a^n)(X - 1/a^n)(X - \alpha_1^n) \dots (X - \alpha_{2d-2}^n) \in \mathbb{Z}[X] \ .$$

As the roots of $P_n(X)$ are all Galois conjugate, the irredicible decomposition of $P_n(X)$ over $\mathbb{Z}$, or equivalently over $\mathbb{Q}$ by Gauss' lemma, is of the form $P_n(X) = Q_n(X)^k$ for some $k \ge 1$. As a^n is a unique simple root of $P_n(X)$ geater than 1, it follows that $k = 1$, i.e., $P_n(X)$ is irreducible over $\mathbb{Z}$. Thus $a^n > 1$ is also a Salem number of the same degree as a. □

Salem numbers naturally appear in the topological entropy of surface automorphisms:

Proposition 4.4. *Let S be a smooth projective surface and $f \in \mathrm{Aut}\,(S)$. Then, the characteristic polynomial of $f^*|H^2(S, \mathbb{Z})$ is the product of cyclotomic polynomials and at most one Salem polynomial of degree $\le \rho(S)$ counted with multiplicities. Here $\rho(S)$ is the Picard number of S. In particular, by Theorem 2.2, the first dynamical degree $d_1(f)$ is either a Salem number or 1 and the entropy $h_{\mathrm{top}}(f)$ is the logarithm of a Salem number if it is positive.*

Proof. This is well-known. See eg. [Og10] for a self-contained proof. □

Salem numbers also naturally appear as the dynamical degrees of automorphisms of projective hyperkähler manifolds:

Proposition 4.5. *Let M be a (not necessarily projective) hyperkähler manifold of dimension $2m$ and $f \in \mathrm{Aut}\,(M)$. Then, the p-th dynamical degree $d_p(f)$ is either 1 or a Salem number of degree $\le b_2(M)$. Moreover, $d_p(f) = d_{2m-p}(f) = d_1(f)^p$, $0 \le p \le m$, and $h_{\mathrm{top}}(f) = m \log d_1(f)$.*

Proof. See [Og09]. □

The main result of this section is the following:

Theorem 4.6. *Let M be a projective hyperkähler manifold of dimension 4 and $f \in \mathrm{Aut}\,(M)$. Assume that $d_1(f)$ is a Salem number of degree ≥ 6. Then f is primitive of positive topological entropy.*

Proof. By Theorem 2.2, $h_{\mathrm{top}}(f) > 0$ if $d_1(f)$ is a Salem number. So, it suffices to show that $d_1(f) = 1$ or $d_1(f)$ is of degree ≤ 4 if f is imprimitive.

Let $\pi : M \dashrightarrow B$ be an f-equivariant dominant rational map to a smooth projective variety, of connected fibers such that $0 < \dim B < \dim M = 4$. We denote by f_B the birational automorphism of B induced by f. Let $\mu : \tilde{M} \to M$ be a resolution of indeterminacy of π and $\tilde{\pi} := \pi \circ \mu : \tilde{M} \to B$ be the induced surjective morphism and $\tilde{f} := \mu^{-1} \circ f \circ \mu \in \mathrm{Bir}\,(\tilde{M})$ be the birational automorphism of $\tilde{M}$ induced by f. Then $\tilde{\pi}$ is $\tilde{f}$-equivariant fibrations.

Lemma 4.7. (1) *If $\dim B = 1$ or 3, then $d_1(f) = 1$.*
(2) *If $\dim B = 2$, then $d_1(f) = d_1(f_B) = d_1(\tilde{f}|\tilde{\pi})$.*

Proof. Put $d = d_1(f) \ge 1$. Note that $d_p(f) = d_p(\tilde{f})$ by Theorem 2.3.

Assume that $\dim B = 1$. Then, by Proposition 4.5 and the product formula (Theorem 2.7), we have

$$1 = d_4(f) = d_4(\tilde{f}) = d_1(f_B)d_3(\tilde{f}|\tilde{\pi}) \ .$$

Thus $d_1(f_B) = d_3(\tilde{f}|\tilde{\pi}) = 1$ as both are greater than or equal to 1. Then, again, by Proposition 4.5, the product formula and $d_p(f) = d_p(\tilde{f})$, we have

$$d = d_3(f) = \mathrm{Max}\,(d_0(f_B)d_3(\tilde{f}|\tilde{\pi}), d_1(f_B)d_2(\tilde{f}|\tilde{\pi})) = \mathrm{Max}\,(1, d_2(\tilde{f}|\tilde{\pi})) = d_2(\tilde{f}|\tilde{\pi}) \ ,$$

$$d = d_1(f) = \mathrm{Max}\,(d_0(f_B)d_1(\tilde{f}|\tilde{\pi}), d_1(f_B)d_0(\tilde{f}|\tilde{\pi})) = \mathrm{Max}\,(1, d_1(\tilde{f}|\tilde{\pi})) = d_1(\tilde{f}|\tilde{\pi}) \ .$$

Thus $d = d_2(\tilde{f}|\tilde{\pi}) = d_1(\tilde{f}|\tilde{\pi})$. Then, again, by Proposition 4.5, the product formula and $d_2(f) = d_2(\tilde{f})$, we have

$$d^2 = d_2(f) = \mathrm{Max}\,(d_0(f_B)d_2(\tilde{f}|\tilde{\pi}), d_1(f_B)d_1(\tilde{f}|\tilde{\pi})) = \mathrm{Max}\,(d_2(\tilde{f}|\tilde{\pi}), d_1(\tilde{f}|\tilde{\pi})) = d \ .$$

Hence $d^2 = d$ and therefore $d = 1$ by $d > 0$, as claimed.

Assume that $\dim B = 3$. Then, by the same computaions as in the case $\dim B = 1$, we have $d_3(f_B) = d_1(\tilde{f}|\tilde{\pi}) = 1$ and

$$d = d_3(f) = \mathrm{Max}\,(d_3(f_B)d_0(\tilde{f}|\tilde{\pi}), d_2(f_B)d_1(\tilde{f}|\tilde{\pi})) = \mathrm{Max}\,(1, d_2(f_B)) = d_2(f_B) \ ,$$

$$d = d_1(f) = \mathrm{Max}\,(d_0(f_B)d_1(\tilde{f}|\tilde{\pi}), d_1(f_B)d_0(\tilde{f}|\tilde{\pi})) = \mathrm{Max}\,(1, d_1(f_B)) = d_1(f_B) \ ,$$

and therefore

$$d^2 = d_2(f) = \mathrm{Max}\,(d_2(f_B)d_0(\tilde{f}|\tilde{\pi}), d_1(f_B)d_1(\tilde{f}|\tilde{\pi})) = \mathrm{Max}\,(d_2(f_B), d_1(f_B)) = d \ .$$

Thus $d = 1$, as claimed.

Assume that $\dim B = 2$. Then, by Proposition 4.5, the product formula and $d_p(f) = d_p(\tilde{f})$, we have

$$1 = d_4(f) = d_2(f_B)d_2(\tilde{f}|\tilde{\pi}) \ .$$

Thus $d_2(f_B) = d_2(\tilde{f}|\tilde{\pi}) = 1$. We also have

$$d = d_1(f) = \mathrm{Max}\,(d_0(f_B)d_1(\tilde{f}|\tilde{\pi}), d_1(f_B)d_0(\tilde{f}|\tilde{\pi})) = \mathrm{Max}\,(d_1(\tilde{f}|\tilde{\pi}), d_1(f_B)) \ ,$$

$$\begin{aligned} d^2 = d_2(f) &= \mathrm{Max}\,(d_0(f_B)d_2(\tilde{f}|\tilde{\pi}), d_1(f_B)d_1(\tilde{f}|\tilde{\pi}), d_2(f_B)d_0(\tilde{f}|\tilde{\pi})) \\ &= \mathrm{Max}\,(1, d_1(f_B)d_1(\tilde{f}|\tilde{\pi}), 1) = d_1(f_B)d_1(\tilde{f}|\tilde{\pi}) \ . \end{aligned}$$

As $d_1(\tilde{f}|\tilde{\pi}) \ge 1$ and $d_1(f_B) \ge 1$, it follows that $d = d_1(f_B) = d_1(\tilde{f}|\tilde{\pi})$ as claimed. $\square$

From now on, we may and will assume that

$$\dim B = 2 \ , \ 1 < d := d_1(f) = d_1(f_B) = d_1(\tilde{f}|\tilde{\pi}) \, .$$

The first essential point is to relate $d_1(f|\pi)$ with the first dynamical defree of a birational automorphism of good fibers of π. For this purpose, the following theorem due to Xie [Xi15, Theorem 1.4] is crucial:

Theorem 4.8. *Let B be a smooth projective surface and $g \in \mathrm{Bir}\,(B)$. We denote by the set of preperiodic point of g by S. More precisely, S is the set of closed points P of S such that there is a positve integer m such that $g^{m'}$ is defined at P for all integers $0 \le m' \le m$ and $g^m(P) = P$. Then S is Zariski dense in B if $d_1(f) > 1$.*

We denote by $\tilde{M}_P$ the fiber of $\tilde{\pi} : \tilde{M} \to B$ over $P \in B$.

Let $I(\tilde{f}^{\pm 1}) \subset \tilde{M}$ be the unioin of the indeterminacy sets of $\tilde{f}, \tilde{f}^{-1} \in \mathrm{Bir}\,(\tilde{M})$. As $\tilde{M}$ is smooth (hence normal), $I(\tilde{f}^{\pm 1})$ is of codimension ≥ 2.

Let Γ be the graph of $\tilde{f}$ and $p_i : \Gamma \to \tilde{M}$ $(i = 1, 2)$ be the natural projections. We define

$$\tilde{f}(I(\tilde{f}^{\pm 1})) := p_2 \circ p_1^{-1}(I(\tilde{f}^{\pm 1}))\ ,\ \tilde{f}^{-1}(I(\tilde{f}^{\pm 1})) := p_1 \circ p_2^{-1}(I(\tilde{f}^{\pm 1}))\ .$$

Finally, we denote by $\mathrm{Exc}(\mu) = \mu^{-1}(I(\pi)) \subset \tilde{M}$ the exceptional set of $\mu : \tilde{M} \to M$. Here $I(\pi) \subset M$ is the indeterminacy set of π and it is of codimension ≥ 2. These three subsets $I(\tilde{f}^{\pm 1})$, $\tilde{f}^{\pm 1}(I(\tilde{f}^{\pm 1}))$ and $\mathrm{Exc}(\mu)$ are proper closed Zariski subsets of $\tilde{M}$.

We can apply Theorem 4.8 to our (B, f_B) as we are now assuming $d_1(f_B) = d_1(f) > 1$, and find a preperiodic point $P \in S$ of f_B such that $\tilde{M}_P$ is a smooth projective surface and

$$\tilde{M}_P \not\subset I(\tilde{f}^{\pm 1}) \cup \tilde{f}^{\pm 1}(I(\tilde{f}^{\pm 1})) \cup \mathrm{Exc}(\mu)\ ,$$

as $\tilde{\pi}$ is proper and S is Zariski dense in B. We choose a positive integer m such that $f_B^m(P) = P$ and all $f_B^{m'}$ $(0 \leq m' \leq m)$ are defined at P as morphisms. As P is a preperiodic point, such m exists. By preperiodicity, we may replace m by its any positive multiples km $(k \in \mathbb{Z}_{>0})$.

As $\tilde{M}_P$ is of codimension 2 in $\tilde{M}$ and m depends on the choice of P, it is not *a priori* clear if one can choose such a P so that $\tilde{M}_P \not\subset I(\tilde{f}^{\pm m})$. However, we can show the following slightly weaker:

Lemma 4.9. *There is a positive integer k such that $\tilde{f}^{mk}|\tilde{M}_P : \tilde{M}_P \dashrightarrow \tilde{M}_P$ is a well-defined birational map.*

Proof. The essential point is that f itself is an automorphism.

Let $M_P := \mu(\tilde{M}_P)$. Then M_P is an irreducible surface on M and $\mu|\tilde{M}_P : \tilde{M}_P \to M_P$ is a birational morphism, as $\tilde{M}_P \not\subset \mu^{-1}(I(\pi)) = \mathrm{Exc}\,(\mu)$. Consider

$$A_k := f^{km}(M_P) \subset M$$

for each positive integer k. As $f \in \mathrm{Aut}\,(M)$, we have $A_k \simeq M_P$ by f^{km}. Note that both A_k and M_P are closed and irreducible.

If $A_k = M_P$, i.e., $f^{km}(M_P) = M_P$ for some positive integer k, then we are done. Indeed, the generic point η of the scheme M_P then maps to η by f^{km}. As $\mu|\tilde{M}_P : \tilde{M}_P \to M_P$ is birational by our choice of P, we have $\mu(\tilde{\eta}) = \eta$ and $\mu^{-1}(\eta) = \tilde{\eta}$. Here $\tilde{\eta}$ is the generic point of the scheme $\tilde{M}_P$. As $\tilde{f}^{km} = \mu^{-1} \circ f^{km} \circ \mu$, it follows that $\tilde{f}^{km}$ is defined at $\tilde{\eta}$ and $\tilde{f}^{km}(\tilde{\eta}) = \tilde{\eta}$. Hence $f^{km}|\tilde{M}_P$ is a well-defined birational automorphism of $\tilde{M}_P$, provided that $A_k = M_P$.

Assume that $A_1 \not\subset I(\pi)$. Then for any general point $Q \in M_P$, we have $Q \notin I(\pi)$ and $f^m(Q) \notin I(\pi)$. Then π is defined at Q so that $\pi(Q) = P$ and

$$\pi(f^m(Q)) = f_B^m(\pi(Q)) = f_B^m(P) = P\ ,$$

by the choice of P and m. As both A_1 and M_P are irreducible, Zariski closed, and of the same dimension, it follows that $A_1 = M_P$ and we are done.

If $A_1 \subset I(\pi)$, we consider A_2. If $A_2 \not\subset I(\pi)$, then the same argument above implies that $A_2 = M_P$ and we are done. If $A_2 \subset I(\pi)$, then we consider A_3 and continue. If this process stops at some k, i.e., if $A_k \not\subset I(\pi)$, then $A_k = M_P$ and we are done.

Now assume that this process never stops, that is, that $A_k \subset I(\pi)$ for all positive integers k. Then, as $\dim I(\pi) \le 4-2 = 2$ and $\dim A_k = 2$, all A_k are irreducible components of the Zariski closed subset $I(\pi)$. As $I(\pi)$ has only finitely many components, it follows that there are positive integers $k_1 < k_2$ such that

$$A_{k_1} = A_{k_2}\ ,\ \text{i.e.}\ ,\ f^{k_1 m}(M_P) = f^{k_2 m}(M_P)\ .$$

As f is an automorphism, it follows that $f^{km}(M_P) = M_P$, that is, $A_k = M_P$, for $k = k_2 - k_1$ and we are done. This completes the proof. □

By replacing f by f^{km} in Lemma 4.9, *we may and will assume that* $\tilde{f}|\tilde{M}_P \in \mathrm{Bir}\,(\tilde{M}_P)$. Remark here that $1 < d_1(f^{km}) = d_1(\tilde{f}^{km}|\tilde{\pi}) = d_1(f_B^{km})$ by Lemma 4.7 (2), applied for f^{km}, and $d_1(f^{km}) = d_1(f)^{km}$ by Theorem 2.3.

Lemma 4.10. $1 < d_1(\tilde{f}|\tilde{\pi}) = d_1(\tilde{f}|\tilde{M}_P)$.

Proof. The first inequality follows from our assumption above and the remark after that.

Let H and H_B be very ample divisors on $\tilde{M}$ and B respectively. As $(H_B^2)\tilde{M}_P \equiv \tilde{\pi}^*(H_B^2)$, we have

$$((\tilde{f}^n)^*(H).H.\tilde{\pi}^*(H_B^2))^{\frac{1}{n}} = (((\tilde{f}^n)^*H)|\tilde{M}_P).(H|\tilde{M}_P))^{\frac{1}{n}} \cdot (H_B^2)^{\frac{1}{n}}\ .$$

As H is very ample, we may choose $D_n \in |H|$, depending on n, such that D_n properly meets the bad loci

$$\cup_{0 \le q \le n}(I(\tilde{f}^{\pm q}) \cup I(\tilde{f}^q|\tilde{M}_P)^{\pm 1}))\ .$$

Then

$$((\tilde{f}^n)^*D_n)|\tilde{M}_P = (\tilde{f}^n|\tilde{M}_P)^*(D_n|\tilde{M}_P) = (\tilde{f}|\tilde{M}_P)^n)^*(D_n|\tilde{M}_P)\ .$$

Hence

$$((\tilde{f}^n)^*(H).H.\tilde{\pi}^*(H_B^2))^{\frac{1}{n}} = (((\tilde{f}|\tilde{M}_P)^n)^*(H|\tilde{M}_P)).(H|\tilde{M}_P))^{\frac{1}{n}} \cdot (H_B^2)^{\frac{1}{n}}\ .$$

By taking the limit under $n \to \infty$, we obtain $d_1(\tilde{f}|\tilde{\pi}) = d_1(\tilde{f}|\tilde{M}_P)$, as claimed. □

Lemma 4.11. $\kappa(F) = 0$ *for general fibers* F *of* $\tilde{\pi}$.

Proof. As $d_1(\tilde{f}|\tilde{M}_P) > 1$ by Lemma 4.10, $\kappa(\tilde{M}_P) = 0$ or $\kappa(\tilde{M}_P) = -\infty$ by Theorem 1.4 and Remark 1.5. As $\tilde{M}_P$ is a smooth fiber of the surjective morphism $\tilde{\pi}$ of relative dimension 2, we have $\kappa(F) = \kappa(\tilde{M}_P)$ for any smooth fiber F of $\tilde{\pi}$. As $\kappa(\tilde{M}) = \kappa(M) = 0$, $\tilde{M}$ is not covered by a rational curves, hence is not covered by surfaces of Kodaira dimension $-\infty$. Hence $\kappa(F) = 0$. □

To conclude the proof, we use the following theorem due to Amerik-Campana ([AC13, Théorème 3.6] in dimension 4) and Matushita-Zhang ([MZ13, Theorem 1.3] in any dimension):

Theorem 4.12. *Let* X *be a projective hyperkähler manifold having a domonant rational map* $p : X \dashrightarrow Y$ *with connected fibers such that* $\dim X/2 \le \dim Y < \dim X$ *and general fibers of* p *are not of general type. Then* $p : X \dashrightarrow Y$ *is birational to* $p' : X' \to Y'$*, where* X' *is a projective hyperkähler manifold and* $p' : X' \to Y'$ *is a holomorphic Lagrangian fibration. In particular, general fibers of* p' *are abelian varieties of dimension* $(\dim X)/2$.

By Lemma 4.11, we can apply Theorem 4.12 to our $\pi : M \dashrightarrow B$. Then general fibers of π, and hence general fibers of $\tilde{\pi}$, are birational to abelian surfaces. As both Kodaira dimension and irregularity are constant for smooth fibers of relative dimension two, the smooth fiber $\tilde{M}_P$ is also birational to an abelian surface. Indeed, a smooth projective variety X is birational to an abelian variety if $\kappa(X) = 0$ and $q(X) = \dim X$ by a fundamental result due to Kawamata ([Ka81]).

Let $\nu : \tilde{M}_P \to A$ be the minimal model of $\tilde{M}_P$. Then A is an abelian surface and the birational automorphism $\tilde{f}|\tilde{M}_P \in \mathrm{Bir}\,(\tilde{M}_P)$ induces a ν-equivariant automorphism $f_A \in \mathrm{Aut}\,(A)$ of A. We have $d_1(\tilde{f}|\tilde{M}_P) = d_1(f_A)$ by Theorem 2.3. By Theorem 2.2 (see also a remark after that), $d_1(f_A)$ is the spectral radius of $f_A^*|N^1(A)$. As $\mathrm{rank}\, N^1(A) = \rho(A) \le 4$ for a complex abelian surface, it follows that $d_1(f_A)$ is of degree ≤ 4.

As already observed, we have $d_1(f) = d_1(\tilde{f}|\tilde{\pi}) = d_1(\tilde{f}|M_P) = d_1(f_A)$. Hence $d_1(f)$ is of degree ≤ 4 and we are almost done.

Very precisely saying, as we replaced f by f^{km}, what we have shown here is that if f is imprimitive and of positive topological entropy, then $d_1(f^{km})$ is of degree ≤ 4 for some positive integers m, k.

However, $d_1(f^{km}) = d_1(f)^{km}$ by Theorem 2.3. As f is an automorphism and $d_1(f) > 1$, we know that $d_1(f)$ is a Salem number by Proposition 4.5. Thus, by Lemma 4.3, $d_1(f^{km}) = d_1(f)^{km}$ is also a Salem number of the same degree as $d_1(f)$. As $d_1(f^{km}) = d_1(f)^{km}$ is of degree ≤ 4, then so is $d_1(f)$. This completes the proof. □

5. Hilbert schemes of points and hyperkähler fourfolds with primitive automorphisms of positive topological entropy.

Let $S^{[n]} = \mathrm{Hilb}^n(S)$ be the Hilbert scheme of the 0-dimensional closed subsechemes of lenghts n of a smooth projective surface S. Then $S^{[n]}$ is a smooth projective variety of dimension $2n$ by Fogarty [Fo68]. Let $f \in \mathrm{Aut}\,(S)$. Then f naturally induces an automorphism, denoted by $f^{[n]} \in \mathrm{Aut}\,(S^{[n]})$, of $S^{[n]}$.

It is well known that $S^{[n]}$ is a projective hyperkähler manifold of dimension $2n$ if S is a projective K3 surface ([Fu83] for $n = 2$, [Be83] for arbitrary n). The universal cover $\widetilde{S^{[n]}}$ of $S^{[n]}$, which is of covering degree 2, is a (projective) Calabi-Yau manifold of dimension $2n$ if S is an Enriques surface ([OS11]). If S is a smooth rational surface, then $S^{[n]}$ is a smooth projective rational variety of dimension $2n$, as $S^{[n]}$ is birational to the symmetric product $\mathrm{Sym}^n\,(\mathbb{P}^2)$, the later of which is rational by classical invariant theory (see eg. [GKZ94, Chap. 4, Theorem 2.2] for details).

Proposition 5.1. *Let S be a projective K3 surface with an automorphism $f \in \mathrm{Aut}\,(S)$ such that $d_1(f)$ is a Salem number of degree ≥ 6. Then the automorphism $f^{[2]}$ of $S^{[2]}$ is primitive and of positive topological entropy.*

Proof. Let E be the exceptional divisor of the Hilbert-Chow morphism $S^{[2]} \to \mathrm{Sym}^2 S$. By [Be83], we have an isomorphism

$$H^2(S^{[2]}, \mathbb{Z}) \simeq H^2(S) \oplus \mathbb{Z}(E/2)$$

compatible with Hodge decomposition and the actions of $f^{[2]}$ and $f \oplus id_{\mathbb{Z}(E/2)}$. Thus, by Theorem 2.2,

$$d_1(f^{[2]}) = r_1(f^{[2]}) = r_1(f) = d_1(f)\ .$$

Thus $f^{[2]}$ is of positive topological entropy and it is primitive by Theorem 4.6. □

The following consequence may be of its own interest:

Theorem 5.2. (1) *Let M be a (not necessarily projective) hyperkähler fourfold and $f \in \mathrm{Aut}\,(M)$. Then*

$$h_{\mathrm{top}}(f) \ge 2 \log \lambda_{\mathrm{Lehmer}}\ ,$$

unless $h_{\mathrm{top}}(f) = 0$. Here $\lambda_{\mathrm{Lehmer}}$ is the Lehmer number (See Example 4.2).

(2) *There is a projective hyperkähler fourfold M with a primitive automorphism $f \in \mathrm{Aut}\,(M)$ of the smallest possible positive topological entropy*

$$h_{\mathrm{top}}(f) = 2 \log \lambda_{\mathrm{Lehmer}}\ .$$

Proof. By Proposition 4.5, $d_1(f) = 1$ or $d_1(f)$ is a Salem number. Guan [Gu01] shows that $3 \le b_2(M) \le 8$ or $b_2(M) = 23$. As $d_1(f) = r_1(f)$ by Theorem 2.2, $d_1(f)$ is then of degree ≤ 23. Thus, as remarked in Example 4.2, it follows from [Mo03] that

$$d_1(f) \ge \lambda_{\mathrm{Lehmer}}\ ,\ \text{unless}\ d_1(f) = 1\ .$$

Thus, again by Proposition 4.5, $h_{\mathrm{top}}(f) \ge 2 \log \lambda_{\mathrm{Lehmer}}$ unless $h_{\mathrm{top}}(f) = 0$. This proves (1).

As remarked in Example 4.2, McMullen [Mc16] shows that there is a projective K3 surface S with automorphism $f \in \mathrm{Aut}\,(S)$ such that $d_1(f)$ is Lehmer number. By Proposition 5.1, $(S^{[2]}, f^{[2]})$ gives then a desired example in (2). □

In the view of Proposition 4.5 and Theorem 5.2, it is interesting to ask the following:

Question 5.3. Let M be a (not necessarily projective) hyperkähler manifold of dimension $2m \ge 6$ and $f \in \mathrm{Aut}\,(M)$. Is then

$$h_{\mathrm{top}}(f) \ge m \log \lambda_{\mathrm{Lehmer}}\ ,$$

unless $h_{\mathrm{top}}(f) = 0$?

6. Smooth rational fourfolds with primitive automorphisms of positive topological entropy.

In this section, we show the following theorem fairly in an explicit way by reducing to certain projective hyperkähler fourfolds:

Theorem 6.1. *There is a 4-dimensional smooth projective rational variety with a primitive automorphism of positive topological entropy.*

Proof. Let E and F be mutually non-isogenous elliptic curves and $S := \mathrm{Km}\,(E \times F)$ be the Kummer K3 surface associated with the product abelain surface $E \times F$. We denote by ω_S a nowhere vanishing holomorphic 2-form on S. Let $\iota \in \mathrm{Aut}\,(S)$ be the automorphism of S of order 2, induced by $(-1_E, 1_F) \in \mathrm{Aut}\,(E \times F)$.

We call a surjective morphism $\varphi : S \to \mathbb{P}^1$ a Jacobian fibration if general fibers of φ are elliptic curves and φ admits at least one global section. We denote by $r(\varphi)$ the rank of the Mordell-Weil group $\mathrm{MW}\,(\varphi)$ of a Jacobian fibration $\varphi : S \to \mathbb{P}^1$. Note that $\mathrm{MW}\,(\varphi)$ is an abelian subgroup of $\mathrm{Aut}\,(S)$ of finite rank. We use the following properites of S:

Proposition 6.2. (1) *The involution ι is in the center of* $\mathrm{Aut}\,(S)$.
(2) *S admits two different Jacobian fibration $\varphi_i : S \to \mathbb{P}^1$ with $r(\varphi_i) = 4$ ($i = 1, 2$).*

Proof. See [Og89]. ☐

We consider the quotient surface $T = S/\langle\iota\rangle$.

Corollary 6.3. *The surface T is a smooth rational surface.*

Proof. Notice that ι is an involution with $\iota^*\omega_S = -\omega_S$. Then, by the local linearlization of ι at any fixed point shows that T is smooth. The fixed locus of ι consists of 8 smooth disjoint rational curves. Thus, $\kappa(T) = -\infty$ by the ramification formula. The iregularity of T is zero, as so is S. Hence T is rational by Castelnouvo's criterion of rationality of surfaces. ☐

Corollary 6.4. *S damits an automorphism $f_S \in \mathrm{Aut}\,(S)$ such that $d_1(f_S)$ is a Salem number of degree ≥ 6.*

Proof. This follows from Proposition 6.2 (2) and Theorem 6.5 below. Note that the degree of a Salem number is even so that if it is greater than 4, then greater than or equal to 6. Theorem 6.5 is stronger than [EOY14, Theorem 1.2] but the idea of proof is the same as [EOY14, Theorem 1.2]. We will give a proof of Theorem 6.5 in Section 8. ☐

Theorem 6.5. *Let V be a projective K3 surface. Assume that V admits a Jacobian fibration $\varphi : V \to \mathbb{P}^1$ with $r(\varphi) \geq r > 0$ and a different Jacobian fibration $\psi : V \to \mathbb{P}^1$ with $r(\psi) > 0$. Then V admits an automorphism $g \in \mathrm{Aut}\,(V)$ such that $d_1(g)$ is a Salem number of degree $> r$, strictly larger than r.*

Choose an automorphism $f_S \in \mathrm{Aut}\,(S)$ in Corollary 6.4. Then f_S induces an automorphism of $f_T \in \mathrm{Aut}\,(T)$ which is equivariant under the quotient morphism.

Now consider the Hilbert scheme $T^{[2]}$ and the automorphism $f_T^{[2]} \in \mathrm{Aut}\,(T^{[2]})$ induced by f_T.

Proposition 6.6. (1) *$T^{[2]}$ is a smooth projective rational fourfold.*
(2) *$f_T^{[2]} \in \mathrm{Aut}\,(T^{[2]})$ is primitive and of positive topological entropy.*

Proof. The first assertion (1) follows from Corollary 6.3 (See also Section 5).

We reduce the proof of (2) to the hyperkähler fourfold $S^{[2]}$. We denote by $f_S^{[2]} \in \mathrm{Aut}\,(S^{[2]})$ the automorphism of $S^{[2]}$ induced by f_S.

The quotient morphism $S \to T$ induces a dominant rational map $\nu : S^{[2]} \dashrightarrow T^{[2]}$. The map ν is equivariant under the automorphisms $f_S^{[2]}$ and $f_T^{[2]}$. As $d_1(f_S)$ is Salem number of degree ≥ 6, the automorphism $f_S^{[2]}$ is primitive and of positive entropy by Proposition 5.1. Thus so is $f_T^{[2]}$ by Lemma 3.5 and Corollary 2.8. ☐

Theorem 6.1 now follows from Proposition 6.6. ☐

7. Calabi-Yau fourfolds with primitive automorphisms of positive topological entropy.

In this section, we show the following theorem fairly in an explicit way, again by reducing to certain projective hyperkähler fourfolds:

Theorem 7.1. *There is a 4-dimensional Calabi-Yau manifold with a primitive automorphism of positive topological entropy.*

Proof. We construct a desired Calabi-Yau fourfold from an Enriques surface W in the following:

Proposition 7.2. *There is an Enriques surface W with an sutomorphism $f_W \in \mathrm{Aut}\,(W)$ such that $d_1(f_W)$ is a Salem number of degree* 10.

Proof. Let W be a smooth Enriques surface. Then, the lattice $(N^1(W), (*, **)_W)$ is isomorphic to the even unimodular root lattice E_{10} of signature $(1, 9)$. We identify

$$(N^1(W), (*, **)_W) = E_{10}$$

under some fixed isomorphism. We have then the natural group homomorphism

$$\rho : \mathrm{Aut}\,(W) \to \mathrm{O}^+(N^1(W)) = \mathrm{O}^+(E_{10}) \ , \ f \mapsto (f^{-1})^* \ .$$

Here $\mathrm{O}^+(E_{10})$ is the group of isometries of E_{10} preserving the positive cone, corresponding to the positive cone of $N^1(W)$, i.e., the connected component of $\{x \in N^1(W) \otimes \mathbb{R} \,|(x^2)_W > 0\}$ containing the ample classes.

Let W be a generic Enriques surface in the sense of Barth-Peters [BP83]. The period points of generic W form a dense subset of 10-dimensional period domain of Enriques surfaces (so there are plenty of generic W) and, under ρ, we have an isomorphism

$$\mathrm{Aut}\,(W) \simeq \mathrm{O}^+(E_{10})[2] \ .$$

Here $\mathrm{O}^+(E_{10})[2]$ is the 2-congruence subgroup of $\mathrm{O}^+(E_{10})$, i.e.,

$$\mathrm{O}^+(E_{10})[2] = \mathrm{Ker}\,(\mathrm{O}^+(E_{10}) \to \mathrm{GL}\,(E_{10} \otimes_{\mathbb{Z}} \mathbb{F}_2) \ ; \ f \mapsto f \bmod 2) \ .$$

We show that any generic Enriques surface has a desired automorphism.

Let κ is the (conjugate class of) Coexter element of $\mathrm{O}(E_{10})$, i.e., the product of the reflections R_{e_i} $(1 \le i \le 10)$, corresponding to (-2)-elements e_i $(1 \le i \le 10)$ forming the root basis of E_{10}. Then the spectral radius $r_1(\kappa)$ of κ on E_{10} is the Lehmer number, a Salem number of degree 10 (See eg. [Mc07]). As $[\mathrm{O}(E_{10}) : \mathrm{O}^+(E_{10})[2]] < \infty$, there is a positive integer such that

$$\kappa^n \in \mathrm{O}^+(E_{10})[2] \ .$$

We fix such an n. Then there is $f \in \mathrm{Aut}\,(W)$ such that $f^* = \kappa^n$ on $N^1(W) = E_{10}$, as W is generic. As $r_1(\kappa)$ is a Salem number of degree 10, so is $d_1(f) = r_1(f) = r_1(\kappa)^n$ by Lemma 4.3. This completes the proof. □

In what follows, we choose and fix W and $f_W \in \mathrm{Aut}\,(W)$ in Proposition 7.2.

Let us consider the Hilbert scheme $W^{[2]}$ and the induced automorphism $f_W^{[2]} \in \mathrm{Aut}\,(W^{[2]})$ by f_W. Let $u : M \to W^{[2]}$ be the universal cover of $W^{[2]}$, of covering degree 2. Then:

Proposition 7.3. (1) *M is a smooth projective Calabi-Yau fourfold.*
(2) *M adimits a primitive automorphism of positive entropy.*

Proof. The first assertion (1) follows from [OS11], as remarked in Section 5.

Let $\tau : S \to W$ be the universal cover of W. Then S is a projective K3 surface and τ is an étale covering of degree 2. As in Proposition 6.6, we reduce the proof of (2) to the hyperkähler fourfold $S^{[2]}$.

We denote by $\iota \in \mathrm{Aut}\,(S)$ the covering involution of τ. Let $f_S \in \mathrm{Aut}\,(S)$ be any one of the two lifts of $f_W \in \mathrm{Aut}\,(W)$. We denote by $f_S^{[2]} \in \mathrm{Aut}\,(S^{[2]})$ the automorphism of $S^{[2]}$ induced by f_S.

Lemma 7.4. $f_S^{[2]} \in \mathrm{Aut}\,(S^{[2]})$ *is a primitive automorphism of positive topological entropy.*

Proof. As τ is equivariant under f_S and f_W, we have $d_1(f_S) = d_1(f_W)$ by Corollary 2.8. Thus $d_1(f_S)$ is a Salem number of degree 10. Hence $d_1(f_S^{[2]})$ is primitive and of positive topological entropy by Proposition 5.1. □

Let $\mathrm{Sym}^2\,(S)$ be the symmetric product of S. We denote by

$$\overline{\Delta}_S \subset \mathrm{Sym}^2\,(S)\ ,\ \overline{\Gamma}_S \subset \mathrm{Sym}^2\,(S)$$

the image of the digonal $\Delta_S \subset S \times S$ and the image of the graph $\Gamma_S \subset S \times S$ of ι under the quotient morphism $S \times S \to \mathrm{Sym}^2\,(S)$ respectively. Note that

$$\overline{\Delta}_S \cap \overline{\Gamma}_S = \emptyset\ ,$$

as ι is a fixed point free involution. We denote by

$$a_1\ ,\ a_2 \in \mathrm{Aut}\,(\mathrm{Sym}^2\,(S))$$

the automorphisms of $\mathrm{Sym}^2\,(S)$ induced by $(\iota, id_S) \in \mathrm{Aut}\,(S \times S)$ and $(\iota, \iota) \in \mathrm{Aut}\,(S \times S)$ respectively. We define $\overline{\Delta}_W \subset \mathrm{Sym}^2\,(W)$ in the same way as $\overline{\Delta}_S$.

Then the natural morphism

$$\overline{\tau} : \mathrm{Sym}^2\,(S) \to \mathrm{Sym}^2\,(W)$$

is a finite Galois cover with Galois group $\langle a_1, a_2\rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$. From the construction, we have

$$\overline{\tau}^{-1}(\overline{\Delta}_W) = \overline{\Delta}_S \cup \overline{\Gamma}_S\ .$$

The following four morphisms

$$a_1|\overline{\Delta}_S : \overline{\Delta}_S \to \overline{\Gamma}_S\ ,\ a_1|\overline{\Gamma}_S : \overline{\Gamma}_S \to \overline{\Delta}_S\ ,\ a_2|\overline{\Delta}_S : \overline{\Delta}_S \to \overline{\Delta}_S\ ,\ a_2|\overline{\Gamma}_S : \overline{\Gamma}_S \to \overline{\Gamma}_S$$

are isomorphisms, the morphism

$$\overline{\tau}|\overline{\Delta}_S : \overline{\Delta}_S \to \overline{\Delta}_W = \overline{\Delta}_S/\langle a_2\rangle$$

is étale of degree 2, the morphism

$$\overline{\tau}|\overline{\Gamma}_S : \overline{\Gamma}_S \to \overline{\Delta}_W$$

is an isomorphism and

$$\overline{\tau}|\mathrm{Sym}^2\,(S) \setminus (\overline{\Delta}_S \cup \overline{\Gamma}_S) : \mathrm{Sym}^2\,(S) \setminus (\overline{\Delta}_S \cup \overline{\Gamma}_S) \to \mathrm{Sym}^2\,(W) \setminus \overline{\Delta}_W$$

is an étale Galois cover with Galois group $\langle a_1, a_2\rangle$.

Recall that the Hilbert-Chow morphism $\nu_S : S^{[2]} \to \mathrm{Sym}^2\,(S)$ is the blow up along $\overline{\Delta}_S$ and similarly for $W^{[2]}$. Then, the morphism $\overline{\tau}$ induces a rational dominant map

$$\tau^{[2]} : S^{[2]} \dashrightarrow W^{[2]}\ .$$

The indterminacy locus of $\tau^{[2]}$ is precisely $\nu_S^{-1}(\overline{\Gamma}_S) \simeq \overline{\Gamma}_S$.

Let $\tilde{M} \to S^{[2]}$ be the blow up of $S^{[2]}$ along $\nu_S^{-1}(\overline{\Gamma}_S)$. As $\overline{\Delta}_S \cup \overline{\Gamma}_S$ is stable under $\langle a_1, a_2 \rangle$, by the universal properity of the blow up, the automorphisms a_1 and a_2 of $\mathrm{Sym}^2(S)$ lift to automorphisms $\tilde{a}_1$ and $\tilde{a}_2$ of $\tilde{M}$ respectively.

The rational map $\tilde{\tau} : \tilde{M} \to W^{[2]}$ induced by $\tau^{[2]}$ is then a morphism. Moreover, $\tilde{\tau}$ is a finite Galois cover with Galois group $\langle \tilde{a}_1, \tilde{a}_2 \rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$, and the induced morphism $\tilde{M}/\langle \tilde{a}_2 \rangle \to W^{[2]}$ is étale of degree 2, by the description of the action of a_1 and a_2 above. Hence $\tilde{M}/\langle \tilde{a}_2 \rangle$ is also the universal cover of $W^{[2]}$ and we may and will identify

$$M = \tilde{M}/\langle \tilde{a}_2 \rangle \ .$$

As f_S is a lift of f_W to S, we have $f_S \circ \iota = \iota \circ f_S$, and therefore $f_S^{[2]} \circ \tilde{a}_2 = \tilde{a}_2 \circ f_S^{[2]}$. Thus $f_S^{[2]}$ decends to an automorphism $f_M \in \mathrm{Aut}\,(M)$ so that the quotient morphism $\tilde{M} \to M$ is equivariant under $f_S^{[2]}$ and f_M. By Corollary 2.8 and Lemma 3.8, f_M is primitive and of positive topological entropy, as so is $f_S^{[2]}$ by Lemma 7.4. This completes the proof of Proposition 7.3. □

Theorem 7.1 now follows from Proposition 7.3. □

Remark 7.5. The universal covering Calabi-Yau $2n$-folds of the Hilbert scheme $W^{[n]}$ of an Enriques surface W may have rich geometric structures. See [Ha15] for some recent interesting progress.

8. Appendix. Proof of Theorem 6.5.

Before entering the proof, we make a few words needed in the proof clear.

Definition 8.1. (1) We denote by $P_m \subset \mathbb{R}[X]$ the set of real monic polynomials of degree m. P_m is then isomorphic to the real affine algebraic set $\mathbb{R}^m$.

(2) A real cyclotomic polynomial (resp. a real Salem polynomial) is a monic polynomial, irreducible over $\mathbb{R}$, which divides a cyclotomic polynomial $C(X) \in \mathbb{Z}[X]$ (resp. a Salem polynomial $S(X) \in \mathbb{Z}[X]$) in $\mathbb{R}[X]$. Note that if a polynomial $P(X) \in \mathbb{Z}[X]$ is divided by a real cyclotomic polynomial (resp. a real Salem polynomial) in $\mathbb{R}[X]$, then $P(X)$ is divided by the corresponding cyclotomic polynomial $C(X)$ (resp. the corresponding Salem polynomial $S(X)$) in $\mathbb{Z}[X]$.

Let

$$G_r := \mathrm{SO}(N^1(V) \otimes_{\mathbb{Z}} \mathbb{R}) \cap \mathrm{Aut}\,(V)^* \ .$$

Here $\mathrm{Aut}\,(V)^* = \{f^*|N^1(V) \,|\, f \in \mathrm{Aut}\,(V)\}$. Note that G_r is a finite index subgroup of $\mathrm{Aut}^*(V)$ and therfore of infinite order. We define A^* similarly for a subgroup $A \subset \mathrm{Aut}\,(V)$.

The essential new point in our proof is the replacement of the pair $\mathcal{S} \subset N^1(V) = \mathrm{NS}\,(V)$ in [EOY14] to the following more refined pair $\mathcal{S}_r \subset L$ and work over $\mathbb{R}$ rather than $\mathbb{Z}$.

Here $\mathcal{S}_r$ is the subset of $N^1(V)$ consisting of the smooth fiber classes $e \in N^1(V)$ of Jacobian fibrations of Mordell-Weil rank $\geq r$ on V, and L is the primitive hull of the $\mathbb{Z}$-submodule of $N^1(V)$ generated by $\mathcal{S}_r$. We set $d := \mathrm{rank}\, L$.

The lattice L is G_r-stable and $L_{\mathbb{R}} := L \otimes_{\mathbb{Z}} \mathbb{R}$ is hyperbolic with respect to the intersection form $(*, **)_V$. Indeed, as $L_{\mathbb{R}}$ contains the fiber class f of the Jacobian fibration $\varphi : V \to \mathbb{P}^1$ and the fiber classes f_n of the Jacobian fibrations $r^{-n} \circ \varphi \circ r^n : V \to \mathbb{P}^1$, where $r \in \mathrm{MW}(\psi)$

is of infinite order and n is an integer. Then $f, f_n \in L$ are not proprtional for some n, as r is of infinite order.

There is a real hyperbolic vector subspace M of rank ≥ 2, of $L_{\mathbb{R}}$, such that M is irreducible under the action of

$$G_r \cap \mathrm{SO}^+(M) \ .$$

That is, M is $G_r \cap \mathrm{SO}^+(M)$-stable and has no real $G_r \cap \mathrm{SO}^+(M)$-stable vector subspace other than $\{0\}$ and M. Here $\mathrm{SO}^+(M)$ is the subgroup of $\mathrm{SO}(M)$ of finite index, preserving the postive cone of $M_{\mathbb{R}}$.

Lemma 8.2. *Such M actually exists.*

Proof. Choose an irreducible M. If M is of positive definite, then it is of rank 1 and $M^{\perp}_{L_{\mathbb{R}}}$ is of negative definite and it is G_r-stable possibly after replacing G_r by a finite index subgroup. However, then G_r would be of finite order, a contradiction. If M is of negative definite, then consider $M^{\perp}_{L_{\mathbb{R}}}$. This is hyperbolic, as it is not of positive definite observed above. Then we find a new irreducible hyperbolic M in $M^{\perp}_{L_{\mathbb{R}}}$ by induction of the rank. If M is parabolic, then there is a unique

$$\{0\} \neq \mathbb{R} \cdot x \subset M$$

such that $(x^2) = 0$. As $\mathbb{R} \cdot x$ is then $\mathrm{Aut}\,(V)^*$-stable, it follows that $M = \mathbb{R} \cdot x$. Choose $f \in \mathcal{S}_r$ such that x and f are linearly independent. This is possible as $|\mathcal{S}_r| = \infty$ by our assumption. Then for any $g \in \mathrm{MW}\,(\varphi_f)$, we have $g^*f = f$ and therefore $g^*x = x$, as $(x, f) \neq 0$. Thus $\mathrm{MW}\,(\varphi_f)^* = id$ on $\mathbb{R}\langle x, f\rangle$. As the orthogonal coplement is of negative definite and it is also $\mathrm{MW}\,(\varphi_f)^*$-stable, it follows that $|\mathrm{MW}\,(\varphi_f)| < \infty$, a contradiction to the fact that it is of rank $r > 0$. □

We choose and fix such an M. Note that in general M is not defined over $\mathbb{Z}$, i.e., there are no sublattice $M' \subset L$ such that $M = M' \otimes_{\mathbb{Z}} \mathbb{R}$.

Lemma 8.3. *There is $e \in \mathcal{S}_r$ such that $M \not\subset e^{\perp}_{L_{\mathbb{R}}} := \{x \in L_{\mathbb{R}} \,|\, (x, e)_V = 0\}$.*

Proof. As $L_{\mathbb{R}} = \mathbb{R}\langle \mathcal{S}\rangle$ and $L_{\mathbb{R}}$ is hyperbolic, inparticular, the intersection form $(*, **)$ on $L_{\mathbb{R}}$ is non-degenerate, we have $\cap_{e \in \mathcal{S}_r} e^{\perp}_{L_{\mathbb{R}}} = \{0\}$ and the result follows. □

We also choose and fix $e \in \mathcal{S}_r$ in Lemma 8.3. As $e \in \mathcal{S}_r$, we have

$$s := \mathrm{rank}\,\mathrm{MW}\,(\varphi_e) \geq r \ .$$

Here $\varphi_e : V \to \mathbb{P}^1$ is the Jacobian fibration $\varphi_e : V \to \mathbb{P}^1$ whose fiber class is e.

Lemma 8.4. *There are an integral basis*

$$\langle e, v_1, \cdots, v_{d-2}\rangle$$

of $e^{\perp} := \{x \in L \,|\, (x, e)_V = 0\} \subset L$, a $\mathbb{Q}$-basis

$$\langle e, v_1, \cdots, v_{d-2}, u\rangle$$

of $L_{\mathbb{Q}}$, with $(e, u) = 1$, and a finite index subgroup

$$H := \langle h_1, \cdots, h_s\rangle \simeq \mathbb{Z}^s$$

of $(\mathrm{MW}\,(\varphi_e)^* \cap \mathrm{SO}(L))$*, such that, under the* $\mathbb{Q}$*-basis of* $L_{\mathbb{Q}}$ *above,*

$$h_i = \begin{pmatrix} 1 & \mathbf{a}_i^t & c_i \\ \mathbf{0} & I_{d-2} & q_i\mathbf{e}_i \\ 0 & \mathbf{0}^t & 1 \end{pmatrix} ,$$

such that $q_i \neq 0$ *for all* i *with* $1 \le i \le s$*. Here* $\mathbf{e}_i$ *is the* i*-th unit vector and* I_{d-2} *is the* $(d-2) \times (d-2)$ *identity matrix.*

Moreover, there are $1 \le i, j \le s$ *such that* $\mathbf{a}_j^t \cdot \mathbf{e}_i \neq 0$*, and* $h_i|M \in \mathrm{SO}^+(M)$ *for all* i *with* $1 \le i \le s$*.*

Proof. Proof of [EOY14, Lemma 4.7, Lemma 4.8] works under obvious modifications. □

Lemma 8.5. $\dim M \ge s+2$.

Proof. As $M \not\subset e^{\perp}_{L_{\mathbb{R}}}$, there is $v \in M$ such that $v \notin e^{\perp}_{L_{\mathbb{R}}}$. As $v \in M \subset L_{\mathbb{R}}$ and $v \notin e^{\perp}_{L_{\mathbb{R}}}$, replacing v by non-zero real multiple if necessary, we can write

$$v = xe + \sum_{k=1}^{d-2} y_k v_k + u \ ,$$

under the basis in Lemma 8.4. Here x and y_k are real numbers. Then operaing h_i $(1 \le i \le s)$ in Lemma 8.4, we obtain

$$h_i(v) = v + (\mathbf{a}_i^t \cdot \mathbf{y} + c_i)e + q_i v_i \ .$$

Here $\mathbf{y} = (y_1, \dots, y_{d-2})^t$. As M is $G_r \cap \mathrm{SO}^+(M)$-stable, $h_i|M \in G_r \cap \mathrm{SO}^+(M)$ and $v \in M$, it follows that $h_i(v) - v \in M$, i.e.,

$$k_i := (\mathbf{a}_i^t \cdot \mathbf{y} + c_i)e + q_i v_i \in M \ .$$

Hence $h_j(k_i) - k_i \in M$, that is,

$$q_i(\mathbf{a}_j^t \cdot \mathbf{e}_i)e \in M \ ,$$

for all integers i, j such that $1 \le i, j \le s$. As $q_i \neq 0$ for all $1 \le i \le s$ and $\mathbf{a}_j^t \cdot \mathbf{e}_i \neq 0$ for some $1 \le i, j \le s$ by Lemma 8.4, it follows that $e \in M$. Then, by $k_i \in M$, $e \in M$ and again by $q_i \neq 0$, we have $v_i \in M$ for all $1 \le i \le s$ as well. Thus

$$e, v_1, v_2 \dots, v_s, v \in M \ .$$

These $s+2$ vectors are linearly independent, as $v \notin e^{\perp}_{L_{\mathbb{R}}}$. This implies the result. □

Let $g \in G_r \cap \mathrm{SO}^+(M)$. We denote by $\Phi_g(X) \in \mathbb{R}[X]$ the characteristic polynomial of the action of g on M. By definition of G_r, there is $\tilde{g} \in \mathrm{Aut}\,(V)$ such that $g = \tilde{g}^*|M$. We call such $\tilde{g}$ a lift of g. We denote by $\Phi_{\tilde{g}^*}(X) \in \mathbb{Z}[X]$ the characteristic polynomial of the action of $\tilde{g}$ on $N^1(V)$. By Proposition 4.4, the irreducible factors of $\Phi_{\tilde{g}^*}(X)$ over $\mathbb{Z}$ are cyclotomic polynomials of degree ≤ 20 and at most one Salem polynomial. Note that there are only finitely many cyclotomic polynomials of degree ≤ 20, say,

$$C_1(X), \dots, C_n(X)$$

and therefore only finitely many real cyclotomic polynomials which divide some $C_i(X)$ $(1 \le i \le n)$, say,

$$R_1(X), \dots, R_N(X) \ .$$

As $\Phi_g(X)|\Phi_{\tilde{g}^*}(X)$ in $\mathbb{R}[X]$ for any lift $\tilde{g}$ of g, the polynomial $\Phi_g(X)$ is the product of $R_i(X)$ $(1 \le i \le N)$ of multiplicity ≥ 1 and some real factors of a Salem polynomial $S(X)$, of multiplicity 1.

Now assume that dim $M = 2m$ is even. Then, thanks to [EOY14, Proposition 4.3], the same proof as [EOY14, Theorem 4.1] applied for the real algebraic morphism

$$\mathrm{SO}(M) \ni g \mapsto \Phi_g(X) \in P_{2m}$$

with irreducibility of M shows that there is $g \in G_r \cap \mathrm{SO}^+(M)$ such that $\Phi_g(X)$ has no real cyclotomic factor. Then we have a decomposition

$$\Phi_g(X) = \prod_{i=1}^{k} S_i(X) \ ,$$

where $S_i(X)$ $(1 \le i \le k)$ are mutually different real Salem polynomials. Let $\tilde{g} \in \mathrm{Aut}\,(V)$ be any lift of this g. Then $\Phi_{\tilde{g}^*}(X)$ has a Salem polynomial factor $S(X) \in \mathbb{Z}[X]$ such that

$$\prod_{i=1}^{k} S_i(X)|S(X)$$

in $\mathbb{R}[X]$. Thus, $S(X)$ is of degree $\ge 2m$, whence, $d_1(\tilde{g})$ is a Salem number of degree $> r$, as $2m \ge s + 2 \ge r + 2$.

Assume that dim $M = 2m + 1$ is odd. Then for each $g \in \mathrm{SO}(M)$, the characteristic polynomial $\Phi_g(X)$ is always divisible by $(X - 1)$. Then we can consider the modified real algebraic morphism

$$\mathrm{SO}(M) \ni g \mapsto \Phi_g(X)/(X - 1) \in P_{2m} \ .$$

Then, again thanks to [EOY14, Proposition 4.3], in the same way as in the case dim $M = 2m$ above, we finds $g \in G_r \cap \mathrm{SO}^+(M)$ such that $\Phi_g(X)/(X - 1)$ is the product of real Salem polynomials of multiplicity 1. Let $\tilde{g} \in \mathrm{Aut}\,(V)$ be any lift of this g. Then for the same reason as above, the characteristic polynomial $\Phi_{\tilde{g}^*}(X)$ has a Salem polynomial factor $S(X) \in \mathbb{Z}[X]$ of degree $\ge 2m$, and therefore, $d_1(\tilde{g})$ is a Salem number of degree $> r$, as $2m \ge s + 1 \ge r + 1$.

This completes the proof of Theorem 6.5.

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