EXTREMAL VARIETIES 3-RATIONALLY CONNECTED BY CUBICS, QUADRO-QUADRIC CREMONA TRANSFORMATIONS AND RANK 3 JORDAN ALGEBRAS

LUC PIRIO AND FRANCESCO RUSSO

ABSTRACT. For any $n \geq 3$, we prove that there are equivalences between

- irreducible n-dimensional non degenerate projective varieties $X \subset \mathbb{P}^{2n+1}$ different from rational normal scrolls and 3-covered by rational cubic curves, up to projective equivalence;
- quadro-quadric Cremona transformations of $\mathbb{P}^{n-1}$, up to linear equivalence;
- n-dimensional complex Jordan algebras of rank three, up to isotopy.

We also provide some applications to the classification of particular classes of varieties in the class defined above and of quadro-quadric Cremona transformations, proving also a structure theorem for these birational maps and for varieties 3-covered by twisted cubics by reinterpreting for these objects the solvability of the radical of a Jordan algebra.

INTRODUCTION

In this paper we continue the study began in [31] of the unexpected relations between the following three sets: n-dimensional complex Jordan algebras of rank three modulo isotopy; irreducible n-dimensional projective varieties $X \subset \mathbb{P}^{2n+1}$ such that through three general points there passes a twisted cubic contained in it modulo projective equivalence; quadro-quadric Cremona transformations in $\mathbb{P}^{n-1}$ modulo linear equivalence.

Jordan algebras have been introduced by physicists around 1930 in the attempt of discovering a non-associative algebraic setting for quantum mechanics. These algebras found later applications in many different areas of mathematics, spanning from Lie algebras and group theory to real and complex differential geometry, see for example [26, Part I] for a general panorama. In algebraic geometry, complex Jordan algebras of rank three were used to construct projective varieties with notable geometric properties either by considering some determinantal varieties associated to the simple finite dimensional ones such as Severi varieties, see [40, IV.4.8], or by defining the so called *twisted cubic over a rank three Jordan algebra*, see [18], [27] [31, Section 4], [22],[23] and Section 3 below. These last objects are examples of projective varieties such that through three general points there passes a twisted cubic contained in it and they also appear as the first exceptional examples to the classification of extremal varieties m–covered by rational curves of fixed degree, see [33] and [31] for definitions and examples and also Section 1 and Section 3. Moreover, twisted cubic over rank three Jordan algebras are also examples of varieties with one apparent double point, see [27], [31, Corollary 5.4] and [10], and the smooth ones are also *Legendrian* varieties, see [27] and [23].

Quadro-quadric Cremona transformations can be considered as the simplest examples of birational maps of a projective space different from linear automorphisms. In the plane these transformations are completely classified and together with projective automorphisms generate the group of birational maps of $\mathbb{P}^2$. In low dimension they were studied classically by the Italian school, see for example [13] and the references therein, and soon later by Semple [35]. These results were reconsidered recently in [28], where the classification in $\mathbb{P}^3$ originally outlined in [13] is completed, see also [5]. In [15] it is proved the surprising and nice result that there are only four examples of quadro-quadric Cremona transformations with smooth irreducible base locus. These four examples are related to the so called Severi varieties and are linked to the four simple complex Jordan algebras of hermitian 3×3 matrices with coefficients in the complexification of the four real division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$, see [15], [40], and also [17], [9] and Corollary 5.9 here.

The main results of the paper, collected in Theorem 4.1 and in the related diagram, assert that the three sets described above are in bijection and that the composition of two of these bijections is the identity map. This correspondence, which we call "XJC*-correspondence*", was conjectured in the final remarks of [31] and it is based on the following results: every quadro–quadric Cremona transformation of $\mathbb{P}^{n-1}$ is linearly equivalent to an involution which is the adjoint of a rank 3 Jordan algebra of dimension n (Theorem 3.4); every irreducible n-dimensional variety $X^n \subset \mathbb{P}^{2n+1}$ which is 3–covered by twisted cubics and different from a rational normal scroll is projectively equivalent to a twisted

cubic over a rank three complex Jordan algebra (Theorem 3.7). Some particular versions of the XJC-correspondence are the following: cartesian products of varieties 3–covered by twisted cubics correspond to direct product Jordan algebras of rank three and to the so called elementary quadratic transformations (Proposition 4.3); smooth varieties 3–covered by twisted cubics, modulo projective equivalence, are in bijection with semi-simple rank three Jordan algebras, modulo isotopy, and with *semi–special* quadro-quadric Cremona transformations, modulo linear equivalence (Theorem 4.4).

The XJC–correspondence is extended in Section 4.3 to cover some degenerated cases: rational normal scrolls, Jordan algebras with a cubic norm and 'fake' quadro-quadric Cremona transformations, respectively. Moreover, the XJC–correspondence leads us to some new constructions and definitions. The theory of the radical and the semi–simple part of a Jordan algebra suggested the definitions of semi–simple part, semi–simple rank and semi-simple dimension of a quadro-quadric Cremona transformation, or of an extremal variety 3–covered by twisted cubics, providing for instance a general Structure Theorem for these maps, see Theorem 5.16. As an application we prove in Corollary 5.9 that every homaloidal polynomial f of degree 3 defining a quadro-quadric Cremona transformation whose ramification locus scheme is cut out by f is, modulo linear equivalence, the norm of a rank 3 semi-simple Jordan algebra, providing a new short proof of [17, Theorem 3.10] and of [9, Theorem 2, Corollary 4].

The paper is organized as follows. In Section 1 we introduce some notation which is not standard. In Section 2 we define precisely the objects studied giving some examples: the X-world consisting of extremal varieties 3–covered by twisted cubics; the C–world consisting of quadro-quadric Cremona transformations and the J–world consisting of rank three complex Jordan algebras. Moreover the natural equivalence relations: projective equivalence, linear equivalence, respectively isotopy are introduced as well the notion of cubic Jordan pair. In Section 3 we define the correspondences between the three sets modulo equivalences. First from the J–world to the X and C worlds. Then from the C–world to the X–world. We prove the equivalence between the C and J worlds in Theorem 3.4 while the equivalence between the X and J worlds is proved in Theorem 3.7. The XJC-correspondence and its particular forms recalled above are stated in Section 4 while Section 5 is devoted to the Structure Theorem of quadro-quadric Cremona transformation, Theorem 5.16, and to the reinterpretation of the theory of the solvability of the radical of a Jordan algebra in the C–world and in the X–world.

1. Notation

If V is a complex vector space of finite dimension and if $A \subset V$ is a subset, then $\langle A \rangle$ denotes the smallest linear subspace of V containing A, analogous notions being defined in $\mathbb{P}(V)$. The projective equivalence class of $x \in V \setminus \{0\}$ is the element $[x] \in \mathbb{P}(V)$. Let P_1, P_2 be two projective subspaces in $\mathbb{P}^N$. When $P_1 \cap P_2 = \emptyset$, we define their *direct sum* as $P_1 \oplus P_2 = \langle P_1, P_2 \rangle \subset \mathbb{P}^N$.

We shall consider (irreducible) algebraic varieties defined over the complex field. If X is an irreducible algebraic variety and if $n = \dim(X)$, we shall write $X = X^n$ or simply X^n. We denote by $[X]$ the projective equivalence class of an irreducible projective variety $X \subset \mathbb{P}^N$. We shall indicate by $(X)^m$ the m-times cartesian product $X \times \cdots \times X$. We denote by $T_x X$ the embedded projective tangent space to $X \subset \mathbb{P}^N$ at a smooth point x o f X while $T_{X,x}$ indicates the abstract tangent space to X at x.

The irreducible quadric hypersurface in $\mathbb{P}^{r+1}$ is denoted by Q^r while $v_3(\mathbb{P}^1) \subset \mathbb{P}^3$ is the twisted cubic curve.

2. The objects

2.1. **The X-world: varieties $X^n(3,3)$.** An irreducible projective variety $X = X^n \subset \mathbb{P}^N$ is said to be 3-*rationally connected by cubic curves* (*3-RC by cubics* for short) if for a general 3-uplet of points $x = (x_i)_{i=1}^3 \in (X)^3$, there exists an irreducible rational cubic curve included in X that passes through x_1, x_2 and x_3.

If $X \subset \mathbb{P}^N$ is 3–RC by cubics, then projecting X from a general projective tangent space $T_x X$ we get an irreducible variety $Y^{n-\delta} \subset \mathbb{P}^{N-n-1}$, $\delta \geq 0$, such that through two general points there passes a line contained in $Y^{n-\delta}$. This immediately implies $Y = \mathbb{P}^{n-\delta}$ so that:

$$\dim\langle X \rangle \leq 2n + 1 - \delta \leq 2n + 1, \tag{1}$$

see also [33, Section 1.2] for more general results and formulations.

We will say that a variety $X \subset \mathbb{P}^N$ 3–RC by cubics is *extremal* if $N = \dim\langle X \rangle = 2n + 1$. In what follows, we shall use the notation $X = X^n(3,3)$ when $X \subset \mathbb{P}^{2n+1}$ is an extremal variety which is 3–RC by cubics.

Thus for $X = X^n(3,3) \subset \mathbb{P}^{2n+1}$ and for two general points $x_1, x_2 \in X$, we have

$$\mathbb{P}^{2n+1} = \langle X \rangle = T_{x_1} X \oplus T_{x_2} X,$$

see also [33, Lemme 1.3].

Example 2.1. (1) There exists a unique 3-RC curve $X^1(3,3)$: the twisted cubic cubic curve $v_3(\mathbb{P}^1) \subset \mathbb{P}^3$;

(2) Let Q be an irreducible hyperquadric in $\mathbb{P}^n$. It is well-known that Q is 3-RC by conics and since $\mathbb{P}^1$ is 3-covered by lines(!), it immediately follows that the Segre product $\mathrm{Seg}(\mathbb{P}^1 \times Q) \subset \mathbb{P}^{2n+1}$ is 3-RC by cubics so that $\mathrm{Seg}(\mathbb{P}^1 \times Q) = X^n(3,3)$. These examples produce a family of $X^n(3,3)$ for every $n \geq 2$;

(3) Let $(\Pi_i)_{i=1}^3$ be a 3-uple of elements of the grassmannian variety $\mathbb{G}(2,5) = G_3(\mathbb{C}^6) \subset \mathbb{P}^{19}$, Plücker embedded. If the Π_i's are general, one can find a basis $(u_i)_{i=1}^6$ of $\mathbb{C}^6$ such that $\Pi_1 = u_1 \wedge u_2 \wedge u_3$, $\Pi_2 = u_4 \wedge u_5 \wedge u_6$ and $\Pi_3 = (u_1 + u_4) \wedge (u_2 + u_5) \wedge (u_3 + u_6)$. Then $s \mapsto (u_1 + su_4) \wedge (u_2 + su_5) \wedge (u_3 + su_6)$ extends to a morphism $\varphi : \mathbb{P}^1 \to G_3(\mathbb{C}^6)$ such that $\varphi(0) = \Pi_1, \varphi(\infty) = \Pi_2$ and $\varphi(1) = \Pi_3$. The curve $\varphi(\mathbb{P}^1) \subset \mathbb{G}(2,5) \subset \mathbb{P}^{19}$ in the Plücker embedding is a twisted cubic, showing that $\mathbb{G}(2,5) = X^9(3,3)$;

(4) The n-dimensional rational normal scrolls $S_{1\ldots13} \subset \mathbb{P}^{2n+1}$, $n \geq 1$, and $S_{1\ldots122} \subset \mathbb{P}^{2n+1}$, $n \geq 2$, are classical examples of $X^n(3,3)$, which we shall call *degenerated examples*, see [33] for the explanation of the terminology and also Section 3.3 below.

We shall denote by $\boldsymbol{X}^n(3,3)$ the set of irreducible non-degenerate varieties $X^n \subset \mathbb{P}^{2n+1}$ which are 3-RC by twisted cubics and which are not degenerated in the above sense, *i.e.* that are different from $S_{1\ldots13}$ or $S_{1\ldots122}$. The description of the projective equivalence classes of elements in $\boldsymbol{X}^n(3,3)$ is a natural geometrical problem already considered in [31], see also [33] for general classification results of this kind. Indeed, this problem naturally appears when trying to solve the question on which maximal rank webs are algebraic, a central problem in web geometry, see [34].

Remind that if $X \in \boldsymbol{X}^n(3,3)$, we shall denote by $[X]$ its projective equivalence class.

2.2. **The C-world: Cremona transformations of bidegree $(2,2)$.** Let $f : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-1}$ be a rational map. There exist a unique integer $d \geq 1$ and $f_i \in |\mathcal{O}_{\mathbb{P}^{n-1}}(d)|$, $i = 1, \ldots, n$, with $\gcd(f_1, \ldots, f_n) = 1$ such that

$$f(x) = \big[f_1(x) : \cdots : f_n(x)\big]$$

for $x \in \mathbb{P}^{n-1}$ outside the *base locus scheme* $\mathcal{B} = V(f_1, \ldots, f_n) \subset \mathbb{P}^{n-1}$ of f. By definition, the *degree* of f is $\deg(f) = d$. We will denote by $F : \mathbb{C}^n \to \mathbb{C}^n$ the homogeneous affine polynomial map defined by $F(x) = (f_1(x), \ldots, f_n(x))$ for $x \in \mathbb{C}^n$. Note that the projectivization of F is of course the rational map f and that F depends on f only up to multiplication by a nonzero constant.

A rational map $f : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-1}$ is *birational* (or is a *Cremona transformation*) if it admits a rational inverse $f^{-1} : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-1}$. In this case, one defines the *bidegree* of f as $\mathrm{bideg}(f) = (\deg f, \deg f^{-1})$. In this paper we will mainly consider *quadro-quadric* Cremona transformations, that is Cremona transformations of bidegree $(2,2)$. The set of such birational maps of $\mathbb{P}^{n-1}$ will be indicated by $\mathbf{Bir}_{2,2}(\mathbb{P}^{n-1})$.

Example 2.2. (1) The *standard involution* of $\mathbb{P}^{n-1}$ is the birational map

$$[x_1 : x_2 : \ldots : x_n] \longmapsto \big[x_2x_3\ldots x_n : x_1x_3\ldots x_n : \ldots : x_1x_2\ldots x_{n-1}\big].$$

It has bidegree $(n-1, n-1)$ and it is an *involution*, that is $f = f^{-1}$ or equivalently $f \circ f$ is equal to the identity of $\mathbb{P}^{n-1}$ as a rational map;

(2) Assume that $x \mapsto (\ell_0(x), \ldots, \ell_n(x))$ is a linear automorphism of $\mathbb{C}^n$. Then for any nonzero linear form $\ell : \mathbb{C}^n \to \mathbb{C}$, the map $x \mapsto [\ell(x)\ell_0(x) : \cdots : \ell(x)\ell_n(x)]$ is a birational map. With the previous definitions it is a birational map of bidegree $(1,1)$ but we shall consider such a map as a *fake quadro-quadric Cremona transformation*, see Section 4.3;

(3) Let $Q^{n-1} \subset \mathbb{P}^n$ be an irreducible hyperquadric. Given $p \in Q_{\mathrm{reg}}$, the projection from p induces a birational map $\pi_p : Q \dashrightarrow \mathbb{P}^{n-1}$. For $p, p' \in Q_{\mathrm{reg}}$ with $p' \notin T_pQ$, the composition $\pi_{p'} \circ \pi_p^{-1} : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-1}$ is a birational map of bidegree $(2,2)$, called an *elementary quadratic transformation*;

(4) Let J be a finite dimensional power-associative algebra. The *inversion* $x \dashrightarrow x^{-1}$ induces a birational involution $j : \mathbb{P}(J) \dashrightarrow \mathbb{P}(J)$. If J has rank r, see 2.3 for the definitions, then j is of bidegree $(r-1, r-1)$.

For simplicity, we denote by V the vector space $\mathbb{C}^n$ in the lines below.

Let $f_1, \dots, f_n$ and $g_1, \dots, g_n$ be quadratic forms on V defining the affine polynomial maps $F = (f_1, \dots, f_n) : V \to V$, respectively $G = (g_1, \dots, g_n) : V \to V$. Let $f : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-1}$, respectively $g : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-1}$, be the induced rational maps. Then $g = f^{-1}$ as rational maps if and only if there exist homogeneous cubic forms $N, M \in \mathrm{Sym}^3(V^*)$ such that, for every $x, y \in V$:

$$G\big(F(x)\big) = N(x)\, x \qquad \text{and} \qquad F\big(G(y)\big) = M(y)\, y. \tag{2}$$

In the previous case, one easily verifies that for every $x, y \in V$, we also have

$$M\big(F(x)\big) = N(x)^2 \qquad \text{and} \qquad N\big(G(y)\big) = M(y)^2. \tag{3}$$

Two Cremona transformations $f, \tilde{f} : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-1}$ are said to be *linearly equivalent* (or just *equivalent* for short) if there exist projective transformations $\ell_1, \ell_2 : \mathbb{P}^{n-1} \to \mathbb{P}^{n-1}$ such that $\tilde{f} = \ell_1 \circ f \circ \ell_2$. This is an equivalence relation on $\mathbf{Bir}_{2,2}(\mathbb{P}^{n-1})$ and in the sequel we shall investigate the quotient space $\mathbf{Bir}_{2,2}(\mathbb{P}^{n-1})/_{\substack{linear\\ equivalence}}$ and its various incarnations.

If $f \in \mathbf{Bir}_{2,2}(\mathbb{P}^{n-1})$ we will denote by $[f]$ its linear equivalence class.

2.3. The J-world: Jordan algebras and Jordan pairs of degree 3.

By definition, a *Jordan algebra* is a commutative complex algebra $\mathbb{J}$ with a unity e such that the *Jordan identity*

$$x^2(xy) = x(x^2y) \tag{4}$$

holds for every $x, y \in \mathbb{J}$ (see [20, 26]). Here we shall also assume that $\mathbb{J}$ is finite dimensional. It is well known that a Jordan algebra is power-associative. By definition, the *rank* $\mathrm{rk}(\mathbb{J})$ of $\mathbb{J}$ is the complex dimension of the (associative) subalgebra $\langle x \rangle$ of $\mathbb{J}$ spanned by the unity e and by a general element $x \in \mathbb{J}$. A general element $x \in \mathbb{J}$ is invertible, *i.e.* for x in an open nonempty subset of $\mathbb{J}$, there exists a unique $x^{-1} \in \langle x \rangle$ such that $xx^{-1} = e = x^{-1}x$.

Example 2.3. (1) Let A be a non-necessarily commutative associative algebra with a unity. Denote by A^+ the vector space A with the symmetrized product $a \cdot a' = \frac{1}{2}(aa' + a'a)$. Then A^+ is a Jordan algebra. Note that $A^+ = A$ if A is commutative.

(2) Let $q : W \to \mathbb{C}$ be a quadratic form on the vector space W. For $(\lambda, w), (\lambda', w') \in \mathbb{C} \oplus W$, the product $(\lambda, w) * (\lambda', w') = (\lambda\lambda' - q(w, w'), \lambda w' + \lambda' w)$ induces a structure of rank 2 Jordan algebra on $\mathbb{C} \oplus W$ with unity $e = (1, 0)$.

(3) Let $\mathbb{A}$ be the complexification of one of the four Hurwitz's algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$ and denote by $\mathrm{Herm}_3(\mathbb{A})$ the algebra of Hermitian 3×3 matrices with coefficients in $\mathbb{A}$:

$$\mathrm{Herm}_3(\mathbb{A}) = \left\{ \begin{pmatrix} r_1 & \overline{x_3} & \overline{x_2} \\ x_3 & r_2 & \overline{x_1} \\ x_2 & x_1 & r_3 \end{pmatrix} \;\middle|\; \begin{array}{c} x_1, x_2, x_3 \in \mathbb{A} \\ r_1, r_2, r_3 \in \mathbb{C} \end{array} \right\}.$$

Then the symmetrized product $M \bullet N = \frac{1}{2}(MN + NM)$ induces on $\mathrm{Herm}_3(\mathbb{A})$ a structure of rank 3 Jordan algebra.

A Jordan algebra of rank 1 is isomorphic to $\mathbb{C}$ (with the standard multiplicative product). It is a classical result that any rank 2 Jordan algebra is isomorphic to an algebra as in Example 2.3.(2). In this paper, we will mainly consider Jordan algebras of rank 3. These are the simplest Jordan algebras which have not been yet classified in arbitrary dimension.

Let $\mathbb{J}$ be a rank 3 Jordan algebra. The general theory specializes in this case and ensures the existence of a linear form $T : \mathbb{J} \to \mathbb{C}$ (the *generic trace*), of a quadratic form $S \in \mathrm{Sym}^2(\mathbb{J}^*)$ and of a cubic form $N \in \mathrm{Sym}^3(\mathbb{J}^*)$ (the *generic norm*) such that

$$x^3 - T(x)x^2 + S(x)x - N(x)e = 0 \tag{5}$$

for every $x \in \mathbb{J}$. Moreover, x is invertible in $\mathbb{J}$ if and only if $N(x) \neq 0$ and in this case $x^{-1} = N(x)^{-1}x^{\#}$, where $x^{\#}$ stands for the *adjoint* of x defined by $x^{\#} = x^2 - T(x)x + S(x)e$. The adjoint satisfies the identity:

$$\big(x^{\#}\big)^{\#} = N(x)x.$$

Example 2.4. (1) The algebra $M_3(\mathbb{C})$ of 3×3 matrices with complex entries is associative. Then $M_3(\mathbb{C})^+$ is a rank 3 Jordan algebra. If $M \in M_3(\mathbb{C})$, the generic trace of M is the usual trace, the norm is the determinant of M and the adjoint is the classical one, that is the transpose of the cofactor matrix of M.

(2) Let $\mathbb{C} \oplus W$ be a rank 2 Jordan algebra as defined in Example 2.3.(2). For $x = (\lambda, w) \in \mathbb{C} \oplus W$, one has a trace $T(x) = 3\lambda$ and a quadric norm $N(x) = \lambda^2 + q(w)$ such that $x^2 - T(x)x + N(x)e = 0$. Then one defines the adjoint by $x^\# = (\lambda, -w)$. In the rank 2 case, one has $(x^\#)^\# = x$.

(3) Let $\mathbb{A}$ be as in Example 2.3.(3). Since $\mathbb{A}$ is the complexification of a Hurwitz's algebra, it comes with a non-degenerate quadratic form $\|\cdot\|^2 : \mathbb{A} \to \mathbb{C}$ that is multiplicative. If $\langle \cdot, \cdot \rangle$ stands for its polarization, then the generic norm on $\mathrm{Herm}_3(\mathbb{A})$ can be computed obtaining:

$$N\begin{pmatrix} r_1 & \overline{x_3} & \overline{x_2} \\ x_3 & r_2 & \overline{x_1} \\ x_2 & x_1 & r_3 \end{pmatrix} = r_1 r_2 r_3 + 2\langle x_1 x_2, x_3 \rangle - r_1 \|x_1\|^2 - r_2 \|x_2\|^2 - r_3 \|x_3\|^2 \tag{6}$$

for every $x_1, x_2, x_3 \in \mathbb{A}$, $r_1, r_2, r_3 \in \mathbb{C}$.

(4) Let J be a power-associative algebra. Also in this case one can define the notions of rank, adjoint $x^\#$, norm $N(x)$ and trace and the theory is completely analogous to the previous one. Let $r = \mathrm{rk}(J) \geq 2$. The adjoint satisfies the identity $\left(x^\#\right)^\# = N(x)^{r-2}x$ thus its projectivization is a birational involution of bidegree $(r-1, r-1)$ of $\mathbb{P}(J)$, see also Example 2.2, 4.

The inverse map $x \mapsto x^{-1} = N(x)^{-1}x^\#$ on J naturally induces a birational involution $\widetilde{\jmath} : \mathbb{P}(J \times \mathbb{C}) \dashrightarrow \mathbb{P}(J \times \mathbb{C})$ of bidegree (r, r), defined by $\widetilde{\jmath}([x, r]) = [rx^\#, N(x)]$. Such maps were classically investigated by N. Spampinato and C. Carbonaro Marletta, see [37, 7, 8], producing examples of interesting Cremona involutions in higher dimensional projective spaces. It is easy to see that letting $\widetilde{J} = J \times \mathbb{C}$, then for $(x, r) \in \widetilde{J}$ one has $(x, r)^\# = (rx^\#, N(x))$ so that the map $\widetilde{\jmath}$ is the adjoint map of the algebra $\widetilde{J}$. A Cremona transformation of bidegree (r, r) will be called *of Spampinato type* if it is linearly equivalent to the adjoint of a direct product $J \times \mathbb{C}$ where J is a power-associative algebra of rank r.

The previous construction will be used in [32] to produce some interesting Cremona involutions and to describe differently some known examples. In Section 3.1 and in Section 3.3.1 maps of this type will naturally appear in relation to tangential projections of twisted cubics over rank three Jordan algebras, respectively extremal varieties 3–covered by twisted cubics.

The set of complex Jordan algebras of dimension n will be denoted by $\boldsymbol{Jordan}^n$ while $\boldsymbol{Jordan}_3^n$ will indicate the subset formed by the elements having rank equal to 3. Here we will focus on the description of $\boldsymbol{Jordan}_3^n$ up to a certain equivalence relation that we now introduce.

2.3.1. **Isotopy.** Let $\mathbb{J}$ be a Jordan algebra. By definition, the *quadratic operator* associated to an element $x \in \mathbb{J}$ is the endomorphism $U_x = 2\, L_x \circ L_x - L_{x^2}$ of $\mathbb{J}$ where L_x stands for the multiplication by x in $\mathbb{J}$. If $u \in \mathbb{J}$ is invertible, one defines the u-*isotope* $\mathbb{J}^{(u)}$ of $\mathbb{J}$ as the algebra structure on $\mathbb{J}$ induced by the product $\bullet^{(u)}$ defined by

$$x \bullet^{(u)} y = \frac{1}{2} U_{x,y}(u),$$

where as usual $U_{x,y} = U_{x+y} - U_x - U_y$ is the linearization of the *quadratic representation* $P : V \to \mathrm{End}(V)$, $x \mapsto P(x) = U_x$ of $\mathbb{J}$ (the name is justified by the fact that P is a homogenous polynomial map of degree 2). Then u^{-1} is a unity for the new product $\bullet^{(u)}$ and moreover $\mathbb{J}^{(u)}$ is a Jordan algebra, the u-*isotope* of $\mathbb{J}$. Let us recall that $x \in \mathbb{J}$ is invertible if and only if U_x is invertible; moreover $x^{-1} = U_x^{-1}(x)$ and $U_{x^{-1}} = U_x^{-1}$ in this case.

Two Jordan algebras $\mathbb{J}$ and $\mathbb{J}'$ are called *isotopic* if $\mathbb{J}'$ is isomorphic to an isotope $\mathbb{J}^{(u)}$ of $\mathbb{J}$. One immediately proves that the rank is invariant by isotopy. The norm $N^{(u)}(x)$ and the adjoint $x^{\#(u)}$ of an element $x \in \mathbb{J}^{(u)}$ are expressed in terms of the norm $N(x)$ and the adjoint $x^\#$ in the algebra $\mathbb{J}$ in the following way, see [26, II.7.4]:

$$N^{(u)}(x) = N(u)N(x) \qquad \text{and} \qquad x^{\#(u)} = N(u)^{-1} U_{u^\#}(x^\#). \tag{7}$$

If $\mathbb{J}$ is a Jordan algebra, then we shall denote by $[\mathbb{J}]$ its isotopy class.

Of course, *isotopy* defines a equivalence relation on $\boldsymbol{Jordan}^n$ and hence on $\boldsymbol{Jordan}_3^n$ since the rank is isotopy-invariant. In this paper, we are interested in the description of the quotient space $\boldsymbol{Jordan}_3^n/_{isotopy}$.

The concept of 'Jordan pair' is a useful notion to deal with Jordan algebras up to isotopy. We introduce it in the next section. This notion will be used later in section 3.3.2.

2.3.2. **Jordan pairs.** By definition (see [24]), a *Jordan pair* is a pair $V = (V^+, V^-)$ of complex vector spaces together with quadratic maps (for $\sigma = \pm$)

$$Q^\sigma : V^\sigma \to \mathrm{Hom}(V^{-\sigma}, V^\sigma),$$

satisfying the following relations for every $(x, y) \in V^\sigma \times V^{-\sigma}$:

$$D^\sigma_{x,y} Q^\sigma_x = Q^\sigma_x D^{-\sigma}_{y,x}, \qquad D^\sigma_{Q^\sigma_x(y),y} = D^\sigma_{x,Q^{-\sigma}_x(y)} \qquad \text{and} \qquad Q^\sigma_{Q^\sigma_x(y)} = Q^\sigma_x Q^{-\sigma}_y Q^\sigma_x,$$

where $D^\sigma_{x,y} \in \mathrm{End}(V^\sigma)$ is defined by $D^\sigma_{x,y}(z) = Q^\sigma_{x+z}(y) - Q^\sigma_x(y) - Q^\sigma_z(y)$ for every $z \in V^\sigma$.

Example 2.5. (1) Let $\mathbb{J}$ be a Jordan algebra. Then $V = (\mathbb{J}, \mathbb{J})$ with quadratic operators $Q^\pm_x = U_x$ for every $x \in \mathbb{J}$, is a Jordan pair. By definition, it is the *Jordan pair associated to* $\mathbb{J}$;

(2) Given integers $p, q > 0$, the pair $V = (M_{p,q}(\mathbb{C}), M_{q,p}(\mathbb{C}))$ together with the quadratic operators defined by $Q^\sigma_x(y) = x \cdot y \cdot x$ (where $\cdot$ designates the usual matrix product) is a Jordan pair.

By definition, a *morphism* between two Jordan pairs (V^+, V^-) and $(\overline{V}^+, \overline{V}^-)$ with respective associated quadratic operators Q^σ and $\overline{Q}^\sigma$, is a pair $h = (h^+, h^-)$ of linear maps $h^\sigma : V^\sigma \to \overline{V}^\sigma$ such that, for all $\sigma = \pm$ and every $(x, y) \in V^\sigma \times V^{-\sigma}$, one has

$$h^\sigma\big(Q^\sigma_x(y)\big) = \overline{Q}^\sigma_{h^\sigma(x)}\big(h^{-\sigma}(y)\big).$$

Isomorphisms and *automorphisms* of Jordan pairs are defined in the obvious way.

An element $u \in V^{-\sigma}$ is said to be *invertible* if $Q^{-\sigma}_u$ is invertible (as a linear map from V^σ into $V^{-\sigma}$). In this case, one verifies that the product

$$x \bullet x' := \frac{1}{2} Q^\sigma_{x,x'}(u) = \frac{1}{2}\Big(Q^\sigma_{x+x'}(u) - Q^\sigma_x(u) - Q^\sigma_{x'}(u)\Big)$$

induces on V^σ a Jordan algebra structure with unit $(Q^{-\sigma}_u)^{-1}(u) \in V^\sigma$. This Jordan algebra is noted by V^σ_u. Then it can be proved that V is isomorphic to the Jordan pair associated to V^σ_u. This gives an equivalence between Jordan algebras up to isotopies and Jordan pairs admitting invertible elements up to isomorphisms, see [24].

3. Equivalences

In this section, we establish some equivalences between the three mathematical worlds introduced above.

3.1. **Starting from the J-world.** Let $\mathbb{J}$ be a Jordan algebra of dimension n and of rank 3. Following Freudenthal in [18], one defines the *twisted cubic* over $\mathbb{J}$ as the Zariski closure $X_\mathbb{J}$ of the image of the affine embedding

$$\begin{aligned} \mu_\mathbb{J} : \mathbb{J} &\longrightarrow \mathbb{P}\big(\mathbb{C} \oplus \mathbb{J} \oplus \mathbb{J} \oplus \mathbb{C}\big) \\ x &\longmapsto \big[1 : x : x^\# : N(x)\big]. \end{aligned}$$

It is known that $X_\mathbb{J} \subset \mathbb{P}^{2n+1}$ belongs to the class $\boldsymbol{X}^n(3,3)$, see for example [31, Section 4.3]. We shall provide below other proofs of this fact, see Proposition 3.3.

Let $\mathbb{J}^{(u)}$ be the u-isotope of $\mathbb{J}$ relatively to an invertible element $u \in \mathbb{J}$. Let ℓ_u be the linear automorphism of $\mathbb{P}(\mathbb{C} \oplus \mathbb{J} \oplus \mathbb{J} \oplus \mathbb{C}) = \mathbb{P}^{2n+1}$ defined by

$$\ell_u\big([s : X : Y : t]\big) = \big[s : X : N(u)^{-1} U_{u^\#}(Y) : N(u) t\big].$$

It follows from (7) that, as affine maps from $\mathbb{J} = \mathbb{J}^{(u)}$ to $\mathbb{P}^{2n+1}$, one has $\mu_{\mathbb{J}^{(u)}} = \ell_u \circ \mu_\mathbb{J}$. Hence the projective varieties $X_\mathbb{J}$ and $X_{\mathbb{J}^{(u)}}$ are projectively equivalent. Therefore the association $\mathbb{J} \to X_\mathbb{J}$ factorizes and induces a well defined application

$$\begin{array}{ccc} \boldsymbol{Jordan}^n_3 \big/_{isotopy} & \longrightarrow & \boldsymbol{X}^n(3,3) \big/_{\substack{projective \\ equivalence}} \\ \big[\mathbb{J}\big] & \longmapsto & \big[X_\mathbb{J}\big]. \end{array}$$

Similarly, since $x^{\#(u)} = N(u)^{-1} U_{u^\#}(x^\#)$, the linear equivalence class of the birational map $\#_\mathbb{J} : [x] \dashrightarrow [x^\#]$ of $\mathbb{P}^{n-1}$ does not depend on the isotopy class of $\mathbb{J}$. Hence we also get a well-defined map

$$\begin{array}{ccc} \boldsymbol{Jordan}^n_3 \big/_{isotopy} & \longrightarrow & \mathbf{Bir}_{2,2}(\mathbb{P}^{n-1}) \big/_{\substack{linear \\ equivalence}} \\ \big[\mathbb{J}\big] & \longmapsto & \big[\#_\mathbb{J}\big]. \end{array}$$

Remark 3.1. The tools used above to construct the 'twisted cubics over Jordan algebras of rank three' are the adjoint $x^{\#}$ and the norm $N(x)$. These notions has been introduced for every unital power-associative algebra so that one can ask if it were possible to define 'twisted cubics over commutative power-associative algebras of rank three with unity.' Since a commutative power-associative algebra of rank three with unity is necessarily a Jordan algebra of the same rank, according to [16, Corollary 13], this generalization would not produce new examples.

In the same vein, one could define a map associating to a rank three power-associative algebra with unity the quadro-quadro Cremona transformation given by the linear equivalence class of its adjoint. As we shall in Theorem 3.4 below this generalization is useless since the restriction of this map to Jordan algebras of rank three will be surjective. Moreover, by applying Theorem 3.4 to the adjoint of a commutative power-associative algebra of rank three with unity one could deduce a new proof of [16, Corollary 13] mentioned above.

One verifies easily that $\infty_{\mathbb{J}} = [0 : \mathbf{0} : \mathbf{0} : 1]$ is a smooth point of $X_{\mathbb{J}}$ and that the homogeneization of $\mu_{\mathbb{J}}$ is the inverse of the birational map $\pi_{\infty_{\mathbb{J}}} : X_{\mathbb{J}} \dashrightarrow \mathbb{P}^n$ given by the restriction to X of the projection from $T_{\infty_{\mathbb{J}}}X_{\mathbb{J}}$, see for example [31, Section 4]. It is immediate to verify that $0_{\mathbb{J}} = \mu_{\mathbb{J}}(\mathbf{0}) = [1 : \mathbf{0} : \mathbf{0} : 0] \in X_{\mathbb{J}}$ is also a smooth point and that $T_{0_{\mathbb{J}}}X_{\mathbb{J}}$ is the closure of the locus of points of the form $[1 : x : \mathbf{0} : 0]$ with $x \in \mathbb{J}$. Thus the birational map $\psi : \mathbb{P}(\mathbb{J} \times \mathbb{C}) \dashrightarrow \mathbb{P}(\mathbb{J} \times \mathbb{C})$ given by $\psi([x_0 : x]) = [x_0 x^{\#} : N(x)]$ is a birational involution of type $(3, 3)$ of Spampinato type (see Example 2.3.(4)) and it is clearly the composition of the homogenization of $\mu_{\mathbb{J}}$ with the (restriction to $X_{\mathbb{J}}$ of the) linear projection $\pi_{0_{\mathbb{J}}}$ from $T_{0_{\mathbb{J}}}X_{\mathbb{J}}$, that is $\psi = \pi_{\infty_{\mathbb{J}}} \circ \pi_{0_{\mathbb{J}}}^{-1}$ as rational maps. We shall return on this in Section 3.3.2.

3.2. **Starting from the C-world.** Let $f \in \mathbf{Bir}_{2,2}(\mathbb{P}^{n-1})$, let $g \in \mathbf{Bir}_{2,2}(\mathbb{P}^{n-1})$ be its inverse, let $F, G : \mathbb{C}^n \to \mathbb{C}^n$ be associated quadratic lifts and let N, M be the associated cubic forms, see Section 2.2.

3.2.1. **From the C-world to the X-world.** Let us consider the following affine embedding

$$\begin{aligned} \mu_f : \mathbb{C}^n &\longrightarrow \mathbb{P}\big(\mathbb{C} \oplus \mathbb{C}^n \oplus \mathbb{C}^n \oplus \mathbb{C}\big) = \mathbb{P}^{2n+1} \\ x &\longmapsto \big[1 : x : F(x) : N(x)\big]. \end{aligned} \tag{8}$$

The Zariski-closure X_f of its image is a non-degenerate irreducible n-dimensional subvariety of $\mathbb{P}^{2n+1}$ containing $0_f = \mu_f(\mathbf{0}) = [1 : \mathbf{0} : \mathbf{0} : 0]$.

In order to prove that X_f is 3-covered by twisted cubics, we shall use in different ways the following crucial result whose incarnations in the three worlds we defined till now will be the starting points of the bridges connecting these apparently different universes.

Lemma 3.2. *Let notation be as above. There exists a bilinear form $B_F : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ such that*

$$dN_x = B_F(F(x), dx),$$

for every $x \in \mathbb{C}^n$.

Proof. In coordinates, the relation $G(F(x)) = N(x)x$ translates into

$$g_i\big(f_1(x), \ldots, f_n(x)\big) = x_i N(x)\,, \qquad i = 1, \ldots, n. \tag{9}$$

Let $I = \langle f_1, \ldots, f_n \rangle \subset \mathbb{C}[x_1, \ldots, x_n] = S = \oplus_{d \geq 0} S_d$ and let $I = \oplus_{d \geq 0} I_d$. Let us recall that the biggest homogeneous ideal of S defining the scheme $\mathcal{B} = V(I)$ is the saturated ideal

$$I^{\mathrm{sat}} = \oplus_{d \geq 0} I_d^{\mathrm{sat}} = \oplus_{d \geq 0} H^0\big(\mathcal{I}_{\mathcal{B}}(d)\big).$$

It follows from (9) that

$$x_i N(x) \in (I_2)^2 \subseteq I_4 \qquad \text{for every } i = 1, \ldots, n. \tag{10}$$

By derivation of (9) with respect to x_j for j distinct from i, we deduce that $x_i(\partial N/\partial x_j) \in I_3$ yielding

$$x_i^2 \frac{\partial N}{\partial x_j} \in I_4 \qquad \text{for every } i, j = 1, \ldots, n,\ i \neq j. \tag{11}$$

By derivation of (9) with respect to x_i we obtain $N(x)+x_i(\partial N/\partial x_i)\in I_3$ for every $i=1,\dots,n$. Multiplying by x_i and using (10) we deduce $x_i^2(\partial N/\partial x_i)\in I_4$ for every i. Combined with (11), this shows that $x_i^2(\partial N/\partial x_j)\in I_4$ for every $i,j=1,\dots,n$. Then by definition

$$\frac{\partial N}{\partial x_i}\in I_2^{\rm sat} \tag{12}$$

for every $i=1,\dots,n$. Since $I_2^{\rm sat}=H^0(\mathcal{I}_{\mathcal{B}}(2))=\text{span}\{f_1,\dots,f_n\}$, the last equality being an immediate consequence of the birationality of f, there exist constants $b_{ij}\in\mathbb{C}$ such that $\partial N/\partial x_i=\sum_{j=1}^n b_{ij}f_j$ for every i. Then letting $B_F(x,y)=\sum_{i,j=1}^n b_{ij}x_iy_j$, we have $dN_x=B_F(F(x),dx)$ for every $x\in\mathbb{C}^n$. □

We now provide below two different proofs that $X_f\in\boldsymbol{X}^n(3,3)$. Both are interesting in our opinion: the first one is more elementary but computational; the second one is algebro-geometric and the computations are hidden in some elementary well known facts.

Proposition 3.3. *Let notation be as above. The variety X_f belongs to the class $X_f\in\boldsymbol{X}^n(3,3)$: X_f in non–degenerate in $\mathbb{P}^{2n+1}$, is 3–RC by twisted cubics and is different from a rational normal scroll.*

First proof. For $a,b\in\mathbb{C}^n$ with $M(b)\neq 0$, $\gamma_{a,b}:t\longmapsto\frac{G(a+tb)}{M(a+tb)}$ is a well-defined rational map and it follows from (2) and (3) that for t generic, one has

$$\mu_f\big(\gamma_{a,b}(t)\big)=\left[1:\frac{G(a+t\,b)}{M(a+t\,b)}:\frac{F(G(a+t\,b))}{M(a+t\,b)^2}:\frac{N(G(a+t\,b))}{M(a+t\,b)^3}\right]=\big[M(a+t\,b):G(a+t\,b):a+t\,b:1\big].$$

Thus $\mu_f\circ\gamma_{a,b}(\mathbb{P}^1)$ is a twisted cubic curve passing through $0_f=\mu_f\big(\gamma_{a,b}(\infty)\big)$ and moreover X_f is 2-covered by twisted cubics passing through 0_f: for $(p,p')\in(X_f)^2$ general, there exists a twisted cubic included in X_f and containing the points p,p' and 0_f.

Now let $x_\star\in\mathbb{C}^n$ be such that $N(x_\star)\neq 0$, let $\tau_{x_\star}$ be the translation by $x_\star$ in $\mathbb{C}^n$ and consider the linear automorphism of $\mathbb{P}^{2n+1}$ defined by

$$\ell_{x_\star}(\omega)=\big[s:x+s\,x_\star:y+dF_{x_\star}(x)+sF(x_\star):t+B_F(y,x_\star)+dN_{x_\star}(x)+sN(x_\star)\big]$$

for $\omega=[s:x:y:t]\in\mathbb{P}(\mathbb{C}\oplus\mathbb{C}^n\oplus\mathbb{C}^n\oplus\mathbb{C})=\mathbb{P}^{2n+1}$, where B_F stands for the bilinear form given by Lemma 3.2. One verifies immediately that

$$\ell_{x_\star}\circ\mu_f=\mu_f\circ\tau_{x_\star}.$$

This shows that the pair $(X_f,\mu_f(x_\star))$ is projectively equivalent to $(X_f,0_f)$ hence X_f is also 2-covered by twisted cubics passing through $\mu_f(x_\star)$. Since this holds for any $x_\star\in\mathbb{C}^n$ such that $N(x_\star)\neq 0$, this implies that $X_f=X^n(3,3)$. The variety X_f is not a rational normal scroll since the linear system of quadrics defining the so called second fundamental form at a general point has no fixed component since it is naturally identified with the linear system defining f, see [31, Section 5] for definitions and details. □

Second proof of Proposition 3.3. Let notation be as above. Consider $\mathbb{C}^n$ as the hyperplane $\mathbb{P}^n\setminus V(x_0)$ so that $[x_0:x_1\dots:x_n]$ are projective coordinates on $\mathbb{P}^n$ and $\mu_f:\mathbb{P}^n\dashrightarrow X_f$ is a rational map defined on $\mathbb{C}^n$. Since f is not fake we can suppose $n\geq 3$. Consider three general points $p=(p_1,p_2,p_3)\in(\mathbb{P}^n)^3$ and let $\Pi_p\subset\mathbb{P}^n$ be their linear span. We claim that the line $L_p=\Pi_p\cap V(x_0)$ determines a plane Π'_p cutting the base locus scheme of f in length three subscheme $\mathcal{P}$ spanning Π'_p. Indeed $D_p=f(L_p)\subset\mathbb{P}^{n-1}$ is a conic cutting the base locus scheme of g in length three subscheme $\mathcal{P}'$ spanning a plane $\overline{\Pi}_p$ because $g(D_p)=L_p$. Then taking $\Pi'_p=g(\overline{\Pi}_p)$ the claim is proved. The length six scheme $\{p_1,p_2,p_3,\mathcal{P}\}$ spans the 3-dimensional space $\langle\Pi_p,\Pi'_p\rangle$ so that it determines a unique twisted cubic $C_p\subset\mathbb{P}^{n-1}$ containing it. By Lemma 3.2 the birational map $\mu_f:\mathbb{P}^n\dashrightarrow X_f$ is given by a linear system of cubic hypersurfaces having points of multiplicity at least 2 along its base locus scheme $V(x_0,N(x))\subset\mathbb{P}^n$. Then $\mu_f(C_p)\subset X_f$ is a twisted cubic passing through the three general points $\mu_f(p_i)$, $i=1,2,3$. This shows that $X_f=X^n(3,3)$ while to verify that X_f is not a rational normal scroll one can argue as in the end of the previous proof. □

One immediately verifies that the projective equivalence class of X_f does not depend on f but only on its linear equivalence class. Hence there exists a well-defined map

$$\begin{array}{ccc}\mathbf{Bir}_{2,2}(\mathbb{P}^{n-1})_{/\substack{linear\\ equivalence}} & \longrightarrow & \boldsymbol{X}^n(3,3)_{/\substack{projective\\ equivalence}}\\ \big[f\big] & \longmapsto & \big[X_f\big].\end{array}$$

3.2.2. **From the C-world to the J-world.** Let us now explain how to associate in a direct and algebraic way a rank 3 Jordan algebra to $f \in \mathbf{Bir}_{2,2}(\mathbb{P}^{n-1})$. Assume that $\mathbb{P}^{n-1} = \mathbb{P}(V)$ for a n-dimensional vector space V. On the open set defined by $N(x) \neq 0$ we define

$$j_f(x) = \frac{F(x)}{N(x)}.$$

Then $j_f : V \dashrightarrow V$ is a birational map which is homogeneous of degree -1. Following [25], we say that the map $j_f : V \dashrightarrow V$ is an *inversion* and that the elements $x \in V$ with $N(x) \neq 0$ are *invertible*. For $x \in V$ invertible, one sets

$$P_f(x) = -d(j_f)_x^{-1}.$$

We defined in this way a rational map $P_f : V \dashrightarrow \mathrm{End}(V)$ which is homogeneous of degree 2. Similarly one defines $j_g : V \dashrightarrow V$ and $P_g : V \dashrightarrow \mathrm{End}(V)$.

Theorem 3.4. *Let notation be as above. For every linear equivalence class $[f]$, $f \in \mathbf{Bir}_{2,2}(\mathbb{P}^{n-1})$, there exists a Jordan algebra $\mathbb{J}_f$ of rank 3 such that $[\#_{\mathbb{J}_f}] = [f]$.*

In particular every quadro-quadric Cremona transformation is linearly equivalent to an involution which is the adjoint of a rank 3 Jordan algebra.

Proof. Replacing F by $P_f(e) \circ F$ if necessary, we can assume that there exists an invertible element $e \in V$ such that $P_f(e) = \mathrm{Id}_V$. Euler's Formula and the homogeneity of j_f imply $P_f(x)(j_f(x)) = x$ for every invertible x so that, without loss of generality, we can also assume $j_f(e) = e$. Similarly, one sets $j_g(y) = M(y)^{-1}G(y)$ for y such that $M(y) \neq 0$. Taking the exterior derivative of the relation $j_g \circ j_f(x) = x$, we deduce $P_f(x) = -d(j_g)_{j_f(x)}$ for any invertible x. The differentials dG_y and dM_y are homogeneous of degree 1, respectively of degree 2, in y. Hence the substitution $y = j_f(x) = N(x)^{-1}F(x)$ in

$$d(j_g)_y = M(y)^{-1}dG_y - M(y)^{-2}G(y)dM_y$$

yields

$$\begin{aligned} P_f(x) =& -dG_{F(x)} + xN(x)^{-1}dM_{F(x)} && \text{(by (2) and (3))} \\ =& -dG_{F(x)} + xN(x)^{-1}B_G\big(G(F(x)), dx\big) && \text{(by Lemma 3.2)} \\ =& -dG_{F(x)} + x\,B_G\big(x, dx\big). \end{aligned}$$

Thus the rational map $P_f : V \dashrightarrow \mathrm{End}(V)$ extends to a polynomial quadratic affine morphism $P_f : V \to \mathrm{End}(V)$. Therefore

$$P_f(x, y) = P_f(x + y) - P_f(x) - P_f(y) \in \mathrm{End}(V)$$

is bilinear in x and y and the results of [25] (in particular Theorem 4.4 and Remark 4.5 therein) assure that the product $\bullet_f$ on V defined by

$$x \bullet_f y = \frac{1}{2}P_f(x, y)(e)$$

satisfies the Jordan identity (4), admits e as a unital element and induces on V a structure of Jordan algebra noted by $\mathbb{J}_f$. For a $x \in V$ invertible element, the inverse for this product is given by $x^{-1} = j_f(x)$ hence the adjoint of x is $x^{\#} = F(x)$, yielding $\mathrm{rk}(\mathbb{J}_f) = 3$, see Example 2.2.(4). □

It can be verified that the isotopy equivalence class of $\mathbb{J}_f$ depends only on the linear equivalence class of f hence one obtains a well-defined map

$$\begin{array}{ccc} \mathbf{Bir}_{2,2}(\mathbb{P}^{n-1})\big/_{\substack{linear\\ equivalence}} & \longrightarrow & \boldsymbol{Jordan}_3^n\big/_{isotopy} \\ \big[f\big] & \longmapsto & \big[\mathbb{J}_f\big]. \end{array}$$

Remark 3.5. In the previous proof we choose a point e such that $P_f(e) = \mathrm{Id}_V$, which is not natural from an intrinsic point of view. More generally one can consider the source space and the target space of f as distinct n-dimensional projective spaces associated to two vector spaces V_F and V_G of dimension n. We can consider f as a birational map $f : \mathbb{P}(V_F) \dashrightarrow \mathbb{P}(V_G)$ with inverse $g : \mathbb{P}(V_G) \dashrightarrow \mathbb{P}(V_F)$. Reasoning as in the proof of Theorem 3.4, one proves that $-d(j_f)_x^{-1}$ (resp. $-d(j_g)_y^{-1}$) depends quadratically on $x \in V_F$ (resp. on $y \in V_G$), defining a quadratic map $P_f : V_F \to \mathrm{End}(V_G, V_F)$ (resp. $P_g : V_G \to \mathrm{End}(V_F, V_G)$). Then (V_F, V_G) together with the pair of quadratic operators (P_f, P_g) is a Jordan pair.

3.3. **Starting from the X-world.** In this section, we describe how to associate to a $X \in \boldsymbol{X}^n(3,3)$ an equivalence class of quadro-quadric Cremona transformations of $\mathbb{P}^{n-1}$ and also how to produce directly (and not through the previous construction!) a rank 3 Jordan algebra $\mathbb{J}$ of dimension n, defined modulo isotopy, such that X is projectively equivalent to $X_{\mathbb{J}}$.

Let $X \subset \mathbb{P}^{2n+1}$ be an element of $\boldsymbol{X}^n(3,3)$. Let $x \in X_{\mathrm{reg}}$ be a general point such that X is 2-RC by a family $\Sigma_x \subset \mathrm{Hilb}^{3t+1}(X,x)$ of twisted cubics included in X and passing through x. Denote by $\pi_x : X \dashrightarrow \mathbb{P}^n$ the restriction to X of the tangential projection with center $T_xX \subset \mathbb{P}^{2n+1}$. It is known that π_x is birational, see [33, 31]. Let $\beta_x : \widetilde{X} \to X$ be the blow-up of X at x and let $E_x = \beta_x^{-1}(x)$ be the associated exceptional divisor. Let $\varphi_{X,x}$ be the restriction to E_x of the lift of π_x to $\widetilde{X}$:

$$\varphi_{X,x} = (\pi_x \circ \beta_x)\big|_{E_x} : E_x \dashrightarrow \mathbb{P}^n. \tag{13}$$

In [31, Section 5], we proved that:

(a) $\varphi_{X,x}$ is birational onto its image that is a hyperplane $H_x \subset \mathbb{P}^n$;
(b) $\varphi_{X,x}$ is induced by the second fundamental form $|II_{X,x}| \subset |\mathcal{O}_{E_x}(2)|$;
(c) as a scheme, the base locus scheme of $\varphi_{X,x}$ coincides with the Hilbert scheme of lines passing through x and contained in X;
(d) $(\varphi_{X,x})^{-1} : H_x \dashrightarrow E_x$ is also induced by a linear system of hyperquadrics in H_x.
(e) $\varphi_{X,x}$ is a fake quadro-quadric transformation if an only if $X \subset \mathbb{P}^{2n+1}$ is a rational normal scroll.

From (c) and (d) one could deduce another proof of Lemma 3.2, see *loc. cit.*, or equivalently one can say that (d) is the incarnation in the X-world of the result proved in Lemma 3.2, see [31, Theorem 5.2] for details.

Remark 3.6. The map $v \mapsto \tilde{v}$ considered in the proof of Theorem 3.7 below is an alternative geometrical definition of (a quadratic lift of) $\varphi_{X,x}$ which is more intrinsic than the preceding one since it does not depend on the embedding of X in the projective space $\mathbb{P}^{2n+1}$.

3.3.1. **From the X-world to the C-world.** From now on we shall assume that $X \in \boldsymbol{X}^n(3,3)$ so that X is not a rational normal scroll. The results listed above imply that, after identifying E_x and H_x with $\mathbb{P}^{n-1}$, the map $\varphi_{X,x}$ is a Cremona transformation of bidegree $(2,2)$ of $\mathbb{P}^{n-1}$. Moreover, in [31, Theorem 5.2] it is proved that X is projectively equivalent to the variety $X_{\varphi_{X,x}}$ associated to $\varphi_{X,x}$ via the construction in section 3.2. We leave to the reader to verify that the linear equivalence class of $\varphi_{X,x}$ does not depend on x but only on the projective equivalence class of X. Therefore we have a well-defined application

$$\begin{array}{ccc} \boldsymbol{X}^n(3,3)_{/\substack{projective\\ equivalence}} & \longrightarrow & \mathbf{Bir}_{2,2}(\mathbb{P}^{n-1})_{/\substack{linear\\ equivalence}} \\ \big[X\big] & \longmapsto & \big[\varphi_{X,x}\big]. \end{array}$$

The results of the previous sections show that this map is a bijection.

3.3.2. **From the X-world to the J-world.** The results of [31, Section 5] recalled above and Theorem 3.4 immediately imply that any $X \in \boldsymbol{X}^n(3,3)$ is of *Jordan type*, that is there exists a rank three Jordan algebra $\mathbb{J}$ such that X is projectively equivalent to $X_{\mathbb{J}}$.

There is also a direct way to recover geometrically the underlying structure of Jordan algebra from X. Since there is no real difficulty here, we will leave to the interested readers to fill up some details of the proof of the next result, which was conjectured firstly in [31, Section 5].

Theorem 3.7. *If $X = X^n(3,3) \subset \mathbb{P}^{2n+1}$ is not a rational normal scroll, then there exists a rank three Jordan algebra $\mathbb{J}_X$ such that X is projectively equivalent to $X_{\mathbb{J}_X}$.*

Proof. Let x^+, x^- denote two general points of X_{reg} such that X is 1-RC by the family $\Sigma_{x^+x^-}$ of twisted cubics included in X passing through x^+ and x^-. One has $\mathbb{P}^{2n+1} = T_{x^+}X \oplus T_{x^-}X$ and for $\sigma = \pm$, let $\pi^\sigma = \pi_{x^\sigma}$ be the restriction to X of the tangential projection with center $T_{x^\sigma}X$ onto the projective tangent space $T_{x^{-\sigma}}X$ at the other point. This map is defined at $x^{-\sigma}$ and by definition $x^{-\sigma} = \pi^\sigma(x^{-\sigma})$.

Define V^σ as the complex tangent space T_{X,x^σ}. For $v \in V^\sigma$ generic, there exists a unique twisted cubic curve C_v included in X, joining x^σ to $x^{-\sigma}$ and having $[v]$ as tangent direction at x^σ. More precisely, there exists a unique isomorphism $\alpha_v : \mathbb{P}^1 \to C_v$ such that $\alpha_v(0:1) = x^\sigma$, $\alpha_v(1:0) = x^{-\sigma}$ and $d\alpha_v(s:1)/ds|_{s=0} = v$. The map

$$v \mapsto \exp(v) := \alpha_v(1:1)$$

can be extended to the whole V^σ since, after some natural identifications, it is nothing but the affine embedding μ_{ψ_σ} defined in (8), where ψ_σ is the inverse of the quadro-quadratic birational map $\varphi_{X,x^{-\sigma}}$ associated to $\pi^{-\sigma}$ through formula (13) above. We thus defined geometrically an *exponential map*

$$\exp : V^\sigma \longrightarrow X$$

whose image is denoted by X^σ. Being an affine embedding, its differential

$$d\exp_v : T_{V^\sigma,v} \to T_{X,\exp(v)}$$

is an isomorphism for every $v \in V^\sigma$. Using the linear structure of V^σ, one can (canonically) identify $T_{V^\sigma,v}$ with V^σ itself obtaining a linear isomorphism $\delta_v^\sigma : V^\sigma \to T_{X,\exp(v)}$. For v general, $\exp(v) \in X^{-\sigma}$ so that there exists a unique $\tilde{v} \in V^{-\sigma}$ such that $\exp(v) = \exp(\tilde{v})$ (moreover $\tilde{v} = d\alpha_v(1:t)/dt|_{t=0}$). Thus we can define a linear isomorphism by setting

$$Q_v^\sigma = -\left(\delta_v^\sigma\right)^{-1} \circ \delta_{\tilde{v}}^{-\sigma} : V^{-\sigma} \longrightarrow V^\sigma.$$

The linear map Q_v^σ depends quadratically on $v \in V^\sigma$ and this association extends to the whole V^σ yielding a quadratic polynomial map

$$Q^\sigma : V^\sigma \to \mathrm{Hom}(V^{-\sigma}, V^\sigma).$$

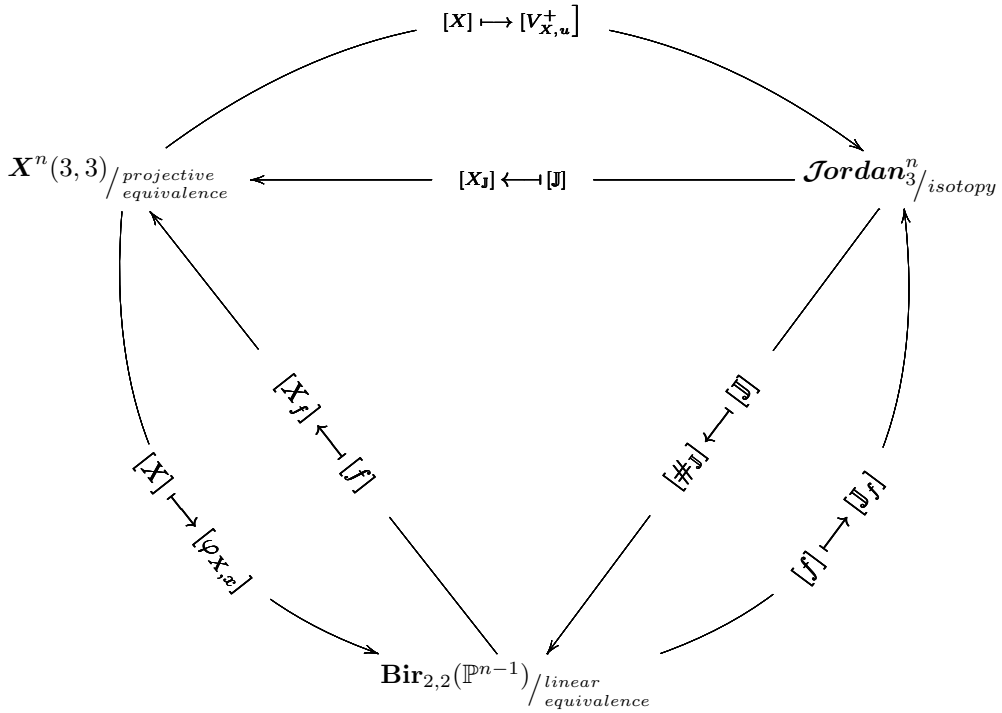
The quadratic maps $Q^\pm$ thus defined induce a structure of Jordan pair on (V^+, V^-) admitting invertible elements. For $u \in V^-$ invertible, the Jordan algebra V_u^+ has rank 3 and the initially considered variety X is projectively equivalent to $X_{V_u^+}$. □

Remark 3.8. Let $X \in \boldsymbol{X}^n(3,3)$. The previous proof shows that the Jordan avatar of the geometrical data formed by X together with two general points x^+, x^- on it is the Jordan pair (V^+, V^-). Similarly, the geometrical object corresponding precisely to a rank three Jordan algebra $\mathbb{J}$ is not really $X_{\mathbb{J}}$ but rather the geometrical data formed by $X_{\mathbb{J}}$ together with three points on it. These two remarks lead to the following heuristic question: what are the Jordan-theoretic counterparts of the data of X alone, or of a pair (X, x) where x is a general point on X? Some of the notions introduced in [2] seem to be relevant to study this question.

We shall now briefly outline another geometrical way of recovering the algebra $\mathbb{J}_X$ naturally associated to $X \in \boldsymbol{X}^n(3,3)$. Let notation be as in Section 3.3, let $x_1, x_2 \in X$ be general points and let $\psi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be the birational map $\pi_{x_1} \circ \pi_{x_2}^{-1}$, see also end of Section 3.1. From the results in [31, Section 5] recalled above, it is not difficult to see that the birational map ψ, which is clearly of bidegree $(3,3)$, is of Spampinato type. Indeed, arguing as in the proof of Theorem 3.4 and based on that analysis, one proves that ψ is linearly equivalent to the involution of a rank 4 Jordan algebra $\widetilde{J}_X$, which is clearly isomorphic to $\mathbb{J}_X \times \mathbb{C}$. We leave to the reader the details of the proof of this claim. In conclusion from a geometrical point of view the passage from the X-world to the C and the J worlds is completely determined by the general tangential projections.

4. Statement of the main theorem and of a general principle

The constructions of the previous sections are all represented in the diagram below, which we will call the 'XJC-*diagram*':



Then the main results of this paper can be formulated in concise terms by making reference to this diagram:

Theorem 4.1. *The above diagram is commutative, all maps appearing in it are bijections and the composition of two of these maps, when possible, is the identity.*

Once the maps in the XJC-diagram have been introduced, the proof of the preceding theorem reduces to straightforward verifications left to the reader.

Theorem 4.1 says in some sense that (up to certain well-understood equivalence relations) there are correspondences between the objects of these three distinct worlds. The X-world is a world of particular projective algebraic varieties sharing deep geometrical properties and it can be considered as a 'geometrical world'. The J-world is a world of particular algebras so that it is an 'algebraic world' while we consider the C-world of another nature, which we will call 'cremonian'.

A consequence of the preceding main theorem is the following general principle:

XJC-Principle. *Any notion, construction or result concerning one of the X, J or C-world admits a counterpart in the other two worlds.*

Remark 4.2. The XJC–Principle is not a mathematical result in the classical sense and it has to be considered as a kind of meta-theorem. Theorem 4.1 and the XJC-Principle are manifestations of a deeper phenomena that could be formulated in terms of equivalences of categories. We plan to come back to this point of view in the near future and we will not deal with this here, although it is very interesting and natural. In the sequel we prefer to present some different applications regarding classification results for particular classes of objects in the different worlds. Other applications will be obtained in [32].

Maybe the better way to realize that such a principle holds consists in presenting some archetypal examples.

4.1. A first occurence of the XJC-principle.

Assume that $X, \mathbb{J}$ and f are corresponding objects.

Proposition 4.3. *The following assertions are equivalent:*

(i) *the variety X is a cartesian product;*
(ii) *the algebra $\mathbb{J}$ is a direct product;*

(iii) *the Cremona map f is an elementary quadratic transformation, see Example 2.2.*

Moreover, the objects satisfying these properties are respectively: the Segre embeddings $\mathrm{Seg}(\mathbb{P}^1 \times Q^{n-1})$, the direct products $\mathbb{C} \times \mathbb{J}'$ where $\mathbb{J}'$ is a Jordan algebra of rank 2, the elementary quadratic transformations.

Proof. Clearly (iii) implies (ii) and (i). If (i) holds and if $X = X_1 \times X_2 \subset \mathbb{P}^{2n+1}$, then we can suppose that through three general points of $X_1 \subset \mathbb{P}^{2n+1}$ there passes a line and that through three general points of $X_2 \subset \mathbb{P}^{2n+1}$ there passes a conic. Then X_1 is a line and X_2 is a quadric hypersurface in its linear span. Thus X is projectively equivalent to the Segre embedding of $\mathbb{P}^1 \times Q^{n-1}$. The other implications/conclusions easily follows. □

4.2. **A second occurrence of the XJC-principle.** In this subsection, we relate the smoothness property in the X-world to an algebraic one in the J-world and to another one in the C-world. We introduce these properties.

By definition, the *radical* R of a Jordan algebra $\mathbb{J}$, indicated by $\mathrm{Rad}(\mathbb{J})$, is defined as the biggest solvable ideal of $\mathbb{J}$ (see also Property 5.3 below for a characterization of the radical when $\mathbb{J}$ has rank 3). Then $\mathbb{J}$ is said to be *semi-simple* if $\mathrm{Rad}(\mathbb{J}) = 0$. In this case, a classical result of the theory asserts that $\mathbb{J}$ is isomorphic to a finite direct product $\mathbb{J}_1 \times \cdots \times \mathbb{J}_m$ where the $\mathbb{J}_k$'s are *simple* Jordan algebras, that is Jordan algebras without any non-trivial ideal.

Following [36] and [15], a Cremona transformation $f : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-1}$ is called *semi–special* if the base locus scheme of f is smooth. A Cremona transformation is said to be *special* if the base locus scheme is smooth and irreducible. Thus special Cremona transformations $f : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-1}$ can be solved, as rational maps, by a single blow-up along an irreducible smooth variety while semi–special Cremona transformations can be solved by blowing–up smooth irreducible subvarieties of $\mathbb{P}^{n-1}$, that is there are no "infinitely near base points". In conclusion the semi–special Cremona transformations are the simplest objects from the point of view of Hironaka's resolutions of rational maps.

Assume that $X, \mathbb{J}$ and f are corresponding objects.

Theorem 4.4. *The following assertions are equivalent:*

(i) *the variety X is smooth;*
(ii) *the algebra $\mathbb{J}$ is semi-simple;*
(iii) *the Cremona transformation f is semi-special.*

Moreover, the classification of the objects satisfying these properties is given in the table below and f is semi–special but not special if and only if it is an elementary quadratic transformation associated to a smooth quadric.

Semi-simple rank 3 Jordan algebra	**Smooth variety $X^n \subset \mathbb{P}^{2n+1}$, 3-RC by cubics, not of Castelnuovo type**	**Special Cremona transformation**
direct product $\mathbb{C} \times \mathbb{J}$ with $\mathbb{J}$ rank 2 Jordan algebra	Segre embedding $\mathrm{Seg}(\mathbb{P}^1 \times Q^{n-1})$ with Q^{n-1} smooth hyperquadric	elementary quadratic
$\mathrm{Herm}_3(\mathbb{R}_\mathbb{C}) \simeq \mathrm{Sym}_3(\mathbb{C})$	6-dimensional Lagrangian grassmannian $LG_3(\mathbb{C}^6) \subset \mathbb{P}^{13}$	$[x] \dashrightarrow [x^\#]$
$\mathrm{Herm}_3(\mathbb{C}_\mathbb{C}) \simeq M_3(\mathbb{C})$	9-dimensional Grassmannian manifold $G_3(\mathbb{C}^6) \subset \mathbb{P}^{19}$	$[x] \dashrightarrow [x^\#]$
$\mathrm{Herm}_3(\mathbb{H}_\mathbb{C}) \simeq \mathrm{Alt}_6(\mathbb{C})$	15-dimensional orthogonal Grassmannian $OG_6(\mathbb{C}^{12}) \subset \mathbb{P}^{31}$	$[x] \dashrightarrow [x^\#]$
$\mathrm{Herm}_3(\mathbb{O}_\mathbb{C})$	27-dimensional E_7-variety in $\mathbb{P}^{55}$	$[x] \dashrightarrow [x^\#]$

Proof. Semi–special Cremona transformations are classified and they correspond to semi–simple Jordan algebras of rank 3, see for example [31, Proposition 5.6], showing the equivalence between (ii) and (iii). It is known that the twisted cubics associated to semi–simple Jordan algebras are smooth and they are described in the table above. We proved the remaining implications in [31, Theorem 5.7]. □

4.3. **A generalization of the XJC-equivalence covering some degenerate cases.** In order to formulate our main result we did not consider some extremal varieties 3-RC by cubics as well as quadro-quadric Cremona transformations equivalent (as rational maps) to linear projective automorphisms. In fact, the XJC-equivalence can be extended in order to cover these "degenerated objects" as we shall see briefly in this subsection.

4.3.1. Let $\overline{\boldsymbol{X}}^n(3,3)$ be the set of projective equivalence classes of extremal n-dimensional irreducible varieties $X \subset \mathbb{P}^{2n+1}$ that are 3-RC by twisted cubics. It is just the union of $\boldsymbol{X}^n(3,3)$ with the projective equivalence classes of the scrolls $S_{1,\dots,1,3}$ (with $n \geq 1$) and $S_{1,\dots,1,2,2}$ (with $n > 1$).

4.3.2. By definition, a *norm* on a Jordan algebra $\mathbb{J}$ is a homogeneous form $\eta \in \mathrm{Sym}(\mathbb{J}^*)$ verifying $\eta(e) = 1$ and which decomposes as a product of powers of the irreducible components of the generic norm N of $\mathbb{J}$, see [3]. Then one defines $\widetilde{\boldsymbol{Jordan}_3}^n$ as the set of *Jordan algebras with a cubic norm*, which by definition is the set of pairs $(\mathbb{J}, \eta)$ where η is a cubic norm on the Jordan algebra $\mathbb{J}$. Since a rank 3 Jordan algebra admits a single cubic norm (the generic one), $\boldsymbol{Jordan}_3^n$ can be considered as a subset of $\widetilde{\boldsymbol{Jordan}_3}^n$. A Jordan algebra with a cubic norm is necessarily of rank less than or equal to 3 and if the rank is less than 3, then it is isomorphic to one of the following Jordan algebras:

- the rank 1 Jordan algebra $\mathbb{C}$, denoted by $\mathbb{J}_0^1$;
- the rank 2 Jordan algebra of Example 2.3 (2) with $q = 0$, denoted by $\mathbb{J}_0^n$;
- the rank 2 Jordan algebra of Example 2.3 (2) with q of rank 1, denoted by $\mathbb{J}_1^n$.

The notation $\mathbb{J}_0^n$ is consistent since the 'Jordan algebra' $\mathbb{C}$ can be described as in Example 2.3 (2) by taking W of dimension 0.

For any $n \geq 1$, the algebra $\mathbb{J}_0^n$ admits a unique cubic norm, namely $\eta(\lambda, w) = \lambda^3$. For any $n > 1$, the algebra $\mathbb{J}_1^n$ admits a 1-dimensional family of cubic norms. Indeed, the generic norm on $\mathbb{J}_1^n$ is given by $N(\lambda, w) = \lambda^2 + q(w)$. Since q has rank 1, there exits a linear form $\ell \in W^*$ such that $q = \ell^2$ so that $N = \ell_+ \ell_-$ with $\ell_\pm(\lambda, w) = \lambda \pm i\ell(w)$ for $(\lambda, w) \in \mathbb{J}_1^n$. Then for every nonzero $(a, b) \in \mathbb{C}^2$, $\eta_{a,b} = (a\ell_+ + b\ell_-)\ell_+\ell_-$ is a cubic norm on $\mathbb{J}_1^n$.

One verifies that modulo isomorphism, $\widetilde{\boldsymbol{Jordan}_3}^n \setminus \boldsymbol{Jordan}_3^n$ consists of the two pairs $(\mathbb{J}_0^n, \lambda^3)$ and $(\mathbb{J}_1^n, \eta_{a,b})$ when $n > 1$, and reduces to $(\mathbb{J}_0^1, \lambda^3)$ when $n = 1$. Let us define $\overline{\boldsymbol{Jordan}_3}^n$ as the set of pairs $(\mathbb{J}, \eta)$ verifying the compatibility relation in Lemma 3.2, that is the partial derivatives of η belong to the ideal generated by the quadratic forms defining the adjoint map. Thus when $n > 1$, $\overline{\boldsymbol{Jordan}_3}^n$ is the union of $\boldsymbol{Jordan}_3^n$ with the isomorphism classes of $(\mathbb{J}_0^n, \lambda^3)$ and of $(\mathbb{J}_1^n, \eta_{1,0})$. We shall indicate by $\widetilde{\mathbb{J}}_1^n$ the Jordan algebra $\mathbb{J}_1^n$ with cubic norm $\eta_{1,0}$.

4.3.3. Finally, let us return to the corresponding Cremona transformations to be considered in order to complete the picture. Consider the set of *normed quadro-quadratic Cremona transformation* of $\mathbb{P}^{n-1}$, that is of pairs $(f, [\eta])$ where $f = [f_1 : \cdots : f_n]$ is a birational map of $\mathbb{P}^{n-1}$ defined by quadratic forms f_i and $[\eta] = \mathbb{C}^*\eta$ is the class of a non-trivial cubic form η such that there exists a quadratic map G satisfying $G\big(f_1(x), \dots, f_n(x)\big) = \eta(x)x$ for every x. Clearly, given $f \in \mathbf{Bir}_{2,2}(\mathbb{P}^{n-1})$, there exists a unique $[\eta]$ as above such that $(f, [\eta])$ is a normed quadro-quadratic Cremona transformation and such that $(f, [\eta])$ satisfies the condition of Lemma 3.2. Therefore $\mathbf{Bir}_{2,2}(\mathbb{P}^{n-1})$ can be considered as a subset of $\overline{\mathbf{Bir}_{2,2}}(\mathbb{P}^{n-1})$, which by definition it is the set of pairs $(f, [\eta])$ satisfying the compatibility relation in Lemma 3.2. One verifies easily that if $(f, [\eta]) \in \overline{\mathbf{Bir}_{2,2}}(\mathbb{P}^{n-1})$ but with f not of bidegree $(2, 2)$, then $(f, [\eta])$ is linearly equivalent to one of the following:

- $(\mathrm{Id}_{\mathbb{P}^{n-1}}, [\ell^3])$ where ℓ is a nonzero linear form, $n \geq 1$;
- $(\mathrm{Id}_{\mathbb{P}^{n-1}}, [\ell^2\ell'])$ where ℓ and ℓ' are linearly independent linear forms, $n > 1$.

Then the XJC-correspondence extends: there are bijection extending the ones in the XJC-diagram such that one has a commutative triangle of equivalences between the sets introduced above:

$$\begin{array}{ccc} \overline{\boldsymbol{X}}^n(3,3)/_{\substack{projective\\equivalence}} & \longleftrightarrow & \overline{\boldsymbol{Jordan}_3}^n/_{isotopy} \\ & \searrow \quad \swarrow & \\ & \overline{\mathbf{Bir}_{2,2}}(\mathbb{P}^{n-1})/_{\substack{linear\\equivalence}} & \end{array} .$$

For instance, let us explain how to associate an extremal variety 3-RC by cubics in $\mathbb{P}^{2n+1}$ to a degenerate element $(f, [\eta])$ of $\overline{\mathbf{Bir}_{2,2}}(\mathbb{P}^{n-1})$. Let F be a quadratic affine lift of f. Then as in section 3.2.1, one defines $X_{f,[\eta]}$ as the Zariski closure of the image of the affine map $x \mapsto [1 : x : F(x) : \eta(x)]$. This variety belongs to $\overline{\boldsymbol{X}}^n(3,3)$: we let the reader verify that the proofs of section 3.2.1 apply if one takes for G the unique affine quadratic map such that $G(F(x)) = \eta(x)x$ for every x.

By the way let us remark that for $(\alpha, \beta) \in \mathbb{C}^2$ such that $\alpha\beta \neq 0$, setting $\ell_{\alpha,\beta} = \alpha\ell_+ + \beta\ell_-$, one associates a non-degenerate n-dimensional variety in $\mathbb{P}^{2n+1}$ to the pair $(\mathbb{J}_1^n, \eta_{\alpha,\beta})$ by defining it as the closure of the image of the

affine map $(\lambda, w) \mapsto [1 : \lambda : w : \ell_{\alpha,\beta}(\lambda, w)\lambda : -\ell_{\alpha,\beta}(\lambda, w)w : \ell_{a,b}(\lambda, w)(\lambda^2 + q(w))]$. However, this variety is not 3-covered by twisted cubics since the compatibility relation in Lemma 3.2 is not satisfied by $\eta_{\alpha,\beta}$.

In fact, this generalization of the XJC-correspondence does not present a very deep interest since it covers only two new cases when $n > 1$, namely the ones described in the following two tryptics:

$$\begin{array}{ccc} \big[S_{1,\dots,1,3}\big] & \longleftrightarrow & \big[\mathbb{J}_0^n\big] \\ & \searrow \quad \swarrow & \\ & \big[(\mathrm{Id}, [\ell^3])\big] & \end{array} \qquad \begin{array}{ccc} \big[S_{1,\dots,1,2,2}\big] & \longleftrightarrow & \big[\widetilde{\mathbb{J}}_1^n\big] \\ & \searrow \quad \swarrow & \\ & \big[\big(\mathrm{Id}, [\ell^2\ell']\big)\big]. & \end{array}$$

In dimension $n = 1$, one has

$$\overline{\boldsymbol{X}}^1(3,3)/proj. = \big[v_3(\mathbb{P}^1)\big], \quad \overline{\boldsymbol{Jordan}}_3^1/isot. = \big[\mathbb{C}\big] \quad \text{and} \quad \overline{\mathbf{Bir}_{2,2}}(\mathbb{P}^0)/lin. = \big[(\mathrm{Id}, x^3)\big]$$

hence the generalized XJC-equivalence reduces in this case to the following trivial tryptic

(14)
$$\begin{array}{ccc} \big[v_3(\mathbb{P}^1)\big] & \longleftrightarrow & \big[\mathbb{C}\big] \\ & \searrow \quad \swarrow & \\ & \big[(\mathrm{Id}_{\mathbb{P}^0}, x^3)\big]. & \end{array}$$

Despite the very small number of new cases covered by the generalized XJC-correspondence, we inserted this extension into the discussion in order to show that the elementary case (14) can be included in the whole picture. Moreover, the notion of *normed quadro-quadric birational map* introduced in 4.3.3 will be used also to describe the general structure of Cremona transformations of bidegree $(2, 2)$ in the next section.

5. Further applications

The theory of Jordan algebras is now well established. We recall some general results on the structure of Jordan algebras, focusing especially on rank 3 algebras.

5.1. **Some results on the structure of Jordan algebras.** Let $\mathbb{J}$ be a fixed Jordan algebra of arbitrary rank $r \geq 1$. For any subset $A \subset \mathbb{J}$, one defines inductively the subsets $A^{(n)} \subset \mathbb{J}$ for any integer $n > 0$ by setting $A^{(1)} = A$ and $A^{(k+1)} = (A^{(k)})^2$ for every $k > 0$. If A is a subalgebra of $\mathbb{J}$, the $A^{(n)}$ form a decreasing sequence of subalgebras $A = A^{(1)} \supset A^{(2)} \supset A^{(3)} \supset \cdots$. By definition, A is *solvable* if $A^{(t)} = 0$ for a positive integer t.

If I_1, I_2 are two solvable ideals of $\mathbb{J}$, it can be verified that $I_1 + I_2$ is solvable too. Since $\mathbb{J}$ is finite dimensional, the union of all the solvable ideals of $\mathbb{J}$ is a solvable ideal of $\mathbb{J}$, which is maximal for inclusion and which is called the *radical* of $\mathbb{J}$ and denoted by $\mathrm{Rad}(\mathbb{J})$, or just by R if there is no risk of confusion.

The notion of solvability introduced above is not the most useful when working with Jordan algebras. Indeed, it can occur that for an ideal $I \subset \mathbb{J}$, the subsets $I^{(k)}$ are not ideals for some $k > 2$. Hence it is not possible to construct inductively a solvable ideal I from its derived series $I = I^{(1)} \supset I^{(2)} \supset \cdots \supset I^{(r-1)} \supset I^{(r)} = 0$. To bypass this technical difficulty, Penico introduced in [29] the nowadays called *Penico's series* of an ideal I as the family $I^{[k]}$, $k \geq 0$ defined inductively by

$$I^{[0]} = \mathbb{J}, \quad I^{[1]} = I \quad \text{and} \quad I^{[k+1]} = \big(I^{[k]}\big)^2 + \big(I^{[k]}\big)^2\mathbb{J} \quad \text{for } k \geq 1.$$

The interest of this notion is twofold. First of all, it can be proved that I is solvable if and only if it is *Penico-solvable*, that is if $I^{[s]} = 0$ for a positive integer s. Moreover, $I^{[k]}$ is an ideal for any $k \geq 1$, see [29].

The notions introduced by Penico are more relevant than the classical ones to describe the structure of Jordan algebras. Since R is solvable, there exists a positive integer $t \geq 1$ such that $R^{[t]} = 0$ and $R^{[t-1]} \neq 0$. Since the $R^{[k]}$'s are ideals in $\mathbb{J}$, the quotients $\mathbb{J}^{[k]} = \mathbb{J}/R^{[k]}$ are Jordan algebras for every $k \geq 1$, yielding, for $\ell = 2, \dots, t$, the exact sequences:

(15)
$$0 \to R^{[\ell-1]}/R^{[\ell]} \longrightarrow \mathbb{J}^{[\ell]} \longrightarrow \mathbb{J}^{[\ell-1]} \to 0.$$

Remark that the left hand side in these exact sequences is an ideal with trivial product because $(R^{[\ell-1]}/R^{[\ell]})^2 = 0$ for every ℓ. In the terminology of Jordan algebras, one says that $\mathbb{J}^{[\ell]}$ is a *null extension* of $\mathbb{J}^{[\ell-1]}$. We can now recall the following important result:

Theorem 5.1 (Albert [1], Penico [29])**.**

(1) The quotient $\mathbb{J}_{\rm ss} = \mathbb{J}^{[1]} = \mathbb{J}/R$ is semi-simple, that is $\mathrm{Rad}(\mathbb{J}_{\rm ss}) = 0$ or equivalently $\mathbb{J}_{\rm ss}$ is isomorphic to a direct product of simple Jordan algebras.

(2) The exact sequence of (non-unital) Jordan algebras $0 \to R \to \mathbb{J} \to \mathbb{J}_{\rm ss} \to 0$ splits: there exists an embedding of Jordan algebras $\sigma : \mathbb{J}_{\rm ss} \hookrightarrow \mathbb{J}$ such that $\mathbb{J} = \sigma(\mathbb{J}_{\rm ss}) \rtimes R$. Moreover the embedding σ is unique up to composition to the left by an automorphism of $\mathbb{J}$.

(3) The Jordan algebra $\mathbb{J}$ is obtained from its semi-simple part $\mathbb{J}_{\rm ss}$ by the series of successive null radical extensions (15).

Simple Jordan algebras are completely classified so that the first part of the previous result ensures that the semi-simple parts of an arbitrary Jordan algebra can be completely described. The second part says that the structure of a general Jordan algebra $\mathbb{J}$ is given by its radical R and by the structure of $\mathbb{J}_{\rm ss}$-module on it. Finally, it comes from (3) that the Jordan product on R as well as its structure of $\mathbb{J}_{\rm ss}$-module can be constructed inductively starting from $\mathbb{J}_{\rm ss}$, by successive extensions of a very simple kind.

Example 5.2. Being associative and commutative, the algebra $A = \mathbb{C}[\varepsilon]/(\varepsilon^3)$ can also be viewed as a 3-dimensional Jordan algebra. One has $R_A = \mathrm{Rad}(A) = \langle \epsilon, \epsilon^2 \rangle$, $R_A^{[2]} = \langle \epsilon^2 \rangle$ and $R_A^{[3]} = 0$. Thus the semi-simple part $A_{\rm ss} = A/\mathrm{Rad}(A)$ has rank 1 and is isomorphic to $\mathbb{C}$.

In the next section, using the XJC-correspondence, we shall state a version of Theorem 5.1 for quadro-quadric Cremona transformations and for twisted cubics over Jordan algebras. We will use the following facts showing that the radical can be determined from the generic norm.

Proposition 5.3. ([38, 0.15 and 9.10]) *For any Jordan algebra $\mathbb{J}$, one has*

$$\mathrm{Rad}(\mathbb{J}) = \big\{ x \in \mathbb{J} \,|\, N(x + \mathbb{J}) = N(x) \big\}.$$

We finish these reminders on Jordan algebras by stating some remarks on the rank 3 case. Assume in what follows that $\mathbb{J}$ has rank 3 and for $x, y \in \mathbb{J}$, set

$$T(x,y) = T(xy) \qquad \text{and} \qquad x\#y = (x+y)^{\#} - x^{\#} - y^{\#}.$$

By Proposition 5.3 (see also [30]), for a rank 3 Jordan algebra one has

$$\mathrm{Rad}(\mathbb{J}) = \big\{ x \in \mathbb{J} \,|\, N(x) = T(x,\mathbb{J}) = T(x^{\#},\mathbb{J}) = 0 \big\} = \big\{ x \in \mathbb{J} \,|\, d^2 N_x = 0 \big\}. \tag{16}$$

Moreover, it can be verified that for $x \in \mathbb{J}$, the quadratic operator $U_x = 2\,L_x^2 - L_{x^2}$ is given by

$$U_x(y) = T(x,y)x - x^{\#}\#y. \tag{17}$$

It follows from (16) that $T(r,\mathbb{J}) = 0$ for every $r \in R$. Furthermore, one has $x\#y = d\#_x(y) = d\#_y(x)$ for every $x, y \in \mathbb{J}$. Since $I^{[2]} = U_I(\mathbb{J})$ for any ideal I, the Penico series can also be defined inductively by

$$R^{[k+1]} = d\#\big(R^{[k]}\big) = \mathbb{J}\#R^{[k]} = \big\langle d\#_x(R^{[k]}) \,\big|\, x \in \mathbb{J} \,\big\rangle.$$

Finally, if $u \in \mathbb{J}$ is invertible, then the quadratic operator $U_x^{(u)}$ in the isotope $\mathbb{J}^{(u)}$ is given by $U_x^{(u)} = U_x U_u$ for every element x. Using this, one verifies easily that the Penico series depends only on the isotopy class of $\mathbb{J}$.

5.2. **The general structure of quadro-quadric Cremona transformations.** A consequence of the equivalence between $\overline{\mathbf{Bir}}_{2,2}(\mathbb{P}^{n-1})/^{lin.}_{equiv.}$ and $\overline{\boldsymbol{Jordan}}_3^{\,n}/_{isot.}$ is a general structure theorem for quadro-quadric Cremona transformations, obtained by translating in the C-world the structure results for Jordan algebras presented above.

The assertions below can be verified without difficulty and their proofs are left to the reader.

5.2.1. **The radical.** Let f be a quadro-quadric Cremona transformation of $\mathbb{P}^{n-1} = \mathbb{P}(V)$ with baselocus scheme $\mathcal{B}_f \subset \mathbb{P}^{n-1}$. The secant scheme $\mathrm{Sec}(\mathcal{B}_f)$ of $\mathcal{B}_f$ is the cubic hypersurface $V(N(x)) \subset \mathbb{P}^{n-1}$, where $N(x)$ is the cubic form appearing in (2). This scheme can be also considered as the *ramification scheme of f*, the name being justified by the fact that the locus of points where the differential of the birational map f is not of maximal rank is exactly $V(N(x))$, see also [10, Section 1.3]. The *radical* of f is the set R_f of points of multiplicity 3 of $\mathrm{Sec}(\mathcal{B}_f)$ and it has a natural scheme structure given by $R_f = V(d^2N_x) \subset \mathbb{P}^{n-1}$. The support of R_f, if not empty, is clearly a linear subspace of $\mathbb{P}^{n-1}$, contained in $\mathrm{Sec}(\mathcal{B}_f)$, and it is the *vertex of the cone* $\mathrm{Sec}(\mathcal{B}_f) = V(N(x)) \subset \mathbb{P}^{n-1}$. We remark that R_f can have any dimension between -1 and $n-2$ (with the usual convention that the empty set is a subspace

of dimension -1). The case when R_f is empty corresponds to the semi-simple case and, at the opposite side, R_f is a hyperplane if and only if $N(x) = L(x)^3$ with $L(x)$ linear form.

5.2.2. **The JC-correspondence in action.** Let g be the quadratic inverse of f. Let F, G be some quadratic lifts of f, respectively g and let R_F and R_G be the affine cones over R_f and R_g respectively. According to the XJC-equivalence (Theorem 4.1), there exist two linear maps $L_1, L_2 \in GL(V)$ such that $F = L_1^{-1} \circ \#_{\mathbb{J}} \circ L_2$ where $\#_{\mathbb{J}}$ denotes the adjoint map of a rank 3 Jordan algebra $\mathbb{J}$. Then $G = L_2^{-1} \circ \#_{\mathbb{J}} \circ L_1$ and $\mathrm{Rad}(\mathbb{J}) = L_2(R_F) = L_1(R_G)$. It is well known that $(x+r)^{\#} - x^{\#} \in \mathrm{Rad}(\mathbb{J})$ for every $x \in \mathbb{J}$ and every $r \in \mathrm{Rad}(\mathbb{J})$, see [38] for instance. In this setting, this gives us the following result.

Lemma 5.4. *If $x \in V$ and if $r \in R_F$, then $F(x+r) - F(x) \in R_G$.*

From the previous Lemma, it follows that F and G pass to the quotient by R_F, respectively by R_G, inducing quadratic affine morphisms $\overline{F} : V/R_F \to V/R_G$, respectively $\overline{G} : V/R_G \to V/R_F$. To understand what these maps are, let $\mathbb{J}_{\mathrm{ss}} = \mathbb{J}/\mathrm{Rad}(\mathbb{J})$ be the semi-simple part of $\mathbb{J}$, consider L_1 and L_2 as isomorphisms between V and $\mathbb{J}$ inducing quotient maps $\overline{L}_1 : V/R_G \simeq \mathbb{J}_{\mathrm{ss}}$ and $\overline{L}_2 : V/R_F \simeq \mathbb{J}_{\mathrm{ss}}$. The following diagram, in which all the vertical arrows are the natural quotient maps, is commutative:

(18)
$$\begin{array}{ccccccc}
 & & & F & & & \\
 & \overset{\frown}{\quad} & & & & & \\
V & \xrightarrow[L_1]{} & \mathbb{J} & \xrightarrow[\#_{\mathbb{J}}]{} & \mathbb{J} & \xrightarrow[L_2^{-1}]{} & V \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
V/R_F & \xrightarrow[\overline{L}_1]{} & \mathbb{J}_{\mathrm{ss}} & \xrightarrow[\#_{\mathbb{J}_{\mathrm{ss}}}]{} & \mathbb{J}_{\mathrm{ss}} & \xrightarrow[\overline{L}_2^{-1}]{} & V/R_G. \\
 & \underset{\smile}{\quad} & & & & & \\
 & & & \overline{F} & & &
\end{array}$$

5.2.3. **The semi-simple part.** Of course, $\overline{F}$ and $\overline{G}$ are quadratic maps, each one being the inverse of the other in the sense used till now. Indeed, if N is the cubic form such that (2) holds, it passes to the quotient and induces a well-defined cubic form $\overline{N}$ on V/R_F defined by $\overline{N}(\overline{x}) = N(x)$ for $x \in V$ (where $\overline{x}$ stands for the class of x modulo R_F). Moreover, $\overline{G}(\overline{F}(\overline{x})) = \overline{N}(\overline{x})\,\overline{x}$ for every $x \in V$ so that the pair $(\overline{F}, [\overline{N}])$ is an element of $\overline{\mathbf{Bir}_{2,2}}(\mathbb{P}(V/R_F))$ (the pair $(\overline{F}, [\overline{N}])$ satisfies the statement of Lemma 3.2). By definition, it is the *semi-simple part* of F and it is denoted by F_{ss}. In practice, one identifies F_{ss} with $\overline{F}$, which is not a big deal since the cubic form $\overline{N}$ is always (essentially) determined.

Example 5.5 (continuation of Example 5.2). The adjoint and the generic norm in $A = \mathbb{C}[\varepsilon]/(\varepsilon^3)$ are given by $(a,b,c)^{\#} = (a^2, -ab, b^2 - ac)$ and $N(a,b,c) = a^3$ if (a,b,c) stands for the coordinates of an element of A relatively to the basis $(1, \varepsilon, \varepsilon^2)$. The semi-simple part of $\#_A$ is the quadratic map $a \mapsto a^2$, which is a lift of the normed quadro-quadratic map $(a^2, [N])$ where N is the cubic norm on A_{ss} induced by the generic norm of A, i.e. $N(a) = N(a,0,0) = a^3$ for every $a \in A_{\mathrm{ss}} \simeq \mathbb{C}$.

We define the *semi-simple rank* $r_{\mathrm{ss}}(\mathbb{J})$ of a Jordan algebra $\mathbb{J}$ as the rank of its semi-simple part $\mathbb{J}_{\mathrm{ss}} = \mathbb{J}/\mathrm{Rad}(\mathbb{J})$ and the *semi–simple dimension* $\dim_{\mathrm{ss}}(\mathbb{J}) = \dim(\mathbb{J}_{\mathrm{ss}})$. These notions are invariant up to isotopies so that we can define the *semi-simple rank* $r_{\mathrm{ss}}(f) = r_{\mathrm{ss}}(F)$ of $f \in \mathbf{Bir}_{2,2}(\mathbb{P}(V))$, respectively the *semi–simple dimension* $\dim_{\mathrm{ss}}(f)$ (or of any affine lift $F \in \mathrm{Sym}^2(V^*) \otimes V$ of f), as the semi-simple rank of the associated isotopy class $[\mathbb{J}_f]$ of Jordan algebras, respectively as the semi-simple dimension of $[\mathbb{J}_f]$. In this way two new invariants (relatively to linear equivalence) of quadro-quadric Cremona transformations naturally appear. Let us see how these definitions work in the simplest cases.

Example 5.6. Modulo linear equivalence there are exactly three equivalence classes of quadro-quadric Cremona transformations on $\mathbb{P}^2$, corresponding to the three isotopy classes of cubic Jordan algebra of dimension three. Let us summarize the JC-correspondence and the semisimple parts of these three classes in the following table:

Semi-simple rank $r_{\mathbf{ss}}$	**Algebra $\mathbb{J}_f$**	**Cremona transformation $f:\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$**	**Semi-simple part of $\mathbb{J}_f$**	**Semi-simple part of f**	**Norm $\overline{N}$**
1	$\frac{\mathbb{C}[\varepsilon]}{(\varepsilon^3)}$	$(x_1^2, -x_1x_2, x_2^2 - x_1x_3)$	$\mathbb{C}$	x_1^2	x_1^3
2	$\mathbb{C} \times \frac{\mathbb{C}[\varepsilon]}{(\varepsilon^2)}$	$(x_2^2, x_1x_2, -x_1x_3)$	$\mathbb{C} \times \mathbb{C}$	(x_2^2, x_1x_2)	$x_1x_2^2$
3	$\mathbb{C} \times \mathbb{C} \times \mathbb{C}$	(x_2x_3, x_1x_3, x_1x_2)	$\mathbb{C} \times \mathbb{C} \times \mathbb{C}$	(x_2x_3, x_1x_3, x_1x_2)	$x_1x_2x_3$

5.2.4. **Classification of the semi-simple part.** Since a semisimple Jordan algebra is a direct product of simple ones and since the classification of all simple Jordan algebras was obtained by Jacobson, see [20], the JC-correspondence provides the complete classification of the semisimple parts of quadro-quadric Cremona transformations, yielding the following result.

Proposition 5.7. *Let $F : V \to V$ be a lift of a normed quadro-quadric Cremona transformation $f : \mathbb{P}(V) \dashrightarrow \mathbb{P}(V)$. Then*

(1) F is semi-simple (ie. $F = F_{\mathrm{ss}}$) if and only if f is semispecial;
(2) if F is semi-simple, then f is linearly equivalent to one of the quadro-quadric Cremona transformations listed in the third column of the table below.

Semi-simple rank r_{ss}	**Semi-simple dim. $\dim_{\mathbf{ss}}$**	**Ambient space $\overline{V}$**	**Semi-simple part**	
			$\overline{F}$	$\overline{N}$
1	1	$\mathbb{C}$	$\lambda \longmapsto \lambda^2$	λ^3
2	$1+\dim(W)$	$\mathbb{C} \oplus W$	$(\lambda, w) \longmapsto (\lambda^2, -\lambda w)$	$\lambda(\lambda^2 + q(w))$
2	2	$\mathbb{C} \times \mathbb{C}$	$(\rho, \lambda) \longmapsto (\lambda^2, \rho\lambda)$	$\rho\lambda^2$
3	$2+\dim(W)$	$\mathbb{C} \times (\mathbb{C} \oplus W)$	$(\rho, \lambda, w) \longmapsto \big(\lambda^2 + q(w), \rho\lambda, -\rho w\big)$	$\rho(\lambda^2 + q(w))$
3	6	$\mathrm{Sym}_3(\mathbb{C})$	$M \longmapsto \mathrm{Adj}(M)$	$\det(M)$
3	9	$\mathrm{M}_3(\mathbb{C})$	$M \longmapsto \mathrm{Adj}(M)$	$\det(M)$
3	15	$\mathrm{Alt}_6(\mathbb{C})$	$M \longmapsto M^{\#}$	$\mathrm{Pf}(M)$
3	27	$\mathrm{Herm}_3(\mathbb{O} \otimes \mathbb{C})$	$M \longmapsto M^{\#}$	*cf.* (6)

TABLE 1. Explicit classification of semi-simple parts of Cremona transformations of bidegree $(2,2)$. In this table, q stands for a nondegenerate quadratic form on a non-trivial vector space W, $\mathrm{Adj}(M)$ is the usual adjoint matrix while $M^{\#}$ is the adjoint in the corresponding algebra; $\det(M)$ is the usual determinant while $\mathrm{Pf}(M)$ is the Pfaffian of an antisymmetric matrix; the last line is expressed using the general formalism of the theory of Jordan algebras (see section 2.3 and also Table 4.2).

As an application of the previous Proposition, we deduce two classification results. Let us recall that a homogeneous polynomial $P \in \mathbb{C}[x_1, \ldots, x_n]$ is called *homaloidal* if the associated polar map

$$P' = \left[\frac{\partial P}{\partial x_1} : \cdots : \frac{\partial P}{\partial x_n}\right] : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-1}$$

is birational. Let notation be as in Section 5.2.1, and set $\mathcal{B}_P = \mathcal{B}_{P'} = V(\frac{\partial P}{\partial x_1}, \ldots, \frac{\partial P}{\partial x_n}) \subset \mathbb{P}^{n-1}$. Assuming that P' has bidegree $(2,2)$, we know that there exists a cubic form N such that $V(N) \subset \mathbb{P}^{n-1}$ is the secant scheme of $\mathcal{B}_P$. Let us remark that since P' is birational, the partial derivates of P are linearly independent so that $V(P) \subset \mathbb{P}^{n-1}$ is not a cone. In particular if P has degree three, then it is necessarily a reduced polynomial. We first classify reducible homaloidal polynomials of degree three defining quadro-quadric Cremona transformations.

Corollary 5.8. *Let P be a cubic homaloidal polynomial in $n \geq 3$ variables such that P' is a quadro-quadric Cremona transformation. If P is reducible, then one of the following holds:*

- *$V(P) = Sec(\mathcal{B}_P)$ and P is linearly equivalent to the norm of the semi–simple (but not simple) rank 3 complex Jordan algebra of the fourth line in Table 1 above;*
- *$V(P) \neq Sec(\mathcal{B}_P)$ and $V(P)$ is the union of a smooth hyperquadric in $\mathbb{P}^{n-1}$ with a tangent hyperplane; in some coordinates, one has $P(x) = x_1(x_2^2 + \cdots + x_{n-1}^2 - x_1x_n)$ and $N(x) = x_1^3$.*

Proof. If P is the product of three distinct linear forms, then necessarily $n = 3$ and we are in the first case, see also the third case in Example 5.6.

Suppose that P is the product of a linear form ℓ with a quadratic form q. Without loss of generality we can assume $\ell = x_1$. Let $Q = V(q) \subset \mathbb{P}^{n-1}$. There is an inclusion of schemes $V(x_1) \cap Q \subset \mathcal{B}_P$ from which it follows that $V(x_1)$ is contained in the secant locus scheme of $\mathcal{B}_P$ so that it is contracted by P' and x_1 is an irreducible factor of $N(x)$.

If the hyperquadric Q is also contracted by P', then it is necessarily a cone with vertex a point and we can suppose, modulo constants, $N(x) = x_1 q(x) = P(x)$. Since $V(N) = V(P) \subset \mathbb{P}^{n-1}$ is not a cone, the rank 3 Jordan algebra $\mathbb{J}_{P'}$ has trivial radical by Proposition 5.3 hence is semi-simple. Since $N = P$ is reducible, $\mathbb{J}_{P'}$ is not simple hence we are in the case corresponding to the fourth line of Table 2.

Finaly, if Q is not contracted by P', then it is a smooth hyperquadric. In this case the hyperplane $V(x_1) \subset \mathbb{P}^{n-1}$ is necessarily tangent to Q at a point (otherwise $\mathcal{B}_P$ would be a degenerated smooth quadric $Q^{n-3} \subset \mathbb{P}^{n-2}$, which is impossible) and we are in the second case. In the isotopy class $[\mathbb{J}_{P'}]$ we can choose a representative such that $x^{\#} = (x_1^2, -x_1x_2, \ldots, -x_1x_{n-1}, x_2^2 + \cdots + x_{n-1}^2 - x_1x_n)$ and $N(x) = x_1^3$. □

Following [14], we will say that a homogeneous polynomial $P \in \mathbb{C}[x_1, \ldots, x_n]$ such that $\det(\mathrm{Hess}(\ln P)) \neq 0$ is *EKP–homaloidal* if its multiplicative Legendre transform P_* is again polynomial. In this case P_* is a homogeneous polynomial function too and $\deg(P) = \deg(P_*)$, see also [17] where this condition was investigated and studied. By the preliminary results of [17], a EKP–homaloidal polynomial is homaloidal and, after having identified $\mathbb{C}^n$ with its bidual, we have

$$\frac{P'_*}{P_*} \circ \frac{P'}{P} = Id_{\mathbb{C}^n}. \tag{19}$$

Therefore such a EKP–homaloidal polynomial of degree d defines a Cremona transformation of type (d, d). If moreover $d = 3$, it follows from (19) (combined with (3)) that we have $V(N) = V(P)$, that is $V(P)$ is the ramification locus scheme of P'. On the contrary as we shall see in the proof of Corollary 5.9 below, if P is a homaloidal cubic polynomial such that $V(N) = V(P)$, then it is EKP–homaloidal. Note that the EKP condition defined above is not satisfied by the reducible polynomials of the type described in the first case of Corollary 5.8 where $V(N)$ is a cubic hypersurface supported on the tangent hyperplane.

Corollary 5.9. *Let P be a homogeneous polynomial in $n \geq 3$ variables. The following assertions are equivalent:*

1. *P is a cubic EKP–homaloidal polynomial;*
2. *P is homaloidal, P' has bidegree $(2, 2)$ and $V(P) = V(N)$;*
3. *P is the norm of a semi–simple rank 3 complex Jordan algebra.*

When these assertions are verified, P is linearly equivalent to one of the norms in the last five lines of Table 5.2.4.

Proof. We have seen before that (1) implies (2). Assume that the latter is satisfied by P. Since P' has bidegree $(2, 2)$, the JC-correspondence ensures that (modulo composition by linear automorphisms), one can assume that P' is nothing but the adjoint map of a rank three complex Jordan algebra noted by $\mathbb{J}_P$. Since P is homaloidal, $V(N) = V(P) \subset \mathbb{P}^{n-1}$ is not a cone. By Proposition 5.3, this implies that the radical of $\mathbb{J}_P$ is trivial. Thus $\mathbb{J}_P$ is semi–simple and the conclusion follows from the classification recalled in Table 2. □

Remark 5.10. For $n = 3$ the two examples described in Corollary 5.8, modulo linear equivalence, are the unique homaloidal polynomials by a result of Dolgachev without any assumption on $\deg(P)$ and/or on P', see [14, Theorem 4]. For $n = 4$ there exists irreducible homaloidal polynomials of degree 3 whose associated Cremona transformation is of type $(2, 3)$. One such example is given by the equation of a special projection of the cubic scroll in $\mathbb{P}^4$ from a point lying in a plane generated by the directrix line and one of the lines of the ruling, see [11] for details and generalizations of this construction. For $n \geq 4$ there exist irreducible homaloidal polynomials of any degree $d \geq 2n - 5$, see [11].

More related to the above results is a very interesting series of irreducible cubic homaloidal polynomials communicated to us by A. Verra. The associated polar map is an involution and hence of type $(2, 2)$ but the ramification locus of these maps is different from the associated cubic hypersurface. The construction of these polynomials is described in [4] but the details about the geometry of their polar maps will be probably treated elsewhere.

5.2.5. **The general structure of quadro-quadric Cremona transformations.** In this section, we translate Theorem 5.1 into the C-world as explicitely as possible. We continue to use the notation introduced in 5.2.2.

It will be useful to denote by V_F (respectively V_G) the space V considered as the source space of the map F (respectively G). If A_f is a subset of $\mathbb{P}(V_F)$, we will denote by A_F the affine cone over A_f in V_F and we shall use the

analogue notation for subsets in $\mathbb{P}(V_G)$ and in V_G. One denotes by $\overline{V_F}$, respectively $\overline{V_G}$, the quotient space V_F/R_F, respectively V_G/R_G, and by $\pi_f : \mathbb{P}(V_F) \dashrightarrow \mathbb{P}(\overline{V_F})$, respectively $\pi_g : \mathbb{P}(V_G) \dashrightarrow \mathbb{P}(\overline{V_G})$, the rational map induced by the canonical linear projection.

The interpretation of part (1) in Theorem 5.1 has been explained in Section 5.2.2, see also part (1) in Theorem 5.14 below for a precise statement. In order to reinterpret part (2) and (3) of Theorem 5.1 it is necessary to introduce some notions and to recall some definitions.

Let $\mathrm{Str}(f)$ be the *structure group* of f (or rather of F) defined as in [38, Section 1.1]: by definition, $\mathrm{Str}(f)$ is the set of linear automorphisms $\theta \in GL(V_F)$ such that $F \circ \theta = \theta^{\#} \circ F$ for a certain $\theta^{\#} \in GL(V_F)$. Of course, $\mathrm{Str}(f)$ depends only on f and one verifies that it is an algebraic subgroup of $GL(V_F)$. Moreover, since f is invertible, $\theta^{\#}$ is uniquely determined by θ and one verifies easily that the map $\theta \mapsto \theta^{\#}$ is an isomorphism of algebraic groups from $\mathrm{Str}(f)$ onto $\mathrm{Str}(g)$.

Example 5.11 (continuation of Example 5.2). The structure group of $F : (a, b, c) \to (a^2, -ab, b^2 - ac)$ is the subgroup of invertible triangular inferior complex matrices $(m_{ij})_{i,j=1}^3$, whose diagonal entries satisfy $m_{11}m_{33} = (m_{22})^2$. Since $F \circ F(a, b, c) = a^3(a, b, c)$, the map $\theta \mapsto \theta^{\#}$ is an automorphism of $\mathrm{Str}(F)$. It is given by

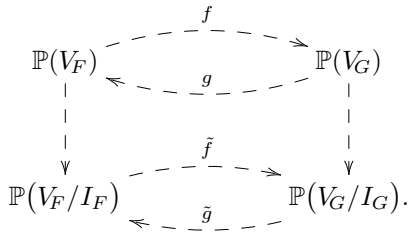
$$\begin{pmatrix} \frac{{m_{22}}^2}{m_{33}} & 0 & 0 \\ m_{21} & m_{22} & 0 \\ m_{31} & m_{32} & m_{33} \end{pmatrix} \longmapsto \begin{pmatrix} \frac{{m_{22}}^4}{{m_{33}}^2} & 0 & 0 \\ -\frac{{m_{22}}^2 m_{21}}{m_{33}} & \frac{{m_{22}}^3}{m_{33}} & 0 \\ \frac{-{m_{22}}^2 m_{31} + {m_{21}}^2 m_{33}}{m_{33}} & -\frac{m_{22}(-m_{22}m_{32} + 2\, m_{21}m_{33})}{m_{33}} & {m_{22}}^2 \end{pmatrix}.$$

Inspired by [38, Section 9], we define an *ideal of* f as a pair (I_f, I_g) of projective subspaces $I_f \subset \mathbb{P}(V_F)$ and $I_g \subset \mathbb{P}(V_G)$ (necessarily of the same dimension) such that

$$j_f(x + I_F) - j_f(x) \subset I_G \qquad \text{and} \qquad j_g(y + I_G) - j_g(y) \subset I_F \tag{20}$$

for $x \in V_F$ and $y \in V_G$ generic, where $j_f : V_F \dashrightarrow V_G$ and $j_g : V_G \dashrightarrow V_F$ are the rational maps considered in Section 3.2.2. In this case, j_f and j_g factor through I_F and I_G and their associated projectivization $\tilde{f}$ and $\tilde{g}$ are (normed) quadro-quadric Cremona transformations such that the following diagram commutes:

(21)



If (I_f, I_g) is an ideal then $f\big(\langle x, I_f\rangle\big) = \langle f(x), I_g\rangle$ for generic x in $\mathbb{P}(V_f)$. This implies that I_g is completely determined by I_f and vice-versa. Thus we can say that $I_f \subset \mathbb{P}(V_F)$ is an ideal of f and that $I_g \subset \mathbb{P}(V_G)$ is an ideal of g. We will say that I_f and I_g are *corresponding* ideals.

An ideal $I_f \subset \mathbb{P}(V_F)$ is *radical* if $I_f \subset R_f$. It is equivalent to the fact that $I_g \subset R_g$. Although the definition of ideal of a f as above was formulated in the affine setting, there is a projective characterization of radical ideals. If E_f and E_g are two projective subspaces of $\mathbb{P}(V_F)$ and $\mathbb{P}(V_G)$ respectively, one defines $df(E_f) \subset \mathbb{P}(V_G)$ as the projectivization of $dF_{V_F}(E_F) = F(V_F, E_F) \subset V_G$[1] and in the analogue way one defines $dg(E_g) \subset \mathbb{P}(V_F)$.

Proposition 5.12. *Assume that $E_f \subset R_f$ and $E_g \subset R_g$. Then the following assertions are equivalent:*

(1) E_f and E_g are corresponding radical ideals for f and g respectively;

(2) one has $df(E_f) \subset E_g$ and $dg(E_g) \subset E_f$.

Proof. Let $P_f : V_F \to \mathrm{End}(V_G, V_F)$ and $P_g : V_G \to \mathrm{End}(V_F, V_G)$ be the quadratic maps considered in Remark 3.5. For any subset $A_G \subset V_G$, one defines $P_f(A_G) \subset V_F$ as the span of the images of A_G by the maps $P_f(x)$ for x varying in V_F. We use the corresponding notation for $P_g(A_F)$ with $A_F \subset V_F$.

Adapting (17) to our setting and using (16), we deduce that $P_f(E_F) \subset E_G$ (resp. $P_g(E_G) \subset E_F$) if and only if $dF(E_F) \subset E_G$ (resp. $dG(E_G) \subset E_F$). Proposition 9.6 of [38], translated in our setting, ensures that E_f and E_g are

[1] When $dF_{V_F}(E_F) = 0$, one sets $df(E_f) = \emptyset$.

corresponding ideals if and only if $P_f(E_F) \subset E_G$ and $P_g(E_G) \subset E_F$, proving the equivalence of conditions *(1)* and *(2)*. □

Given a (normed) quadro-quadric Cremona transformation $\tilde{f}$, any quadro-quadric Cremona map f inducing $\tilde{f}$ on the quotient by the corresponding radical ideal $I_f \subset R_f$ and $I_g \subset R_g$ will be called a *radical extension* of $\tilde{f}$ by I_f (or by (I_f, I_g)). This (radical) extension is *null* if the restriction of f to a generic fiber of the canonical projection $\mathbb{P}(V_F) \dashrightarrow \mathbb{P}(V_F/I_F)$ is equivalent (as a rational map) to a linear map.

Example 5.13 (continuation of Example 5.2). Let f be the projectivization of the map $F : \mathbb{C}^3 \to \mathbb{C}^3$ defined by $F(a, b, c) = (a^2, -ab, b^2 - ac)$, that is nothing but the adjoint map of the algebra $\mathbb{C}[\varepsilon]/(\varepsilon^3)$ expressed in the basis $(1, \varepsilon, \varepsilon^2)$. Then f is a null radical extension of the normed quadro-quadric Cremona transformation $\tilde{f} : \mathbb{P}^1 \dashrightarrow \mathbb{P}^1$ defined as the projectivization of the quadratic map $\tilde{F} : (a, b) \mapsto (a^2, -ab)$ with associated cubic norm $\tilde{N}(a, b) = a^3$.

Let us define inductively a family of projective subspaces of $\mathbb{P}(V_F)$ and $\mathbb{P}(V_G)$ by setting $R_f^{[1]} = R_f$, $R_g^{[1]} = R_g$ and

$$R_f^{[k+1]} = dg\big(R_g^{[k]}\big) \subset \mathbb{P}\big(V_F\big)\,, \quad R_g^{[k+1]} = df\big(R_f^{[k]}\big) \subset \mathbb{P}\big(V_G\big) \qquad \text{for } k \geq 1.$$

By definition, $(R_f^{[k]})_{k\geq 0}$ is the *'Penico series'* of f. It follows from Proposition 5.12 that this is a decreasing series of radical ideals of $\mathbb{P}(V_f)$. Moreover, $R_f^{[k]}$ and $R_g^{[k]}$ are corresponding ideals for every $k \geq 1$. Then, passing to the quotients, the map f induces a normed quadro-quadric Cremona transformation

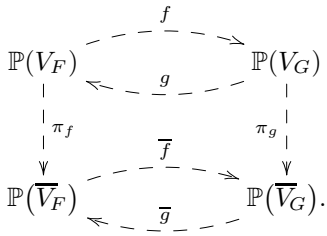
$$f^{[k]} : \mathbb{P}\big(\overline{V}_F^{[k]}\big) \dashrightarrow \mathbb{P}\big(\overline{V}_G^{[k]}\big)$$

for every $k \geq 1$, where $\overline{V}_F^{[k]}$ and $\overline{V}_G^{[k]}$ stand for the quotients spaces $V_F/R_F^{[k]}$ and $V_G/R_G^{[k]}$ respectively.

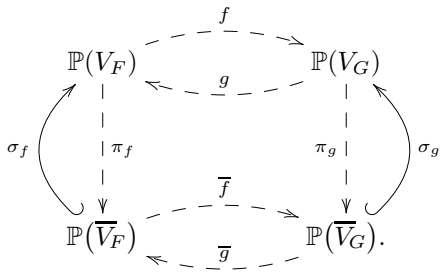
We can now state the 'C-version' of Theorem 5.1:

Theorem 5.14.

(1) The Cremona transformations f and g factor through R_f and R_g: there are semi-simple (normed) quadro-quadric Cremona transformations $\overline{f}$ and $\overline{g}$ such that the following diagram commutes



(2) There exist linear embeddings $\sigma_f : \mathbb{P}(\overline{V}_F) \hookrightarrow \mathbb{P}(V_F)$ and $\sigma_g : \mathbb{P}(\overline{V}_G) \hookrightarrow \mathbb{P}(V_G)$ whose images are linear spaces supplementary to R_f and R_g respectively, such that the diagram below commutes:



Moreover, the pair (σ_f, σ_g) is unique modulo the action of the structure group given by

$$\gamma \cdot (\sigma_f, \sigma_g) = (\gamma \circ \sigma_f, \gamma^{\#} \circ \sigma_g) \quad \textit{for } \gamma \in \mathrm{Str}(f).$$

(3) The radical R_f of f is solvable: there exists $t > 0$ such that $R_f^{[t]}$ is empty. Moreover, $f^{[\ell]}$ is a radical null extension of $f^{[\ell-1]}$ for $\ell = 2, \dots, t$ so that f can be obtained from its semi-simple part $\overline{f}$ by the successive series of non-trivial null radical extensions represented by the commutative diagram

$$\begin{array}{ccccccccc}
\mathbb{P}(V_F) & \dashrightarrow \cdots \dashrightarrow & \mathbb{P}(\overline{V}_F^{[\ell]}) & - - - \dashrightarrow & \mathbb{P}(\overline{V}_F^{[\ell-1]}) & \dashrightarrow \cdots \dashrightarrow & \mathbb{P}(\overline{V}_F) \\
\downarrow f=f^{[t]} & & \downarrow f^{[\ell]} & & \downarrow f^{[\ell-1]} & & \downarrow f^{[1]}=\overline{f} \\
\mathbb{P}(V_G) & \dashrightarrow \cdots \dashrightarrow & \mathbb{P}(\overline{V}_G^{[\ell]}) & - - - \dashrightarrow & \mathbb{P}(\overline{V}_G^{[\ell-1]}) & \dashrightarrow \cdots \dashrightarrow & \mathbb{P}(\overline{V}_G)\,,
\end{array}$$

where all the horizontal maps are the ones induced by the canonical linear projections $\overline{V}_F^{[\ell]} \to \overline{V}_F^{[\ell-1]}$.

Example 5.15 (continuation of Example 5.2). We apply the third part of the previous result to the projectivization f of the adjoint map $F(a,b,c) = (a^2, -ab, b^2 - ac)$ of $A = \mathbb{C}[\epsilon]/(\epsilon^3)$. The associated Penico series is $\emptyset = R_f^{[3]} \subset R_f^{[2]} = \mathbb{P}\langle \varepsilon^2 \rangle \subset R_f^{[1]} = \mathbb{P}R_A = \mathbb{P}\langle \epsilon, \epsilon^2 \rangle$. Hence one has $f^{[3]} = f$, $f^{[1]} = \overline{f}$ (see Example 5.5) and $f^{[2]} : \mathbb{P}^1 \dashrightarrow \mathbb{P}^1$ is nothing but the normed quadro-quadric Cremona transformation $\tilde{f}$ of Example 5.13.

We also point out an immediate consequence of the previous result in the affine setting.

Corollary 5.16. *Let F be an affine lift of a quadro-quadric Cremona transformation f. Set $R = R_F$ and $\overline{V} = V/R$. Then F is linearly equivalent to a quadratic map of the form*

$$\begin{array}{rcl}
\overline{V} \oplus R & \longrightarrow & \overline{V} \oplus R \\
(x, r) & \longmapsto & \big(\overline{F}(x), \mathcal{F}(x,r) + \mathscr{F}(r)\big)
\end{array}$$

where

- *$\overline{F}$ is the semi-simple part of F (so is equivalent to one of the quadratic maps in Table 5.2.4);*
- *$\mathcal{F} : \overline{V} \times R \to R$ is a bilinear map;*
- *$\mathscr{F} : R \to R$ is a quadratic map such that $\mathscr{G} \circ \mathscr{F} \equiv 0$ for another nontrivial quadratic map $\mathscr{G} : R \to R$.*

Moreover, f is a null extension of its semi-simple part if and only if the quadratic map $\mathscr{F}$ vanishes identically.

Example 5.17 (continuation of Example 5.2). Let us consider again the quadratic map $F(a,b,c) = (a^2, -ab, b^2 - ac)$. With the notation of the previous corollary, one has $V = A = \mathbb{C}[\epsilon]/(\epsilon^3)$, $\overline{V} = \mathbb{C}\,1$, $R = \epsilon\, A \simeq \mathbb{C}^2$, $\overline{F}(a) = a^2$, $\mathcal{F}(a,(b,c)) = (-ab, -ac)$ and $\mathscr{F}(b,c) = (0, b^2)$ for every $(a,b,c) \in V = \mathbb{C}^3$.

5.3. **The general structure of twisted cubics over Jordan algebras.** In this section, $X \subset \mathbb{P}^{2n+1}$ stands for a fixed element of $\boldsymbol{X}^n(3,3)$ with $n \geq 3$. According to the 'XJ-correspondence', one can assume that there exists a rank 3 Jordan algebra $\mathbb{J}$ of dimension n such that $X = X_{\mathbb{J}}$. In what follows, we will write $Z_2(\mathbb{J}) = \mathbb{C} \oplus \mathbb{J} \oplus \mathbb{J} \oplus \mathbb{C}$ for simplicity and (α, x, y, β) will stand for linear coordinates on $Z_2(\mathbb{J})$ corresponding to the decomposition in direct sum of the complex vector space $Z_2(\mathbb{J})$. According to our hypothesis, X is the closure of the image of the affine embedding $\mu = \mu_{\mathbb{J}} : \mathbb{J} \hookrightarrow \mathbb{P}Z_2(\mathbb{J}) : x \mapsto [1 : x : x^{\#} : N(x)]$ considered in Section 3.1. Moreover, there exists a family Σ_X of twisted cubics included in X, which is 3-covering and unique.

In order to state the 'X-version' of Theorem 5.1 we shall introduce some terminology and recall some preliminary results.

5.3.1. **The conformal group.** The *structure group* $\mathrm{Str}(\mathbb{J})$ of $\mathbb{J}$ is the algebraic subgroup $\mathrm{Str}(\#_{\mathbb{J}})$ of $GL(\mathbb{J})$ associated to the adjoint map of $\mathbb{J}$ defined in Section 5.2.5 above. It can be verified that there exists a non-trivial character $\gamma \mapsto \eta_\gamma$ on $\mathrm{Str}(\mathbb{J})$ such that $N(\gamma(x)) = \eta_\gamma N(x)$ for every $\mathbb{J}$ and every $\gamma \in \mathrm{Str}(\mathbb{J})$. Then one defines the *conformal group* of $\mathbb{J}$, denoted by $\mathrm{Conf}(\mathbb{J})$, as the subgroup of the group of affine birational transformations of $\mathbb{J}$ generated by $\mathrm{Str}(\#_{\mathbb{J}})$, by the inversion $j : x \dashrightarrow x^{-1}$ and by the translations $t_w : x \mapsto x + w$ (with $w \in \mathbb{J}$).

The projective representation $\rho : \mathrm{Conf}(\mathbb{J}) \to PGL(Z_2(\mathbb{J}))$ is defined in the following way:

$$\begin{aligned}
\rho(j) \cdot \theta &= [\beta : y : x : \alpha] \\
\rho(\gamma) \cdot \theta &= [\alpha : \gamma(x) : \gamma^{\#}(y) : \eta_\gamma \beta] \\
\text{and } \rho(t_w) \cdot \theta &= [\alpha : x + \alpha w : y + x\#w + \alpha w^{\#} : \beta + T(y,w) + T(x,w^{\#}) + \alpha N(w)]
\end{aligned}$$

for every $\theta = [\alpha : x : y : \beta] \in \mathbb{P}Z_2(\mathbb{J})$, $\gamma \in \mathrm{Str}(\mathbb{J})$ and $w \in \mathbb{J}$. It can be verified that $\rho(j) \cdot \mu = \mu \circ j$, $\rho(\gamma) \cdot \mu = \mu \circ \gamma$ and $\rho(t_w) \cdot \mu = \mu \circ t_w$ for every structural transformation γ and every translation t_w. This implies that the image of ρ in $PGL(Z_2(\mathbb{J}))$ is contained in the group $\mathrm{Aut}(X)$ of projective automorphisms of X.

Proposition 5.18. *The representation $\rho : \mathrm{Conf}(\mathbb{J}) \to PGL(Z_2(\mathbb{J}))$ is faithful and $\rho(\mathrm{Conf}(\mathbb{J})) = \mathrm{Aut}(X)$.*

Proof. Let $\varphi \in \mathrm{Aut}(X)$. Being projective, it induces an automorphism of Σ_X, again denoted by φ. The twisted cubic $C_0 = \{[s^3 : s^2te : st^2\, e : t^2] \in \mathbb{P}Z_2(\mathbb{J}) \,\big|\, [s : t] \in \mathbb{P}^1\}$ is included in X and passes through the three points $0_X = \mu(0) = [1 : 0 : 0 : 0]$, $e_X = \mu(e) = [1 : e : e : 1]$ and $\infty_X = \mu(\infty) = [0 : 0 : 0 : 1]$ of X. Since $\mathrm{Conf}(\mathbb{J})$ acts transitively on generic 3-uples of points in X (see [31, Proposition 4.7] for instance), it comes that the orbit $\Gamma = \mathrm{Conf}(\mathbb{J}) \cdot C_0$ of C_0 is dense in the 3-covering family Σ_X of cubics included in X. Thus there exists $C \in \Gamma$ such that $\varphi(C) \in \Gamma$ and, modulo compositions on the left and on the right by conformal automorphisms of X, we can assume $\varphi(C_0) = C_0$. Moreover, since the subgroup of conformal transformations of X fixing C_0 acts as $\mathrm{Aut}(C_0) \simeq PGL_2(\mathbb{C})$ on $C_0 \simeq \mathbb{P}^1$, we can also suppose that φ fixes the points 0_X and ∞_X of C_0. It follows that φ induces an automorphism of the subfamily Σ_X^* of Σ_X formed by the twisted cubics included in X and passing through 0_X and ∞_X.

Let us denote by V^+ and V^- the abstract tangent spaces of X at 0_X and ∞_X respectively. The differential $\varphi_0 = d\varphi_{0_X}$ (resp. $\varphi_\infty = d\varphi_{\infty_X}$) of φ at 0_X (resp. at ∞_X) is a linear automorphism of V^+ (resp. of V^-). Then reasoning as in the proof of Theorem 3.7 we deduce that $(\varphi_0, \varphi_\infty)$ is an automorphism of the Jordan pair (V^+, V^-) constructed there. Taking $u = \exp^{-1}(e_X) \in V^-$ as invertible element (see the notation at the end of the proof of Theorem 3.7), one obtains that $V_u^+ = \mathbb{J}$ as Jordan algebras. It follows then from [24, Proposition 1.8] that $\varphi_0 \in \mathrm{Str}(\mathbb{J})$ and $\varphi_\infty = (\varphi_0^\#)^{-1}$.

Let us now prove that the action of $\psi_0 = \rho(\varphi_0)$ on X coincides with that of φ. Let $x \in X$ be a general point. By hypothesis, there exists a twisted cubic $C_x \in \Sigma_X^*$ passing through x that is unique according to [33] or [31, Theorem 2.4 (1)]. This implies that the tangent map $\Sigma_X^* \to \mathbb{P}(V^+)$ that associates to C_x its projective tangent line at 0_X is 1-1 onto its image. In particular, this gives us that $\varphi(C_x) = \psi_0(C_x)$. Since $\varphi_x = \varphi|_{C_x} \colon C_x \to \varphi(C_x)$ is a projective isomorphism that lets 0_X and ∞_X fixed, it is completely determined by its differential at one point, for instance at 0_X. Since $d(\varphi)_{0_X} = d(\psi_0)_{0_X} = \varphi_0$, this shows that φ_x and $\psi_0|_{C_x}$ coincides so that $\psi_0(x) = \varphi_x(x) = \varphi(x)$. From the generality of $x \in X$, we deduce that $\varphi = \psi_0$. Since $\psi_0 = \rho(\varphi_0)$ with $\varphi_0 \in \mathrm{Str}(\mathbb{J})$, this gives us that $\rho(\mathrm{Conf}(\mathbb{J})) = \mathrm{Aut}(X)$.

Finally, let $\nu \in \mathrm{Conf}(\mathbb{J})$ such that $\rho(\nu) = \mathrm{Id}_X$. From $\mu = \rho(\nu) \cdot \mu = \mu \circ \nu$ one gets that $x = \nu(x)$ for $x \in \mathbb{J}$ generic, that is $\nu = \mathrm{Id}_{\mathbb{J}}$. Thus ρ is faithful and the result is proved. □

A different proof of the previous result is given in [18] for the case when $\mathbb{J} = \mathrm{Herm}_3(\mathbb{O}_{\mathbb{C}})$. The proof therein clearly applies to all twisted cubics over semi-simple rank 3 Jordan algebras but it is not clear wheter it can be applied to the general case. We have included the proof above because we were unaware of any proof of Proposition 5.18 in the literature, despite this result is certainly well-known to the experts of this field.

Remark 5.19. The proof of Proposition 5.18 also shows that the subgroup of projective automorphisms of X fixing two (resp. three) general points of X is isomorphic to the structure group (resp. to the automorphism group) of the Jordan algebra $\mathbb{J}$. This is related to the considerations in Remark 3.3.2.

5.3.2. **The radical and the semi-simple part.** We use again here some notation and construction introduced in the proof of Theorem 3.7: x^+ and x^- are two general points on X such that X is 1-RC by the family $\Sigma_{x^+x^-}$ of twisted cubics included in X passing through x^+ and x^-. For $\sigma = \pm$, one defines a rational map $F^\sigma : V^\sigma \dashrightarrow V^{-\sigma}$ by setting

$$F^\sigma(v) = d\alpha_v^\sigma(1 : t)/dt\big|_{t=0}$$

for $v \in V^\sigma$, where $\alpha_v^\sigma : \mathbb{P}^1 \to X$ is the projective parametrization of a twisted cubic belonging to $\Sigma_{x^+x^-}$ such that $\alpha_v^\sigma(0 : 1) = x^\sigma$, $\alpha_v^\sigma(1 : 0) = x^{-\sigma}$ and $d\alpha_v^\sigma(s : 1)/ds|_{s=0} = v$. The map F^σ is homogeneous of degree -1 and $F^\sigma \circ F^{-\sigma} = \mathrm{Id}_{V^\sigma}$ for every $\sigma = \pm$. Thus the associated projectivization $f^\sigma : \mathbb{P}(V^\sigma) \dashrightarrow \mathbb{P}(V^{-\sigma})$ of F^σ is a Cremona transformation with inverse $f^{-\sigma} : \mathbb{P}(V^{-\sigma}) \dashrightarrow \mathbb{P}(V^\sigma)$. It can be verified that f^σ has bidegree $(2, 2)$ and that it is nothing but the map φ_{X,x^σ} defined in Section 3.3 (up to linear equivalence).

Let $R_{f^\sigma} \subset \mathbb{P}(V^\sigma)$ be the radical of f^σ as defined in Section 5.2.1. Since $\mathbb{P}(V^\sigma)$ identifies canonically with the projective quotient $T_{x^\sigma}X/\langle x^\sigma \rangle$, one can define the cone $R_{x^\sigma} \subset T_{x^\sigma}X$ over R_{f^σ} with vertex x^σ (alternatively, R_{x^σ}

can be defined as the closure of the radical $R_{F^\sigma} \subset V^\sigma$ of F^σ in the natural affine embedding $V^\sigma \subset T_{x^\sigma}X$). By definition, the radical $R_{x^+x^-}$ of X relatively to the pair (x^+, x^-) is the direct sum of R_{x^+} and R_{x^-} in $\mathbb{P}^{2n+1}$:

$$R_{x^+x^-} = R_{x^+} \oplus R_{x^-} \subset T_{x^+}X \oplus T_{x^-}X = \langle X \rangle = \mathbb{P}^{2n+1}.$$

A straightforward verification proves the following result.

Lemma 5.20. *The radical $R_{x^+x^-}$ does not depend on the pair (x^+, x^-) but only on X.*

We can thus define the *radical of* X as the projective subspace $R_X = R_{x^+x^-} \subset \mathbb{P}^{2n+1}$ for any generic pair (x^+, x^-) of elements of X. If r is the dimension of the radical of $\mathbb{J}$ then R_X is a projective subspace of dimension $2r-1$ in $\mathbb{P}^{2n+1}$ that is projectively attached to X, that is one has $\varphi(R_X) = R_X$ for every $\varphi \in \mathrm{Aut}(X)$.

The preceding definition of the radical of X makes quite explicit the link with the corresponding notion in the C-world (hence in the J-world). Notwithstanding we think it is interesting to provide a purely projective definition of R_X. To this end, we first remark that since two generic projective tangent spaces of X are in direct sum, $X \subset \mathbb{P}^{2n+1}$ has the secant variety $\sigma(X)$ filling the whole space by Terracini Lemma, *i.e.* $\sigma(X) = \mathbb{P}^{2n+1}$. Moreover X is also tangentially non-degenerate, *i.e.* the tangent variety $\tau(X) \subset \mathbb{P}^{2n+1}$ of X, defined as the closure of the union of the lines tangent to the smooth locus of X, is a hypersurface in $\mathbb{P}^{2n+1}$.

Lemma 5.21. *The tangent variety $\tau(X)$ is the hypersurface in $\mathbb{P}^{2n+1}$ cut out by the irreducible quartic form*

$$Q\big(\alpha, x, y, \beta\big) = T(x^\#, y^\#) - \beta N(x) - \alpha N(y) - \frac{1}{4}\big(T(x,y) - \alpha\beta\big)^2.$$

Proof. Since $\tau(X)$ is irreducible and singular along X and since $\sigma(X) = \mathbb{P}^{2n+1}$, we have $\deg(\tau(X)) \geq 4$. Indeed, if $\deg(\tau(X)) = 2$, then $X \subset \mathbb{P}^{2n+1}$ would be degenerated being contained in $\mathrm{Sing}(\tau(X))$. If $\deg(\tau(X)) = 3$, then the secant variety of X would be contained in $\tau(X)$ because $\tau(X)$ is singular along X. Since $V(Q) \subset \mathbb{P}^{2n+1}$ is a quartic hypersurface to prove that Q is irreducible and that $\tau(X) = V(Q)$ it will be sufficient to show that $\tau(X) \subset V(Q)$.

The quartic form Q is invariant for the action of the conformal group of X on $\mathbb{P}^{2n+1}$ (the proofs given in [18] or in [12, Section 7] concern a priori only the semi-simple cases but can be applied in full generality). Since the orbit of $0_X \in X$ under the action of $\mathrm{Aut}(X)$ is Zariski-open in X, it is enough to prove that a line in $\mathbb{P}^{2n+1}$ tangent to X at 0_X is included in $V(Q)$. A point of such a line has homogeneous coordinates $p_v = (1, e+v, e+e\#v, 1+T(v))$ with $v \in \mathbb{J}$. A straightforward (but a bit lenght) computation implies that $Q(p_v) = 0$ for every v, proving the result. □

Proposition 5.22. *The tangent hypersurface $\tau(X)$ is a quartic cone of vertex R_X.*

Proof. It follows from (16) that for any $r_x, r_y \in R = \mathrm{Rad}(\mathbb{J})$, one has $Q(\alpha, x + r_x, y + r_y, \beta) = Q(\omega)$ for every $\omega = (\alpha, x, y, \beta) \in Z_2(\mathbb{J})$. This proves that R_X is included in the vertex $V(d^3Q)$ of $V(Q) = \tau(X)$.

Conversely, let $(\partial_{x_1}, \ldots, \partial_{x_n})$ (resp. $(\partial_{y_1}, \ldots, \partial_{y_n})$) be the system of partial derivatives naturally associated to a system of linear coordinates on the first (resp. on the second) $\mathbb{J}$-summand of $Z_2(\mathbb{J})$. The relations $\partial_\alpha^2 \partial_\beta Q(\omega) = \partial_\alpha \partial_\beta^2 Q(\omega) = 0$ imply that $\alpha = \beta = 0$. The set of relations $\partial_\alpha \partial^2_{x_i x_j} Q(\omega) = -\partial_{x_i}\partial_{x_j} N(x) = 0$, $i, j = 1, \ldots, n$, can be summarized by $d^2 N_x = 0$, that is $x \in R = \mathrm{Rad}(\mathbb{J})$ according to (16). Arguing similarly for y, one obtains that $d^3 Q_\omega = 0$ implies that $\omega = (0, x, y, 0)$ with $x, y \in R$. This proves that the vertex $V(d^3Q)$ of $\tau(X)$ is included in R_X and finishes the proof. □

In what follows, let $\mathbb{J} = \mathbb{J}_{\mathrm{ss}} \oplus R$ be the decomposition given in point *(2)* of Theorem 5.1, where the embedding of $\mathbb{J}_{\mathrm{ss}} = \mathbb{J}/R \hookrightarrow \mathbb{J}$ has been fixed once for all (hence is not indicated to simplify). A straightforward verification gives that R_X is nothing but the projectivization $\mathbb{P}(0 \oplus R \oplus R \oplus 0) \subset \mathbb{P}Z_2(\mathbb{J})$. Hence setting $\mu_{\mathrm{ss}} = \mu_{\mathbb{J}_{\mathrm{ss}}}$ and $X_{\mathrm{ss}} = X_{\mathbb{J}_{\mathrm{ss}}}$, we obtain the commutative diagram

$$\begin{array}{ccc} \mathbb{J} & \xrightarrow{\mu = \mu_{\mathbb{J}}} & X \subset \mathbb{P}Z_2(\mathbb{J}) \\ {\scriptstyle \pi_R}\downarrow & & \downarrow{\scriptstyle \pi_{R_X}} \\ \mathbb{J}_{\mathrm{ss}} & \xrightarrow{\mu_{\mathrm{ss}}} & X_{\mathrm{ss}} \subset \mathbb{P}Z_2(\mathbb{J}_{\mathrm{ss}}), \end{array}$$

where π_R stands for the canonical linear projection $\mathbb{J} \to \mathbb{J}_{\mathrm{ss}} = \mathbb{J}/R$ and where π_{R_X} denotes the restriction to X of the linear projection $\mathbb{P}Z_2(\mathbb{J}) \dashrightarrow \mathbb{P}Z_2(\mathbb{J}_{\mathrm{ss}})$ from the radical R_X of X. Since π_R is surjective, this shows that $\pi_{R_X}(X) = X_{\mathrm{ss}}$. By definition, X_{ss} is the *semi-simple part* of X.

We have the following result, based on the classification of smooth varieties $X \in \overline{\boldsymbol{X}}(3,3)$.

Proposition 5.23. *The semi-simple part $X_{\rm ss}$ of X belongs to $\overline{\boldsymbol{X}}(3,3)$ and is smooth. Moreover*

- *$r_{\rm ss}(\mathbb{J})=3$ if and only if $X_{\rm ss}\in\boldsymbol{X}(3,3)$ hence is one of the varieties of the table of Theorem 4.4;*
- *$r_{\rm ss}(\mathbb{J})=2$ if and only if $X_{\rm ss}$ is a scroll $S_{1\dots122}$ of $S_{1\dots13}$ (in particular, $\dim(X)>1$);*
- *$r_{\rm ss}(\mathbb{J})=1$ if and only if $X_{\rm ss}$ is the twisted cubic $v_3(\mathbb{P}^1)\subset\mathbb{P}^3$.*

Example 5.24 (continuation of Example 5.2). The radical of the cubic curve $X_A\subset\mathbb{P}Z_2(A)=\mathbb{P}^7$ over $A=\mathbb{C}[\epsilon]/(\epsilon^3)$ is the 3-dimensional linear subspace $R_{X_A}=\{[0:r:r':0]\,|\,r,r'\in R_A\}\subset\mathbb{P}^7$. The projection from R_{X_A} induces a dominant rational map from X_A onto the twisted cubic curve $v_3(\mathbb{P}^1)\subset\mathbb{P}^3$ which is thus the semi-simple part of X_A.

Example 5.25. Let $\mathbb{H}_{\mathbb{C}}$ be the complexification of the (real) algebra $\mathbb{H}$ of quaternions. Then $\mathbb{H}_{\mathbb{C}}$ can (and will) be identified with the complex algebra $M_2(\mathbb{C})$ of 2×2 complex matrices. For $M\in\mathbb{H}_{\mathbb{C}}$, let $\overline{M}$ stands for the adjoint $M-T(M)\mathbf{Id}$. By definition, the algebra of (complex) *sextonions* is the vector space $\mathbb{S}_{\mathbb{C}}=\mathbb{H}_{\mathbb{C}}\oplus M_{2\times1}(\mathbb{C})$ together with the product defined by $(M,u)\cdot(N,v)=(MN,\overline{M}v+Nu)$ for $M,N\in\mathbb{H}_{\mathbb{C}}$ and $u,v\in M_{2\times1}(\mathbb{C})$, see [22, 39]. This algebra can be embedded in the complexification $\mathbb{O}_{\mathbb{C}}$ of the octonions so that it is alternative and it has an involution given by $\overline{(M,u)}=(\overline{M},-u)$ for $(M,u)\in\mathbb{S}_{\mathbb{C}}$. One then defines the algebra $\mathrm{Herm}_3(\mathbb{S}_{\mathbb{C}})$ (see Example 2.3.(3)), which is a rank 3 Jordan algebra of dimension 21.

The cubic curve $X=X_{\mathrm{Herm}_3(\mathbb{S}_{\mathbb{C}})}\subset\mathbb{P}^{43}$ over $\mathrm{Herm}_3(\mathbb{S}_{\mathbb{C}})$ is singular along a smooth 10-dimensional quadric hypersurface, which spans the radical R_X of X. Thus $\dim(R_X)=11$ (see [22, Corollary 8.14]). The semi-simple part of $X_{\mathrm{Herm}_3(\mathbb{S})}$ is the orthogonal grassmannian variety $X_{\mathrm{Herm}_3(\mathbb{H}_{\mathbb{C}})}=OG_6(\mathbb{C}^{12})\subset\mathbb{P}^{31}$.

5.3.3. **Radical ideals and extensions.** Let $\boldsymbol{I}$ be a proper projective subspace of $\mathbb{P}^{2n+1}$ and denote by $\pi_{\boldsymbol{I}}:\mathbb{P}^{2n+1}\dashrightarrow\mathbb{P}^m$ the linear projection from $\boldsymbol{I}$. By definition, $\boldsymbol{I}$ is a *radical ideal* for (or of) X if $\boldsymbol{I}$ is included in the radical R_X of X and if the restriction of $\pi_{\boldsymbol{I}}$ to X, again denoted by $\pi_{\boldsymbol{I}}$, is such that $\pi_{\boldsymbol{I}}(X)=\tilde{X}$ is still 3-RC by twisted cubics and extremal. Since $X_{\rm ss}\in\overline{\boldsymbol{X}}(3,3)$, R_X itself is a radical ideal for X.

Let $I\subset\mathbb{J}$ be a radical ideal of the Jordan algebra $\mathbb{J}$. Then $\overline{\mathbb{J}}=\mathbb{J}/I$ is a cubic Jordan algebra hence one can define the associated twisted cubic $X_{\overline{\mathbb{J}}}\subset\mathbb{P}Z_2(\overline{\mathbb{J}})\in\overline{\boldsymbol{X}}(3,3)$. Then $I\!I=\mathbb{P}(0\oplus I\oplus I\oplus0)\subset\mathbb{P}Z_2(\mathbb{J})$ is a radical ideal of $X_{\overline{\mathbb{J}}}$. More generally, the image of such a $I\!I$ by any element of $\mathrm{Conf}(X)$ is again a radical ideal for X.

Proposition 5.26. *Any radical ideal of X comes from a radical ideal of $\mathbb{J}$ by the construction presented above.*

Proof. We continue to use the notation introduced above for X and the tilded versions of these will stand for the corresponding notation for $\tilde{X}$.

Let $I\!I\subset\mathbb{P}^{2n+1}$ be a non-trivial radical ideal for X. Let x^+ and x^- be two general points on X. For $\sigma=\pm$, one denotes by π^σ the differential of $\pi_{I\!I}$ at x^σ: since x^σ is general, it is a well-defined surjective linear map from V^σ onto $\tilde{V}^\sigma$ whose kernel I^σ has dimension i. We want to prove that $I=(I^+,I^-)$ is an ideal of the Jordan pair $V=(V^+,V^-)$ such that $V/I=(V^+/I^+,V^-/I^-)$ is isomorphic to $\tilde{V}=(\tilde{V}^+,\tilde{V}^-)$ (as Jordan pairs).

Let Σ (resp. $\tilde{\Sigma}$) be the 3-covering family of twisted cubics on X (resp. on $\tilde{X}$). Then for $C\in\Sigma$ general, its image $\pi_{I\!I}(C)$ by the linear projection $\pi_{I\!I}$ is an irreducible rational curve of degree ≤3 included in $\tilde{X}$. Since the family $\pi_{I\!I}(\Sigma)=\{\pi_{I\!I}(C)\}_{C\in\Sigma}$ is also 3-covering, it comes from [33, Lemme 2.1] that $\pi_{I\!I}(\Sigma)=\tilde{\Sigma}$. Since x^+ and x^- are general points of X, we deduce $\pi_{I\!I}(\Sigma_{x^+x^-})=\{\pi_{I\!I}(C)\,|\,C\in\Sigma_{x^+x^-}\}=\tilde{\Sigma}_{\tilde{x}^+\tilde{x}^-}$. Let $v\in V^\sigma$ be such that F^σ and $\tilde{F}^\sigma$ are defined at v and $\tilde{v}=\pi^\sigma(v)$ respectively. Let $\alpha_v:\mathbb{P}^1\to X$ be the projective parametrization of the twisted cubic element of $\Sigma_{x^+x^-}$ such that $d\alpha_v(s:1)/ds|_{s=0}=v$ (see Section 5.3.2). Then $\tilde{\alpha}_v=\pi_{I\!I}\circ\alpha_v:\mathbb{P}^1\to\tilde{X}$ is a projective parametrization of a twisted cubic in $\tilde{X}$ such that $\tilde{\alpha}_v(0)=\tilde{x}^\sigma$, $\tilde{\alpha}_v(\infty)=\tilde{x}^{-\sigma}$ and $d\tilde{\alpha}_v(s:1)/ds|_{s=0}=\pi^\sigma(v)=\tilde{v}$: with the notation of Section 5.3.2, one has $\tilde{\alpha}_v=\alpha_{\tilde{v}}$. This implies that $\tilde{F}^\sigma\circ\pi^\sigma=\pi^{-\sigma}\circ F^\sigma$ for $\sigma=\pm$. Taking total derivatives, one obtains $d(\tilde{F}^\sigma)_{\pi^\sigma(\cdot)}\circ\pi^\sigma=\pi^{-\sigma}\circ dF^\sigma$. Combined with the fact that $F^{-\sigma}\circ F^\sigma=\mathrm{Id}_{V^\sigma}$ and $\tilde{F}^{-\sigma}\circ\tilde{F}^\sigma=\mathrm{Id}_{\tilde{V}^\sigma}$, this gives that for $v\in V^\sigma$ general:

$$\begin{aligned}\big(d(\tilde{F}^\sigma)_{\pi^\sigma(v)}\big)^{-1}\circ\pi^{-\sigma}&=d(\tilde{F}^{-\sigma})_{\tilde{F}^\sigma\circ\pi^\sigma(v)}\circ\pi^{-\sigma}\\&=d(\tilde{F}^{-\sigma})_{\pi^{-\sigma}\circ F^\sigma(v)}\circ\pi^{-\sigma}\\&=d(\tilde{F}^{-\sigma}\circ\pi^{-\sigma})_{F^\sigma(v)}=d(\pi^\sigma\circ F^{-\sigma})_{F^\sigma(v)}=\pi^\sigma\circ\big(d(F^\sigma)_v\big)^{-1}.\end{aligned}$$

By density, this series of equalities implies that for $\sigma=\pm$, one has

$$\tilde{P}^\sigma_{\pi^\sigma(v)}\circ\pi^{-\sigma}=\pi^\sigma\circ P^\sigma_v$$

for every $v \in V^\sigma$, where P^σ and $\tilde{P}^\sigma$ stand for the quadratic operators of the Jordan pairs V and $\tilde{V}$ respectively (see Remark 3.5). According to Definition 1.3 in [24], this means that $\pi = (\pi^+, \pi^-) : V \to \tilde{V}$ is a surjective morphism of Jordan pairs. Consequently, $\ker(\pi) = (I^+, I^-)$ is an ideal of V and $V/I \simeq \tilde{V}$. For $\sigma = \pm$, let $\boldsymbol{I}^\sigma$ be the closure of I^σ in $V^\sigma \subset T_{x^\sigma}X \subset \mathbb{P}^{2n+1}$. We let the reader verify that the radical ideal $\mathit{I\!I}$ from the beginning is nothing but the direct sum $\boldsymbol{I}^+ \oplus \boldsymbol{I}^- \subset T_{x^+}X \oplus T_{x^-}X = \mathbb{P}^{2n+1}$, concluding the proof. □

If $\mathit{I\!I} \subset \mathbb{P}^{2n+1}$ is a radical ideal for X, we will say that X is a *(radical) extension* of $\tilde{X} = \pi_{\mathit{I\!I}}(X) \in \overline{\boldsymbol{X}}(3,3)$ by $\mathit{I\!I}$. This extension is *null* if the generic fiber of $\pi_{\mathit{I\!I}} : X \dashrightarrow \tilde{X}$ is a linear subspace in $\mathbb{P}^{2n+1}$. It is *split* if there exists a linear embedding $\boldsymbol{\iota} : \langle \tilde{X} \rangle = \mathbb{P}^{2\tilde{n}+1} \hookrightarrow \mathbb{P}^{2n+1}$ the image of which is supplementary to $\mathit{I\!I}$ and is such that $\pi_{\mathit{I\!I}} \circ \sigma$ induces the identity when restricted to $\tilde{X}$.

Proposition 5.27. *The XJ-correspondence induces correspondences between null (respectively split) extensions in the J-world and in the X-world.*

Proof. Let $\pi_{\mathit{I\!I}} : X \dashrightarrow \tilde{X}$ be a (radical) extension in the X-world corresponding to a radical extension $0 \to I \to \mathbb{J} \to \tilde{\mathbb{J}} = \mathbb{J}/I \to 0$ in the J-world (we use here Proposition 5.26). One can assume that $X = X_{\mathbb{J}}$ and $\tilde{X} = X_{\tilde{\mathbb{J}}}$ and that the projection from $\mathit{I\!I}$ is induced by the linear map $Z_2(\mathbb{J}) \to Z_2(\tilde{\mathbb{J}})$ coming from the canonical projection $\mathbb{J} \to \mathbb{J}/I$. Let us denote by $\tilde{x}$ the class in $\tilde{\mathbb{J}}$ of an element $x \in \mathbb{J}$. Since I is assumed to be a radical ideal, $x\#i$ and $i^\#$ belong to I for every $x \in \mathbb{J}$ and $i \in I$. In particular the class of $x^\#$ in $\mathbb{J}/I$ is $\tilde{x}^\#$. From this it follows that $\pi_{\mathit{I\!I}} : X_{\mathbb{J}} \dashrightarrow X_{\mathbb{J}/I}$ is given by $[1 : x : x^\# : N(x)] \longmapsto [1 : \tilde{x} : \tilde{x}^\# : N(x)]$ hence for any $x \in \mathbb{J}$, the fiber of $\pi_{\mathit{I\!I}} : X \dashrightarrow \tilde{X}$ over $\tilde{p}_x = [1 : \tilde{x} : \tilde{x}^\# : N(x)] \in \tilde{X}$ is

$$\pi_{\mathit{I\!I}}^{-1}(\tilde{p}_x) = \overline{\left\{ [1 : x + i : x^\# + x\#i + i^\# : N(x)] \,\middle|\, i \in I \right\}} \subset \mathbb{P}Z_2(\mathbb{J}).$$

This fiber is a linear subspace in $\mathbb{P}Z_2(\mathbb{J})$ if and only if $i^\# = 0$ for every $i \in I$. Since I is radical, $i^\# = i^2$ for every $i \in I$ so that $i^\# = 0$ for every $i \in I$ if and only if $I^2 = 0$ (remember that any product in I can be expressed as a linear combination of squares). This proves the proposition for the case of null extensions.

We now consider the case of split extensions. Clearly a split extension in the J-world yields a split extension in the X-world. On the contrary, assume that $\boldsymbol{\iota} : \langle \tilde{X} \rangle \hookrightarrow \mathbb{P}^{2n+1}$ is a splitting of an extension $\pi_{\mathit{I\!I}} : X \dashrightarrow \tilde{X}$. If $\tilde{x}^+$ and $\tilde{x}^-$ are two general points of $\tilde{X}$, then $x^+ = \boldsymbol{\iota}(\tilde{x}^+)$ and $x^- = \boldsymbol{\iota}(\tilde{x}^-)$ are two points of X for which the construction of the Jordan pair (V^+, V^-) described in the proof of Proposition 5.26 can be performed. For $\sigma = \pm$, let $\iota^\sigma : \tilde{V}^\sigma \hookrightarrow V^\sigma$ be the differential at $\tilde{x}^\sigma$ of the restriction of $\boldsymbol{\iota}$ to $\tilde{X}$. Since $\boldsymbol{\iota}$ is a linear embedding, it sends any twisted cubic included in $\tilde{X}$ onto a twisted cubic in X. From this, one deduces that $\iota = (\iota^+, \iota^-) : \tilde{V} \to V$ is an injective morphism of Jordan pairs. Finally, from the fact that $\mathrm{Im}(\boldsymbol{\iota})$ and $\mathit{I\!I}$ are supplementary in $\mathbb{P}Z_2(\mathbb{J})$, one deduces that $\iota : \tilde{V} \to V$ gives a splitting of the extension of Jordan pairs $I \hookrightarrow V \overset{\pi}{\twoheadrightarrow} \tilde{V}$, concluding the proof of all the assertions. □

Example 5.28 (continuation of Example 5.25). The decomposition $\mathbb{S}_{\mathbb{C}} = \mathbb{H}_{\mathbb{C}} \oplus \mathbb{U}_{\mathbb{C}}$ (with $\mathbb{U}_{\mathbb{C}} = M_{2\times 1}(\mathbb{C})$) induces a decomposition in direct sum $\mathrm{Herm}_3(\mathbb{S}_{\mathbb{C}}) = \mathrm{Herm}_3(\mathbb{H}_{\mathbb{C}}) \oplus \mathrm{Alt}_3(\mathbb{U}_{\mathbb{C}})$ where $\mathrm{Alt}_3(\mathbb{U}_{\mathbb{C}})$ is the space of antisymmetric 3×3 matrices with coefficients in $\mathbb{U}_{\mathbb{C}}$. One has $\mathrm{Rad}(\mathrm{Herm}_3(\mathbb{S}_{\mathbb{C}})) = \mathrm{Alt}_3(\mathbb{U}_{\mathbb{C}})$ and $\mathrm{Herm}_3(\mathbb{S}_{\mathbb{C}})$ is a split and null extension of $\mathrm{Herm}_3(\mathbb{H}_{\mathbb{C}})$ by $\mathrm{Alt}_3(\mathbb{U}_{\mathbb{C}})$. The geometrical interpretation of this is that $X = X_{\mathrm{Herm}_3(\mathbb{S}_{\mathbb{C}})} \subset \mathbb{P}^{43}$ is a split and null extension of $X_{\mathrm{Herm}_3(\mathbb{H}_{\mathbb{C}})} = OG_6(\mathbb{C}^{12})$. In this particular case, the linear projection $\pi_{R_X} : X \dashrightarrow X_{\mathrm{Herm}_3(\mathbb{H}_{\mathbb{C}})}$ is surjective and any of its fibers is a linear subspace of dimension 6 in $\mathbb{P}^{43}$.

5.3.4. **The Penico series of X.** We use in this subsection the notation introduced in the proof of Proposition 5.26: x^+ and x^- are two general points on X, etc. Let $\mathit{I\!I} \subset R_X$ be a radical ideal associated to the radical ideal (I^+, I^-) of the Jordan pair $V = (V^+, V^-)$. For $\sigma = \pm$, define $\mathcal{P}(I^\sigma)$ as $P^\sigma_{I^\sigma}(V^{-\sigma}) \subset V^\sigma$. Then $\mathcal{P}(\mathit{I\!I}) = \mathbb{P}(\mathcal{P}(I^+) \oplus \mathcal{P}(I^-)) \subset \mathbb{P}^{2n+1}$ is a radical ideal of X.

We can now define the *Penico series* of X as the decreasing family of projective subspaces $\mathbb{P}^{2n+1} \supset R^{[1]}_X \supset R^{[2]}_X \supset \cdots \supset R^{[\ell-1]}_X \supset R^{[\ell]}_X \supset \cdots$ defined inductively by

$$R^{[1]}_X = R_X \quad \text{and} \quad R^{[k+1]}_X = \mathcal{P}\big(R^{[k]}_X\big) \quad \text{for } k \geq 1.$$

One verifies that the $R^{[k]}_X$'s do not depend on the base points x^+ and x^- but only on X and that they are projectively attached to X. It would be interesting to give a purely geometrical characterization of the Penico series of X, in the same spirit of the characterization of the radical of X given in Proposition 5.22.

Example 5.29 (continuation of Example 5.2). We have seen in Example 5.24 that the radical R_{X_A} of the cubic curve $X_A \subset \mathbb{P}Z_2(A) = \mathbb{P}^7$ over $A = \mathbb{C}[\epsilon]/(\epsilon^3)$ is a 3-dimensional projective subspace in $\mathbb{P}^7$. One verifies easily $R^{[2]}_{X_A}$ is the projective line $\{[0 : \lambda\,\epsilon^2 : \lambda'\,\epsilon^2 : 0] \,|\, [\lambda, \lambda'] \in \mathbb{P}^1\} \subset R_{X_A}$, whereas $R^{[\ell]}$ is empty for every $\ell \geq 3$.

For any $\ell \geq 1$, let $p^{[\ell]}$ be the restriction to X of the linear projection from $R^{[\ell]}_X$ and denote by $X^{[\ell]}$ its image (note that $X^{[\ell]} = X$ and $p^{[\ell]} = \mathrm{Id}_X$ if $R^{[\ell]}_X$ is empty). Since $R^{[\ell]}_X$ is a radical ideal, $X^{[\ell]}$ belongs to $\overline{\boldsymbol{X}}(3,3)$. Moreover, it is not difficult to verify that for $k < \ell$, the linear subspace $p^{[\ell]}(R^{[k]}_X)$ is a radical ideal for $X^{[\ell]}$ that is nothing but $R^{[k-\ell]}_{X^{[\ell]}}$. If one denotes by $\pi^{[\ell]}$ the restriction to $X^{[\ell]}$ of the linear projection from $p^{[\ell]}(R^{[\ell-1]}_X)$, there is a commutative diagram

$$\begin{array}{ccc} X & \overset{p^{[\ell]}}{\dashrightarrow} & X^{[\ell]} \\ & \searrow^{p^{[\ell-1]}} & \downarrow \pi^{[\ell]} \\ & & X^{[\ell-1]}, \end{array}$$

where the maps in it are (restrictions of) dominant linear projections sending isomorphically a general twisted cubic curve in a source space onto a general twisted cubic curve in the target space.

A good reason to consider the projections $\pi^{[\ell]}$ is certainly given by the following result, whose proof is left to the reader.

Proposition 5.30. *For any $\ell \geq 1$, the generic fiber of the rational map $\pi^{[\ell]} : X^{[\ell]} \dashrightarrow X^{[\ell-1]}$ is a linear subspace.*

5.3.5. **The general structure of twisted cubics over Jordan algebras.** We are now in position of stating the translation in the X-world of Theorem 5.1.

Theorem 5.31. *Assume that $X \in \boldsymbol{X}(3,3)$ is not semi-simple (or equivalently that X is not smooth). Then*

(1) the restriction to X of the linear projection from its radical R_X induces a dominant rational map

$$\pi_{R_X} : X \dashrightarrow X_{\mathrm{ss}}$$

over a semi-simple 3-RC variety $X_{\mathrm{ss}} \in \overline{\boldsymbol{X}}(3,3)$, the restriction of which to a general twisted cubic $C \subset X$ is an isomorphism onto its image $\pi_{R_X}(C)$, which is then a twisted cubic curve in X_{ss};

(2) there exists a linear embedding $\sigma : \mathbb{P}Z_2(\mathbb{J}_{\mathrm{ss}}) \hookrightarrow \mathbb{P}Z_2(\mathbb{J})$ whose image is supplementary to R_X such that

$$\sigma\big(X_{\mathrm{ss}}\big) \subset X \qquad \textit{and} \qquad \pi_{R_X} \circ \sigma \in \mathrm{Aut}\big(X_{\mathrm{ss}}\big).$$

Moreover, σ is unique, up to composition to the left by a projective automorphism of X;

(3) the radical R_X is solvable: there exists a positive integer t such that $R^{[t]}_X$ is empty. Moreover, $X^{[\ell]}$ is a radical null extension of $X^{[\ell-1]}$ for $\ell = 2, \ldots, t$ so that $X = X^{[t]}$ can be obtained from its semi-simple part $X_{\mathrm{ss}} = X^{[1]}$ by the successive series of null radical extensions represented below

$$X \overset{\pi^{[t]}}{\dashrightarrow} X^{[t-1]} \overset{\pi^{[t-1]}}{\dashrightarrow} \cdots \overset{\pi^{[\ell+1]}}{\dashrightarrow} X^{[\ell]} \overset{\pi^{[\ell]}}{\dashrightarrow} X^{[\ell-1]} \overset{\pi^{[\ell-1]}}{\dashrightarrow} \cdots \overset{\pi^{[3]}}{\dashrightarrow} X^{[2]} \overset{\pi^{[2]}}{\dashrightarrow} X_{\mathrm{ss}}.$$

5.3.6. **Null extension and Verra construction.** It follows from part *(3)* of Theorem 5.31 that the notion of radical null extension is particularly relevant when dealing with varieties in the class $\boldsymbol{X}(3,3)$. Notwithstanding the considerations and results of the preceding sections are not fully satisfying from the intrinsic point of view. For instance the construction of all radical null extensions of a given $X \in \boldsymbol{X}(3,3)$ shows immediately that it is desirable to have an intrinsic geometric characterization of such objects, the term 'intrinsic' meaning here 'in term of X alone'. This section is dedicated to this purpose.

Let us recall that if $X \in \boldsymbol{X}(3,3)$, then $X \subset \mathbb{P}^{2n+1}$ is a *variety with one apparent double point*, briefly an *OADP–variety*, meaning that through a general point of $\mathbb{P}^{2n+1}$ there passes a unique secant line to X, see for example [31, Corollary 5.4] for a proof[2]. If $\pi : X' \dashrightarrow X$ is a radical null extension in the X-world then the general fiber of π is a linear subspace. This shows that X' is an OADP-variety obtained from X by the so called *Verra construction* of new OADP-varieties from a given one. We recall brievely this geometric construction below, referring to [10, Section 3] for more details and proofs.

[2]The name 'OADP–variety' comes from the fact that the projection of X from a general point acquires only one double point as (further) singularities (see also [10] for relations between twisted cubics over Jordan algebras and OADP-varieties).

Let $Y \subset \mathbb{P}^{2(n+r)+1}$ be a degenerate OADP–variety of dimension n, which spans a linear space V of dimension $2n+1$. Let $\mathcal{C}_W(Y)$ be the cone over Y with vertex a linear space $W \subset \mathbb{P}^{2(n+r)+1}$ of dimension $2r-1$ in direct sum with V. Assume that $Y' \subset \mathcal{C}_W(Y)$ is an irreducible non–degenerate variety of dimension $n+r$, that is not secant defective and which intersects the general ruling $\Pi \simeq \mathbb{P}^{2r}$ of $\mathcal{C}_W(Y)$ along a linear subspace of dimension r. Then the linear projection of $\mathbb{P}^{2(n+r)+1}$ from W onto V restricts to Y' to a dominant map $\pi : Y' \dashrightarrow Y$ having linear fibers of dimension r that are generically disjoint. We shall say that Y' *is obtained from* Y *via Verra's construction* or also that Y' is a *Verra variety*. It is not difficult to prove that Verra varieties are OADP–varieties.

Let Y' be a Verra variety as above. Then $\pi^{-1}(y)$ is a linear subspace of dimension $r-1$ of W for $y \in Y$ general. Therefore $y \mapsto \pi^{-1}(y)$ defines a rational map $\gamma_{Y'} : Y \dashrightarrow \mathbb{G}(r-1, W) = G_r(\mathbb{C}^{2r})$. Moreover, $\gamma_{Y'}(y_1)$ and $\gamma_{Y'}(y_2)$ are skew subspaces of W when y_1 and y_2 are two general points of Y. Conversely, let $\mathcal{V}_Y^r$ be the set of rational maps $\gamma : Y \dashrightarrow \mathbb{G}(r-1, \mathbb{P}^{2r-1})$ satisfying the condition that $\gamma(y_1)$ and $\gamma(y_2)$ are skew if y_1 and y_2 are general in Y. It can be verified that for any $\gamma \in \mathcal{V}_Y^r$, if Y_0 stands for the open subset of Y on which γ is defined, then

$$Y_\gamma = \overline{\bigcup_{y \in Y_0} \langle y, \gamma(y) \rangle} \subset \mathbb{P}^{2(n+r)+1}$$

is an OADP–variety that is obtained from Y by Verra's construction. This gives an identification between the set of $(n+r)$-dimensional Verra varieties constructed from Y up to projective equivalence and the quotient of the set $\mathcal{V}_Y^r$ by a certain relation of equivalence that the interested reader could make explicit without difficulty.

Since a radical null extension $\pi : X' \dashrightarrow X$ is obtained by Verra's construction from X, there exists $\gamma_{X'} \in \mathcal{V}_X^r$ such that $X' = X_{\gamma_{X'}}$. Nevertheless one verifies easily that not every $\gamma \in \mathcal{V}_X^r$ is such that X_γ is a radical null extension of X, see also Example 5.33 below. A necessary and sufficient assuring that $X' \in \boldsymbol{X}(3,3)$ is given by the following result:

Theorem 5.32. *Let $X \in \boldsymbol{X}(3,3)$ and let $\gamma \in \mathcal{V}_X^r$. The following conditions are equivalent:*

(1) the Verra variety X_γ belongs to $\boldsymbol{X}(3,3)$ so that in particular is a radical null extension of X

(2) the restriction of γ to a general cubic curve $C \subset X$ is an embedding and $\gamma(C)$ is a line in $G_r(\mathbb{C}^{2r})$.

Proof. Clearly *(1)* implies *(2)* and we now prove the converse. Let $X' = X_\gamma$ where $\gamma \in \mathcal{V}_X^r$ is such that *(2)* holds and denote by $\pi : X' \dashrightarrow X$ the (restriction of the) linear projection defining X' as a Verra variety over X. If x_1', x_2' and x_3' are three general points on X' then the $x_i = \pi(x_i')$'s are three general points on X. Let C be the twisted cubic included in X passing trough the x_i's. Clearly, $\pi^{-1}(C)$ is nothing but the Verra variety C_{γ_C} over C, where γ_C stands for the restriction of γ to C. Since C is a general cubic in X, it follows from *(2)* that C_{γ_C} is the $(r+1)$-dimensional rational normal scroll $S_{1\dots13}$ in $\mathbb{P}^{2r+3}$. The later being an element of the class $\overline{\boldsymbol{X}}(3,3)$, there exists a twisted cubic $C' \subset C_{\gamma_C}$ passing trough x_1', x_2' and x_3' and such $\pi(C') = C$. This shows that $X' = X_\gamma$ is 3-RC by cubic curves. Then X_γ is a radical extension of X. Since the general fiber of $\pi : X_\gamma \dashrightarrow X$ is linear, this radical extension is null by the definition, concluding the proof. □

The following examples show that there exist $\gamma \in \mathcal{V}_X^r$ such that $X_\gamma \notin \overline{\boldsymbol{X}}(3,3)$.

Example 5.33. Let $\mathbb{J}$ be a rank 3 Jordan algebra of dimension $n \geq 1$ with generic norm $N(x)$. Let $\mathbb{J}'$ be a split radical extension of $\mathbb{J}$ by a Jordan bimodule R of dimension 1.

First of all, since $R^2 \subset R$ and $r^3 = 0$ for every $r \in R$ (and since $R \subset \mathrm{Rad}(\mathbb{J}')$), it follows that $R^2 = 0$. Because the extension $R \hookrightarrow \mathbb{J}' \twoheadrightarrow \mathbb{J}$ is split, one can assume that $\mathbb{J}' = \mathbb{J} \oplus R$ with product given by

$$(x_1, r_1) * (x_2, r_2) = (x_1 * x_2, r_1\varphi(x_2) + r_2\varphi(x_1)) \qquad \text{for } x_1, x_2 \in \mathbb{J},\ r_1, r_2 \in R,$$

where $\varphi : \mathbb{J} \to \mathbb{C}$ is a certain (fixed) linear form. The unity of $\mathbb{J}'$ is $e' = (e, 0)$, yielding $\varphi(e) = 1$.

Reasoning as in the proof of Theorem 5.16 and recalling that $r^2 = 0$ for every $r \in R'$, we deduce that there exists also a linear form $f : \mathbb{J} \to \mathbb{C}$ such that $(x,r)^\# = (x^\#, rf(x))$ for every $(x,r) \in \mathbb{J}'$. By definition of radical we have $N(x') = N(x,r) = N(x)$ for $x' = (x,r) \in \mathbb{J}'$ so that the identity $N(x')x' = (x'^\#)^\#$ is equivalent to

$$N(x) = f(x)f(x^\#) \tag{22}$$

for every $x \in \mathbb{J}$.

Now if $X_\mathbb{J} \subset \mathbb{P}^{2n+1}$ is a twisted cubic over a simple rank three Jordan algebra $\mathbb{J}$ (so with $n \in \{6, 9, 15, 27\}$) and if $Z^{n+1} \subset \mathbb{P}^{2n+3}$ is obtained from $X_\mathbb{J}$ via Verra construction, then $Z \notin \overline{\boldsymbol{X}}^{n+1}(3,3)$. Indeed, otherwise Z would be projectively equivalent to $X_{\mathbb{J}'}$ for a certain 1-dimensional null radical extension $\mathbb{J}'$ of $\mathbb{J}$, that is necessarily split (this

follows from the fact that $\mathbb{J}$ is simple). Then (22) would imply that the norm $N(x)$ of $\mathbb{J}$ is a reducible polynomial of degree 3, which is not the case.

Part *(3)* of Theorem 5.31 points out that among radical null extensions the split ones are the most interesting. Accordingly to the general principle of the XJC-correspondence, given $X \in \boldsymbol{X}(3,3)$, it would be interesting to get a characterization in geometric terms of the rational maps $\gamma \in \mathcal{V}_X^r$ satisfying condition *(2)* of the preceding theorem and such that the radical extension $X_\gamma \dashrightarrow X$ is not only null but also split. We intend to return on this and on other related questions in the future.

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LUC PIRIO, IRMAR, UMR 6625 DU CNRS, UNIVERSITÉ RENNES1, CAMPUS DE BEAULIEU, 35000 RENNES, FRANCE
E-mail address: luc.pirio@univ-rennes1.fr

FRANCESCO RUSSO, DIPARTIMENTO DI MATEMATICA E INFORMATICA, UNIVERSITÀ DEGLI STUDI DI CATANIA, VIALE A. DORIA 6, 95125 CATANIA, ITALY
E-mail address: frusso@dmi.unict.it