JORDAN CONSTANT FOR CREMONA GROUP OF RANK 3

YURI PROKHOROV AND CONSTANTIN SHRAMOV

ABSTRACT. We give explicit bounds for Jordan constants of groups of birational automorphisms of rationally connected threefolds over fields of zero characteristic, in particular, for Cremona groups of ranks 2 and 3.

CONTENTS

1. Introduction

1.1. **Jordan property.** The *Cremona group of rank n* is the group $\mathrm{Cr}_n(\Bbbk)$ of birational transformations of the projective space $\mathbb{P}^n$ over a field $\Bbbk$. It has been actively studied from various points of view for many years (see [Hud27], [CL13], [Dés12], [DI09a], [Ser09a], [Can16], and references therein). One of the approaches to this huge group is to try to understand its finite subgroups. It appeared that it is possible to obtain a complete classification of finite subgroups of $\mathrm{Cr}_2(\Bbbk)$ over an algebraically closed field $\Bbbk$ of characteristic 0 (see [BB00], [BB04], [Bla09], [DI09a], [Tsy11], and [Pro15c]), and to obtain partial classification results for $\mathrm{Cr}_3(\Bbbk)$ (see [Pro12], [Pro11], [Pro14], [Pro13a], and [Pro15b]). Some results are also known for non algebraically closed fields, see e.g. [Ser09b], [DI09b], and [Yas15]. In general, it is partially known and partially expected that the collection of finite subgroups of a Cremona group shares certain features with the collection of finite subgroups of a group $\mathrm{GL}_m(\Bbbk)$.

Theorem 1.1.1 (C. Jordan, see e.g. [CR62, Theorem 36.13]). *There is a constant $I = I(n)$ such that for any finite subgroup $G \subset \mathrm{GL}_n(\mathbb{C})$ there exists a normal abelian subgroup $A \subset G$ of index at most I.*

This leads to the following definition (cf. [Pop11, Definition 2.1]).

Definition 1.1.2. A group Γ is called *Jordan* (alternatively, we say that Γ *has Jordan property*) if there is a constant J such that for any finite subgroup $G \subset \Gamma$ there exists a normal abelian subgroup $A \subset G$ of index at most J.

Theorem 1.1.1 implies that all linear algebraic groups over an arbitrary field $\Bbbk$ of characteristic 0 are Jordan. Jordan property was also studied recently for groups of birational automorphisms of algebraic varieties. The starting point here was the following result of J.-P. Serre.

Theorem 1.1.3 (J.-P. Serre [Ser09b, Theorem 5.3], [Ser09a, Théorème 3.1]). *The Cremona group $\mathrm{Cr}_2(\Bbbk)$ over a field $\Bbbk$ of characteristic 0 is Jordan.*

Remark 1.1.4. Note that the assumption about characteristic is indispensable. Indeed, the group $\mathrm{Cr}_2(\Bbbk)$ contains $\mathrm{PGL}_2(\Bbbk)$, so that if the characteristic of the field $\Bbbk$ equals $p > 0$ and $\Bbbk$ is algebraically closed, then $\mathrm{Cr}_2(\Bbbk)$ contains a series of simple subgroups $\mathrm{PSL}_2(\mathbf{F}_{p^k})$ of increasing order.

It also appeared that there are surfaces with non-Jordan groups of birational selfmaps (see [Zar14b]). V. Popov managed to give a complete classification of surfaces with Jordan groups of birational automorphisms.

Theorem 1.1.5 (V. Popov [Pop11, Theorem 2.32])**.** *Let S be a surface over a field $\Bbbk$ of characteristic 0. Then the group $\mathrm{Bir}(S)$ of birational automorphisms of S is Jordan if and only if S is not birational to $E \times \mathbb{P}^1$, where E is an elliptic curve.*

In dimension 3 Jordan property is known for groups of birational automorphisms of rationally connected varieties (see e. g. [Kol96, § IV.3] for definition and basic background).

Theorem 1.1.6 (see [PS16a, Theorem 1.8])**.** *Fix a field $\Bbbk$ of characteristic 0. Then there is a constant J such that for any rationally connected variety X of dimension 3 defined over $\Bbbk$ and any finite subgroup $G \subset \mathrm{Bir}(X)$ there exists a normal abelian subgroup $A \subset G$ of index at most J. In particular, for any rationally connected threefold X the group $\mathrm{Bir}(X)$ is Jordan.*

Actually, by [PS16a, Theorem 1.8] the assertion of Theorem 1.1.6 holds in arbitrary dimension modulo boundedness of terminal Fano threefolds (see e. g. [Bor96] or [PS16a, Conjecture 1.7]); the latter boundedness was recently proved in [**?**, Theorem 1.1]. For other results (in particular, for birational automorphisms of non rationally connected varieties, and for automorphisms of varieties of different types) see [PS14], [Pop11], [Pop14], [Zar14a], [BZ15a], [BZ15b], [MZ15], [Zim14], [MT15], and [Mun16].

1.2. **Jordan constants.** Given a Jordan group Γ, one may get interested in the minimal value of the constant involved in Definition 1.1.2, and in the values of other relevant constants.

Definition 1.2.1. Let Γ be a Jordan group. The *Jordan constant $J(\Gamma)$* of the group Γ is the minimal number J such that for any finite subgroup $G \subset \Gamma$ there exists a normal abelian subgroup $A \subset G$ of index at most J. The *weak Jordan constant $\bar{J}(\Gamma)$* of the group Γ is the minimal number $\bar{J}$ such that for any finite subgroup $G \subset \Gamma$ there exists a (not necessarily normal) abelian subgroup $A \subset G$ of index at most $\bar{J}$.

Remark 1.2.2. It is more traditional to study Jordan constants than weak Jordan constants of Jordan groups, although there is no big difference between them. Indeed, one has $\bar{J}(\Gamma) \leqslant J(\Gamma)$ for any Jordan group Γ for obvious reasons. Moreover, if G is a finite group and A is an abelian subgroup of G, then by [Isa08, Theorem 1.41] one can find a normal abelian subgroup N of G such that

$$[G : N] \leqslant [G : A]^2.$$

Therefore, if Γ is a Jordan group, one always has $J(\Gamma) \leqslant \bar{J}(\Gamma)^2$. On the other hand, the advantage of the weak Jordan constant is that it allows easy estimates using subgroups of the initial group. Namely, if Γ_1 is a subgroup of finite index in a group Γ_2, and Γ_1 is Jordan, then Γ_2 is Jordan with

$$\bar{J}(\Gamma_2) \leqslant [\Gamma_2 : \Gamma_1] \cdot \bar{J}(\Gamma_1).$$

Also, if Δ_1 and Δ_2 are Jordan groups, the group $\Delta_1 \times \Delta_2$ is Jordan with

$$\bar{J}(\Delta_1 \times \Delta_2) = \bar{J}(\Delta_1) \times \bar{J}(\Delta_2).$$

In particular, if Γ is a subgroup of $\Delta \times A$, where Δ is a Jordan group and A is an abelian group, then Γ is Jordan with $\bar{J}(\Gamma) \leqslant \bar{J}(\Delta)$.

Jordan constants are known for example for the groups $\mathrm{GL}_n(\mathbb{C})$ (see [Col07]). In [Ser09b] J.-P. Serre gave an explicit bound for the Jordan constant of the Cremona group $\mathrm{Cr}_2(\Bbbk)$ (see Remark 1.2.6 below). Our first result also concerns the group $\mathrm{Cr}_2(\Bbbk)$.

Proposition 1.2.3. *Suppose that the field* $\Bbbk$ *has characteristic* 0. *Then one has*

$$\bar{J}\big(\operatorname{Cr}_2(\Bbbk)\big) \leqslant 288, \quad J\big(\operatorname{Cr}_2(\Bbbk)\big) \leqslant 82944.$$

The first of these bounds becomes an equality if $\Bbbk$ *is algebraically closed.*

The main goal of this paper is to present a bound for Jordan constants of the groups of birational automorphisms of rationally connected threefolds, in particular, for the group $\operatorname{Cr}_3(\Bbbk) = \operatorname{Bir}(\mathbb{P}^3)$.

Theorem 1.2.4. *Let* X *be a rationally connected threefold over a field* $\Bbbk$ *of characteristic* 0. *Then one has*

$$\bar{J}\big(\operatorname{Bir}(X)\big) \leqslant 10\,368, \quad J\big(\operatorname{Bir}(X)\big) \leqslant 107\,495\,424.$$

If moreover X *is rational and* $\Bbbk$ *is algebraically closed, then the first of these bounds becomes an equality.*

It is known (see [PS16a, Theorem 1.10]) that if X is a rationally connected threefold over a field of characteristic 0, then there is a constant L such that for any prime $p > L$ any finite p-group $G \subset \operatorname{Bir}(X)$ is abelian. An immediate consequence of Theorem 1.2.4 is an explicit bound for the latter constant L.

Corollary 1.2.5. *Let* X *be a rationally connected threefold over a field* $\Bbbk$ *of characteristic* 0, *and let* $p > 10\,368$ *be a prime. Let* $G \subset \operatorname{Bir}(X)$ *be a finite* p-*group. Then* G *is abelian.*

We believe that one can significantly improve the bound given by Corollary 1.2.5.

Remark 1.2.6. J.-P. Serre showed (see the remark made after Theorem 5.3 in [Ser09b]) that any finite subgroup G of the Cremona group $\operatorname{Cr}_2(\Bbbk)$ over a field $\Bbbk$ of characteristic 0 has a normal abelian subgroup $A \subset G$ such that the index $[G : A]$ *divides* the number $2^{10} \cdot 3^4 \cdot 5^2 \cdot 7$. The result of Theorem 1.2.4 is not that precise: we cannot say much about the primes that divide the index $[G : A]$ in our case. This is explained by the fact that to obtain the bound we have to deal with terminal singularities on threefolds as compared to smooth surfaces. See Remark 8.2.1 for our expectations on possible improvements of the bounds given by Proposition 1.2.3 and Theorem 1.2.4, and Remark 8.2.2 for a further disclaimer in higher dimensions.

The plan of the paper is as follows. In §2 we compute weak Jordan constants for some linear groups. In §3 we compute certain relevant constants for rational surfaces, and in particular prove Proposition 1.2.3. In §4 we study groups of automorphisms of three-dimensional terminal singularities and estimate their weak Jordan constants; then we use these estimates to bound weak Jordan constants for groups of automorphisms of non-Gorenstein terminal Fano threefolds. In §5 we estimate weak Jordan constants for groups acting on three-dimensional G-Mori fiber spaces. In §6 and §7 we bound weak Jordan constants for groups of automorphisms of Gorenstein terminal (and in particular smooth) Fano threefolds. Finally, in §8 we summarize the above partial results and complete the proof of Theorem 1.2.4, and also make concluding remarks. In Appendix A we collect some information about automorphism groups of two particular classes of smooth Fano varieties: complete intersections of quadrics, and complete intersections in weighted projective spaces; these results are well known to experts, but we decided to include them because we do not know proper references.

Notation and conventions. In what follows we denote by $\boldsymbol{\mu}_m$ a cyclic group of order m. We denote by $m.\Gamma$ a central extension of a group Γ by a group isomorphic to $\boldsymbol{\mu}_m$. Starting from this point we always work over an *algebraically closed* field of characteristic 0.

2. Linear groups

Now we are going to find weak Jordan constants $\bar{J}\big(\mathrm{GL}_n(\Bbbk)\big)$ for small values of n. Note that the values of Jordan constants $J\big(\mathrm{GL}_n(\Bbbk)\big)$ were computed in [Col07] for any n.

2.1. **Preliminaries.** The following remark is elementary but rather useful.

Remark 2.1.1. Suppose that Γ_1 is a Jordan group, and there is a surjective homomorphism $\Gamma_1 \to \Gamma_2$. Then Γ_2 is also Jordan. Moreover, one has $\bar{J}(\Gamma_1) \geqslant \bar{J}(\Gamma_2)$ and $J(\Gamma_1) \geqslant J(\Gamma_2)$. In particular, for any n the group $\mathrm{PGL}_n(\Bbbk)$ is Jordan with

$$\bar{J}\big(\mathrm{PGL}_n(\Bbbk)\big) \leqslant \bar{J}\big(\mathrm{GL}_n(\Bbbk)\big), \quad J\big(\mathrm{PGL}_n(\Bbbk)\big) \leqslant J\big(\mathrm{GL}_n(\Bbbk)\big).$$

We will also need the following well-known observation. Let U be an arbitrary variety and P be a point of U. Denote by $\mathrm{Aut}_P(U)$ the stabilizer of P in $\mathrm{Aut}(U)$. Let $T_P(U)$ be the Zariski tangent space to the variety U at the point P.

Lemma 2.1.2 (see e.g. [BB73, Lemma 2.4], [Pop14, Lemma 4]). *Suppose that U is an irreducible variety. For any finite group $G \subset \mathrm{Aut}_P(U)$ the natural representation $G \to \mathrm{GL}\big(T_P(U)\big)$ is faithful. In particular, one has*

$$\bar{J}\big(\mathrm{Aut}_P(U)\big) \leqslant \bar{J}\Big(\mathrm{GL}\big(T_P(U)\big)\Big), \quad J\big(\mathrm{Aut}_P(U)\big) \leqslant J\Big(\mathrm{GL}\big(T_P(U)\big)\Big).$$

Remark 2.1.3. One does not necessarily have an embedding $\Gamma \hookrightarrow \mathrm{GL}\big(T_P(U)\big)$ for a non-reductive subgroup $\Gamma \subset \mathrm{Aut}_P(U)$. This is not the case already for $U \cong \mathbb{A}^2$ and $\Gamma = \mathrm{Aut}_P(U)$.

2.2. **Dimension** 2. The following easy result will be used both to find the weak Jordan constant of the group $\mathrm{GL}_2(\Bbbk)$, and also later in the proof of Corollary 7.2.3.

Lemma 2.2.1. *Let G be a group that fits into an exact sequence*

$$1 \to \Gamma \longrightarrow G \xrightarrow{\phi} \mathrm{PGL}_2(\Bbbk),$$

where $\Gamma \cong \boldsymbol{\mu}_2$. Then G is Jordan with $\bar{J}(G) \leqslant 12$.

Proof. Note that Γ is contained in the center of the group G. We may assume that G is finite. By the well-known classification of finite subgroups in $\mathrm{PGL}_2(\Bbbk)$, we know that the group $\bar{G} = \phi(G)$ is either cyclic, or dihedral, or isomorphic to one of the groups $\mathfrak{A}_4$, $\mathfrak{S}_4$, or $\mathfrak{A}_5$.

If $\bar{G}$ is cyclic, then the group G is abelian.

If $\bar{G}$ is dihedral, then the group G contains an abelian subgroup of index 2.

If $\bar{G} \cong \mathfrak{A}_4$, then $\bar{G}$ contains a cyclic subgroup of order 3, so that $\bar{J}(G) \leqslant 4$; the inequality here is due to the fact that in the case when $G \cong \boldsymbol{\mu}_2 \times \mathfrak{A}_4$ one has $\bar{J}(G) = 3$, but for a non-trivial central extension $G \cong 2.\mathfrak{A}_4$ one has $\bar{J}(G) = 4$.

If $\bar{G} \cong \mathfrak{S}_4$, then $\bar{G}$ contains a cyclic subgroup of order 4, and $\bar{J}(G) = 6$.

Finally, if $\bar{G} \cong \mathfrak{A}_5$, then $\bar{G}$ contains a cyclic subgroup of order 5, and $\bar{J}(G) = 12$. $\qquad \square$

As an easy application of Lemma 2.2.1, we can find the weak Jordan constants of the groups $\mathrm{GL}_2(\Bbbk)$ and $\mathrm{Aut}(\mathbb{P}^1) \cong \mathrm{PGL}_2(\Bbbk)$.

Corollary 2.2.2. *One has*
$$\bar{J}\big(\mathrm{GL}_2(\Bbbk)\big) = \bar{J}\big(\mathrm{PGL}_2(\Bbbk)\big) = 12.$$

Proof. Let V be a three-dimensional vector space over $\Bbbk$, and let $G \subset \mathrm{GL}(V)$ be a finite subgroup. It is enough to study the weak Jordan constant $\bar{J}(G)$. Moreover, for this we may assume that $G \subset \mathrm{SL}(V) \cong \mathrm{SL}_2(\Bbbk)$, and that G contains the scalar matrix acting by -1 on V. Therefore, the bound $\bar{J}(G) \leqslant 12$ follows from Lemma 2.2.1, so that $\bar{J}(\mathrm{GL}_2(\Bbbk)) \leqslant 12$. The inequality
$$\bar{J}\big(\mathrm{PGL}_2(\Bbbk)\big) \leqslant \bar{J}\big(\mathrm{GL}_2(\Bbbk)\big)$$
holds by Remark 2.1.1. The value $\bar{J}(\mathrm{PGL}_2(\Bbbk)) = 12$ is given by the group $\mathfrak{A}_5 \subset \mathrm{PGL}_2(\Bbbk)$, and the value $\bar{J}(\mathrm{GL}_2(\Bbbk)) = 12$ is given by the group $2.\mathfrak{A}_5 \subset \mathrm{GL}_2(\Bbbk)$. $\qquad \square$

Remark 2.2.3. Suppose that C is an irreducible curve such that the normalization $\hat{C}$ of C has genus g. Since the action of the group $\mathrm{Aut}(X)$ lifts to $\hat{C}$, one has
$$\bar{J}\big(\mathrm{Aut}(C)\big) \leqslant \bar{J}\big(\mathrm{Aut}(\hat{C})\big).$$
On the other hand, it is well known that $\bar{J}\big(\mathrm{Aut}(\hat{C})\big) \leqslant 6$ if $g = 1$, and the Hurwitz bound implies that
$$\bar{J}\big(\mathrm{Aut}(\hat{C})\big) \leqslant |\mathrm{Aut}(\hat{C})| \leqslant 84(g-1)$$
if $g \geqslant 2$.

We can use a classification of finite subgroups in $\mathrm{PGL}_2(\Bbbk)$ to find the weak Jordan constant of the automorphism group of a line and a smooth two-dimensional quadric. More precisely, we have the following result.

Lemma 2.2.4. *The following assertions hold.*
  (i) *Let $G \subset \mathrm{Aut}(\mathbb{P}^1)$ be a finite group. Then there exists an abelian subgroup $A \subset G$ of index at most 12 acting on $\mathbb{P}^1$ with a fixed point.*
  (ii) *Let $G \subset \mathrm{Aut}\big(\mathbb{P}^1 \times \mathbb{P}^1\big)$ be a finite group. Then there exists an abelian subgroup $A \subset G$ of index at most 288 that acts on $\mathbb{P}^1 \times \mathbb{P}^1$ with a fixed point, and does not interchange the rulings of $\mathbb{P}^1 \times \mathbb{P}^1$.*
  (iii) *One has*
$$\bar{J}\Big(\mathrm{Aut}\big(\mathbb{P}^1 \times \mathbb{P}^1\big)\Big) = 288.$$

Proof. Assertion (i) follows from the classification of finite subgroups of $\mathrm{PGL}_2(\Bbbk)$. Observe that
$$\mathrm{Aut}\big(\mathbb{P}^1 \times \mathbb{P}^1\big) \cong \big(\mathrm{PGL}_2(\Bbbk) \times \mathrm{PGL}_2(\Bbbk)\big) \rtimes \boldsymbol{\mu}_2.$$
Therefore, assertion (i) implies assertion (ii). In particular, we get the bound
$$\bar{J}\Big(\mathrm{Aut}\big(\mathbb{P}^1 \times \mathbb{P}^1\big)\Big) \leqslant 288.$$

The required equality is given by the group

$$\left(\mathfrak{A}_5 \times \mathfrak{A}_5\right) \rtimes \boldsymbol{\mu}_2 \subset \operatorname{Aut}\left(\mathbb{P}^1 \times \mathbb{P}^1\right).$$

This proves assertion (iii). □

2.3. Dimension 3.

Lemma 2.3.1. *One has*

$$\bar{J}\left(\operatorname{PGL}_3(\Bbbk)\right) = 40, \qquad \bar{J}\left(\operatorname{GL}_3(\Bbbk)\right) = 72.$$

Proof. Let V be a three-dimensional vector space over $\Bbbk$, and let $G \subset \operatorname{GL}(V)$ be a finite subgroup. It is enough to study the weak Jordan constant $\bar{J}(G)$. Moreover, for this we may assume that $G \subset \operatorname{SL}(V) \cong \operatorname{SL}_3(\Bbbk)$. Recall that there are the following possibilities for the group G (see [MBD16, Chapter XII] or [Fei71, §8.5]):

  (i) the G-representation V is reducible;
 (ii) there is a transitive homomorphism $h\colon G \to \mathfrak{S}_3$ such that V splits into a sum of three one-dimensional representations of the subgroup $H = \operatorname{Ker}(h)$;
(iii) the group G is generated by some subgroup of scalar matrices in $\operatorname{SL}_3(\Bbbk)$ and a group $\hat{G}$ that is one of the groups $\mathfrak{A}_5$ or $\operatorname{PSL}_2(\mathbf{F}_7)$;
 (iv) one has $G \cong 3.\mathfrak{A}_6$;
  (v) one has $G \cong \mathcal{H}_3 \rtimes \Sigma$, where $\mathcal{H}_3$ is the Heisenberg group of order 27, and Σ is some subgroup of $\operatorname{SL}_2(\mathbf{F}_3)$.

Let us denote by $\bar{G}$ the image of G in the group $\operatorname{PGL}_3(\Bbbk)$. One always has $\bar{J}(\bar{G}) \leqslant \bar{J}(G)$.

In case (i) there is an embedding $G \hookrightarrow A \times \Gamma$, where A is a finite abelian group and Γ is a finite subgroup of $\operatorname{GL}_2(\Bbbk)$. Thus

$$\bar{J}(\bar{G}) \leqslant \bar{J}(G) = \bar{J}(\Gamma) \leqslant \bar{J}\left(\operatorname{GL}_2(\Bbbk)\right) = \bar{J}\left(\operatorname{PGL}_2(\Bbbk)\right) = 12$$

by Corollary 2.2.2.

In case (ii) the group H is an abelian subgroup of G, so that

$$\bar{J}(\bar{G}) \leqslant \bar{J}(G) \leqslant [G : H] \leqslant |\mathfrak{S}_3| = 6.$$

In case (iii) it is easy to check that $\bar{J}(\bar{G}) \leqslant \bar{J}(G) = \bar{J}(\hat{G}) \leqslant 24$.

In case (iv) one has $G \cong 3.\mathfrak{A}_6$ and $\bar{G} \cong \mathfrak{A}_6$. The abelian subgroup of maximal order in $\bar{G}$ is a Sylow 3-subgroup, so that $\bar{J}(\bar{G}) = 40$. The abelian subgroup of maximal order in G is $\boldsymbol{\mu}_{15}$ that is a preimage of a subgroup is a Sylow 5-subgroup with respect to the natural projection $G \to \bar{G}$. This gives $\bar{J}(G) = 72$.

In case (v) one has

$$\bar{J}(\bar{G}) \leqslant \bar{J}(G) \leqslant \bar{J}\left(\mathcal{H}_3 \rtimes \operatorname{SL}_2(\mathbf{F}_3)\right) = 24.$$

Therefore, we see that $\bar{J}\left(\operatorname{PGL}_3(\Bbbk)\right) = 40$ and $\bar{J}\left(\operatorname{GL}_3(\Bbbk)\right) = 72$. □

Lemma 2.3.2. *Let $\bar{G} \subset \operatorname{PGL}_3(\Bbbk)$ be a finite subgroup of order $|\bar{G}| > 360$. Then $\bar{J}(\bar{G}) \leqslant 12$.*

Proof. Let $G \subset \operatorname{SL}_3(\Bbbk)$ be a preimage of $\bar{G}$ with respect to the natural projection

$$\operatorname{SL}_3(\Bbbk) \to \operatorname{PGL}_3(\Bbbk).$$

Then one has $|\bar{G}| = |G|/3$, and $\bar{J}(\bar{G}) \leqslant \bar{J}(G)$.

Let us use the notation introduced in the proof of Lemma 2.3.1. If G is a group of type (i) or (ii), then $\bar{J}(G) \leqslant 12$. If G is a group of type (iii) or (iv), then

$$|G| \leqslant |3.\mathfrak{A}_6| = 1080,$$

and $|\bar{G}| \leqslant 360$. Finally, if G is a group of type (v), then

$$|G| \leqslant |\mathcal{H}_3 \rtimes \mathrm{SL}_2(\mathbf{F}_3)| = 648,$$

and $|\bar{G}| \leqslant 216$. $\qquad\square$

Lemma 2.3.3. *Let B be a (non-trivial) finite abelian subgroup of $\mathrm{PGL}_3(\Bbbk)$. Then B is generated by at most three elements.*

Proof. Recall that a finite abelian subgroup of $\mathrm{GL}_n(\Bbbk)$ is generated by at most n elements. Let $\tilde{B} \subset \mathrm{SL}_3(\Bbbk)$ be the preimage of B with respect to the natural projection $\mathrm{SL}_3(\Bbbk) \to \mathrm{PGL}_3(\Bbbk)$. Let $\tilde{A} \subset \tilde{B}$ be a maximal abelian subgroup and let $A \subset B$ be its image. Then A has an isolated fixed point on $\mathbb{P}^2$, and the number of its isolated fixed points is at most 3. Therefore, the group B has an orbit of length at most 3 on $\mathbb{P}^2$. Let P be a point of such orbit, and let $B' \subset B$ be the stabilizer of P. By Lemma 2.1.2 there is a faithful representation of the group B' in the Zariski tangent space $T_P(\mathbb{P}^2) \cong \Bbbk^2$, so that B' is generated by at most two elements. The group B is generated by its subgroup B' and an arbitrary element from $B \setminus B'$, if any. $\qquad\square$

The following fact is a refinement of [PS14, Lemma 2.8] (cf. [PS14, Remark 2.4]).

Lemma 2.3.4. *Let G be a group that fits into an exact sequence*

$$1 \to \Gamma \longrightarrow G \xrightarrow{\phi} \mathrm{PGL}_3(\Bbbk),$$

where $\Gamma \cong \boldsymbol{\mu}_2^m$ with $m \leqslant 2$. Then G is Jordan with

$$\bar{J}(G) \leqslant 2304.$$

Proof. We may assume that G is finite. If the order of the group $\phi(G) \subset \mathrm{PGL}_3(\Bbbk)$ is at most 360, then one has

$$\bar{J}(G) \leqslant [G : \Gamma] = |\phi(G)| \leqslant 360.$$

Therefore, we may assume that $|\phi(G)| > 360$. By Lemma 2.3.2 we can find an abelian subgroup B in $\phi(G)$ of index $[\phi(G) : B] \leqslant 12$. Put $\tilde{G} = \phi^{-1}(B)$. Then

$$[G : \tilde{G}] = [\phi(G) : B] \leqslant 12,$$

so that by Remark 1.2.2 we are left with the task to bound $\bar{J}(\tilde{G})$.

We have an exact sequence of groups

$$1 \to \Gamma \to \tilde{G} \to B \to 1.$$

For an element $g \in \tilde{G}$ denote by $Z(g)$ the centralizer of g in $\tilde{G}$. Since B is an abelian quotient of $\tilde{G}$, we see that the commutator subgroup of $\tilde{G}$ has order at most $|\Gamma|$, so that for any $g \in \tilde{G}$ one has $[G : Z(g)] \leqslant |\Gamma|$.

Since B is an abelian subgroup of $\mathrm{PGL}_3(\Bbbk)$, it is generated by at most three elements by Lemma 2.3.3. Choose three generators of B, and let g_1, g_2 and g_3 be elements of $\tilde{G}$ that project to these three generators. Put

$$I = Z(g_1) \cap Z(g_2) \cap Z(g_3).$$

Then the index
$$[\tilde{G} : I] \leqslant |\Gamma|^3 \leqslant 64.$$

Let C be the centralizer of Γ in $\tilde{G}$. Since Γ is a normal subgroup of $\tilde{G}$, we see that C is a normal subgroup of $\tilde{G}$ as well. Moreover, since $\Gamma \subset C$, we have an inclusion $\tilde{G}/C \subset B$, so that $\tilde{G}/C$ is an abelian group generated by three elements. Also, one has an inclusion
$$\tilde{G}/C \subset \text{Aut}(\Gamma) \subset \text{GL}_2(\mathbf{F}_2) \cong \mathfrak{S}_3.$$

Therefore, we conclude that $|\tilde{G}/C| \leqslant 3$.

Let Z be the center of $\tilde{G}$. Then Z contains the intersection $C \cap I$, so that
$$\bar{J}(\tilde{G}) \leqslant J(\tilde{G}) \leqslant [\tilde{G} : Z] \leqslant [\tilde{G} : C \cap I] \leqslant [\tilde{G} : C] \cdot [\tilde{G} : I] \leqslant 3 \cdot 64 = 192,$$

and thus
$$\bar{J}(G) \leqslant [G : \tilde{G}] \cdot \bar{J}(\tilde{G}) \leqslant 2304. \qquad \square$$

2.4. Dimension 4.

Lemma 2.4.1. *One has*
$$\bar{J}\big(\text{PGL}_4(\Bbbk)\big) = \bar{J}\big(\text{GL}_4(\Bbbk)\big) = 960.$$

Proof. Let V be a four-dimensional vector space over $\Bbbk$, and let $G \subset \text{GL}(V)$ be a finite subgroup. It is enough to study the weak Jordan constant $\bar{J}(G)$. Moreover, for this we may assume that $G \subset \text{SL}(V) \cong \text{SL}_4(\Bbbk)$. Then there are the following possibilities for the group G (see [Bli17, Chapter VII] or [Fei71, §8.5]):

(i) the G-representation V is reducible;
(ii) there is a transitive homomorphism $h \colon G \to \mathfrak{S}_k$ such that V splits into a sum of k representations of the subgroup $H = \text{Ker}(h)$ of dimension $4/k$ for some $k \in \{2, 4\}$;
(iii) the group G contains a subgroup H of index at most 2, such that H is a quotient by a certain central subgroup of a group $\Gamma_1 \times \Gamma_2$, where Γ_1 and Γ_2 are finite subgroups of $\text{GL}_2(\Bbbk)$;
(iv) the group G is generated by some subgroup of scalar matrices in $\text{SL}_4(\Bbbk)$ and a group $\hat{G}$ that is one of the groups $\mathfrak{A}_5$, $\mathfrak{S}_5$, $2.\mathfrak{A}_5$, $2.\mathfrak{S}_5$, or $\text{SL}_2(\mathbf{F}_7)$;
(v) the group G is generated by some subgroup of scalar matrices in $\text{SL}_4(\Bbbk)$ and a group $\hat{G}$ that is one of the groups $2.\mathfrak{A}_6$, $2.\mathfrak{S}_6$, $2.\mathfrak{A}_7$, or $\text{Sp}_4(\mathbf{F}_3)$;
(vi) the group G contains an extra-special group $\mathcal{H}_4$ of order 32 and is contained in the normalizer of $\mathcal{H}_4$ in $\text{SL}(V)$.

In case (i) there is an embedding $G \hookrightarrow \Gamma_1 \times \Gamma_2$, where Γ_i is a finite subgroup of $\text{GL}_{n_i}(\Bbbk)$ for $i \in \{1, 2\}$, and $n_1 \leqslant n_2$ are positive integers such that $n_1 + n_2 = 4$. One has
$$\bar{J}(G) \leqslant \bar{J}(\Gamma_1 \times \Gamma_2) \leqslant \bar{J}\big(\text{GL}_{n_1}(\Bbbk)\big) \cdot \bar{J}\big(\text{GL}_{n_1}(\Bbbk)\big).$$

If $(n_1, n_2) = (1, 3)$, this gives $\bar{J}(G) \leqslant 72$ by Lemma 2.3.1. If $(n_1, n_2) = (2, 2)$, this gives
$$\bar{J}(G) \leqslant 12 \cdot 12 = 144$$

by Corollary 2.2.2.

In case (ii) the group H is a subgroup of G of index
$$[G : H] \leqslant |\mathfrak{S}_k| = k!$$

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Moreover, there is an embedding $H \hookrightarrow \Gamma_1 \times \ldots \times \Gamma_k$, where Γ_i are finite subgroups of $\mathrm{GL}_{4/k}(\Bbbk)$. Thus

$$\bar{J}(G) \leqslant [G : H] \cdot \bar{J}(H) \leqslant k! \cdot \bar{J}(\Gamma_1) \cdot \ldots \cdot \bar{J}(\Gamma_k) \leqslant k! \cdot \bar{J}\big(\mathrm{GL}_{4/k}(\Bbbk)\big)^k.$$

If $k = 2$, this gives $\bar{J}(G) \leqslant 288$ by Corollary 2.2.2. If $k = 4$, this gives $\bar{J}(G) \leqslant 24$.

In case (iii) we obtain the bound $\bar{J}(G) \leqslant 288$ in a similar way.

In case (iv) one has

$$\bar{J}(G) = \bar{J}\big(\hat{G}\big) \leqslant |\hat{G}| \leqslant 336.$$

In case (v) one has

$$\bar{J}(G) = \bar{J}\big(\hat{G}\big) \leqslant \bar{J}\big(\mathrm{Sp}_4(\mathbf{F}_3)\big) = 960.$$

In case (vi) one has $\bar{J}(G) \leqslant \bar{J}(N)$, where N is the normalizer of $\mathcal{H}_4$ in $\mathrm{SL}(V)$. The group N fits into the exact sequence

$$1 \to \tilde{\mathcal{H}}_4 \to N \to \mathfrak{S}_6 \to 1,$$

where $\tilde{\mathcal{H}}_4$ is a group generated by $\mathcal{H}_4$ and a scalar matrix

$$\sqrt{-1} \cdot \mathrm{Id} \in \mathrm{SL}(V).$$

Recall that

$$\mathcal{H}_4 \cong Q_8 \times Q_8/\boldsymbol{\mu}_2,$$

where Q_8 is a quaternion group of order 8. Being viewed as a subgroup of $\mathrm{SL}_2(\Bbbk)$, the group Q_8 is normalized by a binary octahedral group $2.\mathfrak{S}_4$. Thus the group N contains a subgroup

$$R \cong 2.\mathfrak{S}_4 \times 2.\mathfrak{S}_4/\boldsymbol{\mu}_2,$$

and also a subgroup $\tilde{R}$ generated by R and $\sqrt{-1} \cdot \mathrm{Id}$. One has

$$\bar{J}\big(\tilde{R}\big) = \bar{J}(R) = \bar{J}(2.\mathfrak{S}_4 \times 2.\mathfrak{S}_4) = \bar{J}(2.\mathfrak{S}_4)^2 = 36.$$

On the other hand, we compute the index $[N : \tilde{R}] = 20$. This gives

$$\bar{J}(N) \leqslant [N : \tilde{R}] \cdot \bar{J}\big(\tilde{R}\big) = 20 \cdot 36 = 720.$$

Therefore, we see that $\bar{J}(G) \leqslant 960$, and thus $\bar{J}\big(\mathrm{GL}_4(\Bbbk)\big) \leqslant 960$. The inequality

$$\bar{J}\big(\mathrm{PGL}_4(\Bbbk)\big) \leqslant \bar{J}\big(\mathrm{GL}_4(\Bbbk)\big)$$

holds by Remark 2.1.1. The value $\bar{J}(\mathrm{PGL}_4(\Bbbk)) = 960$ is given by the group $\mathrm{PSp}_4(\mathbf{F}_3) \subset \mathrm{PGL}_4(\Bbbk)$ whose abelian subgroup of maximal order is $\boldsymbol{\mu}_3^3$ (cf. [Vdo01, Table 2]). The value $\bar{J}(\mathrm{GL}_4(\Bbbk)) = 960$ is given by the group $\mathrm{Sp}_4(\mathbf{F}_3) \subset \mathrm{GL}_4(\Bbbk)$ whose abelian subgroup of maximal order is $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_3^3$ that is a preimage of a subgroup $\boldsymbol{\mu}_3^3 \subset \mathrm{PSp}_4(\mathbf{F}_3)$ with respect to the natural projection $\mathrm{Sp}_4(\mathbf{F}_3) \to \mathrm{PSp}_4(\mathbf{F}_3)$. $\square$

Remark 2.4.2. The group $2.\mathfrak{S}_5$ listed in case (iv) of Lemma 2.4.1 is omitted in the list given in [Fei71, §8.5]. It is still listed by some other classical surveys, see e.g. [Bli17, §119].

Recall that for a given group G with a representation in a vector space V a *semi-invariant* of G of degree n is an eigen-vector of G in $\mathrm{Sym}^n V^\vee$.

Lemma 2.4.3. *Let V be a four-dimensional vector space over $\Bbbk$, and let $G \subset \mathrm{GL}(V)$ be a finite subgroup. If G has a semi-invariant of degree 2, then $\bar{J}(G) \leqslant 288$.*

Proof. Let q be a semi-invariant of G of degree 2. We consider the possibilities for the rank of the quadratic form q case by case.

Suppose that V has a one-dimensional subrepresentation of G. Then $G \subset \Bbbk^* \times \mathrm{GL}_3(\Bbbk)$, so that $\bar{J}(G) \leqslant 72$ by Lemma 2.3.1. Therefore we may assume that the rank of q is not equal to 1 or 3.

Suppose that the rank of q is 2, so that q is a product of two linear forms. Then there is a subgroup $G_1 \subset G$ of index at most 2 such that these linear forms are semi-invariant with respect to G_1. Hence V splits as a sum of a two-dimensional and two one-dimensional representations of G_1. This implies that $G_1 \subset \Bbbk^* \times \Bbbk^* \times \mathrm{GL}_2(\Bbbk)$, so that

$$\bar{J}(G) \leqslant 2 \cdot \bar{J}(G_1) \leqslant 2 \cdot \bar{J}\big(\mathrm{GL}_2(\Bbbk)\big) = 24$$

by Corollary 2.2.2.

Finally, suppose that the rank of q is 4, so that the quadric $Q \subset \mathbb{P}(V) \cong \mathbb{P}^3$ given by the equation $q = 0$ is smooth, i.e. $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$. By Lemma 2.2.4 there is a subgroup $H \subset G$ of index $[G : H] \leqslant 288$ that acts on Q with a fixed point P and does not interchange the lines L_1 and L_2 passing through P on Q. As the representation of H, the vector space V splits as a sum of the one-dimensional representation corresponding to the point P, two one-dimensional representations arising from the lines L_1 and L_2, and one more one-dimensional representation. Therefore, H is an abelian group (note that Lemma 2.2.4 asserts only that the image of H in $\mathrm{PGL}_4(\Bbbk)$ is abelian). This shows that $\bar{J}(G) \leqslant 288$ and completes the proof of the lemma. $\qquad\square$

2.5. Dimension 5.

Lemma 2.5.1. *One has*

$$\bar{J}\big(\mathrm{PGL}_5(\Bbbk)\big) = \bar{J}\big(\mathrm{GL}_5(\Bbbk)\big) = 960.$$

Proof. Let V be a five-dimensional vector space over $\Bbbk$, and let $G \subset \mathrm{GL}(V)$ be a finite subgroup. It is enough to study the weak Jordan constant $\bar{J}(G)$. Moreover, for this we may assume that $G \subset \mathrm{SL}(V) \cong \mathrm{SL}_5(\Bbbk)$. Recall that there are the following possibilities for the group G (see [Bra67] or [Fei71, §8.5]):

(i) the G-representation V is reducible;
(ii) there is a transitive homomorphism $h \colon G \to \mathfrak{S}_5$ such that V splits into a sum of five one-dimensional representations of the subgroup $H = \mathsf{Ker}(h)$;
(iii) the group G is generated by some subgroup of scalar matrices in $\mathrm{SL}_5(\Bbbk)$ and a group $\hat{G}$ that is one of the groups $\mathfrak{A}_5$, $\mathfrak{S}_5$, $\mathfrak{A}_6$, $\mathfrak{S}_6$, $\mathrm{PSL}_2(\mathbf{F}_{11})$, or $\mathrm{PSp}_4(\mathbf{F}_3)$;
(iv) one has $G \cong \mathcal{H}_5 \rtimes \Sigma$, where $\mathcal{H}_5$ is the Heisenberg group of order 125, and Σ is some subgroup of $\mathrm{SL}_2(\mathbf{F}_5)$.

In case (i) there is an embedding $G \hookrightarrow \Gamma_1 \times \Gamma_2$, where Γ_i is a finite subgroup of $\mathrm{GL}_{n_i}(\Bbbk)$ for $i \in \{1, 2\}$, and $n_1 \leqslant n_2$ are positive integers such that $n_1 + n_2 = 5$. One has

$$\bar{J}(G) \leqslant \bar{J}(\Gamma_1 \times \Gamma_2) \leqslant \bar{J}\big(\mathrm{GL}_{n_1}(\Bbbk)\big) \cdot \bar{J}\big(\mathrm{GL}_{n_1}(\Bbbk)\big).$$

If $(n_1, n_2) = (1, 4)$, this gives $\bar{J}(G) \leqslant 960$ by Lemma 2.4.1. If $(n_1, n_2) = (2, 3)$, this gives

$$\bar{J}(G) \leqslant 12 \cdot 72 = 864$$

by Corollary 2.2.2 and Lemma 2.3.1.

In case (ii) the group H is an abelian subgroup of G, so that

$$\bar{J}(G) \leqslant [G : H] \leqslant |\mathfrak{S}_5| = 120.$$

In case (iii) it is easy to check that $\bar{J}(G) = \bar{J}(\hat{G}) \leqslant 960$, cf. the proof of Lemma 2.4.1. In case (iv) one has

$$\bar{J}(G) \leqslant \bar{J}\big(\mathcal{H}_5 \rtimes \mathrm{SL}_2(\mathbf{F}_5)\big) = 120.$$

Therefore, we see that $\bar{J}(G) \leqslant 960$, and thus $\bar{J}\big(\mathrm{GL}_5(\Bbbk)\big) \leqslant 960$. The inequality

$$\bar{J}\big(\mathrm{PGL}_5(\Bbbk)\big) \leqslant \bar{J}\big(\mathrm{GL}_5(\Bbbk)\big)$$

holds by Remark 2.1.1. The value $\bar{J}(\mathrm{PGL}_5(\Bbbk)) = 960$ is given by the group $\mathrm{PSp}_4(\mathbf{F}_3) \subset \mathrm{PGL}_5(\Bbbk)$, cf. the proof of Lemma 2.4.1. Similarly, the value $\bar{J}(\mathrm{GL}_5(\Bbbk)) = 960$ is given by the group $\mathrm{PSp}_4(\mathbf{F}_3) \subset \mathrm{GL}_5(\Bbbk)$. $\qquad\square$

2.6. **Dimension 7.** We start with a general observation concerning finite groups with relatively large abelian subgroups.

Lemma 2.6.1. *Let G be a group, and $\tilde{\Gamma} \subset G$ be a normal finite abelian subgroup. Suppose that $\tilde{\Gamma}$ cannot be generated by less than m elements. Let V be an N-dimensional vector space over $\Bbbk$. Suppose that V is a faithful representation of G. Then there exist positive integers $t, m_1, \ldots, m_t, d_1, \ldots, d_t$ such that*

- $m_1 d_1 + \ldots + m_t d_t = N$;
- $m_1 + \ldots + m_t \geqslant m$;
- *the group G is Jordan with*

$$\bar{J}(G) \leqslant \Big(\prod_{i=1}^{t} m_i! \Big) \cdot \Big(\prod_{i=1}^{t} \bar{J}\big(\mathrm{GL}_{d_i}(\Bbbk)\big)^{m_i} \Big).$$

Proof. Let

$$(2.6.2) \qquad\qquad V = V_1 \oplus \ldots \oplus V_s$$

be the splitting of V into isotypical components with respect to $\tilde{\Gamma}$. Since V is a faithful representation of $\tilde{\Gamma}$, and $\tilde{\Gamma}$ is an abelian group, we have an injective homomorphism $\tilde{\Gamma} \hookrightarrow (\Bbbk^*)^s$. By assumption one has $s \geqslant m$. Suppose that the splitting (2.6.2) contains m_1 summands of dimension d_1, m_2 summands of dimension d_2, $\ldots$, and m_t summands of dimension d_t. Then one has $m_1 d_1 + \ldots + m_t d_t = N$. Moreover, the total number of summands in (2.6.2) equals $m_1 + \ldots + m_t = s \geqslant m$.

Since $\tilde{\Gamma} \subset G$ is a normal subgroup, the group G interchanges the summands in (2.6.2). Moreover, G can interchange only those subspaces V_i and V_j that have the same dimension. Therefore, we get a homomorphism

$$\psi \colon G \to \prod_{i=1}^{t} \mathfrak{S}_{m_i}.$$

Let $\Delta \subset G$ be the kernel of the homomorphism ψ. Then each summand of (2.6.2) is invariant with respect to Δ. Since V is a faithful representation of Δ, one has an inclusion

$$\Delta \hookrightarrow \prod_{j=1}^{s} \mathrm{GL}(V_j) \cong \prod_{i=1}^{t} \big(\mathrm{GL}_{d_i}(\Bbbk)\big)^{m_i}.$$

Note that

$$[G : \Delta] \leqslant \Big| \prod_{i=1}^{t} \mathfrak{S}_{m_i} \Big| = \prod_{i=1}^{t} m_i!.$$

Recall that the groups $\mathrm{GL}_{d_i}(\Bbbk)$ are Jordan by Theorem 1.1.1. Thus the group G is Jordan with

$$\bar{J}(G) \leqslant [G : \Delta] \cdot \bar{J}(\Delta) \leqslant \Big(\prod_{i=1}^{t} m_i! \Big) \cdot \Big(\prod_{i=1}^{t} \bar{J}\big(\mathrm{GL}_{d_i}(\Bbbk)\big)^{m_i} \Big)$$

by Remark 1.2.2. $\qquad\square$

Lemma 2.6.1 allows us to provide a bound for Jordan constants of some subgroups of $\mathrm{GL}_7(\Bbbk)$. This bound will be used in the proof of Lemma 7.1.2.

Lemma 2.6.3. *Let G be a group, and $\tilde{\Gamma} \subset G$ be a normal finite abelian subgroup such that $\tilde{\Gamma} \cong \boldsymbol{\mu}_2^m$ with $m \geqslant 4$. Suppose that G has a faithful seven-dimensional representation. Then G is Jordan with*

$$\bar{J}(G) \leqslant 10368.$$

Proof. Since $\tilde{\Gamma} \cong \boldsymbol{\mu}_2^m$ has a faithful seven-dimensional representation, we have $m \leqslant 7$. By Lemma 2.6.1 there exist positive integers t, $m_1, \ldots, m_t$, $d_1, \ldots, d_t$, such that

$$m_1 d_1 + \ldots + m_t d_t = 7,$$

while $m_1 + \ldots + m_t \geqslant m$ and

(2.6.4) $$\bar{J}(G) \leqslant \Big(\prod_{i=1}^{t} m_i! \Big) \cdot \Big(\prod_{i=1}^{t} \bar{J}\big(\mathrm{GL}_{d_i}(\Bbbk)\big)^{m_i} \Big).$$

In particular, one has $4 \leqslant m_1 + \ldots + m_t \leqslant 7$. Also, we may assume that $d_1 < \ldots < d_t$. We consider several possibilities for $m_1 + \ldots + m_t$ case by case.

If $m_1 + \ldots + m_t = 7$, then $t = 1$, $d_1 = 1$ and $m_1 = 7$, so that (2.6.4) gives

$$\bar{J}(G) \leqslant 7! = 5040.$$

If $m_1 + \ldots + m_t = 6$, then $t = 2$, $d_1 = 1$, $m_1 = 5$, $d_2 = 2$, $m_2 = 1$, so that (2.6.4) gives

$$\bar{J}(G) \leqslant 5! \cdot \bar{J}\big(\mathrm{GL}_2(\Bbbk)\big) = 120 \cdot 12 = 1440$$

by Corollary 2.2.2.

If $m_1 + \ldots + m_t = 5$, then $t = 2$, $d_1 = 1$, and either $m_1 = 4$, $d_2 = 3$, $m_2 = 1$, or $m_1 = 3$, $d_2 = 2$, $m_2 = 2$. In the former case (2.6.4) gives

$$\bar{J}(G) \leqslant 4! \cdot \bar{J}\big(\mathrm{GL}_3(\Bbbk)\big) = 24 \cdot 72 = 1728$$

by Lemma 2.3.1. In the latter case (2.6.4) gives

$$\bar{J}(G) \leqslant 3! \cdot 2! \cdot \bar{J}\big(\mathrm{GL}_2(\Bbbk)\big)^2 = 6 \cdot 2 \cdot 12^2 = 1728$$

by Corollary 2.2.2.

Finally, if $m_1 + \ldots + m_t = 4$, then either

$$t = 2, \ d_1 = 1, \ m_1 = 3, \ d_2 = 4, \ m_2 = 1,$$

or

$$t = 2, \ d_1 = 1, \ m_1 = 1, \ d_2 = 2, \ m_2 = 3,$$

or

$$t = 3, \ d_1 = 1, \ m_1 = 2, \ d_2 = 2, \ m_2 = 1, \ d_3 = 3, \ m_3 = 1.$$

In the first case (2.6.4) gives

$$\bar{J}(G) \leqslant 3! \cdot \bar{J}\big(\mathrm{GL}_4(\Bbbk)\big) = 6 \cdot 960 = 5760$$

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by Lemma 2.4.1. In the second case (2.6.4) gives

$$\bar{J}(G) \leqslant 3! \cdot \bar{J}\big(\mathrm{GL}_2(\Bbbk)\big)^3 = 6 \cdot 12^3 = 10368$$

by Corollary 2.2.2. In the third case (2.6.4) gives

$$\bar{J}(G) \leqslant 2! \cdot \bar{J}\big(\mathrm{GL}_2(\Bbbk)\big) \cdot \bar{J}\big(\mathrm{GL}_3(\Bbbk)\big) = 2 \cdot 12 \cdot 72 = 1728$$

by Corollary 2.2.2 and Lemma 2.3.1. $\qquad\square$

3. Surfaces

The goal of this section is to estimate weak Jordan constants for automorphism groups of rational surfaces, as well as some other constants of similar nature. In the sequel for any variety X we will denote by $\Phi(X)$ the minimal positive integer m such that for any finite group $G \subset \mathrm{Aut}(X)$ there is a subgroup $F \subset G$ with $[G : F] \leqslant m$ acting on X with a fixed point. If there does not exist an integer m with the above property, we put $\Phi(X) = +\infty$. Note that $\Phi(X)$ is bounded by some universal constant for rationally connected varieties X of dimension at most 3 by [PS16a, Theorem 4.2].

3.1. **Preliminaries.** We start with the one-dimensional case.

Lemma 3.1.1. *One has* $\Phi(\mathbb{P}^1) = 12$. *Moreover, if T is a finite union of rational curves such that its dual graph $T^\vee$ is a tree, then $\Phi(T) \leqslant 12$.*

Proof. The inequality $\Phi(\mathbb{P}^1) \leqslant 12$ is given by Lemma 2.2.4(i). The equality is given by the icosahedral group $\mathfrak{A}_5 \subset \mathrm{Aut}(\mathbb{P}^1)$. Since for any rational curve C one has

$$\mathrm{Aut}(C) \subset \mathrm{Bir}(C) \cong \mathrm{Bir}(\mathbb{P}^1) = \mathrm{Aut}(\mathbb{P}^1),$$

we see that $\Phi(\mathbb{P}^1) = 12$.

Let T be a finite union of rational curves such that its dual graph $T^\vee$ is a tree. Then there is a natural homomorphism of $\mathrm{Aut}(T)$ to the finite group $\mathrm{Aut}(T^\vee)$. It is easy to show by induction on the number of vertices that either there is an edge of $T^\vee$ that is invariant under $\mathrm{Aut}(T^\vee)$, or there is a vertex of $T^\vee$ that is invariant under $\mathrm{Aut}(T^\vee)$. In the former case there is a point $P \in T$ fixed by $\mathrm{Aut}(T)$, so that $\Phi(T) = 1$. In the latter case there is a rational curve $C \subset T$ that is invariant under $\mathrm{Aut}(T)$, so that

$$\Phi(T) \leqslant \Phi(C) \leqslant 12. \qquad\square$$

Now we proceed with the two-dimensional case. In a sense, we are going to do in a more systematic way the same things that were done in Lemma 2.2.4. For a variety X with an action of a finite group G, we will denote by $\Phi_a(X, G)$ the minimal positive integer m such that there is an *abelian* subgroup $A \subset G$ with $[G : A] \leqslant m$ acting on X with a fixed point. The main advantage of this definition is the following property.

Lemma 3.1.2. *Let X and Y be smooth surfaces acted on by a finite group G. Suppose that there is a G-equivariant birational morphism $\pi \colon Y \to X$. Then $\Phi_a(Y, G) = \Phi_a(X, G)$.*

Proof. The assertion is implied by the results of [KS00] in arbitrary dimension. We give the proof for dimension 2 for the readers convenience.

The inequality $\Phi_a(Y, G) \geqslant \Phi_a(X, G)$ is obvious. To prove the opposite inequality choose an abelian subgroup $A \subset G$ such that there is a point $P \in X$ fixed by A. We are going to produce a point $Q \in Y$ fixed by A such that $\pi(Q) = P$.

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The birational morphism π is a composition of blow ups of smooth points. Since π is G-equivariant and thus A-equivariant, we may replace X by a neighborhood of the point P and thus suppose that π is a sequence of blow ups of points lying over the point P. If π is an isomorphism, then there is nothing to prove. Otherwise, by induction in the number of blow ups, we see that it is enough to consider the case when π is a single blow up of the point P. In this case the exceptional divisor $E = \pi^{-1}(P)$ is identified with the projectivization of the Zariski tangent space $T_P(X)$, and the action of A on E comes from a linear action of A on $T_P(X)$. Since the group A is abelian, it has a one-dimensional invariant subspace in $T_P(X)$, which gives an A-invariant point $Q \in E \subset Y$. $\qquad\square$

3.2. Del Pezzo surfaces.

Lemma 3.2.1. *Let $G \subset \mathrm{Aut}(\mathbb{P}^2)$ be a finite group. Then one has $\Phi_a(\mathbb{P}^2, G) \leqslant 72$.*

Proof. One has $\mathrm{Aut}(X) \cong \mathrm{PGL}_3(\Bbbk)$. By the holomorphic Lefschetz fixed-point formula any cyclic group acting on a rational variety has a fixed point. Now the required bound is obtained from the classification of finite subgroups of $\mathrm{GL}_3(\Bbbk)$ (see [MBD16, Chapter XII] or [Fei71, §8.5], and also the proof of Lemma 2.3.1). $\qquad\square$

Remark 3.2.2. Note that the bound given by Lemma 3.2.1 is actually attained for the group $\mathfrak{A}_6 \subset \mathrm{PGL}_3(\Bbbk)$ whose abelian subgroup of maximal order acting on $\mathbb{P}^2$ with a fixed point is $\boldsymbol{\mu}_5$.

Lemma 3.2.3. *Let X be a smooth del Pezzo surface. Let $G \subset \mathrm{Aut}(X)$ be a finite group. Then one has*
$$\Phi_a(X, G) \leqslant 288.$$
Moreover, if X is not isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, then $\Phi_a(X, G) \leqslant 144$.

Proof. If $X \cong \mathbb{P}^2$, then $\Phi_a(X, G) \leqslant 72$ by Lemma 3.2.1.

Suppose that $X \cong \mathbb{P}^1 \times \mathbb{P}^1$. Then one has $\Phi_a(X, G) \leqslant 288$ by Lemma 2.2.4(ii). Note that this value is attained for the group
$$G \cong \left(\mathfrak{A}_5 \times \mathfrak{A}_5\right) \rtimes \boldsymbol{\mu}_2 \subset \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1).$$

Suppose that X is a blow up $\pi \colon X \to \mathbb{P}^2$ at one or two points. Then π is an $\mathrm{Aut}(X)$-equivariant birational morphism, so that $\Phi_a(X, G) \leqslant 72$ by Lemmas 3.2.1 and 3.1.2.

Put $d = K_X^2$. We may assume that $d \leqslant 6$.

Suppose that $d = 6$. Then
$$\mathrm{Aut}(X) \cong \left(\Bbbk^* \times \Bbbk^*\right) \rtimes \mathrm{D}_6,$$
where D_6 is the dihedral group of order 12 (see [Dol12, Theorem 8.4.2]). The subgroup $\Bbbk^* \times \Bbbk^* \subset \mathrm{Aut}(X)$ acts on X with a fixed point by Borel's theorem (see e.g. [Hum75, VIII.21]). From this one can easily deduce that $\Phi_a(X, G) \leqslant 12$ for any finite subgroup $G \subset \mathrm{Aut}(X)$.

If $d \leqslant 5$, then the group $\mathrm{Aut}(X)$ is finite, and it is enough to show that $\Phi_a\big(X, \mathrm{Aut}(X)\big) \leqslant 144$.

Suppose that $d = 5$. Then $\mathrm{Aut}(X) \cong \mathfrak{S}_5$ (see [Dol12, Theorem 8.5.6]). Hence for any subgroup $G \subset \mathrm{Aut}(X)$ one has
$$\Phi_a\big(X, \mathrm{Aut}(X)\big) \leqslant |\mathrm{Aut}(X)| = 120.$$

Suppose that $d = 4$. Then
$$\mathrm{Aut}(X) \cong \boldsymbol{\mu}_2^4 \rtimes \Gamma,$$

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where $|\Gamma| \leqslant 10$ (see [Dol12, Theorem 8.6.6]). Representing X as an intersection of two quadrics with equations in diagonal form, one can see that there is a subgroup $\boldsymbol{\mu}_2^2 \subset \mathrm{Aut}(X)$ acting on X with a fixed point. Therefore, one has

$$\Phi_a\big(X, \mathrm{Aut}(X)\big) \leqslant \frac{|\mathrm{Aut}(X)|}{|\boldsymbol{\mu}_2^2|} \leqslant \frac{160}{4} = 40.$$

Suppose that $d = 3$. Then either $\mathrm{Aut}(X) \cong \boldsymbol{\mu}_3^3 \rtimes \mathfrak{S}_4$ and X is the Fermat cubic, or $|\mathrm{Aut}(X)| \leqslant 120$ (see [Dol12, Theorem 9.5.6]). In the former case it is easy to see that there is a subgroup $\boldsymbol{\mu}_3^2 \subset \mathrm{Aut}(X)$ acting on X with a fixed point, so that

$$\Phi_a\big(X, \mathrm{Aut}(X)\big) \leqslant \frac{|\mathrm{Aut}(X)|}{|\boldsymbol{\mu}_3^2|} = \frac{648}{9} = 72.$$

In the latter case one has

$$\Phi_a\big(X, \mathrm{Aut}(X)\big) \leqslant |\mathrm{Aut}(X)| \leqslant 120.$$

Suppose that $d = 2$. Then either $|\mathrm{Aut}(X)| \leqslant 96$, or $\mathrm{Aut}(X) \cong \boldsymbol{\mu}_2 \times (\boldsymbol{\mu}_4^2 \rtimes \mathfrak{S}_3)$, or $\mathrm{Aut}(X) \cong \boldsymbol{\mu}_2 \times \mathrm{PSL}_2(\mathbf{F}_7)$ (see [Dol12, Table 8.9]). In the latter case one has

$$\Phi_a\big(X, \mathrm{Aut}(X)\big) \leqslant |\mathrm{Aut}(X)| \leqslant 120.$$

To estimate $\Phi_a\big(X, \mathrm{Aut}(X)\big)$ in the former two cases, recall that the anticanonical linear system $|-K_X|$ defines a double cover

$$\varphi_{|-K_X|} \colon X \to \mathbb{P}^2$$

branched over a smooth quartic curve $C \subset \mathbb{P}^2$. The subgroup $\boldsymbol{\mu}_2$ acts by the Galois involution of the corresponding double cover. In particular, the curve $\varphi_{|-K_X|}^{-1}(C)$ consists of $\boldsymbol{\mu}_2$-fixed points. If $\mathrm{Aut}(X) \cong \boldsymbol{\mu}_2 \times (\boldsymbol{\mu}_4^2 \rtimes \mathfrak{S}_3)$, this gives

$$\Phi_a\big(X, \mathrm{Aut}(X)\big) \leqslant \frac{|\mathrm{Aut}(X)|}{|\boldsymbol{\mu}_2|} = \frac{192}{2} = 96.$$

If $\mathrm{Aut}(X) \cong \boldsymbol{\mu}_2 \times \mathrm{PSL}_2(\mathbf{F}_7)$, then the group $\mathrm{PSL}_2(\mathbf{F}_7) \subset \mathrm{Aut}(X)$ contains a subgroup $\boldsymbol{\mu}_7$, and $\boldsymbol{\mu}_7$ acts on the curve $\varphi_{|-K_X|}^{-1}(C) \cong C$ with a fixed point (this can be easily seen, for example, from the Riemann–Hurwitz formula since C is a smooth curve of genus 3). Thus

$$\Phi_a\big(X, \mathrm{Aut}(X)\big) \leqslant \frac{|\mathrm{Aut}(X)|}{|\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_7|} = \frac{336}{14} = 24.$$

Finally, suppose that $d = 1$. Then

$$\Phi_a\big(X, \mathrm{Aut}(X)\big) \leqslant |\mathrm{Aut}(X)| \leqslant 144$$

(see [Dol12, Table 8.14]). $\qquad\qquad\square$

Remark 3.2.4. In several cases (say, for a del Pezzo surface of degree $d = 5$) one can produce better upper bounds for $\Phi_a(X, G)$ than those given in the proof of Lemma 3.2.3, but we do not pursue this goal.

Lemma 3.2.3 immediately implies the following.

Corollary 3.2.5 (cf. Lemma 2.2.4(iii))**.** *Let X be a smooth del Pezzo surface. Then one has $\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant 288$. Moreover, if X is not isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, then $\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant 144$.*

3.3. **Rational surfaces.** Now we pass to the case of arbitrary rational surfaces.

Lemma 3.3.1. *Let X be a smooth rational surface, and $G \subset \operatorname{Aut}(X)$ be a finite subgroup. Then there exists an abelian subgroup $H \subset G$ of index $[G : H] \leqslant 288$ that acts on X with a fixed point.*

Proof. Let Y be a smooth projective rational surface, and $G \subset \operatorname{Aut}(Y)$ be a finite group. Let $\pi \colon Y \to X$ be a result of a G-Minimal Model Program ran on Y. One has

$$\Phi_a(Y, G) = \Phi_a(X, G)$$

by Lemma 3.1.2. Moreover, X is either a del Pezzo surface, or there is a G-equivariant conic bundle structure on X (see [Isk80b, Theorem 1G]). If X is a del Pezzo surface, then $\Phi_a(X, G) \leqslant 288$ by Lemma 3.2.3, so that $\Phi_a(Y, G) \leqslant 288$.

Therefore, we assume that there is a G-equivariant conic bundle structure

$$\phi \colon X \to B \cong \mathbb{P}^1.$$

There is an exact sequence of groups

$$1 \to G_\phi \longrightarrow G \xrightarrow{u} G_B \to 1,$$

where G_ϕ acts by fiberwise automorphisms with respect to ϕ, and $G_B \subset \operatorname{Aut}(\mathbb{P}^1)$. By Lemma 3.1.1 we find a subgroup $G'_B \subset G_B$ of index $[G_B : G'_B] \leqslant 12$ acting on $\mathbb{P}^1$ with a fixed point $P \in \mathbb{P}^1$. The group

$$G' = u^{-1}(G'_B) \subset G$$

acts by automorphisms of the fiber $C = \phi^{-1}(P)$. Note that C is a reduced conic, i.e. it is either isomorphic to $\mathbb{P}^1$, or is a union of two copies of $\mathbb{P}^1$ meeting at one point.

Suppose that $C \cong \mathbb{P}^1$. Then there is a point $Q \in C$ that is invariant with respect to some subgroup $G'' \subset G'$ of index $[G' : G''] \leqslant 12$ by Lemma 3.1.1. The morphism $\phi \colon X \to B$ is smooth at Q. Hence the map $d\phi \colon T_Q(X) \to T_P(B)$ is surjective. By Lemma 2.1.2 the group G'' acts faithfully on the Zariski tangent space $T_Q(X)$, and the group G'_B acts faithfully on the Zariski tangent space $T_P(B)$. The map $d\phi$ is G''-equivariant and so G'' has one-dimensional invariant subspace $\operatorname{Ker}(d\phi) \subset T_Q(X) \cong \Bbbk^2$. In this case G'' must be abelian with $[G : G''] \leqslant 12 \cdot 12 = 144$.

Now consider the case when C is a reducible conic, i.e. it is a union of two copies of $\mathbb{P}^1$ meeting at one point, say Q. Then Q is G'-invariant. There exists a subgroup $G'' \subset G'$ of index $[G' : G''] \leqslant 2$ such that both irreducible components C_1, $C_2 \subset C$ are invariant with respect to G''. In this case subspaces $T_Q(C_i) \subset T_Q(X)$ are G''-invariant and as above G'' is abelian with $[G : G''] \leqslant 12 \cdot 2 = 24$.

Therefore, one has

$$\Phi_a(Y, G) = \Phi_a(X, G) \leqslant [G : G''] \leqslant 144. \qquad \square$$

Corollary 3.3.2. *Let X be a smooth rational surface. Then one has $\bar{J}\big(\operatorname{Aut}(X)\big) \leqslant 288$.*

Corollary 3.3.3. *One has $\bar{J}\big(\operatorname{Cr}_2(\Bbbk)\big) = 288$.*

Proof. Let $G \subset \operatorname{Cr}_2(\Bbbk)$ be a finite group. It is enough to study the weak Jordan constant $\bar{J}(G)$. Regularizing the action of G and taking an equivariant desingularization (see e.g. [PS14, Lemma-Definition 3.1]), we may assume that $G \subset \operatorname{Aut}(X)$ for a smooth rational surface X. Now the bound $\bar{J}\big(\operatorname{Cr}_2(\Bbbk)\big) \leqslant 288$ follows from Corollary 3.3.2. The equality is due to Lemma 2.2.4(iii). $\qquad \square$

A direct consequence of Corollary 3.3.3 is that the weak Jordan constant of the Cremona group of rank 2 is bounded by 288 for an arbitrary (not necessarily algebraically closed) base field. Together with Remark 1.2.2 this gives a proof of Proposition 1.2.3.

3.4. **Non-rational surfaces.** We conclude this section by three easy observations concerning automorphism groups of certain non-rational surfaces.

Lemma 3.4.1. *Let C be a smooth curve of genus $g \geqslant 2$, and let S be a ruled surface over C. Then the group $\mathrm{Aut}(S)$ is Jordan with $\bar{J}\big(\mathrm{Aut}(S)\big) \leqslant 1008(g-1)$.*

Proof. Let $G \subset \mathrm{Aut}(S)$ be a finite group. It is enough to prove the corresponding bound for $\bar{J}(G)$. There is an exact sequence of groups

$$1 \to G_\phi \longrightarrow G \longrightarrow G_C \to 1,$$

where G_ϕ acts by fiberwise automorphisms with respect to ϕ, and $G_C \subset \mathrm{Aut}(C)$. One has

$$|G_C| \leqslant 84(g-1)$$

by the Hurwitz bound. On the other hand, the group G_ϕ is a subgroup of $\mathrm{Aut}(\mathbb{P}^1) \cong \mathrm{PGL}_2(\Bbbk)$, so that G_ϕ contains an abelian subgroup H of index

$$[G_\phi : H] \leqslant 12$$

by Corollary 2.2.2. Thus one has

$$\bar{J}(G) \leqslant [G : H] = [G : G_\phi] \cdot [G_\phi : H] = [G : G_\phi] \cdot |G_C| \leqslant 12 \cdot 84 \cdot (g-1) = 1008(g-1). \ \square$$

Lemma 3.4.2 (cf. [PS14, Corollary 2.15]). *Let S be an abelian surface. Then the group $\mathrm{Aut}(S)$ is Jordan with $\bar{J}\big(\mathrm{Aut}(S)\big) \leqslant 5760$.*

Proof. One has $\mathrm{Aut}(S) \cong A \rtimes \Gamma$, where A is an abelian group (that is identified with the group of points on S), and Γ is a subgroup of $\mathrm{GL}_4(\mathbb{Z})$. Thus $\mathrm{Aut}(S)$ is Jordan with

$$\bar{J}\big(\mathrm{Aut}(S)\big) \leqslant [\mathrm{Aut}(S) : A] = |\Gamma| \leqslant 5760$$

by the Minkowski bound for $\mathrm{GL}_4(\mathbb{Z})$ (see e. g. [Ser07, §1.1]). $\square$

To obtain a bound for a weak Jordan constant in the last case we will need the following easy group-theoretic fact.

Lemma 3.4.3. *Let G be a finite group with $|G| \leqslant 79380$. Then*

$$\bar{J}(G) \leqslant 9922.$$

Proof. Suppose that $|G|$ is divisible by a prime number p. Then G contains a cyclic subgroup of order p, so that

$$\bar{J}(G) \leqslant \frac{|G|}{p}.$$

In particular, if $|G|$ is divisible by a prime $p \geqslant 11$, then

$$\bar{J}(G) \leqslant \frac{|G|}{11} < 7217.$$

Similarly, suppose that p is a prime such that $|G|$ is divisible by p^2. Let $G_p \subset G$ be a Sylow p-subgroup. Then $|G_p| \geqslant p^2$. If $|G_p| = p^2$, then G_p is abelian, so that

$$\bar{J}(G) \leqslant [G : G_p] = \frac{|G|}{p^2}.$$

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If $|G_p| \geqslant p^3$, then G_p contains an abelian subgroup A of order $|A| \geqslant p^2$ (see Corollary 2 of Theorem 1.17 in Chapter 2 of [Suz82]), and we again have

$$\bar{J}(G) \leqslant [G : A] \leqslant \frac{|G|}{p^2}.$$

In particular, if there is a prime $p \geqslant 3$ such that $|G|$ is divisible by p^2, then

$$\bar{J}(G) \leqslant \frac{|G|}{p^2} \leqslant \frac{|G|}{9} \leqslant 8820.$$

Now suppose that $|G|$ is not divisible by any prime greater than 7, and $|G|$ is not divisible by a square of any prime greater than 2. This means that

$$|G| = 2^\alpha \cdot 3^\beta \cdot 5^\gamma \cdot 7^\delta,$$

where $\beta, \gamma, \delta \in \{0, 1\}$. If $\alpha \leqslant 3$, then

$$\bar{J}(G) \leqslant |G| \leqslant 2^3 \cdot 3 \cdot 5 \cdot 7 = 840.$$

Thus we assume that $\alpha \geqslant 4$. Let $G_2 \subset G$ be a Sylow 2-subgroup. Applying Corollary 2 of Theorem 1.17 in Chapter 2 of [Suz82] once again, we see that G_2 contains an abelian subgroup A of order $|A| \geqslant 8$. Hence one has

$$\bar{J}(G) \leqslant [G : A] \leqslant \frac{|G|}{8} < 9923. \qquad \square$$

Now we are ready to bound a weak Jordan constant for automorphism groups of surfaces of general type of low degree.

Lemma 3.4.4. *Let S be a smooth minimal surface of general type of degree $K_S^2 \leqslant 45$. Then the group $\mathrm{Aut}(S)$ is Jordan with $\bar{J}\big(\mathrm{Aut}(S)\big) \leqslant 9922$.*

Proof. By [Xia95] one has

$$|\mathrm{Aut}(S)| \leqslant 42^2 \cdot K_S^2 \leqslant 79380.$$

Thus the group $\mathrm{Aut}(S)$ is Jordan with

$$\bar{J}\big(\mathrm{Aut}(S)\big) \leqslant 9922$$

by Lemma 3.4.3. $\qquad \square$

4. Terminal singularities

In this section we study Jordan property for automorphism groups of germs of three-dimensional terminal singularities, and derive some conclusions about automorphism groups of non-Gorenstein terminal Fano threefolds.

4.1. **Local case.** Recall from §2.1 that for an arbitrary variety U and a point $P \in U$ we denote by $\mathrm{Aut}_P(U)$ the stabilizer of P in $\mathrm{Aut}(U)$. Now we are going to estimate a weak Jordan constant of a group $\mathrm{Aut}_P(U)$, where $P \in U$ is a three-dimensional terminal singularity.

Lemma 4.1.1. *Let U be a threefold, and $P \in U$ be a terminal singular point of U. Let $G \subset \mathrm{Aut}_P(U)$ be a finite subgroup. Then for some positive integer r there is an extension*

$$(4.1.2) \qquad\qquad 1 \longrightarrow \boldsymbol{\mu}_r \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1$$

such that the following assertions hold.

- (i) *There is an embedding $\tilde{G} \subset \mathrm{GL}_4(\Bbbk)$, and the group $\tilde{G}$ has a semi-invariant of degree 2.*
- (ii) *If (U, P) is a cyclic quotient singularity, then there is an embedding $\tilde{G} \subset \mathrm{GL}_3(\Bbbk)$.*
- (iii) *Let D be a G-invariant boundary on X such that the log pair (U, D) is log canonical and such that a minimal center of log canonical singularities is a G-invariant curve containing P (see [Kaw97, Proposition 1.5]). Then $\tilde{G} \subset \Bbbk^* \times \mathrm{GL}_3(\Bbbk)$.*

Proof. Let $r \geqslant 1$ be the index of $U \ni P$, i. e. r equals the minimal positive integer t such that tK_U is Cartier at P. Replacing U by a smaller G-invariant neighborhood of P if necessary, we may assume that $rK_U \sim 0$. Consider the index-one cover

$$\pi \colon (U^\sharp, P^\sharp) \to (U, P)$$

(see [Rei87, Proposition 3.6]). Then $U^\sharp \ni P^\sharp$ is a terminal singularity of index 1, and $U \cong U^\sharp/\boldsymbol{\mu}_r$. Note that $U^\sharp \ni P^\sharp$ is a hypersurface singularity, i. e. $\dim T_{P^\sharp}(U^\sharp) \leqslant 4$ (see [Rei87, Corollary 3.12(i)]). Moreover, $U^\sharp$ is smooth at $P^\sharp$ if (U, P) is a cyclic quotient singularity.

By construction of the index one cover every element of $\mathrm{Aut}_P(U)$ admits r lifts to $\mathrm{Aut}(U^\sharp, P^\sharp)$. Thus we have a natural exact sequence (4.1.2), where $\tilde{G}$ is some subgroup of $\mathrm{Aut}_{P^\sharp}(U^\sharp)$. Furthermore, by Lemma 2.1.2 we know that $\tilde{G} \subset \mathrm{GL}_3(\Bbbk)$ if $U^\sharp$ is smooth at $P^\sharp$. This gives assertion (ii).

Now suppose that $\dim T_{P^\sharp}(U^\sharp) = 4$. By Lemma 2.1.2 one has an embedding $\tilde{G} \subset \mathrm{GL}_4(\Bbbk)$. Moreover, $U^\sharp \ni P^\sharp$ is a hypersurface singularity of multiplicity 2 by [KM98, Corollary 5.38]. This means that the kernel of the natural map

$$\mathrm{Sym}^2\big(\mathfrak{m}_{P^\sharp, U^\sharp}/\mathfrak{m}^2_{P^\sharp, U^\sharp}\big) \longrightarrow \mathfrak{m}^2_{P^\sharp, U^\sharp}/\mathfrak{m}^3_{P^\sharp, U^\sharp}$$

is generated by an element of degree 2. Therefore, the group $\tilde{G}$ has a semi-invariant polynomial of degree 2. This completes the proof of assertion (i).

Finally, let C, D, and G be as in assertion (iii). Put $D^\sharp = \pi^*D$ and $C^\sharp = \pi^{-1}(C)$. One can show that $C^\sharp$ is again a minimal center of log canonical singularities of $(U^\sharp, D^\sharp)$ (cf. [KM98, Proposition 5.20]). In particular, $C^\sharp$ is smooth (see [Kaw97, Theorem 1.6]). As above, one has an embedding $\tilde{G} \subset \mathrm{GL}\big(T_{P^\sharp}(U^\sharp)\big)$. Moreover, since $C^\sharp$ is $\tilde{G}$-invariant, we have a decomposition of $\tilde{G}$-representations

$$T_{P^\sharp}(U^\sharp) = T_1 \oplus T_3,$$

where $T_1 = T_{P^\sharp}(C^\sharp) \cong \Bbbk$ and $\dim T_3 = 3$. Hence, one has

$$\tilde{G} \subset \mathrm{GL}(T_1) \times \mathrm{GL}(T_3) \cong \Bbbk^* \times \mathrm{GL}_3(\Bbbk),$$

which proves assertion (iii). $\qquad\square$

Corollary 4.1.3. *Let U be a threefold, and $P \in U$ be a terminal singularity. Then the following assertions hold.*

(i) *The group* $\mathrm{Aut}_P(U)$ *is Jordan with*

$$\bar{J}(\mathrm{Aut}_P(U)) \leqslant 288.$$

(ii) *If* (U, P) *is a cyclic quotient singularity, then* $\mathrm{Aut}_P(U)$ *is Jordan with*

$$\bar{J}(\mathrm{Aut}_P(U)) \leqslant 72.$$

(iii) *Let* $C \ni P$ *be a curve contained in* U *and* $\Gamma \subset \mathrm{Aut}_P(U)$ *be a subgroup such that* C *is* Γ*-invariant. Assume that* C *is a minimal center of log canonical singularities of the log pair* (U, D) *for some* Γ*-invariant boundary* D*. Then* Γ *is Jordan with*

$$\bar{J}(\Gamma) \leqslant 72.$$

Proof. Suppose that $G \subset \mathrm{Aut}_P(U)$ is a finite subgroup. It is enough to prove the corresponding bounds for the constant $\bar{J}(G)$. One has $\bar{J}(G) \leqslant \bar{J}(\tilde{G})$, where $\tilde{G}$ is the extension of G given by Lemma 4.1.1. Thus, assertion (i) follows from Lemma 4.1.1(i) and Lemma 2.4.3, while assertion (ii) follows from Lemma 4.1.1(ii) and Lemma 2.3.1.

Suppose that Γ is as in assertion (iii), and $G \subset \Gamma$. Then

$$\bar{J}(G) \leqslant \bar{J}(\tilde{G}) \leqslant \bar{J}\big(\Bbbk^* \times \mathrm{GL}_3(\Bbbk)\big) = \bar{J}\big(\mathrm{GL}_3(\Bbbk)\big)$$

by Lemma 4.1.1(iii). Therefore, assertion (iii) follows from Lemma 2.3.1. $\square$

4.2. **Non-Gorenstein Fano threefolds.** Now we will use Corollary 4.1.3 to study automorphism groups of non-Gorenstein terminal Fano threefolds.

Lemma 4.2.1. *Let* X *be a Fano threefold with terminal singularities. Suppose that* X *has a non-Gorenstein singular point. Then the group* $\mathrm{Aut}(X)$ *is Jordan with*

$$\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant 4608.$$

Proof. We use the methods of [Pro12, §6]. Let P_1 be a non-Gorenstein point and $P_1, \ldots, P_N \in X$ be its $\mathrm{Aut}(X)$-orbit. Let r be the index of points $P_1, \ldots, P_N \in X$. By the orbifold Riemann–Roch theorem and Bogomolov–Miyaoka inequality we have

$$\frac{3}{2}N \leqslant \left(r - \frac{1}{r}\right)N \leqslant 24$$

(see [Kaw92], [KMMT00]). This immediately implies that $N \leqslant 16$.

The subgroup $\mathrm{Aut}_{P_1}(X) \subset \mathrm{Aut}(X)$ stabilizing the point P_1 has index

$$[\mathrm{Aut}(X) : \mathrm{Aut}_{P_1}(X)] \leqslant N.$$

Thus we have

$$\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant [\mathrm{Aut}(X) : \mathrm{Aut}_{P_1}(X)] \cdot \bar{J}\big(\mathrm{Aut}_{P_1}(X)\big) \leqslant N \cdot \bar{J}\big(\mathrm{Aut}_{P_1}(X)\big) \leqslant 16 \cdot 288 = 4608$$

by Corollary 4.1.3(i). $\square$

Remark 4.2.2. It is known that terminal non-Gorenstein Fano threefolds are bounded, i.e. they belong to an algebraic family (see [Kaw92], [KMMT00]). However it is expected that the class of these varieties is huge [B$^+$]. There are only few results related to some special types of these Fanos (see e.g. [BS07], [Pro16]).

5. Mori fiber spaces

Recall that a G-equivariant morphism $\phi \colon X \to S$ of normal varieties acted on by a finite group G is a *G-Mori fiber space*, if X has terminal $G\mathbb{Q}$-factorial singularities, one has $\dim(S) < \dim(X)$, the fibers of ϕ are connected, the anticanonical divisor $-K_X$ is ϕ-ample, and the relative G-invariant Picard number $\rho^G(X/S)$ equals 1. If the dimension of X equals 3, there are three cases:

- S is a point, $-K_X$ is ample; in this case X is said to be a $G\mathbb{Q}$-Fano threefold, and X is a G-Fano threefold provided that the singularities of X are Gorenstein;
- S is a curve, a general fiber of ϕ is a del Pezzo surface; in this case X is said to be a $G\mathbb{Q}$-del Pezzo fibration;
- S is a surface, a general fiber of ϕ is a rational curve; in this case X is said to be a $G\mathbb{Q}$-conic bundle.

The goal of this section is to estimate weak Jordan constants for the automorphism groups of varieties of $G\mathbb{Q}$-conic bundles and $G\mathbb{Q}$-del Pezzo fibrations.

5.1. Conic bundles.
We start with automorphism groups of $G\mathbb{Q}$-conic bundles.

Lemma 5.1.1. *Let G be a finite group, and $\phi \colon X \to S$ be a three-dimensional G-equivariant fibration into rational curves over a rational surface S. Then $\bar{J}(G) \leqslant 3456$.*

Proof. By [Avi14] we may assume that X is smooth, and any fiber of ϕ is a (possibly reducible or non-reduced) conic. There is an exact sequence of groups

$$1 \to G_\phi \longrightarrow G \xrightarrow{\ \gamma\ } G_S \to 1,$$

where G_ϕ acts by fiberwise automorphisms with respect to ϕ, and $G_S \subset \mathrm{Aut}(S)$. By Lemma 3.3.1 there is an abelian subgroup $G'_S \subset G_S$ of index

$$[G_S : G'_S] \leqslant 288$$

such that G'_S acts on $\mathbb{P}^1$ with a fixed point. Let $P \in S$ be one of the fixed points of G'_S, and let

$$C = \phi^{-1}(P) \subset X$$

be the fiber of ϕ over the point P. Put $G' = \gamma^{-1}(G'_S)$. Then G' is a subgroup of G of index

$$[G : G'] = [G_S : G'_S] \leqslant 288,$$

and the fiber C is G'-invariant.

The fiber C is a reduced conic, so that it is either isomorphic to $\mathbb{P}^1$, or is a union of two copies of $\mathbb{P}^1$ meeting at one point.

In the former case there is a point $Q \in C$ that is invariant with respect to some subgroup $G'' \subset G'$ of index

$$[G' : G''] \leqslant 12$$

by Lemma 3.1.1. In the latter case the intersection point Q of the irreducible components C_1 and C_2 of C is invariant with respect to the group G', and there exists a subgroup $G'' \subset G'$ of index $[G' : G''] \leqslant 2$ such that C_1 and C_2 are invariant with respect to G''.

By Lemma 2.1.2 the group G'' acts faithfully on the Zariski tangent space $T_Q(X)$, and the group G'_S acts faithfully on the Zariski tangent space $T_P(S)$. As we have seen, the group G'' preserves the point Q and a tangent direction

$$v \in T_Q(X) \cong \Bbbk^3$$

that lies in the kernel of the natural projection $T_Q(X) \to T_P(S)$. Moreover, there is an embedding

$$G'' \hookrightarrow \Gamma_1 \times \Gamma_2,$$

where $\Gamma_1 \subset \Bbbk^*$, and $\Gamma_2 \subset G'_S$. Since G'_S and $\Bbbk^*$ are abelian groups, we conclude that so is G''. Therefore, one has

$$\bar{J}(G) \leqslant [G : G''] = [G : G'] \cdot [G' : G''] \leqslant 288 \cdot 12 = 3456. \qquad \square$$

5.2. **Del Pezzo fibrations.** Before we pass to the case of $G\mathbb{Q}$-del Pezzo fibrations we will establish some auxiliary results. Recall [KSB88, Definition 3.7] that a surface singularity is said to be *of type* T if it is a quotient singularity and admits a $\mathbb{Q}$-Gorenstein one-parameter smoothing.

Lemma 5.2.1. *Let X be a normal threefold with at worst isolated singularities and let $S \subset X$ be an effective Cartier divisor such that the log pair (X, S) is purely log terminal (see [KM98, §2.3]). Then S has only singularities of type T.*

Proof. Regard X as the total space of a deformation of S. By our assumptions divisors $K_X + S$ and S are $\mathbb{Q}$-Cartier. Hence X is $\mathbb{Q}$-Gorenstein. By the inversion of adjunction (see [KM98, Theorem 5.20]) the surface S has only Kawamata log terminal (i. e. quotient) singularities (see [KM98, Theorem 5.50]). Hence the singularities of S are of type T. $\qquad \square$

Lemma 5.2.2. *Let S be a singular del Pezzo surface with T-singularities. Assume that S has at least one non-Gorenstein point. Then $\mathrm{Aut}(S)$ has an orbit of length at most 2 on S.*

Proof. Assume that $\mathrm{Aut}(S)$ has no orbits of length at most 2 on S. By [HP10, Proposition 2.6] one has

$$\dim |-K_S| = K_S^2 \geqslant 1.$$

Write $|-K_S| = F + |M|$, where $|M|$ is a linear system without fixed components and F is the fixed part of $|-K_S|$, so that

$$\dim |M| = \dim |-K_S| = K_S^2.$$

By [Pro15a, Theorem 4.2] the log pair $(S, M + F)$ is log canonical for a general member $M \in |M|$. In particular, F is reduced. Let $\mathrm{Sing}'(S)$ be the set of non-Du Val points of S. By our assumptions $\mathrm{Sing}'(S) \neq \varnothing$. Clearly, any member of $|-K_S|$ contains $\mathrm{Sing}'(S)$; otherwise $-K_S$ is Cartier at some point of $\mathrm{Sing}'(S)$, so that this point is Du Val on S. Since the log pair $(S, F+M)$ is log canonical and K_S+F+M is Cartier, by the classification of two-dimensional log canonical singularities ([KM98, Theorem 4.15]) the divisor $F + M$ has two analytic branches at each point of $\mathrm{Sing}(X) \cap \mathrm{Supp}(F + M)$. In particular, we have

$$\mathrm{Sing}'(S) \subset \mathrm{Sing}(F + M).$$

Thus by our assumption $\mathrm{Sing}(F + M)$ contains at least three points. Furthermore, since the support of $F + M$ is connected, by adjunction one has $p_a(F + M) = 1$, all irreducible

components of $F + M$ are smooth rational curves and the corresponding dual graph is a combinatorial cycle. Moreover, the number of these irreducible components is at least 3.

First assume that $F \neq 0$. By Shokurov's connectedness theorem (see e.g [KM98, Theorem 5.48]) we know that F is connected. Hence F is a connected chain of rational curves. In this situation $\mathrm{Aut}(S)$ acts on F so that there exists either a fixed point $P \in \mathrm{Sing}\,(F)$ or an invariant irreducible component $F_1 \subset F$ (cf. the proof of Lemma 3.1.1). In the first case we have a contradiction with our assumption and in the second case $\mathrm{Aut}(S)$ permutes two points of intersection of F_1 with $\mathrm{Supp}\,(F - F_1)$, again a contradiction.

Thus $F = 0$ and so $\mathrm{Sing}'(S) \subset \mathrm{Bs}\,|M|$ and $\mathrm{Sing}'(S) \subset \mathrm{Sing}\,(M)$. Since $\mathrm{Sing}'(S)$ contains at least three points and $p_a(M) = 1$, the divisor M is reducible. By Bertini's theorem the linear system $|M|$ is composed of a pencil, which means that there is a pencil $|L|$ such that $|M| = n|L|$ for some $n \geqslant 2$, and $\mathrm{Sing}'(S) \subset \mathrm{Bs}\,|L|$. Since the log pair (S, M) is log canonical, there are exactly two irreducible components of M passing through any point $P \in \mathrm{Sing}'(S)$, see [KM98, Theorem 4.15]. Since $\mathrm{Sing}'(S)$ contains at least three points, the dual graph of M cannot be a combinatorial cycle, a contradiction. $\qquad\square$

Lemma 5.2.3. *Let X be a threefold, and $G \subset \mathrm{Aut}(X)$ be a finite subgroup. Suppose that there is a G-invariant smooth del Pezzo surface S contained in the smooth locus of X. Then $\bar{J}(G) \leqslant 288$.*

Proof. There is an exact sequence of groups

$$1 \to K \longrightarrow G \xrightarrow{\beta} H \to 1,$$

where K acts on S trivially, and $H \subset \mathrm{Aut}(S)$. By Lemma 3.2.3 there is a point $Q \in S$ fixed by an abelian subgroup $H_Q \subset H$ of index $[H : H_Q] \leqslant 288$. Put $G_Q = \beta^{-1}(H_Q)$. Then $G_Q \subset G$ is a subgroup that fixes the point Q, such that the index

$$[G : G_Q] = [H : H_Q] \leqslant 288.$$

By Lemma 2.1.2 the group G_Q acts faithfully on the Zariski tangent space $T_Q(X) \cong \Bbbk^3$. The two-dimensional Zariski tangent space $T_Q(S) \subset T_Q(X)$ is G_Q-invariant, and thus G_Q is contained in a subgroup

$$\Bbbk^* \times \mathrm{GL}\big(T_Q(S)\big) \cong \Bbbk^* \times \mathrm{GL}_2(\Bbbk) \subset \mathrm{GL}_3(\Bbbk) \cong \mathrm{GL}\big(T_Q(X)\big).$$

Hence $G_Q \subset A \times H_Q$, where $A \subset \Bbbk^*$ is some cyclic group. Therefore, the group G_Q is abelian, so that one has

$$\bar{J}(G) \leqslant [G : G_Q] \leqslant 288. \qquad\square$$

Remark 5.2.4. Let $G \subset \mathrm{Aut}(X)$ be a finite subgroup, and $\Sigma \subset X$ be a non-empty finite subset. Then a stabilizer $G_P \subset G$ of a point $P \in \Sigma$ has index $[G : G_P] \leqslant |\Sigma|$, so that by Remark 1.2.2 one has

$$\bar{J}(G) \leqslant |\Sigma| \cdot \bar{J}(G_P) \leqslant |\Sigma| \cdot \bar{J}\big(\mathrm{Aut}_P(X)\big).$$

Now we are ready to finish with weak Jordan constants of rationally connected three-dimensional $G\mathbb{Q}$-del Pezzo fibrations.

Lemma 5.2.5. *Let G be a finite group, and $\phi\colon X \to B \cong \mathbb{P}^1$ be a three-dimensional $G\mathbb{Q}$-del Pezzo fibration. Then $\bar{J}(G) \leqslant 10368$.*

Proof. There is an exact sequence of groups

$$1 \to G_\phi \longrightarrow G \overset{\alpha}{\longrightarrow} G_B \to 1,$$

where G_ϕ acts by fiberwise automorphisms with respect to ϕ, and

$$G_B \subset \mathrm{Aut}(B) \cong \mathrm{PGL}_2(\Bbbk).$$

By Lemma 3.1.1 there is a subgroup $G'_B \subset G_B$ of index $[G_B : G'_B] \leqslant 12$ such that G'_B acts on B with a fixed point.

Let $P \in B$ be one of the fixed points of G'_B, let $F = \phi^*(P)$ be the scheme fiber over P, and let $S = F_{\mathrm{red}}$. Put $G' = \alpha^{-1}(G'_B)$. Then G' is a subgroup of G of index

$$[G : G'] = [G_B : G'_B] \leqslant 12,$$

and the fiber S is G'-invariant. In particular, one has

$$\bar{J}(G) \leqslant [G : G'] \cdot \bar{J}(G').$$

Suppose that F is a multiple fiber of ϕ, i.e. $S \neq F$. Then by [MP09] there is a G-invariant set $\Sigma \subset S$ of singular points of X such that either $|\Sigma| \leqslant 3$, or $|\Sigma| = 4$ and Σ consists of cyclic quotient singularities. In the former case Remark 5.2.4 and Corollary 4.1.3(i) imply that

$$\bar{J}(G) \leqslant 12 \cdot 3 \cdot 288 = 10368.$$

In the latter case Remark 5.2.4 and Corollary 4.1.3(ii) imply that

$$\bar{J}(G) \leqslant 12 \cdot 4 \cdot 72 = 3456.$$

Therefore, we can assume that S is not a multiple fiber of ϕ. In particular, $S = F$ is a Cartier divisor on X.

Suppose that the log pair (X, S) is not purely log terminal (see [KM98, §2.3]). Let c be the log canonical threshold of the log pair (X, S) (cf. the proof of [PS16a, Lemma 3.4]). Let $Z_1 \subset S$ be a minimal center of log canonical singularities of the log pair (X, cS), see [Kaw97, Proposition 1.5]. Since (X, S) is not purely log terminal, we conclude that $c < 1$, so that $\dim(Z) \leqslant 1$. It follows from [PS16a, Lemma 2.5] that Z is G'-invariant. If Z is a point, then

$$\bar{J}(G) \leqslant [G : G'] \cdot \bar{J}(G') \leqslant 12 \cdot 288 = 3456$$

by Corollary 4.1.3(i). Thus we assume that Z is a curve. Using [PS16a, Lemma 2.5] once again, we see that Z is smooth and rational. By Lemma 3.1.1 there is a subgroup $G'' \subset G'$ of index $[G' : G''] \leqslant 12$ such that G'' has a fixed point on Z. Hence

$$\bar{J}(G) \leqslant [G : G''] \cdot \bar{J}(G'') \leqslant 144 \cdot 72 = 10368$$

by Corollary 4.1.3(iii).

Therefore, we may assume that the log pair (X, S) is purely log terminal. Then by [KM98, Theorem 5.50] the surface S is a del Pezzo surface with only Kawamata log terminal singularities. Moreover, the singularities of S are of type T (see Lemma 5.2.1). If K_S is not Cartier, Lemma 5.2.2 implies that there is a G'-orbit of length at most 2 contained in S. In this case we have

$$\bar{J}(G) \leqslant [G : G'] \cdot \bar{J}(G') \leqslant 12 \cdot 2 \cdot 288 = 6912$$

by Remark 5.2.4 and Corollary 4.1.3(i).

Therefore, we may assume that K_S is Cartier and so S has at worst Du Val singularities. Denote their number by $m(S)$. Then by Noether formula applied to the minimal resolution we have

$$m(S) \leqslant 9 - K_S^2 \leqslant 8.$$

Thus by Remark 5.2.4 and Corollary 4.1.3(i) we have

$$\bar{J}(G) \leqslant [G : G'] \cdot \bar{J}(G') \leqslant 2 \cdot 9 \cdot 288 = 5184.$$

Therefore, we are left with the case when S is smooth. Now Lemma 5.2.3 implies that

$$\bar{J}(G) \leqslant [G : G'] \cdot \bar{J}(G') \leqslant 12 \cdot 288 = 3456$$

and completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

6. Gorenstein Fano threefolds

Let X be a Fano threefold with at worst terminal Gorenstein singularities. In this case, the number

$$\mathrm{g}(X) = \frac{1}{2}(-K_X)^3 + 1$$

is called the *genus* of X. By Reimann–Roch theorem and Kawamata–Viehweg vanishing one has

$$\dim |-K_X| = \mathrm{g}(X) + 1$$

(see e. g. [IP99, 2.1.14]). In particular, $\mathrm{g}(X)$ is an integer, and $\mathrm{g}(X) \geqslant 2$. The maximal number $\iota = \iota(X)$ such that $-K_X$ is divisible by ι in $\mathrm{Pic}\,(X)$ is called the *Fano index*, or sometimes just *index*, of X. Recall that $\mathrm{Pic}\,(X)$ is a finitely generated torsion free abelian group, see e.g. [IP99, Proposition 2.1.2]. The rank $\rho(X)$ of the free abelian group $\mathrm{Pic}\,(X)$ is called the *Picard rank* of X. Let H be a divisor class such that $-K_X \sim \iota(X)H$. The class H is unique since $\mathrm{Pic}\,(X)$ is torsion free. Define the *degree* of X as $\mathrm{d}(X) = H^3$. The goal of this section is to bound weak Jordan constants for automorphism groups of singular terminal Gorenstein Fano threefolds.

6.1. **Low degree.** We start with the case of small anticanonical degree. We will use notation and results of §A.2.

Proposition 6.1.1 (cf. [KPS16, Lemma 4.4.1]). *Let X be a Fano threefold with terminal Gorenstein singularities such that $\rho(X) = 1$. Suppose that H is not very ample, i. e. one of the following possibilities holds (see [Shi89, Theorem 0.6], [Shi89, Corollary 0.8], [JR06, Theorem 1.1], [PCS05, Theorem 1.4]):*

   (i) *$\iota(X) = 2$ and $\mathrm{d}(X) = 1$;*
   (ii) *$\iota(X) = 2$ and $\mathrm{d}(X) = 2$;*
   (iii) *$\iota(X) = 1$ and $\mathrm{g}(X) = 2$;*
   (iv) *$\iota(X) = 1$, $\mathrm{g}(X) = 3$, and X is a double cover of a three-dimensional quadric.*

Suppose that $G \subset \mathrm{Aut}(X)$ is a finite group. Then for some positive integer r there is a central extension

$$1 \to \boldsymbol{\mu}_r \to \tilde{G} \to G \to 1$$

such that one has an embedding $\tilde{G} \subset \mathrm{GL}_3(\Bbbk) \times \Bbbk^*$ in case (i)*, an embedding $\tilde{G} \subset \mathrm{GL}_4(\Bbbk)$ in cases* (ii) *and* (iii)*, and an embedding $\tilde{G} \subset \mathrm{GL}_5(\Bbbk)$ in case* (iv)*.*

Proof. According to [Shi89, Corollary 0.8] and [PCS05, Theorem 1.5], in cases (i), (ii) and (iii) our Fano variety X is naturally embedded as a weighted hypersurface in the weighted projective space $\mathbb{P} = \mathbb{P}(a_0, \ldots, a_4)$, where

$$(a_0, \ldots, a_4) = (1^3, 2, 3), (1^4, 2), (1^4, 3),$$

respectively. In case (iv) our X is naturally embedded as a weighted complete intersection of multidegree $(2, 4)$ in $\mathbb{P} = \mathbb{P}(1^5, 2)$. Let $\mathscr{O}_X(1)$ be the restriction of the (non-invertible) divisorial sheaf $\mathscr{O}_\mathbb{P}(1)$ to X (see [Dol82, 1.4.1]). Since X is Gorenstein, in all cases it is contained in the smooth locus of $\mathbb{P}$, and thus $\mathscr{O}_X(1)$ is an invertible divisorial sheaf on X. Moreover, under the above embeddings we have

$$\mathscr{O}_X(1) = \mathscr{O}_X(-K_X)$$

in cases (iii) and (iv), while

$$\mathscr{O}_X(1) = \mathscr{O}_X(-\tfrac{1}{2}K_X)$$

in cases (i) and (ii). Since the group $\mathrm{Pic}\,(X)$ has no torsion, in all cases the class of $\mathscr{O}_X(1)$ in $\mathrm{Pic}\,(X)$ is invariant with respect to the whole automorphism group $\mathrm{Aut}(X)$. Also, the line bundle $\mathscr{O}_X(1)$ is ample, so that the algebra $R(X, \mathscr{O}_X(1))$ is finitely generated. Therefore, by Lemma A.2.13 for any finite subgroup $\Gamma \subset \mathrm{Aut}(X)$ the action of Γ on X is induced by its action on

$$\mathbb{P} \cong \mathrm{Proj}\, R(X, \mathscr{O}_X(1)).$$

Thus the assertion follows from Lemma A.2.8. $\qquad\square$

Remark 6.1.2. Assume the setup of Proposition 6.1.1. Then using the notation of the proof of Lemma A.2.13 one can argue that a central extension of the group G acts on the vector space

$$V = \bigoplus_{m=1}^{N} V_m,$$

which immediately gives its embedding into $\mathrm{GL}_{k_1 + \ldots + k_N}(\Bbbk)$. This would allow to avoid using Lemma A.2.8, but would give a slightly weaker result.

Using a more explicit geometric approach, one can strengthen the assertion of Proposition 6.1.1(i).

Corollary 6.1.3. *In the assumptions of Proposition* 6.1.1(i) *one has* $G \subset \mathrm{GL}_3(\Bbbk)$.

Proof. The base locus of the linear system $|H|$ is a single point P which is contained in the smooth part of X (see e.g. [Shi89, Theorem 0.6]). Clearly, the point P is $\mathrm{Aut}(X)$-invariant. Therefore, Lemma 2.1.2 implies that $G \subset \mathrm{GL}_3(\Bbbk)$. $\qquad\square$

Lemma 6.1.4. *Let* X *be a Fano threefold with Gorenstein terminal singularities. Suppose that* $\rho(X) = 1$, *and one of the following possibilities holds:*

(i) $\iota(X) = 2$ *and* $\mathrm{d}(X) = 1$;
(ii) $\iota(X) = 2$ *and* $\mathrm{d}(X) = 2$;
(iii) $\iota(X) = 1$ *and* $\mathrm{g}(X) = 2$;
(iv) $\iota(X) = 1$, $\mathrm{g}(X) = 3$, *and* X *is a double cover of a three-dimensional quadric.*

Then the group $\mathrm{Aut}(X)$ *is Jordan with* $\bar{J}(\mathrm{Aut}(X)) \leqslant 960$.

Proof. Apply Proposition 6.1.1 together with Lemma 2.5.1. $\qquad\square$

Lemma 6.1.5. *Let $X \subset \mathbb{P}^4$ be a hypersurface of degree at least 2. Then the group $\mathrm{Aut}(X)$ is Jordan with $\bar{J}(\mathrm{Aut}(X)) \leqslant 960$.*

Proof. There is an embedding $\mathrm{Aut}(X) \subset \mathrm{PGL}_5(\Bbbk)$, see e.g. [KPS16, Corollary 3.1.4]. Thus the assertion follows from Lemma 2.5.1. $\qquad\square$

6.2. **Complete intersection of a quadric and a cubic.** Now we will describe some properties of finite subgroups of automorphisms of a complete intersection of a quadric and a cubic in $\mathbb{P}^5$.

Lemma 6.2.1. *Let $X \subset \mathbb{P}^5$ be a Fano threefold with terminal Gorenstein singularities such that $\rho(X) = 1$, $\iota(X) = 1$, and $\mathrm{g}(X) = 4$, i.e. X is a complete intersection of a quadric and a cubic in $\mathbb{P}^5$ (see [Isk80a, Proposition IV.1.4], [PCS05, Theorem 1.6 or Remark 4.2]). Let $Q \subset \mathbb{P}^5$ be the (unique) quadric passing through X. Then one of the following possibilities occurs:*

(i) *the quadric Q is smooth; in this case there is a subgroup $\mathrm{Aut}'(X) \subset \mathrm{Aut}(X)$ of index at most 2 such that $\mathrm{Aut}'(X) \subset \mathrm{PGL}_4(\Bbbk)$;*

(ii) *the quadric Q is a cone with an isolated singularity; in this case for any finite subgroup $G \subset \mathrm{Aut}(X)$ there is an embedding*

$$G \subset \mathrm{SO}_5(\Bbbk) \times \Bbbk^* \subset \mathrm{GL}_5(\Bbbk);$$

(iii) *the quadric Q is a cone whose singular locus is a line; in this case for any finite subgroup $G \subset \mathrm{Aut}(X)$ there is a subgroup $F \subset G$ of index $[G : F] \leqslant 3$ such that there is an embedding*

$$F \subset \Bbbk^* \times (\mathrm{SO}_4(\Bbbk) \times \Bbbk^*/\boldsymbol{\mu}_2) \subset \Bbbk^* \times \mathrm{GL}_4(\Bbbk).$$

Proof. The embedding $X \hookrightarrow \mathbb{P}^5$ is given by the anticanonical linear system on X. Hence there is an action of the group $\mathrm{Aut}(X)$ on $\mathbb{P}^5$ that agrees with the action of $\mathrm{Aut}(X)$ on X, see e.g. [KPS16, Lemma 3.1.2]. The quadric Q is $\mathrm{Aut}(X)$-invariant, and the action of $\mathrm{Aut}(X)$ on Q is faithful. Since the singularities of X are terminal and thus isolated, we see that the singular locus of Q is at most one-dimensional.

Suppose that Q is non-singular. Then Q is isomorphic to the Grassmannian $\mathrm{Gr}(2,4)$, so that

$$\mathrm{Aut}(Q) \cong \mathrm{PGL}_4(\Bbbk) \rtimes \boldsymbol{\mu}_2,$$

which gives case (i).

Therefore, we may assume that Q is singular. Then $\mathrm{Sing}\,(Q)$ is a linear subspace of $\mathbb{P}^5$ of dimension $\delta \leqslant 1$.

Suppose that $\delta = 0$, so that $\mathrm{Sing}\,(Q)$ is a single point P. Then the point P is $\mathrm{Aut}(Q)$-invariant, and thus also $\mathrm{Aut}(X)$-invariant. Let $G \subset \mathrm{Aut}(X)$ be a finite subgroup. By Lemma 2.1.2 there is an embedding

$$G \subset \mathrm{GL}(T_P(Q)) = \mathrm{GL}(T_P(\mathbb{P}^5)) \cong \mathrm{GL}_5(\Bbbk).$$

Moreover, the group G acts by a character on a quadratic polynomial on $T_P(\mathbb{P}^5)$ that corresponds to the quadric Q. Hence G is contained in the subgroup

$$\pi^{-1}(\mathrm{PSO}_5(\Bbbk)) \subset \mathrm{GL}_5(\Bbbk),$$

where $\pi \colon \mathrm{GL}_5(\Bbbk) \to \mathrm{PGL}_5(\Bbbk)$ is the natural projection. This gives case (ii).

Finally, suppose that $\delta = 1$. Let $L \cong \mathbb{P}^1$ be the vertex of Q. Then L is $\mathrm{Aut}(Q)$-invariant, and thus also $\mathrm{Aut}(X)$-invariant. Let $G \subset \mathrm{Aut}(X)$ be a finite subgroup. Note that $X \cap L$

is non-empty and consists of at most three points. Hence there is a subgroup $F \subset G$ of index $[G : F] \leqslant 3$ such that F has a fixed point on L. Denote this point by P. By Lemma 2.1.2 there is an embedding

$$F \hookrightarrow \mathrm{GL}\big(T_P(\mathbb{P}^5)\big) \cong \mathrm{GL}_5(\Bbbk).$$

Moreover, the representation of F in $T_P(\mathbb{P}^5)$ splits as a sum of a one-dimensional and a four-dimensional representations since F preserves the tangent direction $T_P(L)$ to L. Put

$$V = T_P(\mathbb{P}^5)/T_P(L).$$

Then there is an embedding $F \hookrightarrow F_1 \times F_2$, where F_1 is a finite cyclic group, and F_2 is a finite subgroup of $\mathrm{GL}(V) \cong \mathrm{GL}_4(\Bbbk)$. The last thing we need to observe is that F_2 preserves a quadric cone in $\mathbb{P}(V)$ corresponding to an intersection of the tangent cone to Q at P with the subspace $V \hookrightarrow T_P(\mathbb{P}^5)$. Therefore, F_2 is contained in the subgroup

$$\pi^{-1}\left(\mathrm{PSO}_4(\Bbbk)\right) \subset \mathrm{GL}_4(\Bbbk),$$

where $\pi \colon \mathrm{GL}_4(\Bbbk) \to \mathrm{PGL}_4(\Bbbk)$ is the natural projection. Since

$$\pi^{-1}\left(\mathrm{PSO}_4(\Bbbk)\right) \cong \mathrm{SO}_4(\Bbbk) \times \Bbbk^*/\boldsymbol{\mu}_2,$$

this gives case (iii) and completes the proof of the lemma. $\square$

Corollary 6.2.2. *Let X be a Fano threefold with Gorenstein terminal singularities. Suppose that $\rho(X) = 1$, $\iota(X) = 1$, and $\mathrm{g}(X) = 4$. Then the group $\mathrm{Aut}(X)$ is Jordan with*

$$\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant 1920.$$

Proof. By Lemma 6.2.1 one of the following possibilities holds:

(i) there is a subgroup $\mathrm{Aut}'(X) \subset \mathrm{Aut}(X)$ of index at most 2 such that $\mathrm{Aut}'(X) \subset \mathrm{PGL}_4(\Bbbk)$;
(ii) for any finite subgroup $G \subset \mathrm{Aut}(X)$ there is an embedding $G \subset \mathrm{GL}_5(\Bbbk)$;
(iii) for any finite subgroup $G \subset \mathrm{Aut}(X)$ there is a subgroup $F \subset G$ of index $[G : F] \leqslant 3$ such that there is an embedding

$$F \subset \Bbbk^* \times \left(\mathrm{SO}_4(\Bbbk) \times \Bbbk^*/\boldsymbol{\mu}_2\right).$$

In particular, the group $\mathrm{Aut}(X)$ is Jordan. In case (i) one has

$$\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant 2 \cdot \bar{J}\big(\mathrm{Aut}'(X)\big) \leqslant 2 \cdot \bar{J}\big(\mathrm{PGL}_4(\Bbbk)\big) = 2 \cdot 960 = 1920$$

by Lemma 2.4.1. In case (ii) one has

$$\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant \bar{J}\big(\mathrm{PGL}_5(\Bbbk)\big) = 960$$

by Lemma 2.5.1. In case (iii) one has

$$\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant 3 \cdot \bar{J}\left(\Bbbk^* \times \left(\mathrm{SO}_4(\Bbbk) \times \Bbbk^*/\boldsymbol{\mu}_2\right)\right) = 3 \cdot \bar{J}\big(\mathrm{SO}_4(\Bbbk)\big) \leqslant 3 \cdot 288 = 864$$

by Lemma 2.4.3. $\square$

6.3. General case. The results of §6.1 and §6.2 imply the following

Corollary 6.3.1. *Let X be a Fano threefold with Gorenstein terminal singularities. Suppose that $\rho(X) = 1$, $\iota(X) = 1$, and $g(X) \leqslant 4$. Then $\mathrm{Aut}(X)$ is Jordan with*

$$\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant 1920.$$

Proof. Recall that $g(X) \geqslant 2$. If $g(X) = 2$, then $\mathrm{Aut}(X)$ is Jordan with $\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant 960$ by Lemma 6.1.4. If $g(X) = 3$ and $-K_X$ is not very ample, then $\mathrm{Aut}(X)$ is also Jordan with $\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant 960$ by Lemma 6.1.4. If $g(X) = 3$ and $-K_X$ is very ample, then X is a smooth quartic in $\mathbb{P}^4$ (because $\dim |-K_X| = 4$ and $-K_X^3 = 4$), so that $\mathrm{Aut}(X)$ is Jordan with $\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant 960$ by Lemma 6.1.5. Finally, if $g(X) = 4$, then the group $\mathrm{Aut}(X)$ is Jordan with $\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant 1920$ by Corollary 6.2.2. $\qquad\square$

Now we are ready to study automorphism groups of arbitrary *singular* Gorenstein G-Fano threefolds.

Lemma 6.3.2. *Let G be a finite group, and let X be a singular Gorenstein G-Fano threefold. Then the group $\mathrm{Aut}(X)$ is Jordan with*

$$\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant 9504.$$

Proof. Let $P_1, \ldots, P_N \in X$ be all singular points of X. The group $\mathrm{Aut}(X)$ acts on the set $\{P_1, \ldots, P_N\}$. The subgroup $\mathrm{Aut}_{P_1}(X) \subset \mathrm{Aut}(X)$ stabilizing the point P_1 has index

$$[\mathrm{Aut}(X) : \mathrm{Aut}_{P_1}(X)] \leqslant N.$$

We have

$$\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant N \cdot \bar{J}\big(\mathrm{Aut}_{P_1}(X)\big).$$

According to [Nam97] there exists a *smoothing* of X, that is a one-parameter deformation

$$\mathfrak{X} \to B \ni 0$$

such that a general fiber $\mathfrak{X}_b$ is smooth and the central fiber $\mathfrak{X}_0$ is isomorphic to X. One has

(6.3.3) $$N \leqslant 21 - \frac{1}{2}\chi_{\mathrm{top}}(\mathfrak{X}_b) = 20 - \rho(\mathfrak{X}_b) + h^{1,2}(\mathfrak{X}_b)$$

by [Nam97, Theorem 13]. Moreover, there is an identification $\mathrm{Pic}(\mathfrak{X}_b) \cong \mathrm{Pic}(X)$, see [JR11, Theorem 1.4].

Suppose that $\rho(X) \geqslant 2$. Smooth Fano threefolds V whose Picard group admits an action of a finite group G such that $\rho(V)^G = 1$ and $\rho(V) > 1$ are classified in [Pro13b]. Applying this classification to $V = X_b$ we obtain $h^{1,2}(\mathfrak{X}_b) \leqslant 9$.

Suppose that $\rho(X) = 1$. If $\iota(X) = 2$ and $\mathrm{d}(X) \leqslant 2$, then the group $\mathrm{Aut}(X)$ is Jordan with $\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant 960$ by Lemma 6.1.4. If $\iota(X) = 1$ and $g(X) \leqslant 4$, then $\mathrm{Aut}(X)$ is Jordan with $\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant 1920$ by Corollary 6.3.1. In all other cases by the classification of smooth Fano threefolds (see [IP99, §12.2]) we have $h^{1,2}(\mathfrak{X}_b) \leqslant 14$.

Therefore, we are left with several possibilities with $h^{1,2}(\mathfrak{X}_b) \leqslant 14$. In this case (6.3.3) implies that $N \leqslant 33$. Now Corollary 4.1.3(i) implies that $\mathrm{Aut}(X)$ is Jordan with

$$\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant 33 \cdot 288 = 9504. \qquad\square$$

7. Smooth Fano threefolds

In this section we bound weak Jordan constants for automorphism groups of smooth Fano threefolds.

7.1. Complete intersections of quadrics. It appears that we can get a reasonable bound for a weak Jordan constant of an automorphism group of a smooth complete intersection of two quadrics of arbitrary dimension. Here we will use the results of §A.1.

Lemma 7.1.1. *Let $X \subset \mathbb{P}^n$, $n \geqslant 4$, be a smooth complete intersection of 2 quadrics. Then the group $\mathrm{Aut}(X)$ is Jordan with*

$$\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant (n+1)!$$

Proof. By Proposition A.1.3 there is an exact sequence

$$1 \longrightarrow \Gamma \longrightarrow \mathrm{Aut}(X) \longrightarrow G_{\mathcal{P}} \longrightarrow 1$$

where $\Gamma \cong \boldsymbol{\mu}_2^n$ and $G_{\mathcal{P}} \subset \mathfrak{S}_{n+1}$. Therefore, the group $\mathrm{Aut}(X)$ is Jordan with

$$\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant [\mathrm{Aut}(X) : \Gamma] \leqslant |\mathfrak{S}_{n+1}| = (n+1)! \qquad \square$$

In dimension 3 we can also bound weak Jordan constants for automorphism groups of smooth complete intersections of three quadrics.

Lemma 7.1.2. *Let $X \subset \mathbb{P}^6$ be a smooth complete intersection of 3 quadrics. Then the group $\mathrm{Aut}(X)$ is Jordan with*

$$\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant 10368.$$

Proof. There is an exact sequence

$$1 \to \Gamma \to \mathrm{Aut}(X) \to \mathrm{PGL}_3(\Bbbk),$$

where $\Gamma \cong \boldsymbol{\mu}_2^m$ with $m \leqslant 6$, see (A.1.2) and Corollary A.1.7. If $m \leqslant 2$, then $\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant 2304$ by Lemma 2.3.4. Therefore, we assume that $m \geqslant 3$.

Put

$$V = H^0\big(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(H)\big)^{\vee},$$

so that $\mathbb{P}^6$ is identified with $\mathbb{P}(V)$. Since the anticanonical class of X is linearly equivalent to a hyperplane section of X in $\mathbb{P}^6$, the group $\mathrm{Aut}(X)$ acts on V, see e.g. [KPS16, Corollary 3.1.3]. Thus we may assume that $\mathrm{Aut}(X) \subset \mathrm{GL}(V)$.

Let $-\mathrm{Id} \in \mathrm{GL}(V)$ be the scalar matrix $\mathrm{diag}(-1, \ldots, -1)$. Let $\tilde{\Gamma} \subset \mathrm{GL}(V)$ be a group generated by Γ and $-\mathrm{Id}$, and let $G \subset \mathrm{GL}(V)$ be a group generated by $\mathrm{Aut}(X)$ and $-\mathrm{Id}$. Since $\mathrm{Aut}(X) \subset \mathrm{GL}(V)$ acts faithfully on $\mathbb{P}(V)$ and thus does not contain scalar matrices, we see that

$$\tilde{\Gamma} \cong \boldsymbol{\mu}_2 \times \Gamma \cong \boldsymbol{\mu}_2^{m'}$$

with $m' = m + 1 \geqslant 4$. We conclude that $\mathrm{Aut}(X)$ is Jordan with

$$\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant \bar{J}(G) \leqslant 10368$$

by Lemma 2.6.3. $\qquad \square$

Remark 7.1.3. Let $X \subset \mathbb{P}^6$ be a smooth complete intersection of 3 quadrics. Then X is non-rational, see [Bea77a, Theorem 5.6]. Therefore, automorphism groups of varieties of this type cannot provide examples of subgroups in $\mathrm{Cr}_3(\Bbbk)$ whose Jordan constants attain the bounds given by Theorem 1.2.4, cf. Remark 8.2.1 below.

7.2. Fano threefolds of genus 6. Recall that a smooth Fano threefold X with $\rho(X) = 1$, $\iota(X) = 1$, and $g(X) = 6$ may be either an intersection of the Grassmannian $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$ with a quadric and two hyperplanes, or a double cover of a smooth Fano threefold

$$Y = \mathrm{Gr}(2,5) \cap \mathbb{P}^6 \subset \mathbb{P}^9$$

with the branch divisor $B \in |-K_Y|$ (see [Gus82]). We will refer to the former varieties as *Fano threefolds of genus* 6 *of the first type*, and to the latter varieties as *Fano threefolds of genus* 6 *of the second type*.

Remark 7.2.1. In [DK15] these were called ordinary and special varieties, respectively.

Lemma 7.2.2 (cf. [DIM12, Corollary 4.2], [DK15, Proposition 3.12]). *Let X be a smooth Fano threefold with $\rho(X) = 1$, $\iota(X) = 1$, and $g(X) = 6$. If X is of the first type, then there is an embedding*

$$\mathrm{Aut}(X) \hookrightarrow \mathrm{Aut}\big(\mathrm{Gr}(2,5)\big) \cong \mathrm{PGL}_5(\Bbbk).$$

If X is of the second type, then there is a normal subgroup $\Gamma \subset \mathrm{Aut}(X)$ such that $\Gamma \cong \boldsymbol{\mu}_2$ and there is an exact sequence

$$1 \to \Gamma \to \mathrm{Aut}(X) \to \mathrm{PGL}_2(\Bbbk).$$

Proof. By definition, we have a natural morphism $\gamma\colon X \to \mathrm{Gr}(2,5)$. By [DK15, Theorem 2.9] the morphism γ is functorial. Note that γ is completely determined by what is called GM data in [DK15], in particular it is equivariant with respect to the action of the group $\mathrm{Aut}(X)$. Consider the corresponding map

$$\theta\colon \mathrm{Aut}(X) \to \mathrm{Aut}(\mathrm{Gr}(2,5)) \cong \mathrm{PGL}_5(\Bbbk).$$

Suppose that X is a Fano threefold of genus 6 of the first type. Then functoriality of γ implies that θ is an embedding. This proves the first assertion of the lemma.

Now suppose that X is a Fano threefold of genus 6 of the second type. Then the morphism γ is a double cover, and its image is a Fano threefold Y with $\rho(Y) = 1$, $\iota(Y) = 2$, and $d(Y) = 5$, see [DK15, Proposition 2.20]. Let $\Gamma \subset \mathrm{Aut}(X)$ be the subgroup generated by the Galois involution of the double cover $\gamma\colon X \to Y$. Then $\Gamma \cong \boldsymbol{\mu}_2$ is a normal subgroup of $\mathrm{Aut}(X)$, and $\mathrm{Aut}(X)/\Gamma$ embeds into $\mathrm{Aut}(Y)$. On the other hand, one has $\mathrm{Aut}(Y) \cong \mathrm{PGL}_2(\Bbbk)$, see e.g. [Muk88, Proposition 4.4] or [CS16, Proposition 7.1.10]. This gives the second assertion of the lemma. $\qquad\square$

Corollary 7.2.3. *Let X be a smooth Fano threefold with $\rho(X) = 1$, $\iota(X) = 1$ and $g(X) = 6$. Then the group $\mathrm{Aut}(X)$ is Jordan with*

$$\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant 960.$$

Proof. Suppose that X is a Fano threefold of genus 6 of the first type. Then there is an embedding $\mathrm{Aut}(X) \subset \mathrm{PGL}_5(\Bbbk)$ by Lemma 7.2.2, so that $\mathrm{Aut}(X)$ is Jordan with $\bar{J}(\mathrm{Aut}(X)) \leqslant 960$ by Lemma 2.5.1.

Now suppose that X is a Fano threefold of genus 6 of the second type. Then there is an exact sequence

$$1 \to \Gamma \to \mathrm{Aut}(X) \to \mathrm{PGL}_2(\Bbbk)$$

by Lemma 7.2.2. Therefore, $\mathrm{Aut}(X)$ is Jordan with $\bar{J}(G) \leqslant 12$ by Lemma 2.2.1. $\qquad\square$

7.3. **Large degree and index.** Now we consider the cases with large anticanonical degree and large index.

Lemma 7.3.1. *Let X be a smooth Fano threefold with $\iota(X) = 1$ and $g(X) \geqslant 7$. Then the group $\mathrm{Aut}(X)$ is Jordan with*

    (i) $\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant 504$ *if* $g(X) = 7$;
   (ii) $\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant 9922$ *if* $g(X) = 8$;
  (iii) $\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant 2016$ *if* $g(X) = 9$;
  (iv) $\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant 5760$ *if* $g(X) = 10$;
   (v) $\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant 40$ *if* $g(X) = 12$.

Proof. Assertion (i) follows from [KPS16, Corollary 4.3.5(i)] and Remark 2.2.3. Assertion (ii) follows from [KPS16, Corollary 4.3.5(ii)] and Lemma 3.4.4. Assertion (iii) follows from [KPS16, Corollary 4.3.5(iii)] and Lemma 3.4.1. Assertion (iv) follows from [KPS16, Corollary 4.3.5(iv)] and Lemma 3.4.2. Finally, assertion (v) follows from [KPS16, Corollary 4.3.5(v)] and Lemma 2.3.1. $\qquad\square$

Lemma 7.3.2. *Let G be a finite group, and X be a smooth Fano threefold. Suppose that $\rho(X) = 1$ and $\iota(X) > 1$. Then the group $\mathrm{Aut}(X)$ is Jordan with*

$$\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant 960.$$

Proof. It is known that $\iota(X) \leqslant 4$. Moreover, $\iota(X) = 4$ if and only if $X \cong \mathbb{P}^3$, and $\iota(X) = 3$ if and only if X is a quadric in $\mathbb{P}^4$ (see e.g. [IP99, 3.1.15]). In the former case one has $\mathrm{Aut}(X) \cong \mathrm{PGL}_4(\Bbbk)$, so that the group $\mathrm{Aut}(X)$ is Jordan with $\bar{J}\big(\mathrm{Aut}(X)\big) = 960$ by Lemma 2.4.1. In the latter case the group $\mathrm{Aut}(X)$ is Jordan with $\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant 960$ by Lemma 6.1.5.

Thus we may assume that $\iota(X) = 2$. Recall that $1 \leqslant d(X) \leqslant 5$ (see e.g. [IP99, §12.2]).

If $d(X) = 5$, then X is isomorphic to a linear section of the Grassmannian $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$ by a subspace $\mathbb{P}^6 \subset \mathbb{P}^9$, see [IP99, §12.2]. In this case one has

$$\mathrm{Aut}(X) \cong \mathrm{PGL}_2(\Bbbk),$$

see [Muk88, Proposition 4.4] or [CS16, Proposition 7.1.10]. So, the group $\mathrm{Aut}(X)$ is Jordan with $\bar{J}\big(\mathrm{Aut}(X)\big) = 12$ by Corollary 2.2.2.

If $d(X) = 4$, then X is a complete intersection of two quadrics in $\mathbb{P}^5$ (see [IP99, §12.2]). Thus $\mathrm{Aut}(X)$ is Jordan with $\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant 720$ by Lemma 7.1.1.

If $d(X) = 3$, then $X \cong X_3 \subset \mathbb{P}^4$ is a cubic threefold (see [IP99, §12.2]). Thus $\mathrm{Aut}(X)$ is Jordan with $\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant 960$ by Lemma 6.1.5.

Finally, if $d(X) = 2$ or $d(X) = 1$, then $\mathrm{Aut}(X)$ is Jordan with $\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant 960$ by Lemma 6.1.4. $\qquad\square$

7.4. **Large Picard rank.** Finally, we deal with smooth G-Fano threefolds with Picard rank greater than 1. We denote by W_6 a smooth divisor of bidegree $(1,1)$ in $\mathbb{P}^2 \times \mathbb{P}^2$. Clearly, W_6 is a Fano threefold with $\iota(W_6) = 2$ and $\rho(W_6) = 2$.

Lemma 7.4.1. *Let G be a finite group, and X be a smooth G-Fano threefold. Suppose that $\rho(X) > 1$. Then $\mathrm{Aut}(X)$ is Jordan with $\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant 10368$.*

Proof. By [Pro13b] we have the following possibilities.

    (i) $\rho(X) = 2$, $\iota(X) = 2$, and $X \cong W_6$;

(ii) $\rho(X) = 3$, $\iota(X) = 2$, and $X \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$;

(iii) $\rho(X) = 2$, $\iota(X) = 1$, and $X \subset \mathbb{P}^2 \times \mathbb{P}^2$ is a divisor of bidegree $(2, 2)$;

(iv) $\rho(X) = 2$, $\iota(X) = 1$, and X is a double cover of W_6 whose branch divisor $S \subset W_6$ is a member of the linear system $|-K_{W_6}|$;

(v) $\rho(X) = 2$, $\iota(X) = 1$, and X is the blow up of $\mathbb{P}^3$ along a curve $C \subset \mathbb{P}^3$ of degree 6 and genus 3;

(vi) $\rho(X) = 2$, $\iota(X) = 1$, and X is the blow up of a smooth quadric $Q \subset \mathbb{P}^4$ along a rational twisted quartic curve $C \subset Q$;

(vii) $\rho(X) = 3$, $\iota(X) = 1$, and X is a double cover of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ whose branch divisor S is a member of the linear system $|-K_{\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1}|$;

(viii) $\rho(X) = 3$, $\iota(X) = 1$, and X is the blow up of W_6 along a rational curve $C \subset W_6$ of bidegree $(2, 2)$;

(ix) $\rho(X) = 4$, $\iota(X) = 1$, and $X \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is a divisor of multidegree $(1, 1, 1, 1)$; in this case each of four projections $\pi_i \colon X \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is the blow up along an elliptic curve C which is an intersection of two members of the linear system $|-\frac{1}{2}K_{\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1}|$.

In case (i) one has

$$\mathrm{Aut}(X) \cong \mathrm{PGL}_3(\Bbbk) \rtimes \boldsymbol{\mu}_2,$$

so that $\mathrm{Aut}(X)$ is Jordan with

$$\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant |\boldsymbol{\mu}_2| \cdot \bar{J}\big(\mathrm{PGL}_3(\Bbbk)\big) = 2 \cdot 40 = 80$$

by Lemma 2.3.1.

In case (ii) one has

$$\mathrm{Aut}(X) \cong \big(\mathrm{PGL}_2(\Bbbk) \times \mathrm{PGL}_2(\Bbbk) \times \mathrm{PGL}_2(\Bbbk)\big) \rtimes \mathfrak{S}_3,$$

so that $\mathrm{Aut}(X)$ is Jordan with

$$\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant |\mathfrak{S}_3| \cdot \bar{J}\big(\mathrm{PGL}_2(\Bbbk)\big) = 6 \cdot 12^3 = 10368$$

by Corollary 2.2.2.

In case (iii) one has $\rho(X) = 2$, so that the projections

$$p_i \colon X \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^2, \quad i = 1, 2,$$

are all possible Mori contractions from X. Hence the action of $\mathrm{Aut}(X)$ on X lifts to the action on $\mathbb{P}^2 \times \mathbb{P}^2$ and the embedding

$$p_1 \times p_2 \colon X \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^2$$

is $\mathrm{Aut}(X)$-equivariant. Thus

$$\mathrm{Aut}(X) \subset \mathrm{Aut}(\mathbb{P}^2 \times \mathbb{P}^2) \cong \big(\mathrm{PGL}_3(\Bbbk) \times \mathrm{PGL}_3(\Bbbk)\big) \rtimes \boldsymbol{\mu}_2,$$

so that $\mathrm{Aut}(X)$ is Jordan with

$$\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant 2 \cdot \bar{J}\big(\mathrm{PGL}_3(\Bbbk)\big)^2 = 2 \cdot 40^2 = 3200$$

by Lemma 2.3.1.

In case (iv) one has $\rho(X) = 2$, so that two conic bundles

$$\pi_i \colon X \to W_6 \to \mathbb{P}^2, \quad i = 1, 2,$$

are all possible Mori contractions from X. Thus there is a subgroup $\mathrm{Aut}'(X) \subset \mathrm{Aut}(X)$ of index at most 2 such that the conic bundle $\pi_1 \colon X \to \mathbb{P}^2$ is $\mathrm{Aut}'(X)$-equivariant. Let $G \subset \mathrm{Aut}'(X)$ be a finite subgroup. Then one has

$$\rho(X/\mathbb{P}^2)^G = \rho(X/\mathbb{P}^2) = 1,$$

so that $\pi_1 \colon X \to \mathbb{P}^2$ is a G-equivariant conic bundle. Thus $\mathrm{Aut}(X)$ is Jordan with

$$\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant [\mathrm{Aut}(X) : \mathrm{Aut}'(X)] \cdot \bar{J}\big(\mathrm{Aut}'(X)\big) \leqslant 2 \cdot 3456 = 6912$$

by Lemma 5.1.1.

In case (v) one has $\rho(X) = 2$, so that the contraction $\pi \colon X \to \mathbb{P}^3$ is one of the two possible Mori contractions from X. Hence there is a subgroup $\mathrm{Aut}'(X)$ of index at most 2 such that π is $\mathrm{Aut}'(X)$-equivariant. In particular, $\mathrm{Aut}'(X)$ acts on $\mathbb{P}^3$ faithfully, and since the curve $C \subset \mathbb{P}^3$ is not contained in any plane, $\mathrm{Aut}'(X)$ acts faithfully on C as well. Therefore, $\mathrm{Aut}(X)$ is Jordan with

$$\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant [\mathrm{Aut}(X) : \mathrm{Aut}'(X)] \cdot \bar{J}\big(\mathrm{Aut}'(X)\big) \leqslant 2 \cdot \bar{J}\big(\mathrm{Aut}(C)\big) \leqslant 2 \cdot 168 = 336$$

by Remark 2.2.3.

In case (vi) one has $\rho(X) = 2$, so that the contraction $\pi \colon X \to Q$ is one of the two possible Mori contractions from X. Hence there is a subgroup $\mathrm{Aut}'(X)$ of index at most 2 such that π is $\mathrm{Aut}'(X)$-equivariant. In particular, $\mathrm{Aut}'(X)$ acts on Q faithfully. Since all automorphisms of Q are linear, and the curve $C \subset Q \subset \mathbb{P}^4$ is not contained in any hyperplane, $\mathrm{Aut}'(X)$ acts faithfully on C as well. Therefore, $\mathrm{Aut}(X)$ is Jordan with

$$\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant [\mathrm{Aut}(X) : \mathrm{Aut}'(X)] \cdot \bar{J}\big(\mathrm{Aut}'(X)\big) \leqslant 2 \cdot \bar{J}\big(\mathrm{PGL}_2(\Bbbk)\big) = 24$$

by Corollary 2.2.2.

In case (vii) one has $\rho(X) = 3$, and the map $X \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^8$ is given by the anticanonical linear system. Three projections $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ give us three conic bundle structures

$$\pi_i \colon X \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1, \quad i = 1, 2, 3,$$

on X and these projections are permuted by the automorphism group $\mathrm{Aut}(X)$, because the morphism $X \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is $\mathrm{Aut}(X)$-equivariant. Thus there is a subgroup $\mathrm{Aut}'(X) \subset \mathrm{Aut}(X)$ of index at most 3 such that the conic bundle $\pi_1 \colon X \to \mathbb{P}^1 \times \mathbb{P}^1$ is $\mathrm{Aut}'(X)$-equivariant. Let $G \subset \mathrm{Aut}'(X)$ be a finite subgroup. Then one has

$$\rho(X/\mathbb{P}^1 \times \mathbb{P}^1)^G = \rho(X/\mathbb{P}^1 \times \mathbb{P}^1) = 1,$$

so that $\pi_1 \colon X \to \mathbb{P}^1 \times \mathbb{P}^1$ is a G-equivariant conic bundle. Thus $\mathrm{Aut}(X)$ is Jordan with

$$\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant [\mathrm{Aut}(X) : \mathrm{Aut}'(X)] \cdot \bar{J}\big(\mathrm{Aut}'(X)\big) \leqslant 3 \cdot 3456 = 10368$$

by Lemma 5.1.1.

In case (viii) one has $\rho(X) = 3$, and three divisorial contractions

$$\pi_i \colon X \to W_6, \quad i = 1, 2, 3,$$

are all possible birational Mori contractions from X (see [MM82, Table 3, no. 13]). Thus there is a subgroup $\mathrm{Aut}'(X)$ of index at most 3 such that π_1 is $\mathrm{Aut}'(X)$-equivariant. In particular, $\mathrm{Aut}'(X)$ acts on W_6 faithfully. The morphism π_1 is a blow up of a rational curve $C_1 \subset W_6$ of bi-degree $(2, 2)$. Since the images of C_1 under both projections

$$C_1 \hookrightarrow W_6 \to \mathbb{P}^2$$

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span $\mathbb{P}^2$, we see that $\mathrm{Aut}'(X)$ acts on C_1 faithfully as well. Therefore, $\mathrm{Aut}(X)$ is Jordan with

$$\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant [\mathrm{Aut}(X) : \mathrm{Aut}'(X)] \cdot \bar{J}\big(\mathrm{Aut}'(X)\big) \leqslant 3 \cdot \bar{J}\big(\mathrm{PGL}_2(\Bbbk)\big) = 36$$

by Corollary 2.2.2.

In case (ix) one has $\rho(X) = 4$, and four projections

$$\pi_i \colon X \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \quad i = 1, 2, 3, 4,$$

are all possible birational Mori contractions from X (see [MM82, Table 4, no. 1]). Thus there is a subgroup $\mathrm{Aut}'(X) \subset \mathrm{Aut}(X)$ of index at most 4 such that the divisorial contraction

$$\pi_1 \colon X \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$

is $\mathrm{Aut}'(X)$-equivariant. The morphism π_1 is a blow up of an elliptic curve

$$C_1 \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$

of tri-degree $(1, 1, 1)$. Since all three projections

$$C_1 \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$$

are dominant, one can see that $\mathrm{Aut}'(X)$ acts on C_1 faithfully as well. Therefore, $\mathrm{Aut}(X)$ is Jordan with

$$\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant [\mathrm{Aut}(X) : \mathrm{Aut}'(X)] \cdot \bar{J}\big(\mathrm{Aut}'(X)\big) \leqslant 4 \cdot \bar{J}\big(\mathrm{Aut}(C)\big) \leqslant 24$$

by Remark 2.2.3. $\qquad\square$

Remark 7.4.2 (cf. Remark 7.1.3). Let X be a smooth G-Fano threefold with $\rho(X) > 1$, and assume the notation of the proof of Lemma 7.4.1. Then one has $\bar{J}\big(\mathrm{Aut}(X)\big) < 10368$ with an exception of case (ii), and with a possible exception of case (vii). However, if X is like in case (vii), then it is non-rational, see [AB92]. Therefore, automorphism groups of varieties of this type cannot provide examples of subgroups in $\mathrm{Cr}_3(\Bbbk)$ whose Jordan constants attain the bounds given by Theorem 1.2.4, cf. Remark 8.2.1 below.

Remark 7.4.3. In general, studying Fano varieties with large automorphism groups is an interesting problem on its own. In many cases such varieties exhibit intriguing birational properties, see e.g. [CS11], [CS16], [PS16b].

8. Proof of the main theorem

In this section we complete the proof of Theorem 1.2.4.

8.1. **Summary for Fano threefolds.** We summarize the results of §6 and §7 as follows.

Proposition 8.1.1. *Let X be a Fano threefold with terminal Gorenstein singularities. Suppose that $\rho(X) = 1$. Then the group $\mathrm{Aut}(X)$ is Jordan with*

$$\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant 10368.$$

Proof. If X is singular, the group $\mathrm{Aut}(X)$ is Jordan with

$$\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant 9504$$

by Lemma 6.3.2. Therefore, we assume that X is smooth. If $\iota(X) > 1$, then the group $\mathrm{Aut}(X)$ is Jordan with

$$\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant 960$$

by Lemma 7.3.2.

It remains to consider the case when X is a smooth Fano threefold with $\mathrm{Pic}\,(X) = \mathbb{Z} \cdot K_X$. According to the classification (see e.g. [IP99, §12.2]), one has either $2 \leqslant \mathrm{g}(X) \leqslant 10$, or $\mathrm{g}(X) = 12$. If $\mathrm{g}(X) \leqslant 4$, then the group $\mathrm{Aut}(X)$ is Jordan with

$$\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant 1920$$

by Corollary 6.3.1. If $\mathrm{g}(X) = 5$, then the variety X is an intersection of three quadrics in $\mathbb{P}^6$ (see [IP99, §12.2]), so that the group $\mathrm{Aut}(X)$ is Jordan with

$$\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant 10368$$

by Lemma 7.1.2. If $\mathrm{g}(X) = 6$, then the group $\mathrm{Aut}(X)$ is Jordan with

$$\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant 960$$

by Corollary 7.2.3. Finally, if $\mathrm{g}(X) \geqslant 7$, then the group $\mathrm{Aut}(X)$ is Jordan with

$$\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant 9922$$

by Lemma 7.3.1. $\qquad\square$

Corollary 8.1.2. *Let G be a finite group, and X be a (Gorenstein) G-Fano threefold. Then the group $\mathrm{Aut}(X)$ is Jordan with*

$$\bar{J}\big(G\big) \leqslant 10368.$$

Proof. If X is singular, the group $\mathrm{Aut}(X)$ is Jordan with

$$\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant 9504$$

by Lemma 6.3.2. If X is smooth and $\rho(X) > 1$, then the group $\mathrm{Aut}(X)$ is Jordan with

$$\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant 10368$$

by Lemma 7.4.1. Therefore, we may assume that $\rho(X) = 1$, so that the group $\mathrm{Aut}(X)$ is Jordan with

$$\bar{J}\big(\mathrm{Aut}(X)\big) \leqslant 10368$$

by Proposition 8.1.1. $\qquad\square$

Remark 8.1.3 (cf. Remark 3.2.4). In several cases one can produce better bounds for weak Jordan constants of certain Fano threefolds applying a bit more effort. We did not pursue this goal since the current estimates are already enough to prove our main results.

8.2. **Proof and concluding remarks.** Now we are ready to prove Theorem 1.2.4.

Proof of Theorem 1.2.4. Let X be a rationally connected threefold over an arbitrary field $\Bbbk$ of characteristic 0, and let $G \subset \mathrm{Bir}(X)$ be a finite group. It is enough to establish the upper bounds for $\bar{J}(G)$ and $J(G)$. Moreover, to prove the bounds we may assume that $\Bbbk$ is algebraically closed.

Regularizing the action of G and taking an equivariant desingularization (see e.g. [PS14, Lemma-Definition 3.1]), we may assume that X is smooth and $G \subset \mathrm{Aut}(X)$. Applying G-equivariant Minimal Model Program to X (which is possible due to an equivariant version of [BCHM10, Corollary 1.3.3] and [MM86, Theorem 1], since rational connectedness implies uniruledness), we may assume that either there is a $G\mathbb{Q}$-conic

bundle structure $\phi\colon X \to S$ for some rational surface S, or there is a $G\mathbb{Q}$-del Pezzo fibration $\phi\colon X \to \mathbb{P}^1$, or X is a $G\mathbb{Q}$-Fano threefold. Therefore, we have

$$\bar{J}(G) \leqslant 10368$$

by Lemmas 5.1.1 and 5.2.5 and Corollary 8.1.2. Applying Remark 1.2.2, we obtain the inequality

$$J(G) \leqslant 10368^2 = 107495424.$$

If $\Bbbk$ is algebraically closed, then the group $\mathrm{Cr}_3(\Bbbk)$ contains a group

$$\mathrm{Aut}\big(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1\big) \supset \big(\mathfrak{A}_5 \times \mathfrak{A}_5 \times \mathfrak{A}_5\big) \rtimes \mathfrak{S}_3,$$

and the largest abelian subgroup of the latter finite group has order 125. Therefore, one has

$$\bar{J}\big(\mathrm{Cr}_3(\Bbbk)\big) = 10368.$$

$\square$

Remark 8.2.1. We do not know whether the bound for the (usual) Jordan constant for the group $\mathrm{Cr}_3(\Bbbk)$ over an algebraically closed field $\Bbbk$ of characteristic 0 provided by Theorem 1.2.4 is sharp or not. The Jordan constant of the group $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$ is smaller than that, but there may be other automorphism groups of rational varieties providing this value, cf. Lemma 5.2.5. We also do not know the actual value of $J\big(\mathrm{Cr}_2(\Bbbk)\big)$, but we believe that it can be found by a thorough (and maybe a little bit boring) analysis of automorphism groups of del Pezzo surfaces and two-dimensional conic bundles, since in dimension 2 much more precise classification results are available.

Remark 8.2.2. In dimension 4 and higher we cannot hope (at least on our current level of understanding the problem) to obtain results similar to Theorem 1.2.4. Indeed, in dimension 3 we use the classification of terminal singularities to obtain bounds for Jordan constants of automorphism groups of terminal Fano varieties and Mori fiber spaces. The result of [Kol11, Theorem 1] shows that a "nice" classification of higher dimensional terminal singularities is impossible, at least in the setup we used in Lemma 4.1.1 and Corollary 4.1.3, due to unboundedness of the dimensions of Zariski tangent spaces of their index one covers.

APPENDIX A. AUTOMORPHISMS OF SOME COMPLETE INTERSECTIONS

In this section we collect some (well-known) results about automorphisms of complete intersections of quadrics, and complete intersections in weighted projective spaces.

A.1. **Complete intersections of quadrics.** Let $X \subset \mathbb{P}^n = \mathbb{P}(V)$ be a smooth complete intersection of r quadrics. Let I_X be the ideal sheaf of X, so that

$$W = H^0(\mathbb{P}(V), I_X(2))$$

is the r-dimensional vector space of quadrics passing through X. Let

$$q\colon W \hookrightarrow \mathrm{Sym}^2 V^\vee$$

be the natural embedding.

Lemma A.1.1. *Suppose that $n \geqslant 2r$. Then any automorphism of X is induced by an automorphism of $\mathbb{P}(V)$ and induces an automorphism of $\mathbb{P}(W)$.*

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Proof. By adjunction formula one has $-K_X \sim (n + 1 - 2r)H$, where H is the class of a hyperplane section of X. Thus X is Fano, and in particular there is no torsion in the Picard group of X. Therefore, the class of H in $\mathrm{Pic}\,(X)$ is $\mathrm{Aut}(X)$-invariant, and there is a natural embedding

$$\mathrm{Aut}(X) \hookrightarrow \mathrm{PGL}(V).$$

Furthermore, the twisted ideal sheaf $I_X(2)$ is invariant, hence the subspace

$$\mathbb{P}(W) \subset \mathbb{P}(\mathrm{Sym}^2 V^\vee)$$

is invariant under the action of $\mathrm{Aut}(X)$, and so we also have a map

$$\mathrm{Aut}(X) \to \mathrm{PGL}(W). \qquad \square$$

In the remaining part of this section we denote by $\mathrm{Aut}_W(X)$ the image of the morphism $\mathrm{Aut}(X) \to \mathrm{PGL}(W)$ constructed in Lemma A.1.1, and by $\Gamma(X)$ its kernel. Thus we have an exact sequence

(A.1.2) $$1 \to \Gamma(X) \to \mathrm{Aut}(X) \to \mathrm{Aut}_W(X) \to 1.$$

Note that the group $\Gamma(X)$ is the subgroup of $\mathrm{Aut}(X)$ which preserves every quadric in the linear system of quadrics passing through X. In what follows we discuss what one can say about the groups $\Gamma(X)$ and $\mathrm{Aut}_W(X)$ in some special cases.

First, if $r = 1$ then $W = \Bbbk$, so $\mathrm{Aut}_W(X) = 1$ and

$$\Gamma(X) = \mathrm{Aut}(X) \cong \mathrm{PSO}_{n+1}(\Bbbk).$$

Now assume that $r = 2$.

Proposition A.1.3. *Let $X \subset \mathbb{P}^n$, $n \geqslant 4$, be a smooth complete intersection of two quadrics. Then $\Gamma(X) \cong \boldsymbol{\mu}_2^n$ and*

$$\mathrm{Aut}_W(X) \subset \mathrm{PGL}_2(\Bbbk).$$

Moreover, there is also an embedding

$$\mathrm{Aut}_W(X) \subset \mathfrak{S}_{n+1}.$$

Proof. In some homogeneous coordinates $x_0, \ldots, x_n$ on $\mathbb{P}^n$ the variety X is given by equations $f_1 = f_2 = 0$, where

(A.1.4) $$f_1 = x_0^2 + \ldots + x_n^2, \quad f_2 = \lambda_0 x_0^2 + \ldots + \lambda_n x_n^2$$

for some pairwise distinct numbers λ_i (see e. g. [Rei72, Proposition 2.1]). It is easy to see that the singular quadrics in the pencil generated by f_1 and f_2 are given by equations

$$f_2 - \lambda_i f_1 = 0,$$

and their singular loci are the points $(1 : 0 : \ldots : 0), \ldots, (0 : \ldots : 0 : 1)$.

Since the subgroup $\Gamma(X) \subset \mathrm{Aut}(X)$ preserves every quadric in the pencil, it also preserves the singular loci of the singular quadrics. Therefore, it is a subgroup in the standard torus (formed by the diagonal matrices) in $\mathrm{PGL}(V)$. Since it also fixes the quadric $f_1 = 0$, it follows that all diagonal entries of the matrix that represents an element of $\Gamma(X)$ have the same square. So, rescaling the matrices if necessary, we may assume that all diagonal entries are ± 1. Therefore, one has

(A.1.5) $$\Gamma(X) \cong \boldsymbol{\mu}_2^{n+1}/\boldsymbol{\mu}_2 \cong \boldsymbol{\mu}_2^n,$$

because we have to quotient out by the transformation $\pm \mathrm{Id}_V$ acting trivially on $\mathbb{P}(W)$.

Since $\dim(W) = 2$, we have $\operatorname{Aut}_W(X) \subset \operatorname{PGL}(W) \cong \operatorname{PGL}_2(\Bbbk)$. On the other hand, the group $\operatorname{Aut}_W(X)$ permutes the points of $\mathbb{P}(W)$ corresponding to singular quadrics, hence there is a homomorphism $\operatorname{Aut}_W(X) \to \mathfrak{S}_{n+1}$. It is an embedding since any automorphism of the projective line $\mathbb{P}(W) \cong \mathbb{P}^1$ that preserves $n + 1 \geqslant 5$ points is trivial. $\qquad\square$

Remark A.1.6. Assume the notation of Proposition A.1.3. Then the group $\Gamma(X)$ is generated by the $n + 1$ reflections $\gamma_i\colon x_i \mapsto -x_i$ which satisfy the obvious relation

$$\gamma_1 \circ \cdots \circ \gamma_{n+1} = 1,$$

see (A.1.5).

Now consider the case $n = 5$, i.e. $X \subset \mathbb{P}^5$ is a smooth three-dimensional intersection of two quadrics. Let B be the corresponding hyperelliptic curve, see [Rei72, Proposition 2.1], and also [KPS16, Remark 2.2.11]. Let $\Sigma(X)$ be the Hilbert scheme of lines on X (see [KPS16, §2.1]). Then $\Sigma(X)$ is an abelian surface isomorphic to the Jacobian of B, and also isomorphic to the intermediate Jacobian of X (see [GH78, §6]). The group $\Gamma(X)$ consists of elements γ of the types listed in Table 1. Here by $\operatorname{Fix}(\gamma, Z)$ we denote the locus of fixed points of an automorphism γ of a variety Z.

TABLE 1. Elements of $\Gamma(X)$

γ	#	$\operatorname{Fix}(\gamma, X)$	$\operatorname{Fix}(\gamma, \Sigma(X))$
1	1	X	$\Sigma(X)$
γ_i	6	del Pezzo surface of degree 4	16 points
$\gamma_i \circ \gamma_j,\ i \neq j$	15	elliptic curve	$\varnothing$
$\gamma_i \circ \gamma_j \circ \gamma_k,\ i \neq j \neq k \neq i$	10	8 points	16 points

Put

$$\Gamma_0(X) = \{1,\ \gamma_i \circ \gamma_j \mid i \neq j\}.$$
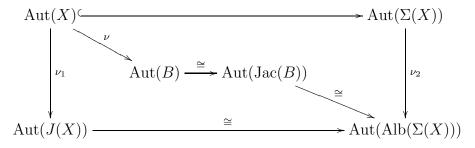
Then $\Gamma_0(X)$ is a subgroup of index 2 in $\Gamma(X)$. Let σ be the homomorphism

$$\sigma\colon \Gamma(X) \to \Gamma(X)/\Gamma_0(X) \cong \{\pm 1\} \cong \boldsymbol{\mu}_2.$$

Since $\Sigma(X)$ is an abelian surface, one has $H^2(\Sigma(X), \mathbb{Z}) \cong \Lambda^2 H^1(\Sigma(X), \mathbb{Z})$. We see from the topological Lefschetz fixed point formula and Table 1 that $\Gamma(X)$ acts on $H^3(X, \mathbb{Z})$ and $H^1(\Sigma(X), \mathbb{Z})$ via

$$\gamma(x) = \sigma(\gamma) \cdot x.$$

Thus we have the following commutative diagram

$$
\begin{array}{ccc}
\operatorname{Aut}(X) \lhook\joinrel\longrightarrow & & \operatorname{Aut}(\Sigma(X)) \\
\Big\downarrow{\scriptstyle\nu_1} \quad {\scriptstyle\nu}\searrow & & \quad \Big\downarrow{\scriptstyle\nu_2} \\
& \operatorname{Aut}(B) \xrightarrow{\cong} \operatorname{Aut}(\operatorname{Jac}(B)) & \\
& \qquad\qquad \searrow{\scriptstyle\cong} & \\
\operatorname{Aut}(J(X)) \xrightarrow{\ \cong\ } & & \operatorname{Aut}(\operatorname{Alb}(\Sigma(X)))
\end{array}
$$

where $J(X)$ is the intermediate Jacobian of X. The elements of the kernel of

$$\nu\colon \operatorname{Aut}(X) \to \operatorname{Aut}(B)$$

act trivially on our pencil of quadrics, and each γ_i switches two families of planes on smooth quadrics. Therefore, one has $\mathsf{Ker}\,\nu = \Gamma_0(X)$, and $\nu(\gamma_i)$ is the hyperelliptic involution of the curve B. Furthermore, $\mathsf{Ker}\,\nu_2$ is the subgroup of translations and

$$\mathrm{Aut}(X) \cap \mathsf{Ker}\,\nu_2 = \mathsf{Ker}\,\nu = \Gamma_0(X).$$

This shows that the kernel of the homomorphism

$$\nu_1 \colon \mathrm{Aut}(X) \to \mathrm{Aut}(J(X))$$

coincides with $\Gamma_0(X)$. See also [Pan15], [CPZ15].

An easy consequence of Proposition A.1.3 is the following result.

Corollary A.1.7. *For any $2 \leqslant r \leqslant n/2$ we have $\Gamma(X) \cong \boldsymbol{\mu}_2^m$ for some $m \leqslant n$.*

Proof. Let Q and Q' be two general quadrics passing through X. Then $Y = Q \cap Q'$ is a complete intersection of Q and Q'. Moreover, Y is non-singular outside X by Bertini's theorem, and Y is non-singular at the points of X since X is a complete intersection of Q, Q', and several other quadrics.

Since $\Gamma(X)$ preserves every quadric passing through X, it also preserves the quadrics in the pencil generated by Q and Q', hence $\Gamma(X) \subset \Gamma(Y)$. So, the claim follows from Proposition A.1.3. $\qquad\square$

We can describe the cases when $\Gamma(X) \neq 1$. This, in fact, is equivalent to "strict semistability" of q, i.e. to the situation when

$$V = V_0 \oplus V_1 \oplus \ldots \oplus V_m$$

and $q = q_0 + q_1 + \ldots + q_m$, where

$$q_i : W \to \mathrm{Sym}^2 V_i^\vee.$$

In the example below all V_i are one-dimensional.

Example A.1.8. Let $X \subset \mathbb{P}^n$ be given by $r \leqslant n$ equations

$$\lambda_{10}x_0^2 + \lambda_{11}x_1^2 + \ldots + \lambda_{1n}x_n^2 = 0, \quad \ldots, \quad \lambda_{r0}x_0^2 + \lambda_{r1}x_1^2 + \ldots + \lambda_{rn}x_n^2 = 0,$$

where $\lambda_{ij} \in \Bbbk$ are sufficiently general. Then X is a smooth complete intersection of r quadrics, and clearly all diagonal matrices with entries ± 1 preserve each of the quadrics. Therefore, in this case one has $\Gamma(X) \cong \boldsymbol{\mu}_2^n$. This shows that the group $\Gamma(X)$ may be nontrivial for any r.

Now we will consider intersections of three quadrics. Let Δ be a reduced connected curve. Recall that Δ is said to be *stable* if its singularities are nodes, and Δ has no infinitesimal automorphisms. The automorphism group of a stable curve is finite [DM69, Theorem 1.11]. Note also that any nodal plane curve of degree at least 4 is stable (see e.g. [Has99, Proposition 2.1]).

Lemma A.1.9. *Let $X \subset \mathbb{P}^n$, $n \geqslant 6$, be a smooth complete intersection of three quadrics. Then $\mathrm{Aut}_W(X)$ acts faithfully on a stable curve. In particular, the group $\mathrm{Aut}(X)$ is finite.*

Proof. Let $\Delta \subset \mathbb{P}^2 = \mathbb{P}(W)$ be the curve that parameterizes degenerate quadrics passing through X. This curve is usually called the *Hesse curve* of X (see [Tyu75, §2.2]). One has

$$\deg \Delta = n + 1 \geqslant 7.$$

The curve Δ is $\mathrm{Aut}(X)$-invariant. Since it is not a line, we conclude that $\mathrm{Aut}_W(X)$ acts faithfully on Δ. It is well known that the curve Δ is nodal; this follows, for example, from [Bea77b, Proposition 1.2(iii)] applied to the quadric bundle over $\mathbb{P}^2$ that is obtained by blowing up $\mathbb{P}^n$ along X. Thus, the curve Δ is stable.

As we noticed above, stability of Δ implies finiteness of $\mathrm{Aut}_W(X)$. On the other hand, $\Gamma(X)$ is finite by Corollary A.1.7. So, finiteness of $\mathrm{Aut}(X)$ follows from exact sequence (A.1.2). $\square$

A.2. **Complete intersections in weighted projective spaces.** In this section we discuss the automorphism groups of complete intersections in weighted projective spaces. Recall that a weighted projective space $\mathbb{P}(a_0, \ldots, a_n)$, $n \geqslant 1$, is defined as

$$\mathbb{P}(a_0, \ldots, a_n) = \mathrm{Proj}\, \mathbb{k}\,[x_0, \ldots, x_n]\,,$$

where the variables $x_0, \ldots, x_n$ have (positive integer) weights $a_0, \ldots, a_n$, respectively (see [Dol82]). Also, it can be constructed as a quotient

$$\mathbb{P}(a_0, \ldots, a_n) = \left(\mathbb{A}^n \setminus \{0\}\right)/\mathbb{k}^*, \quad \lambda \colon (x_0, \ldots, x_n) \mapsto (\lambda^{a_0} x_0, \ldots, \lambda^{a_n} x_n).$$

In a standard way, a weighted projective space $\mathbb{P} = \mathbb{P}(a_0, \ldots, a_n)$ is equipped with rank 1 coherent sheaves $\mathscr{O}_{\mathbb{P}}(m)$, $m \in \mathbb{Z}$. These sheaves are *divisorial* but non-invertible in general (see [Dol82, §1]).

Any weighted projective space is isomorphic to a *well-formed* weighted projective space, i.e. a weighted projective space $\mathbb{P}(a_0, \ldots, a_n)$ such that the greatest common divisor of any n among the $n+1$ weights $a_0, \ldots, a_n$ equals 1 (see [Dol82, 1.3.1] for details).

Let $\mathrm{Cox}(\mathbb{P})$ be the Cox ring of $\mathbb{P}$, see [CLS11, §5.1 and §5.2] or [ADHL15, §I.4.1] for a definition.

Lemma A.2.1. *Suppose that the weighted projective space $\mathbb{P} = \mathbb{P}(a_0, \ldots, a_n)$ is well-formed. Then the following assertions hold.*

(i) *The group $\mathrm{Cl}\,(\mathbb{P}) \cong \mathbb{Z}$ is generated by the class of $\mathscr{O}_{\mathbb{P}}(1)$.*

(ii) *One has a canonical isomorphism of $\mathbb{Z}$-graded rings*

$$\mathrm{Cox}(\mathbb{P}) \cong \bigoplus_{m \geqslant 0} H^0(\mathbb{P}, \mathscr{O}_{\mathbb{P}}(m)) \cong \mathbb{k}[x_0, \ldots, x_n],$$

where the weight of the variable x_i is defined to be a_i.

Proof. We have the standard exact sequence

(A.2.2) $$0 \longrightarrow \mathbb{Z} \cdot D_i \longrightarrow \mathrm{Cl}\,(\mathbb{P}) \longrightarrow \mathrm{Cl}\,(U_0) \longrightarrow 0,$$

where

$$U_0 = \mathbb{P} \setminus D_0 \cong \mathbb{A}^n/\boldsymbol{\mu}_{a_0}$$

and the action of $\boldsymbol{\mu}_{a_0}$ on $\mathbb{A}^n$ is diagonal with weights $(a_1, \ldots, a_n)$. By our well-formedness assumption $\gcd(a_1, \ldots, a_n) = 1$, i.e. the action of $\boldsymbol{\mu}_{a_0}$ on $\mathbb{A}^n$ is free in codimension 1. Therefore, $\mathrm{Cl}\,(U_0) \cong \mathbb{Z}/a_0\mathbb{Z}$ and $\mathrm{Cl}\,(\mathbb{P}) \cong \mathbb{Z} \oplus T$, where T is a finite cyclic group whose order divides a_0. By symmetry the order of T divides a_i for all i and again by our well-formedness assumption $T = 0$. Thus $\mathrm{Cl}\,(\mathbb{P}) \cong \mathbb{Z}$. Let D be the positive generator of $\mathrm{Cl}\,(\mathbb{P})$ and let D_i be the effective Weil divisor given by $x_i = 0$. Since $\mathrm{Cl}\,(U_0) \cong \mathbb{Z}/a_0\mathbb{Z}$, the sequence (A.2.2) shows $D_0 \sim a_0 D$ and similarly $D_i \sim a_i D$ for all i. By definition of sheaves $\mathscr{O}_{\mathbb{P}}(m)$ we have $\mathscr{O}_{\mathbb{P}}(a_i) \cong \mathscr{O}_{\mathbb{P}}(D_i)$. This proves assertion (i).

Now one can show that the Serre homomorphism

$$\Bbbk[x_0, \ldots, x_n]_m \longrightarrow H^0(\mathbb{P}, \mathscr{O}_{\mathbb{P}}(m))$$

is an isomorphism (see e.g. [Dol82, 1.4.1]). Keeping in mind the definition of the grading on $\mathrm{Cox}(\mathbb{P})$, see [CLS11, §5.2], we get the required isomorphism of $\mathbb{Z}$-graded rings, which proves assertion (ii). $\qquad\square$

Remark A.2.3. Lemma A.2.1(ii) is given as [CLS11, Exercise 5.2.2]; the proof is based on [CLS11, Example 5.1.14] and [CLS11, Exercises 4.1.5 and 4.2.11]. Lemma A.2.1(i) is [CLS11, Exercise 4.1.5]. In both cases one can find some details clarified in [CLS]. The proof relies on the description of $\mathbb{P}$ as a toric variety given in [CLS11, Example 3.1.17], which uses a well-formedness assumption. All the rest is a standard techniques of working with the divisor class group of a toric variety based on [CLS11, Theorem 4.1.3].

Consider the polynomial ring $R = \Bbbk[x_0, \ldots, x_n]$ as a graded $\Bbbk$-algebra $R = \bigoplus_{i \geqslant 0} R_i$ with grading given by $\deg x_i = a_i > 0$. In particular, one has $R_0 = \Bbbk$. Denote by $R_{\leqslant m}$ the graded vector subspace $\bigoplus_{i \leqslant m} R_i \subset R$.

Lemma A.2.4. *Let $U_m \subset R_m$ be the intersection of R_m with the subalgebra of R generated by $R_{\leqslant m-1}$, and put*

$$k_m = \dim R_m - \dim U_m.$$

Suppose that R is finitely generated, so that there is a positive N such that $k_m = 0$ for $m > N$. Put

$$\Gamma = \mathrm{GL}_{k_1}(\Bbbk) \times \ldots \times \mathrm{GL}_{k_N}.$$

Then $\mathrm{Aut}(R)$, regarded as the automorphism group of the graded *algebra R, contains the group Γ, and any reductive subgroup of $\mathrm{Aut}(R)$ is isomorphic to a subgroup of Γ.*

Proof. The group $\mathrm{Aut}(R)$ acts on every vector space R_m so that the subspace U_m is $\mathrm{Aut}(R)$-invariant. Choose $V_m \subset R_m$ to be a vector subspace such that

$$U_m \oplus V_m = R_m.$$

One has $k_m = \dim V_m$. This gives an obvious action of Γ on R.

Now let $G \subset \mathrm{Aut}(R)$ be a reductive subgroup. Then one can choose a G-invariant vector subspace $V_m' \subset R_m$ such that

$$U_m \oplus V_m' = R_m.$$

Moreover, the action of G on R is recovered from its action on $\bigoplus V_m'$. Since $V_m' \cong V_m$, this gives the second assertion of the lemma. $\qquad\square$

We will use the abbreviation

$$(a_1^{k_1}, \ldots, a_N^{k_N}) = (\underbrace{a_1, \ldots, a_1}_{k_1 \text{ times}}, \ldots, \underbrace{a_N, \ldots, a_N}_{k_N \text{ times}}),$$

where $k_1, \ldots, k_N$ are allowed to be any non-negative integers.

Proposition A.2.5. *Suppose that the weighted projective space $\mathbb{P} = \mathbb{P}(a_1^{k_1}, \ldots, a_N^{k_N})$ is well-formed. Let R_U be the unipotent radical of the group $\mathrm{Aut}(\mathbb{P})$, so that the quotient*

$$\mathrm{Aut}_{\mathrm{red}}(\mathbb{P}) = \mathrm{Aut}(\mathbb{P})/R_U$$

is reductive. Then

$$\mathrm{Aut}_{\mathrm{red}}(\mathbb{P}) \cong \left(\mathrm{GL}_{k_1}(\Bbbk) \times \ldots \times \mathrm{GL}_{k_N}(\Bbbk)\right)/\Bbbk^*,$$

where $\Bbbk^*$ *embeds into the above product by*

(A.2.6) $$t \mapsto (t^{a_1}\mathrm{Id}_{k_1}, \ldots, t^{a_N}\mathrm{Id}_{k_N}).$$

Here Id_k *denotes the identity* $k \times k$*-matrix.*

Proof. Since $\mathbb{P}$ is well-formed, by Lemma A.2.1(ii) one has an isomorphism of $\mathbb{Z}$-graded rings

$$\mathrm{Cox}(\mathbb{P}) \cong \Bbbk\left[x_1^{(1)}, \ldots, x_1^{(k_1)}, \ldots, x_N^{(1)}, \ldots, x_N^{(k_N)}\right],$$

where the weight of the variable $x_i^{(j)}$ is defined to be a_i. The action of $\Bbbk^*$ on $\mathrm{Cox}(\mathbb{P})$ defined by (A.2.6) agrees with this grading. One has

$$\mathrm{Spec}\,\mathrm{Cox}(\mathbb{P}) \cong \mathbb{A}^{k_1+\ldots+k_N}.$$

Let $\widetilde{\mathrm{Aut}}(\mathbb{P})$ be the normalizer of $\Bbbk^*$ in the group

$$\Gamma = \mathrm{Aut}\big(\mathbb{A}^{k_1+\ldots+k_N} \setminus \{0\}\big) \cong \mathrm{Aut}_0\big(\mathbb{A}^{k_1+\ldots+k_N}\big).$$

Then Γ naturally acts on $\mathrm{Cox}(\mathbb{P})$, that is identified with the ring of regular functions on $\mathbb{A}^{k_1+\ldots+k_N} \setminus \{0\}$. Moreover, $\widetilde{\mathrm{Aut}}(\mathbb{P})$ is actually a centralizer of $\Bbbk^*$ in Γ, since all weights of the action of $\Bbbk^*$ on $\mathrm{Cox}(\mathbb{P})$ are positive (and $\mathrm{Cox}(\mathbb{P})$ splits into a sum of eigen-spaces of $\Bbbk^*$). Thus $\widetilde{\mathrm{Aut}}(\mathbb{P})$ is isomorphic to the group of graded automorphisms of the ring $\mathrm{Cox}(\mathbb{P})$ by [Cox95, Theorem 4.2(iii)].

According to the Levi decomposition there exists a reductive subgroup $\tilde{L} \subset \widetilde{\mathrm{Aut}}(\mathbb{P})$ such that $\tilde{L} \cong \widetilde{\mathrm{Aut}}(\mathbb{P})/\tilde{R}_U$, where $\tilde{R}_U$ is the unipotent radical of $\widetilde{\mathrm{Aut}}(\mathbb{P})$. By Lemma A.2.4 one has

$$\widetilde{\mathrm{Aut}}(\mathbb{P})/\tilde{R}_U \cong \tilde{L} \cong \mathrm{GL}_{k_1}(\Bbbk) \times \ldots \times \mathrm{GL}_{k_N}(\Bbbk).$$

On the other hand, by [Cox95, Theorem 4.2(ii)] one has

$$1 \to \Bbbk^* \to \widetilde{\mathrm{Aut}}(\mathbb{P}) \to \mathrm{Aut}(\mathbb{P}) \to 1,$$

and the assertion follows. $\qquad\square$

Remark A.2.7. The assertion of Proposition A.2.5 fails without the assumption that the weighted projective space $\mathbb{P}$ is well-formed. One can take the weighted projective line

$$\mathbb{P}(1,2) \cong \mathbb{P}^1$$

as a counterexample.

Lemma A.2.8. *Let* $\mathbb{P} = \mathbb{P}(a_1^{k_1}, \ldots, a_N^{k_N})$ *be a well-formed weighted projective space such that* $a_1 < \ldots < a_{N-1} < a_N$. *Let* $G \subset \mathrm{Aut}(\mathbb{P})$ *be a finite subgroup. Assume that* $k_N = 1$. *Then for some positive integer* r *there is a central extension*

(A.2.9) $$1 \to \boldsymbol{\mu}_r \to \tilde{G} \to G \to 1$$

with

$$\tilde{G} \subset \mathrm{GL}_{k_1}(\Bbbk) \times \ldots \times \mathrm{GL}_{k_{N-1}}(\Bbbk).$$

Proof. Let $R_U \subset \mathrm{Aut}(\mathbb{P})$ be the unipotent radical of $\mathrm{Aut}(\mathbb{P})$. Since any nontrivial element of a unipotent group has infinite order, the intersection $G \cap R_U$ is trivial, hence G embeds into the reductive quotient

$$\mathrm{Aut}_{\mathrm{red}}(\mathbb{P}) = \mathrm{Aut}(\mathbb{P})/R_U.$$

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Consider the embedding

$$\Bbbk^* \hookrightarrow \mathrm{GL}_{k_1}(\Bbbk) \times \ldots \times \mathrm{GL}_{k_N}(\Bbbk)$$

given by formula (A.2.6). By Proposition A.2.5 one has

$$\mathrm{Aut}_{\mathrm{red}}(\mathbb{P}) \cong \big(\mathrm{GL}_{k_1}(\Bbbk) \times \ldots \times \mathrm{GL}_{k_N}(\Bbbk)\big)/\Bbbk^*,$$

Consider the subgroup

$$\mathrm{SL}_{k_1,\ldots,k_N}(\Bbbk) = \Big\{ (g_1, \ldots, g_N) \in \mathrm{GL}_{k_1}(\Bbbk) \times \ldots \times \mathrm{GL}_{k_N}(\Bbbk) \mid \prod_{i=1}^{N} \det(g_i) = 1 \Big\}.$$

in $\mathrm{GL}_{k_1}(\Bbbk) \times \ldots \times \mathrm{GL}_{k_N}(\Bbbk)$. This group intersects with the above $\Bbbk^*$ along $\boldsymbol{\mu}_r \subset \Bbbk^*$, where $r = \sum_{i=1}^{N} a_i k_i$. Moreover, we have

$$\mathrm{Aut}_{\mathrm{red}}(\mathbb{P}) \cong \mathrm{SL}_{k_1,\ldots,k_N}(\Bbbk)/\boldsymbol{\mu}_r$$

Denote the preimage of G in $\mathrm{SL}_{k_1,\ldots,k_N}(\Bbbk)$ by $\tilde{G}$. Then there is a central extension (A.2.9). So, it remains to notice that as $k_N = 1$, we have

$$\mathrm{SL}_{k_1,\ldots,k_{N-1},k_N}(\Bbbk) \cong \mathrm{GL}_{k_1}(\Bbbk) \times \ldots \times \mathrm{GL}_{k_{N-1}}(\Bbbk). \qquad \square$$

Let Y be a normal projective variety and let A be a Weil divisor on Y. Put

$$R_m(Y, A) = H^0\big(Y, \mathscr{O}_Y(mA)\big).$$

Then

$$R(Y, A) = \bigoplus_{m=0}^{\infty} R_m(Y, A)$$

has a natural structure of a graded $\Bbbk$-algebra.

Remark A.2.10. If the divisor A is ample, then the algebra $R(Y, A)$ is finitely generated, and $Y \cong \mathrm{Proj}(R(Y, A))$.

As before, define a (graded) vector subspace

$$R_{\leqslant N}(Y, A) = \bigoplus_{m \leqslant N} R_m(Y, A) \subset R(Y, A).$$

Lemma A.2.11. *Let Y be a normal projective variety with an action of a group Γ, and A be an ample Weil divisor on Y. Suppose that the class of A in $\mathrm{Cl}(Y)$ is Γ-invariant. Then for some positive integer r there is a central extension*

(A.2.12) $$1 \to \boldsymbol{\mu}_r \to \tilde{\Gamma} \to \Gamma \to 1$$

such that $\tilde{\Gamma}$ acts on the algebra $R(Y, A)$, and this action induces the initial action of Γ on Y.

Proof. One has

$$Y \cong \mathrm{Proj}\big(R(Y, A)\big).$$

The algebra $R(Y, A)$ is generated by its vector subspace $R_{\leqslant N}(Y, A)$ for some N. Now it suffices to define an action of an appropriate central extension (A.2.12) on $R_m(Y, A)$ for each $1 \leqslant m \leqslant N$ (see e.g. [KPS16, §3.1]). Taking r to be sufficiently divisible, we may assume that $\tilde{\Gamma}$ acts on the whole vector space $R_{\leqslant N}(Y, A)$, which gives the desired action on the algebra $R(Y, A)$. $\qquad \square$

Lemma A.2.13 (cf. Lemma A.2.4). *Let Y be a normal projective variety with an action of a finite group Γ, and A be an ample Weil divisor on Y. Suppose that the class of A in $\mathrm{Cl}(Y)$ is Γ-invariant and $R(Y, A)$ is generated by $R_{\leqslant N}(Y, A)$. For $1 \leqslant m \leqslant N$ let $U_m \subset R_m(Y, A)$ be the intersection of $R_m(Y, A)$ with the subalgebra of $R(Y, A)$ generated by $R_{\leqslant m-1}(Y, A)$, and put*

$$k_m = \dim R_m(Y, A) - \dim U_m.$$

Then there is a natural embedding

$$Y \hookrightarrow \mathbb{P} = \mathbb{P}\big(1^{k_1}, \ldots, N^{k_N}\big)$$

and an action of Γ on $\mathbb{P}$ that induces the initial action of Γ on Y.

Proof. By Lemma A.2.11 there is an action of a finite central extension $\tilde{\Gamma}$ of Γ on $R(Y, A)$ that induces the initial action of Γ on Y. In particular, the group $\tilde{\Gamma}$ acts on every vector space $R_m(Y, A)$. Obviously, the subspace U_m is $\tilde{\Gamma}$-invariant. Choose $V_m \subset R_m(Y, A)$ to be a $\tilde{\Gamma}$-invariant vector subspace such that

$$U_m \oplus V_m = R_m(Y, A).$$

One has $k_m = \dim V_m$. Let $x_m^{(1)}, \ldots, x_m^{(k_m)}$ be a basis in V_m. Then there is a natural surjection

$$\Bbbk\left[x_1^{(1)}, \ldots, x_1^{(k_1)}, \ldots, x_N^{(1)}, \ldots, x_N^{(k_N)}\right] \to R(Y, A)$$

that induces an embedding

$$Y \cong \mathrm{Proj}\big(R(Y, A)\big) \hookrightarrow \mathrm{Proj}\left(\Bbbk\left[x_1^{(1)}, \ldots, x_1^{(k_1)}, \ldots, x_N^{(1)}, \ldots, x_N^{(k_N)}\right]\right) = \mathbb{P}.$$

Note that the action of $\tilde{\Gamma}$ on $\mathbb{P}$ factors through the action of Γ on $\mathbb{P}$, and this action clearly induces the initial action of Γ on Y. $\qquad\square$

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Steklov Mathematical Institute of Russian Academy of Sciences, 8 Gubkina st., Moscow, Russia, 119991

E-mail address: prokhoro@mi.ras.ru, costya.shramov@gmail.com