FINITE GROUPS OF BIRATIONAL SELFMAPS OF THREEFOLDS

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1. Introduction

Finite subgroups of birational automorphism groups are a classical object of study. I. Dolgachev and V. Iskovskikh [DI09] managed to classify all finite subgroups of the birational automorphism group of a plane over an algebraically closed field of characteristic zero. However, for an arbitrary variety X it would be naive to expect any kind of classification for finite subgroups of its birational automorphism group $\mathrm{Bir}(X)$. Thus it is reasonable to find out some general properties that hold for all such subgroups.

As a starting point one can look at common properties of finite subgroups of linear algebraic groups over fields of characteristic zero. The following result is due to H. Minkowski (see e. g. [Ser07, Theorem 5] and [Ser07, §4.3]) and C. Jordan (see [CR62, Theorem 36.13]).

Theorem 1.1. *If $\Bbbk$ is a number field, then there is a constant $B = B(n, \Bbbk)$ such that for any finite subgroup $G \subset \mathrm{GL}_n(\Bbbk)$ one has $|G| \leqslant B$. If $\Bbbk$ is an arbitrary field of characteristic zero, then there is a constant $J = J(n)$ such that for any finite subgroup $G \subset \mathrm{GL}_n(\Bbbk)$ there exists a normal abelian subgroup $A \subset G$ of index at most J.*

This leads to the following definition

Definition 1.2. We say that a group Γ *has bounded finite subgroups* if there exists a constant $B = B(\Gamma)$ such that for any finite subgroup $G \subset \Gamma$ one has $|G| \leqslant B$. A group Γ is called *Jordan* (alternatively, we say that Γ *has Jordan property*) if there is a constant J such that for any finite subgroup $G \subset \Gamma$ there exists a normal abelian subgroup $A \subset G$ of index at most J.

It was noticed by J.-P. Serre that Jordan property sometimes holds for groups of birational automorphisms.

Theorem 1.3 ([Ser09, Theorem 5.3], [Ser10, Théorème 3.1])**.** *The group $\mathrm{Bir}(\mathbb{P}^2)$ over an arbitrary field of characteristic zero is Jordan.*

It appeared that one can generalize Theorem 1.3 to higher dimensions.

Theorem 1.4 (see [PS14, Theorem 1.8] and [Bir16, Theorem 1.1])**.** *Let X be a variety over a field of characteristic zero. Then the following assertions hold.*

(i) *If irregularity of X equals zero, then the group $\mathrm{Bir}(X)$ is Jordan. In particular, this holds if X is rationally connected.*
(ii) *If X is not uniruled, then the group $\mathrm{Bir}(X)$ is Jordan.*
(iii) *If irregularity of X equals zero and X is not uniruled, then the group $\mathrm{Bir}(X)$ has bounded finite subgroups.*

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Finite subgroups of birational automorphism groups of projective spaces and some other varieties were intensively studied from the point of view of their boundedness, see e.g. [PS16a], [PS16b], [Yas16], and references therein. However, Theorem 1.4 tells us nothing about varieties that have a structure of a conic bundle over, say, an abelian variety. To treat this case T. Bandman and Yu. Zarhin proved the following result.

Theorem 1.5 ([BZ15, Corollary 4.8])**.** *Let $\mathbb{K}$ be a field of characteristic zero containing all roots of* 1*. Let C be a conic over $\mathbb{K}$. Assume that C is not $\mathbb{K}$-rational, i.e. that $C(\mathbb{K}) = \varnothing$. Then any finite subgroup of* $\mathrm{Aut}(C)$ *has order at most* 4*.*

The main purpose of this note is to prove a result which is a two-dimensional counterpart of Theorem 1.5.

Theorem 1.6. *Let $\mathbb{K}$ be a field of characteristic zero containing all roots of* 1*, and S be a geometrically rational surface over $\mathbb{K}$. Assume that S is not $\mathbb{K}$-rational but has a $\mathbb{K}$-point. Then* $\mathrm{Bir}_{\mathbb{K}}(S)$ *has bounded finite subgroups.*

It is known (see [Zar14], or Theorem 3.6 below) that there are surfaces whose birational automorphism groups are not Jordan; they are birational to products $E \times \mathbb{P}^1$, where E is an elliptic curve. The following result of V. Popov classifies surfaces with non Jordan birational automorphism groups.

Theorem 1.7 ([Pop11, Theorem 2.32])**.** *Let S be a surface over a field $\Bbbk$ of characteristic zero. Then the group* $\mathrm{Bir}(S)$ *of birational automorphisms of S is Jordan if and only if S is not birational to $E \times \mathbb{P}^1$, where E is an elliptic curve.*

Applying Theorems 1.5 and 1.6, we can immediately obtain a partial generalization of Theorem 1.7 to dimension 3, proving that a threefold with a non Jordan group of birational automorphisms must be birational to a product of $\mathbb{P}^1$ and some surface. However, some additional information about automorphism groups of non uniruled surfaces allows us to give a complete classification of treefolds with non Jordan birational automorphism groups. Namely, we prove the following.

Theorem 1.8. *Let X be a threefold. Then the group* $\mathrm{Bir}(X)$ *is not Jordan if and only if X is birational either to $A \times \mathbb{P}^1$, where A is an abelian surface, or to $S \times \mathbb{P}^1$, where S is a bielliptic surface, or to $E \times \mathbb{P}^1 \times Z$, where E is an elliptic curve and Z is an arbitrary curve.*

The plan of the paper is as follows. In §2 we prove Theorem 1.6. In §3 we collect some auxiliary facts about automorphism groups of minimal non uniruled surfaces. Finally, in §4 we prove Theorem 1.8.

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Until the end of the paper all varieties are assumed to be projective and defined over an algebraically closed field $\Bbbk$ of characteristic zero if the converse is not stated explicitly.

2. Geometrically rational surfaces

In this section we prove Theorem 1.6. Until the end of the section we always assume that $\mathbb{K}$ is a field of characteristic zero that contains all roots of 1.

Recall that a Fano–Mori model S/B of a surface $\bar{S}$ is a smooth surface S birational to $\bar{S}$ endowed with a morphism $\phi\colon S \to B$ with connected fibers, where B is either a curve or a point, and the relative Picard rank $\mathrm{rk}\,\mathrm{Pic}(S/B)$ equals 1.

Definition 2.1. Let S/B and S'/B' be Fano–Mori models of a surface over a field $\mathbb{K}$ and let $\chi\colon S \dashrightarrow S'$ be a $\mathbb{K}$-birational map. We say that χ is *square birational* if either $\dim B = \dim B' = 0$ and χ is an isomorphism, or $\dim B = \dim B' = 1$ and there exists a commutative diagram

$$\begin{array}{ccc} S & \overset{\chi}{\dashrightarrow} & S' \\ \downarrow & & \downarrow \\ B & \xrightarrow{\cong} & B' \end{array}$$

Lemma 2.2. *Let S/B and S'/B' be Fano–Mori models of a surface over a field $\mathbb{K}$. If there exists a square birational map $\chi\colon S/B \dashrightarrow S'/B'$, then $K_S^2 = K_S'^2$.*

Proof. By [Isk96, Theorem 2.6] (case X, $X_1 \in \mathbb{C}$) the map $S \dashrightarrow S'$ is a composition of elementary transformations over $B = B'$. These transformations do not change K_S^2. □

The following results were proved in a series of works [Man74], [Isk67], [Isk70], [Isk73], [Isk80]; see [Isk96] for the modern approach.

Theorem 2.3. *Let S/B be a Fano–Mori model of a geometrically rational surface over a field $\mathbb{K}$. Assume that $S(\mathbb{K}) \neq \varnothing$. Then S is $\mathbb{K}$-rational if and only if $K_S^2 \geqslant 5$.*

Theorem 2.4. *Let S/B and S'/B' be Fano–Mori models of a geometrically rational surface over a field $\mathbb{K}$ and let $\chi\colon S \dashrightarrow S'$ be a $\mathbb{K}$-birational map. Assume that one has $K_S^2 \leqslant 4$, both B and B' are one-dimensional, and $S(\mathbb{K}) \neq \varnothing$. Then $K_S^2 = K_{S'}^2$.*

Proof. If $K_S^2 \leqslant 0$, then χ is square birational and so one has $K_S^2 = K_{S'}^2$, see [Isk96, Proposition 4.2]. If $1 \leqslant K_S^2 \leqslant 4$, then $K_S^2 = K_{S'}^2$ by [Isk96, Theorems 4.6(i) and 4.9]. □

Lemma 2.5. *Let S be a del Pezzo surface over an arbitrary field $\Bbbk$. If $K_S^2 \leqslant 5$, then*

$$|\operatorname{Aut}(S)| \leqslant 696\,729\,600.$$

Proof. We may assume that $\Bbbk$ is algebraically closed. Since $K_S^2 \leqslant 5$, the action of $\operatorname{Aut}(S)$ on $\operatorname{Pic}(S)$ is faithful and so the order of the group $\operatorname{Aut}(S)$ is bounded by the order of the Weyl group $W(E_8)$, see [Dol12, Corollary 8.2.40], [Man67, Theorem 4.5]. □

Now we prove Theorem 1.6.

Proof of Theorem 1.6. Assume that S is not $\mathbb{K}$-rational. Let $G \subset \operatorname{Bir}_{\mathbb{K}}(S)$ be a finite group. Replace S with its G-minimal model [Isk80]. By Theorem 2.3 one has $K_S^2 \leqslant 4$. If S is a del Pezzo surface, then by Lemma 2.5 the order of G is bounded by an absolute constant. From now on we assume that S has a conic bundle structure $f\colon S \to B$.

Denote by $\bar{\mathbb{K}}$ the algebraic closure of $\mathbb{K}$, and for any object $\square$ defined over $\mathbb{K}$ let

$$\bar{\square} = \square \otimes \bar{\mathbb{K}}$$

be the corresponding extension of scalars. Let $\Delta \subset B$ be the discriminant locus of the conic bundle $f\colon S \to B$. Let $\bar{F}_1, \ldots, \bar{F}_n$ be the fibers over $\bar{\Delta}$. Every fiber $\bar{F}_i$ has the form $\bar{F}_i^{(1)} + \bar{F}_i^{(2)}$, where $\bar{F}_i^{(1)}$ and $\bar{F}_i^{(2)}$ are (-1)-curves meeting transversally at one point. Up to permutation we may assume that $\bar{F}_1, \ldots, \bar{F}_m$ are fibers whose irreducible components are $\operatorname{Gal}(\bar{\mathbb{K}}/\mathbb{K})$-conjugate, and $\bar{F}_{m+1}, \ldots, \bar{F}_n$ are ones whose irreducible components are not conjugate. Thus we have a decomposition of Δ into a disjoint union of two G-invariant subsets Δ' and Δ'', where $\Delta' = f(\bar{F}_1 + \ldots + \bar{F}_m)$ and $\Delta'' = f(\bar{F}_{m+1} + \ldots + \bar{F}_n)$. Now run the Minimal Model Program (without group action) on S over B:

$$\begin{array}{ccc} S & \longrightarrow & S' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f'} & \\ & B & \end{array}$$

This means that we contract components of the fibers $\bar{F}_{m+1}, \ldots, \bar{F}_n$ (one in each fiber). We end up with a conic bundle $f'\colon S' \to B$ whose discriminant locus coincides with Δ'. Clearly, one has $K_{S'}^2 \geqslant K_S^2$. Since S' is not $\mathbb{K}$-rational and $S'(\mathbb{K}) \neq \varnothing$, we have $K_{S'}^2 \leqslant 4$ by Theorem 2.3. Hence, one has $m = |\bar{\Delta}'| = 8 - K_{S'}^2 \geqslant 4$.

The group G fits into the following exact sequence

$$1 \longrightarrow G_F \longrightarrow G \longrightarrow G_B \longrightarrow 1,$$

where G_B acts faithfully on the base B, and G_F acts faithfully on the generic fiber of f'. Since S is not $\mathbb{K}$-rational, the generic fiber S'_η of f' has no $\mathbb{K}(B)$-points. Hence by Theorem 1.5 the order of G_F is at most 4. On the other hand, the group G_B preserves the set $\Delta' \subset B$. Therefore its order is bounded by

$$m! = (8 - K_{S'}^2)!$$

Hence

$$|G| \leqslant 4(8 - K_{S'}^2)!$$

By Theorem 2.4 the number $K_{S'}^2$, and thus also the number m, is a birational invariant of S, i.e. it does not depend on the choice of birational model S'. Therefore, the order of G is bounded. □

3. Non-rational surfaces

In this section we collect some results about automorphism groups of non uniruled surfaces. We believe that they are well known to experts, but in some cases we failed to find appropriate references, and thus included the proofs for the reader's convenience.

Lemma 3.1 (cf. [Bea78, Exercise IX.7(1)])**.** *Let S be a minimal surface, and $\phi\colon S \to \mathbb{P}^1$ be an elliptic fibration. Suppose that there exists a fiber F of ϕ such that F_{red} is not a smooth elliptic curve. Then the irregularity of S is zero.*

Proof. Every irreducible component of F is a rational curve, see e.g. [BPVdV84, § V.7]. Hence F is contracted by the Albanese morphism $\alpha\colon S \to \mathrm{Alb}(S)$. Therefore, all other fibers of ϕ are contracted by α as well, which means that α factors through ϕ. Thus the image $\alpha(S)$ is dominated by $\mathrm{Alb}(\mathbb{P}^1)$, which is a point. □

The following result is a version of Mordell–Weil theorem over function fields, known as Lang–Néron theorem; see e.g. [Con06, Theorem 7.1], and also [Con06, §2].

Theorem 3.2. *Let $\phi\colon S \to B$ be an elliptic fibration over a curve, and let $\mathcal{E}$ be the fiber of ϕ over the scheme-theoretic general point of B. Then the group of $\Bbbk(B)$-points of $\mathcal{E}$ is finitely generated, and in particular the torsion subgroup of $\mathrm{Aut}(\mathcal{E})$ is finite, unless S is birational to $B \times E$ over B, where E is a fiber of ϕ.*

We will say that a group *has unbounded finite subgroups* if it fails to have bounded finite subgroups.

Lemma 3.3. *Let S be a smooth minimal surface of Kodaira dimension* 1*. Suppose that the group* $\mathrm{Aut}(S)$ *has unbounded finite subgroups. Then S is birational to a product $B\times E$, where E is an elliptic curve and B is a curve of genus at least* 2*.*

Proof. If S is birational to a product of B and a fiber of ϕ, then $g(B) \geqslant 2$ since the Kodaira dimension of S is 1. Suppose that this is not the case.

Let $\phi\colon S \to B$ be the canonical fibration. The morphism ϕ is $\mathrm{Aut}(S)$-equivariant. Consider the exact sequence

$$1 \to \mathrm{Aut}(S)_\phi \to \mathrm{Aut}(S) \to \Gamma \to 1,$$

where Γ is a subgroup of $\mathrm{Aut}(B)$. The group $\mathrm{Aut}(S)_\phi$ has bounded finite subgroups by Theorem 3.2. Therefore, it is enough to check that Γ has bounded finite subgroups to derive a contradiction. In particular, this holds if $g(B) \geqslant 2$, since the group $\mathrm{Aut}(B)$ is finite in this case. Thus we will assume that $g(B) \leqslant 1$.

Suppose that $g(B) = 1$. If ϕ has at least one multiple or singular fiber, then the group Γ is finite, which gives a contradiction. Thus we may assume that ϕ has no multiple or singular fibers, so that ϕ is a smooth morphism. Since Γ has to be an infinite group, we see that all fibers of ϕ are isomorphic. Hence the surface S is either abelian, or bielliptic, see [BPVdV84, § V.5B]. Both of the latter have Kodaira dimension 0, which again gives a contradiction.

Therefore, we see that $g(B) = 0$. Suppose that ϕ has a fiber F such that F_{red} is not a smooth elliptic curve. Then the irregularity of S equals zero by Lemma 3.1. Since S is not uniruled, Theorem 1.4(iii) implies that the group $\mathrm{Aut}(S)$ has bounded finite subgroups, which is not the case by assumption.

Therefore, we see that all (set-theoretic) fibers of ϕ are smooth elliptic curves; in particular, this applies to set-theoretic fibers F_{red}, where F is a multiple fiber. We may assume that $\Bbbk = \mathbb{C}$. Then the topological Euler characteristic $\chi_{\mathrm{top}}(S)$ equals 0. By the Noether formula one has

$$\chi(\mathscr{O}_S) = \frac{1}{12}\left(K_S^2 + \chi_{\mathrm{top}}(S)\right) = 0.$$

By the canonical bundle formula (see e.g. [BPVdV84, Theorem V.12.1]) we have

$$K_S \sim \phi^*\left(K_B + L + \sum(1 - 1/m_i)P_i\right),$$

where P_i are images of all multiple fibers of ϕ, the fiber $\phi^{-1}(P_i)$ is a multiple fiber of multiplicity m_i, and L is a divisor of degree $\chi(\mathscr{O}_S) = 0$. Since S has Kodaira dimension 1, we see that

$$\deg\left(K_B + L + \sum(1 - 1/m_i)P_i\right) > 0.$$

This implies that $\sum(1 - 1/m_i) \geqslant 2$. Hence ϕ has at least three multiple fibers. This means that the group Γ is finite, which gives a contradiction. □

Lemma 3.4. *Let S be a bielliptic surface, and A be the Albanese variety of S. Then the group* $\mathrm{Aut}(S)$ *contains the group of points of the elliptic curve A.*

Proof. Write $S \cong E \times C/G$, where E and C are elliptic curves, and G acts on C by translations. Then $A \cong C/G$, and a translation given by every element of $g \in C$ commutes with G. □

Lemma 3.5. *Let S be a non uniruled surface. Then the group* $\mathrm{Bir}(S)$ *has unbounded finite subgroups if and only if S is birational either to an abelian surface, or to a bielliptic surface, or to a product $B \times E$ of a curve B of genus $g(B) \geqslant 2$ and an elliptic curve E.*

Proof. By Theorem 1.4(iii) we may assume that the irregularity of S is positive. Replacing S by a minimal model (of its resolution of singularities), we may assume that $\mathrm{Bir}(S) = \mathrm{Aut}(S)$, and S is either an abelian surface, or a bielliptic surface, or a surface of Kodaira dimension 1, or a surface of general type. If S is an abelian surface, then $\mathrm{Aut}(S)$ obviously contains arbitrarily large finite subgroups. If S is a bielliptic surface, $\mathrm{Aut}(S)$ contains arbitrarily large finite subgroups by Lemma 3.4. If S has Kodaira dimension 1, then the assertion follows from Lemma 3.3. Finally, if S is a surface of general type, then $\mathrm{Aut}(S)$ is finite. □

The only source of varieties with non Jordan birational automorphism groups that we are aware of is the following construction of Yu. Zarhin.

Theorem 3.6 ([Zar14])**.** *Let A be a (positive dimensional) abelian variety, and $X \cong A \times \mathbb{P}^1$. Then the group* $\mathrm{Bir}(X)$ *is not Jordan.*

Sketch of the proof. Let L be a very ample line bundle on A, and Y be its total space. Then Y is birational to X. For a given n there is a group $G_L \cong (\mathbb{Z}/n\mathbb{Z})^{2\dim A}$ of points of A such that the corresponding translations preserve L, provided that L is ample enough. Moreover, the group G_L has an extension

$$1 \to \mathbb{Z}/n\mathbb{Z} \to \tilde{G}_L \to G_L \to 1$$

acting on Y, and thus acting by birational automorphisms of X. Moreover, the group $\tilde{G}_L$ does not contain abelian subgroups of index less than n. Going through this construction for arbitrarily large n, one concludes that the group $\mathrm{Bir}(X)$ is not Jordan. We refer the reader to [Zar14] for details. □

Using Theorem 3.6 and keeping in mind Lemma 3.4, we obtain the following result.

Corollary 3.7. *Let X be a variety birational either to a product $A \times \mathbb{P}^1 \times Z$, or to a product $S \times \mathbb{P}^1 \times Z$, where A is a positive dimensional abelian variety, S is a bielliptic surface, and Z is an arbitrary variety. Then the group* $\mathrm{Bir}(X)$ *is not Jordan.*

4. Threefolds

In this section we prove Theorem 1.8.

Definition 4.1 ([PS14, Definition 2.5], [BZ15, Definition 1.1])**.** We say that a group Γ *has finite subgroups of bounded rank* if there exists a constant $R = R(\Gamma)$ such that each finite abelian subgroup $A \subset \Gamma$ is generated by at most R elements.

Lemma 4.2. *Let*

$$1 \to \Gamma' \to \Gamma \to \Gamma''$$

be an exact sequence of groups. Then the following assertions hold.

(i) *If Γ' is Jordan, and Γ'' has bounded finite subgroups, then Γ is Jordan.*
(ii) *If Γ' has bounded finite subgroups, and Γ'' is Jordan and has finite subgroups of bounded rank, then Γ is Jordan.*

Proof. For assertion (i) see [PS14, Lemma 2.3]. For assertion (ii) see [PS14, Lemma 2.8]. □

Proposition 4.3. *Let X be a non uniruled variety. Then* $\mathrm{Bir}(X)$ *has finite subgroups of bounded rank.*

Proof. See the proof of [BZ15, Corollary 3.8], or [PS14, Remark 6.9]. □

Recall that to any variety X one can associate the *maximal rationally connected fibration*

$$\phi_{\mathrm{RC}}\colon X \dashrightarrow X_{\mathrm{nu}},$$

which is a canonically defined rational map with rationally connected fibers and non-uniruled base X_{nu} (see [Kol96, § IV.5], [GHS03, Corollary 1.4]). The maximal rationally connected fibration is equivariant with respect to the group $\mathrm{Bir}(X)$.

Theorem 4.4 ([BZ15, Theorem 1.5])**.** *Let X be a variety, and $\phi\colon X \dashrightarrow Y$ be the maximal rationally connected fibration. Suppose that* $\dim Y = \dim X - 1$*. Then* $\mathrm{Bir}(X)$ *is Jordan unless X is birational to $Y \times \mathbb{P}^1$.*

Corollary 4.5. *Let X be a threefold, and $\phi\colon X \dashrightarrow Y$ be the maximal rationally connected fibration. Suppose that* $\dim Y = 2$*. Then* $\mathrm{Bir}(X)$ *is not Jordan if and only if X is birational to $Y' \times \mathbb{P}^1$, where Y' is either an abelian surface, or a bielliptic surface, or a product of $E \times Z$ of an elliptic curve E and an arbitrary curve Z.*

Proof. By Theorem 4.4 we may assume that X is birational to $Y \times \mathbb{P}^1$. Since ϕ is equivariant with respect to the group $\mathrm{Bir}(X)$, we have an exact sequence

$$1 \to \mathrm{Bir}(X)_\phi \to \mathrm{Bir}(X) \to \mathrm{Bir}(Y),$$

where the action of $\mathrm{Bir}(X)_\phi$ is fiberwise with respect to ϕ. The group $\mathrm{Bir}(X)_\phi$ is a subgroup of $\mathrm{Aut}(\mathbb{P}^1_{\Bbbk(Y)})$, and thus it is Jordan by Theorem 1.4(i).

By construction the surface Y is not uniruled. We know from Lemma 3.5 that $\mathrm{Bir}(Y)$ has bounded finite subgroups unless Y is birational either to an abelian surface, or to a bielliptic surface, or to a product $E \times Z$ of an elliptic curve E and an arbitrary curve Z. If Y is birational to none of the latter surfaces, then the group $\mathrm{Bir}(X)$ is Jordan by Lemma 4.2(i). If on the contrary Y is of one of these three types, then the group $\mathrm{Bir}(X)$ is not Jordan by Corollary 3.7. □

Lemma 4.6. *Let X be a variety, and $\phi\colon X \dashrightarrow Y$ be the maximal rationally connected fibration. Suppose that* $\dim Y = \dim X - 2$*, and that ϕ has a rational section. Then* $\mathrm{Bir}(X)$ *is Jordan unless X is birational to $Y \times \mathbb{P}^2$, and* $\mathrm{Bir}(Y)$ *has unbounded finite subgroups.*

Proof. Let S be the fiber of ϕ over the general scheme-theoretic point of Y. Then S is a geometrically rational surface defined over the field $\mathbb{K} = \Bbbk(Y)$, and S has a smooth $\mathbb{K}$-point by assumption. Since ϕ is equivariant with respect to $\mathrm{Bir}(X)$, we have an exact sequence

$$1 \to \mathrm{Bir}(X)_\phi \to \mathrm{Bir}(X) \to \mathrm{Bir}(Y),$$

where the action of $\mathrm{Bir}(X)_\phi$ is fiberwise with respect to ϕ.

Suppose that X is not birational to $Y \times \mathbb{P}^2$. Thus that S is not rational over $\mathbb{K}$. The group $\mathrm{Bir}(X)_\phi$ is contained in the group $\mathrm{Bir}(S)$, and thus has bounded finite subgroups by Theorem 1.6. On the other hand, the variety Y is not uniruled. Thus the group $\mathrm{Bir}(Y)$ is

Jordan by Theorem 1.4(ii) and has finite subgroups of bounded rank by Proposition 4.3. Hence the group $\mathrm{Bir}(X)$ is Jordan by Lemma 4.2(ii).

Therefore, we see that X is birational to $Y \times \mathbb{P}^2$. The group $\mathrm{Bir}(X)_\phi$ is a subgroup of $\mathrm{Aut}(\mathbb{P}^2_{\Bbbk(Y)})$, and thus it is Jordan by Theorem 1.4(i). This means that if $\mathrm{Bir}(Y)$ has bounded finite subgroups, then the group $\mathrm{Bir}(X)$ is Jordan by Lemma 4.2(i). □

Corollary 4.7. *Let X be a threefold, and $\phi\colon X \dashrightarrow Y$ be the maximal rationally connected fibration. Suppose that $\dim Y = 1$. Then $\mathrm{Bir}(X)$ is not Jordan if and only if X is birational to $Y' \times \mathbb{P}^2$, where Y' is an elliptic curve.*

Proof. The map ϕ has a rational section by [GHS03]. Thus by Lemma 4.6 we may assume that X is birational to $Y \times \mathbb{P}^2$, and the group $\mathrm{Bir}(Y)$ is infinite. Since Y is a non-rational curve, the assertion immediately follows. □

Now we are ready to prove Theorem 1.8.

Proof of Theorem 1.8. Let $\phi\colon X \dashrightarrow Y$ be the maximal rationally connected fibration. If $\dim Y = 0$, then X is rationally connected, so that $\mathrm{Bir}(X)$ is Jordan by Theorem 1.4(i). If $\dim Y = 3$, then X is not uniruled, so that $\mathrm{Bir}(X)$ is Jordan by Theorem 1.4(ii). If $\dim Y = 2$, then the assertion follows from Corollary 4.5. Finally, if $\dim Y = 1$, then the assertion follows from Corollary 4.7. □

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