2-Elementary Subgroups of the Space Cremona Group

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Abstract We give a sharp bound for orders of elementary abelian two-groups of birational automorphisms of rationally connected threefolds.

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1 Introduction

Throughout this paper we work over $\Bbbk$, an algebraically closed field of characteristic 0. Recall that the *Cremona group* $\mathrm{Cr}_n(\Bbbk)$ is the group of birational transformations of the projective space $\mathbb{P}^n_{\Bbbk}$. We are interested in finite subgroups of $\mathrm{Cr}_n(\Bbbk)$. For $n = 2$ these subgroups are classified basically (see [5] and references therein) but for $n \geq 3$ the situation becomes much more complicated. There are only a few, very specific classification results (see e.g. [14, 15, 18]).

Let p be a prime number. A group G is said to be *p-elementary abelian* of rank r if $G \simeq (\mathbb{Z}/p\mathbb{Z})^r$. In this case we denote $\mathrm{r}(G) := r$. A. Beauville [3] obtained a sharp bound for the rank of p-elementary abelian subgroups of $\mathrm{Cr}_2(\Bbbk)$.

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Theorem 1.1 ([3]). *Let $G \subset \mathrm{Cr}_2(\Bbbk)$ be a 2-elementary abelian subgroup. Then $\mathrm{r}(G) \le 4$. Moreover, this bound is sharp and such groups G with $\mathrm{r}(G) = 4$ are classified up to conjugacy in $\mathrm{Cr}_2(\Bbbk)$.*

The author [14] was able to get a similar bound for p-elementary abelian subgroups of $\mathrm{Cr}_3(\Bbbk)$ which is sharp for $p \ge 17$.

In this paper we improve this result in the case $p = 2$. We study 2-elementary abelian subgroups acting on rationally connected threefolds. In particular, we obtain a *sharp* bound for the rank of such subgroups in $\mathrm{Cr}_3(\Bbbk)$. Our main result is the following.

Theorem 1.2. *Let Y be a rationally connected three-dimensional algebraic variety over $\Bbbk$ and let $G \subset \mathrm{Bir}_{\Bbbk}(Y)$ be a 2-elementary abelian group. Then $\mathrm{r}(G) \le 6$.*

Corollary 1.3. *Let $G \subset \mathrm{Cr}_3(\Bbbk)$ be a 2-elementary abelian group. Then $\mathrm{r}(G) \le 6$ and the bound is sharp (see Example 3.4).*

Unfortunately we are not able to classify all the birational actions $G \hookrightarrow \mathrm{Bir}_{\Bbbk}(Y)$ as above attaining the bound $\mathrm{r}(G) \le 6$ (cf. [3]). However, in some cases we get a description of these "extremal" actions.

The structure of the paper is as follows. In Sect. 3 we reduce the problem to the study of biregular actions of 2-elementary abelian groups on Fano-Mori fiber spaces and investigate the case of nontrivial base. A few facts about actions of 2-elementary abelian groups on Fano threefolds are discussed in Sect. 4. In Sect. 5 (resp. Sect. 6) we study actions on non-Gorenstein (resp. Gorenstein) Fano threefolds. Our main theorem is a direct consequence of Propositions 3.2, 5.1, and 6.1.

2 Preliminaries

Notation.

- For a group G, $\mathrm{r}(G)$ denotes the minimal number of generators. In particular, if G is an elementary abelian p-group, then $G \simeq (\mathbb{Z}/p\mathbb{Z})^{r(G)}$.
- $\mathrm{Fix}(G, X)$ (or simply $\mathrm{Fix}(G)$ if no confusion is likely) denotes the fixed point locus of an action of G on X.

Terminal Singularities. Recall that the *index* of a terminal singularity $(X \ni P)$ is a minimal positive integer r such that K_X is a Cartier divisor at P.

Lemma 2.1. *Let $(X \ni P)$ be a germ of a threefold terminal singularity and let $G \subset \mathrm{Aut}(X \ni P)$ be a 2-elementary abelian subgroup. Then $\mathrm{r}(G) \le 4$. Moreover, if $\mathrm{r}(G) = 4$, then $(X \ni P)$ is not a cyclic quotient singularity.*

Proof. Let m be the index of $(X \ni P)$. Consider the index-one cover $\pi\colon (X^\sharp \ni P^\sharp) \to (X \ni P)$ (see [19]). Here $(X^\sharp \ni P^\sharp)$ is a terminal point of index 1 (or smooth) and π is a cyclic cover of degree m which is étale outside of P.

Thus $X \ni P$ is the quotient of $X^\sharp \ni P$ by a cyclic group of order m. If $m = 1$, we take π to be the identity map. We may assume that $\Bbbk = \mathbb{C}$ and then the map $X^\sharp \setminus \{P^\sharp\} \to X \setminus \{P\}$ can be regarded as the topological universal cover. Hence there exists a natural lifting $G^\sharp \subset \mathrm{Aut}(X^\sharp \ni P^\sharp)$ fitting in the following exact sequence

$$1 \longrightarrow C_m \longrightarrow G^\sharp \longrightarrow G \longrightarrow 1, \tag{*}$$

where $C_m \simeq \mathbb{Z}/m\mathbb{Z}$. We claim that $G^\sharp$ is abelian. Assume the converse, then $m \ge 2$. The group $G^\sharp$ permutes the eigenspaces of C_m in the Zariski tangent space $T_{P^\sharp,X^\sharp}$. Let $n := \dim T_{P^\sharp,X^\sharp}$ be the embedded dimension. By the classification of three-dimensional terminal singularities [10, 19] we have one of the following:

$$\begin{aligned}
&(1)\ \tfrac{1}{m}(a,-a,b), \quad n=3, \quad \gcd(a,m)=\gcd(b,m)=1;\\
&(2)\ \tfrac{1}{m}(a,-a,b,0), \quad n=4, \quad \gcd(a,m)=\gcd(b,m)=1;\\
&(3)\ \tfrac{1}{4}(a,-a,b,2), \quad n=4, \quad \gcd(a,2)=\gcd(b,2)=1, \quad m=4,
\end{aligned} \tag{**}$$

where $\frac{1}{m}(a_1,\dots,a_n)$ denotes the diagonal action

$$x_k \longmapsto \exp(2\pi i a_k/m)\cdot x_k, \quad k=1,\dots,n.$$

Put $T = T_{P^\sharp,X^\sharp}$ in the first case and denote by $T \subset T_{P^\sharp,X^\sharp}$ the three-dimensional subspace $x_4 = 0$ in the second and the third cases. Then C_m acts on T freely outside of the origin and T is $G^\sharp$-invariant. By (*) we see that the derived subgroup $[G^\sharp, G^\sharp]$ is contained in C_m. In particular, $[G^\sharp, G^\sharp]$ is abelian and also acts on T freely outside of the origin. Assume that $[G^\sharp, G^\sharp] \ne \{1\}$. Since $\dim T = 3$, this implies that the representation of $G^\sharp$ on T is irreducible (otherwise T has a one-dimensional invariant subspace, say T_1, and the kernel of the map $G^\sharp \to GL(T_1) \simeq \Bbbk^*$ must contain $[G^\sharp, G^\sharp]$). In particular, the eigenspaces of C_m on T have the same dimension. Since T is irreducible, the order of $G^\sharp$ is divisible by $3 = \dim T$ and so $m > 2$. In this case, by the above description of the action of C_m on $T_{P^\sharp,X^\sharp}$ we get that there are exactly three distinct eigenspaces $T_i \subset T$. The action of $G^\sharp$ on the set $\{T_i\}$ induces a transitive homomorphism $G^\sharp \to \mathfrak{S}_3$ whose kernel contains C_m. Hence we have a transitive homomorphism $G \to \mathfrak{S}_3$. Since G is a two-group, this is impossible.

Thus $G^\sharp$ is abelian. Then

$$\mathrm{r}(G) \le \mathrm{r}(G^\sharp) \le \dim T_{P^\sharp,X^\sharp}.$$

This proves our statement. $\square$

Remark 2.2. If in the above notation the action of G on X is free in codimension one, then $\mathrm{r}(G) \le \dim T_{P^\sharp,X^\sharp} - 1$.

For convenience of references, we formulate the following easy result.

Lemma 2.3. *Let G be a 2-elementary abelian group and let X be a G-threefold with isolated singularities.*

(i) *If* $\dim \mathrm{Fix}(G) > 0$, *then* $\dim \mathrm{Fix}(G) + \mathrm{r}(G) \leq 3$.
(ii) *Let $\delta \in G \backslash \{1\}$ and let $S \subset$* $\mathrm{Fix}(\delta)$ *be the union of two-dimensional components. Then S is G-invariant and smooth in codimension* 1.

Sketch of the proof. Consider the action of G on the tangent space to X at a general point of a component of $\mathrm{Fix}(G)$ (resp. at a general point of $\mathrm{Sing}(S)$). □

3 *G*-Equivariant Minimal Model Program

Definition 3.1. Let G be a finite group. A *G-variety* is a variety X provided with a biregular faithful action of G. We say that a normal G-variety X is *$G\mathbb{Q}$-factorial* if any G-invariant Weil divisor on X is $\mathbb{Q}$-Cartier.

The following construction is standard (see e.g. [15]).

Let Y be a rationally connected three-dimensional algebraic variety and let $G \subset \mathrm{Bir}(Y)$ be a finite subgroup. Taking an equivariant compactification and running an equivariant minimal model program we get a G-variety X and a G-equivariant birational map $Y \dashrightarrow X$, where X has a structure a *G-Fano-Mori fibration* $f : X \to B$. This means that X has at worst terminal $G\mathbb{Q}$-factorial singularities, f is a G-equivariant morphism with connected fibers, B is normal, $\dim B < \dim X$, the anticanonical Weil divisor $-K_X$ is ample over B, and the relative G-invariant Picard number $\rho(X)^G$ equals to one. Obviously, in the case $\dim X = 3$ we have the following possibilities:

(C) B is a rational surface and a general fiber $f^{-1}(b)$ is a conic;
(D) $B \simeq \mathbb{P}^1$ and a general fiber $f^{-1}(b)$ is a smooth del Pezzo surface;
(F) B is a point and X is a *$G\mathbb{Q}$-Fano threefold*, that is, X is a Fano threefold with at worst terminal $G\mathbb{Q}$-factorial singularities and such that $\mathrm{Pic}(X)^G \simeq \mathbb{Z}$. In this situation we say that X is *G-Fano threefold* if X is Gorenstein, that is, K_X is a Cartier divisor.

Proposition 3.2. *Let G be a 2-elementary abelian group and let $f : X \to B$ be a G-Fano-Mori fibration with* $\dim X = 3$ *and* $\dim B > 0$. *Then* $\mathrm{r}(G) \leq 6$. *Moreover, if* $\mathrm{r}(G) = 6$ *and $B \simeq \mathbb{P}^1$, then a general fiber $f^{-1}(b)$ is a del Pezzo surface of degree* 4 *or* 8.

Proof. Let $G_f \subset G$ (resp. $G_B \subset \mathrm{Aut}(B)$) be the kernel (resp. the image) of the homomorphism $G \to \mathrm{Aut}(B)$. Thus G_B acts faithfully on B and G_f acts faithfully on the generic fiber $F \subset X$ of f. Clearly, G_f and G_B are 2-elementary groups with $\mathrm{r}(G_f) + \mathrm{r}(G_B) = \mathrm{r}(G)$. Assume that $B \simeq \mathbb{P}^1$. Then $\mathrm{r}(G_B) \leq 2$ by the classification of finite subgroups of $PGL_2(\Bbbk)$. By Theorem 1.1 we have $\mathrm{r}(G_f) \le 4$. If furthermore $\mathrm{r}(G) = 6$, then $\mathrm{r}(G_f) = 4$ and the assertion about F follows by Lemma 3.3 below. This proves our assertions in the case $B \simeq \mathbb{P}^1$. The case $\dim B = 2$ is treated similarly. □

Lemma 3.3 (cf. [3]). *Let F be a del Pezzo surface and let $G \subset \mathrm{Aut}(F)$ be a 2-elementary abelian group with $\mathrm{r}(F) \ge 4$. Then $\mathrm{r}(F) = 4$ and one of the following holds:*

(i) $K_F^2 = 4$, $\rho(F)^G = 1$;
(ii) $K_F^2 = 8$, $\rho(F)^G = 2$.

Proof. Similar to [3, §3]. □

Example 3.4. Let $F \subset \mathbb{P}^4$ be the quartic del Pezzo surface given by $\sum x_i^2 = \sum \lambda_i x_i^2 = 0$ with $\lambda_i \ne \lambda_j$ for $i \ne j$ and let $G_f \subset \mathrm{Aut}(F)$ be the 2-elementary abelian subgroup generated by involutions $x_i \mapsto -x_i$. Consider also a 2-elementary abelian subgroup $G_B \subset \mathrm{Aut}(\mathbb{P}^1)$ induced by a faithful representation $\mathrm{Q}_8 \to GL_2(\Bbbk)$ of the quaternion group Q_8. Then $\mathrm{r}(G_f) = 4$, $\mathrm{r}(G_B) = 2$, and $G := G_f \times G_B$ naturally acts on $X := F \times \mathbb{P}^1$. Two projections give us two structures of G-Fano-Mori fibrations of types (D) and (C). This shows that the bound $\mathrm{r}(G) \le 6$ in Proposition 3.2 is sharp. Moreover, X is rational and so we have an embedding $G \subset \mathrm{Cr}_3(\Bbbk)$.

4 Actions on Fano Threefolds

Main Assumption. From now on we assume that we are in the case (F), that is, X is a $G\mathbb{Q}$-Fano threefold.

Remark 4.1. The group G acts naturally on the space of anticanonical sections $H^0(X, -K_X)$. Assume that $H^0(X, -K_X) \ne 0$. Since G is an abelian group, there exists a decomposition if $H^0(X, -K_X)$ into eigenspaces. Then any eigensection $s \in H^0(X, -K_X)$ defines an invariant member $S \in |-K_X|$.

Lemma 4.2. *Let X be a $G\mathbb{Q}$-Fano threefold, where G is a 2-elementary abelian group with $\mathrm{r}(G) \ge 5$. Let S be an invariant effective Weil divisor such that $-(K_X + S)$ is nef. Then the pair (X, S) is log canonical (lc). In particular, S is reduced. If $-(K_X + S)$ is ample, then the pair (X, S) is purely log terminal (plt).*

Proof. Assume that the pair (X, S) is not lc. Since S is G-invariant and $\rho(X)^G = 1$, we see that S is numerically proportional to K_X. Hence S is ample. We apply quite standard connectedness arguments of Shokurov [22] (see, e.g., [11, Prop. 2.6]): for a suitable G-invariant boundary D, the pair (X, D) is lc, the divisor $-(K_X + D)$ is ample, and the minimal locus V of log canonical singularities is also G-invariant. Moreover, V is either a point or a smooth rational curve.

By Lemma 2.1 we may assume that G has no fixed points. Hence, $V \simeq \mathbb{P}^1$ and we have a map $\varsigma : G \to \mathrm{Aut}(\mathbb{P}^1)$. By Lemma 2.3 $\mathrm{r}(\ker \varsigma) \le 2$. Therefore, $\mathrm{r}(\varsigma(G)) \ge 3$. This contradicts the classification of finite subgroups of $PGL_2(\Bbbk)$.

If $-(K_X + S)$ is ample and (X, S) has a log canonical center of dimension ≤ 1, then by considering $(X, S' = S + \epsilon B)$, where B is a suitable invariant divisor and $0 < \epsilon \ll 1$, we get a non-lc pair (X, S'). This contradicts the above considered case. □

Corollary 4.3. *Let X be a $G\mathbb{Q}$-Fano threefold, where G is a 2-elementary abelian group with $\mathrm{r}(G) \ge 6$ and let S be an invariant Weil divisor. Then $-(K_X + S)$ is not ample.*

Proof. If $-(K_X + S)$ is ample, then by Lemma 4.2 the pair (X, S) is plt. By the adjunction principle [22] the surface S is irreducible, normal and has only quotient singularities. Moreover, $-K_S$ is ample. Hence S is rational. We get a contradiction by Theorem 1.1 and Lemma 2.3(i). □

Lemma 4.4. *Let S be a K3 surface with at worst Du Val singularities and let $\Gamma \subset \mathrm{Aut}(S)$ be a 2-elementary abelian group. Then $\mathrm{r}(\Gamma) \le 5$.*

Proof. Let $\tilde{S} \to S$ be the minimal resolution. Here $\tilde{S}$ is a smooth K3 surface and the action of Γ lists to $\tilde{S}$. Let $\Gamma_{\mathrm{s}} \subset \Gamma$ be the largest subgroup that acts trivially on $H^{2,0}(\tilde{S}) \simeq \mathbb{C}$. The group $\Gamma/\Gamma_{\mathrm{s}}$ is cyclic. Hence, $\mathrm{r}(\Gamma/\Gamma_{\mathrm{s}}) \le 1$. According to [13, Th. 4.5] we have $\mathrm{r}(\Gamma_{\mathrm{s}}) \le 4$. Thus $\mathrm{r}(\Gamma) \le 5$. □

Corollary 4.5. *Let X be a $G\mathbb{Q}$-Fano threefold, where G is a 2-elementary abelian group. Let $S \in |-K_X|$ be a G-invariant member. If $\mathrm{r}(G) \ge 7$, then the singularities of S are worse than Du Val.*

Proposition 4.6. *Let X be a $G\mathbb{Q}$-Fano threefold, where G is a 2-elementary abelian group with $\mathrm{r}(G) \ge 6$. Let $S \in |-K_X|$ be a G-invariant member and let $G_\bullet \subset G$ be the largest subgroup that acts trivially on the set of components of S. One of the following holds:*

- (i) *S is a K3 surface with at worst Du Val singularities, then $S \subset \mathrm{Fix}(\delta)$ for some $\delta \in G \setminus \{1\}$ and $G/\langle\delta\rangle$ faithfully acts on S. In this case $\mathrm{r}(G) = 6$.*
- (ii) *The surface S is reducible (and reduced). The group G acts transitively on the components of S and $G_\bullet$ acts faithfully on each component $S_i \subset S$. There are two subcases:*
  - (a) *any component $S_i \subset S$ is rational and $\mathrm{r}(G_\bullet) \le 4$.*
  - (b) *any component $S_i \subset S$ is birationally ruled over an elliptic curve and $\mathrm{r}(G_\bullet) \le 5$.*

Proof. By Lemma 4.2 the pair (X, S) is lc. Assume that S is normal (and irreducible). By the adjunction formula $K_S \sim 0$. We claim that S has at worst Du Val singularities. Indeed, otherwise by the Connectedness Principle [22, Th. 6.9] S

has at most two non-Du Val points. These points are fixed by an index two subgroup $G' \subset G$. This contradicts Lemma 2.1. Taking Lemma 4.4 into account we get the case (i).

Now assume that S is not normal. Let $S_i \subset S$ be an irreducible component (the case $S_i = S$ is not excluded). If the action on components $S_i \subset S$ is not transitive, there is an invariant divisor $S' < S$. Since X is $G\mathbb{Q}$-factorial and $\rho(X)^G = 1$, the divisor $-(K_X + S')$ is ample. This contradicts Corollary 4.3.

By Lemma 2.3(ii) the action of $G_\bullet$ on each component S_i is faithful.

If S_i is a rational surface, then $\mathrm{r}(G_\bullet) \le 4$ by Theorem 1.1. Assume that S_i is not rational. Let $\nu\colon S' \to S_i$ be the normalization. Write $0 \sim \nu^*(K_X + S) = K_{S'} + D'$, where D' is the *different*, see [22, §3]. Here D' is an effective reduced divisor and the pair is lc [22, 3.2]. Since S is not normal, $D' \neq 0$. Consider the minimal resolution $\mu\colon \tilde{S} \to S'$ and let $\tilde{D}$ be the crepant pull-back of D', that is, $\mu_* \tilde{D} = D'$ and

$$K_{\tilde{S}} + \tilde{D} = \mu^*(K_{S'} + D') \sim 0.$$

Here $\tilde{D}$ is again an effective reduced divisor. Hence $\tilde{S}$ is a ruled surface. Consider the Albanese map $\alpha : \tilde{S} \to C$. Let $\tilde{D}_1 \subset \tilde{D}$ be an α-horizontal component. By the adjunction formula $\tilde{D}_1$ is an elliptic curve and so C is. Let Γ be the image of $G_\bullet$ in $\mathrm{Aut}(C)$. Then $\mathrm{r}(\Gamma) \le 3$ and so $\mathrm{r}(G_\bullet) \le 5$. So, the last assertion is proved. □

5 Non-Gorenstein Fano Threefolds

Let G be a 2-elementary abelian group and let X be $G\mathbb{Q}$-Fano threefold. In this section we consider the case where X is non-Gorenstein, i.e., it has at least one terminal point of index > 1. We denote by $\mathrm{Sing}'(X) = \{P_1, \dots, P_M\}$ the set of non-Gorenstein points.

Recall that any (analytic) threefold terminal singularity $U \ni P$ has a small deformation U_t, where $t \in$ (unit disk) $\subset \mathbb{C}$, such that for $0 < |t| \ll 1$ the threefold $U_t \ni P_{i,t}$ has only cyclic quotient singularities $U_t \ni P_{i,t}$ of the form $\frac{1}{m_i}(1, -1, a_i)$ with $\gcd(m_i, a_i) = 1$ [19]. The collection $\mathbf{B}(U, P) := \left\{\frac{1}{m_i}(1, -1, a_i)\right\}$ does not depend on the choice of deformation and called the *basket* of $U \ni P$. For a threefold X with terminal singularities we denote by $\mathbf{B} = \mathbf{B}(X)$ its *global basket*, the union of baskets of all singular points.

Proposition 5.1. *Let X be a non-Gorenstein Fano threefold with terminal singularities. Assume that X admits a faithful action of a 2-elementary abelian group G with* $\mathrm{r}(G) \ge 6$. *Then* $\mathrm{r}(G) = 6$, *G transitively acts on* $\mathrm{Sing}'(X)$, $|-K_X| \neq \emptyset$, *and the configuration of non-Gorenstein singularities is described below.*

(1) $M = 8$, $\mathbf{B}(X, P_i) = \left\{\frac{1}{2}(1, 1, 1)\right\}$;
(2) $M = 8$, $\mathbf{B}(X, P_i) = \left\{\frac{1}{3}(1, 1, 2)\right\}$;
(3) $M = 4$, $\mathbf{B}(X, P_i) = \left\{2 \times \frac{1}{2}(1, 1, 1)\right\}$;

(4) $M = 4$, $\mathbf{B}(X, P_i) = \left\{2 \times \frac{1}{3}(1,1,2)\right\}$;
(5) $M = 4$, $\mathbf{B}(X, P_i) = \left\{3 \times \frac{1}{2}(1,1,1)\right\}$;
(6) $M = 4$, $\mathbf{B}(X, P_i) = \left\{\frac{1}{4}(1,-1,1),\ \frac{1}{2}(1,1,1)\right\}$.

Proof. Let $P^{(1)}, \dots, P^{(n)} \in \operatorname{Sing}'(X)$ be representatives of distinct G-orbits and let G_i be the stabilizer of $P^{(i)}$. Let $r := \mathrm{r}(G)$, $r_i := \mathrm{r}(G_i)$, and let $m_{i,1}, \dots, m_{i,\nu_i}$ be the indices of points in the basket of $P^{(i)}$. We may assume that $m_{i,1} \ge \dots \ge m_{i,\nu_i}$ By the orbifold Riemann–Roch formula [19] and a form of Bogomolov–Miyaoka inequality [8, 9] we have

$$\sum_{i=1}^{n} 2^{r-r_i} \sum_{j=1}^{\nu_i} \left(m_{i,j} - \frac{1}{m_{i,j}}\right) < 24. \tag{***}$$

If P is a cyclic quotient singularity, then $\nu_i = 1$ and by Lemma 2.1 $r_i \le 3$. If P is not a cyclic quotient singularity, then $\nu_i \ge 2$ and again by Lemma 2.1 $r_i \le 4$. Since $m_{i,j} - 1/m_{i,j} \ge 3/2$, in both cases we have

$$2^{r-r_i} \sum_{j=1}^{\nu_i} \left(m_{i,j} - \frac{1}{m_{i,j}}\right) \ge 3 \cdot 2^{r-4} \ge 12.$$

Therefore, $n = 1$, i.e., G transitively acts on $\operatorname{Sing}'(X)$, and $r = 6$.

If P is not a point of type $cAx/4$ (i.e., it is not as in (3) of (**)), then by the classification of terminal singularities [19] $m_{1,1} = \dots = m_{1,\nu_i}$ and (***) has the form

$$24 > 2^{6-r_1}\nu_1 \left(m_{1,1} - \frac{1}{m_{1,1}}\right) \ge 8\left(m_{1,1} - \frac{1}{m_{1,1}}\right).$$

Hence $r_1 \ge 3$, $\nu_1 \le 3$, $m_{1,1} \le 3$, and $3 \cdot 2^{r_1-3} \ge \nu_1 m_{1,1}$. If $r_1 = 3$, then $\nu_1 = 1$. If $r_1 = 4$, then $\nu_1 \ge 2$ and $\nu_1 m_{1,1} \le 6$. This gives us the possibilities (1)–(5).

Assume that P is a point of type $cAx/4$. Then $m_{1,1} = 4$, $\nu_1 > 1$, and $m_{1,j} = 2$ for $1 < j \le \nu_1$. Thus (***) has the form

$$24 > 2^{6-r_1}\left(\frac{15}{4} + \frac{3}{2}(\nu_1 - 1)\right) = 2^{4-r_1}\left(9 + 6\nu_1\right).$$

We get $\nu_1 = 2$, $r_1 = 4$, i.e., the case (6).

Finally, the computation of $\dim|-K_X|$ follows by the orbifold Riemann–Roch formula [19]

$$\dim|-K_X| = -\frac{1}{2}K_X^3 + 2 - \sum_{P \in \mathbf{B}(X)} \frac{b_P(m_P - b_P)}{2m_P}.$$

6 Gorenstein Fano Threefolds

The main result of this section is the following:

Proposition 6.1. *Let G be a 2-elementary abelian group and let X be a (Gorenstein) G-Fano threefold. Then* $\mathrm{r}(G) \le 6$. *Moreover, if* $\mathrm{r}(G) = 6$, *then* $\mathrm{Pic}(X) = \mathbb{Z} \cdot K_X$ *and* $-K_X^3 \ge 8$.

Let X be a Fano threefold with at worst Gorenstein terminal singularities. Recall that the number

$$\iota(X) := \max\{i \in \mathbb{Z} \mid -K_X \sim iA,\ A \in \mathrm{Pic}(X)\}$$

is called the *Fano index* of X. The integer $g = \mathrm{g}(X)$ such that $-K_X^3 = 2g - 2$ is called the *genus* of X. It is easy to see that $\dim |-K_X| = g+1$ [7, Corollary 2.1.14]. In particular, $|-K_X| \ne \emptyset$.

Notation. Throughout this section G denotes a 2-elementary abelian group and X denotes a Gorenstein G-Fano threefold. There exists an invariant member $S \in |-K_X|$ (see 4.1). We write $S = \sum_{i=1}^N S_i$, where the S_i are irreducible components. Let $G_\bullet \subset G$ be the kernel of the homomorphism $G \to \mathfrak{S}_N$ induced by the action of G on $\{S_1, \dots, S_N\}$. Since G is abelian and the action of G on $\{S_1, \dots, S_N\}$ is transitive, the group $G_\bullet$ coincides with the stabilizer of any S_i. Clearly, $N = 2^{\mathrm{r}(G) - \mathrm{r}(G_\bullet)}$. If $\mathrm{r}(G) \ge 6$, then by Proposition 4.6 we have $\mathrm{r}(G_\bullet) \le 5$ and so $N \ge 2^{\mathrm{r}(G)-5}$.

Lemma 6.2. *Let* $G \subset \mathrm{Aut}(\mathbb{P}^n)$ *be a 2-elementary subgroup and n is even. Then G is conjugate to a diagonal subgroup. In particular,* $\mathrm{r}(G) \le n$.

Proof. Let $G^\sharp \subset SL_{n+1}(\Bbbk)$ be the lifting of G and let $G' \subset G^\sharp$ be a Sylow two-subgroup. Then $G' \simeq G$. Since G' is abelian, the representation $G' \hookrightarrow SL_{n+1}(\Bbbk)$ is diagonalizable. □

Corollary 6.3. *Let* $Q \subset \mathbb{P}^4$ *be a quadric and let* $G \subset \mathrm{Aut}(Q)$ *be a 2-elementary subgroup. Then* $\mathrm{r}(G) \le 4$.

Lemma 6.4. *Let* $G \subset \mathrm{Aut}(\mathbb{P}^3)$ *be a 2-elementary subgroup. Then* $\mathrm{r}(G) \le 4$.

Certainly, the fact follows by Blichfeldt's theorem which asserts that the lifting $G^\sharp \subset SL_4(\Bbbk)$ is a monomial representation (see e.g. [20, §3]). Here we give a short independent proof.

Proof. Assume that $\mathrm{r}(G) \ge 5$. Take any element $\delta \in G \setminus \{1\}$. By Lemma 2.1 the group G has no fixed points. Since the set $\mathrm{Fix}(\delta)$ is G-invariant, $\mathrm{Fix}(\delta) = L_1 \cup L_2$, where $L_1,\ L_2 \subset \mathbb{P}^3$ are skew lines.

Let $G_1 \subset G$ be the stabilizer of L_1. There is a subgroup $G_2 \subset G_1$ of index 2 having a fixed point $P \in L_1$. Thus $\mathrm{r}(G_2) \ge 3$ and the "orthogonal" plane Π is G_2-invariant. By Lemma 6.2 there exists an element $\delta' \in G_2$ that acts trivially on Π, i.e., $\Pi \subset \mathrm{Fix}(\delta')$. But then δ' has a fixed point, a contradiction. □

Lemma 6.5. *If* $\mathrm{Bs}|-K_X| \neq \emptyset$*, then* $\mathrm{r}(G) \leq 4$.

Proof. By Shin [21] the base locus $\mathrm{Bs}|-K_X|$ is either a single point or a rational curve. In both cases $\mathrm{r}(G) \leq 4$ by Lemmas 2.1 and 2.3. □

Lemma 6.6. *If* $-K_X$ *is not very ample, then* $\mathrm{r}(G) \leq 5$.

Proof. Assume that $\mathrm{r}(G) \geq 6$. By Lemma 6.5 the linear system $|-K_X|$ is base point free. It is easy to show that $|-K_X|$ defines a double cover $\phi : X \to Y \subset \mathbb{P}^{g+1}$ (cf. [6, Chap. 1, Prop. 4.9]). Here Y is a variety of degree $g-1$ in $\mathbb{P}^{g+1}$, a variety of minimal degree. Let $\bar{G}$ be the image of G in $\mathrm{Aut}(Y)$. Then $\mathrm{r}(\bar{G}) \geq \mathrm{r}(G) - 1$. If $g = 2$ (resp. $g = 3$), then $Y = \mathbb{P}^3$ (resp. $Y \subset \mathbb{P}^4$ is a quadric) and $\mathrm{r}(G) \leq 5$ by Lemma 6.4 (resp. by Corollary 6.3). Thus we may assume that $g \geq 4$. If Y is smooth, then according to the Enriques theorem (see, e.g., [6, Th. 3.11]) Y is a rational scroll $\mathbb{P}_{\mathbb{P}^1}(\mathscr{E})$, where $\mathscr{E}$ is a rank 3 vector bundle on $\mathbb{P}^1$. Then X has a G-equivariant projection to a curve. This contradicts $\rho(X)^G = 1$. Hence Y is singular. In this case, Y is a projective cone (again by the Enriques theorem). If its vertex $O \in Y$ is zero-dimensional, then $\dim T_{O,Y} \geq 5$. On the other hand, X has only hypersurface singularities. Therefore the double cover $X \to Y$ is not étale over O and so G has a fixed point on X. This contradicts Lemma 2.1. Thus Y is a cone over a curve with vertex along a line L. As above, L must be contained in the branch divisor and so $L' := \phi^{-1}(L)$ is a G-invariant rational curve. Since the image of G in $\mathrm{Aut}(L')$ is a 2-elementary abelian group of rank ≤ 2, by Lemma 2.3 we have $\mathrm{r}(G) \leq 4$. □

Remark 6.7. Recall that for a Fano threefold X with at worst Gorenstein terminal singularities one has $\iota(X) \leq 4$. Moreover, $\iota(X) = 4$ if and only if $X \simeq \mathbb{P}^3$ and $\iota(X) = 3$ if and only if X is a quadric in $\mathbb{P}^4$ [7]. In these cases we have $\mathrm{r}(G) \leq 4$ by Lemma 6.4 and Corollary 6.3, respectively. If $\iota(X) = 2$, then X is so-called *del Pezzo threefold*. The number $d := (-\frac{1}{2}K_X)^3$ is called the *degree* of X.

Lemma 6.8. *Assume that the divisor* $-K_X$ *is very ample,* $\mathrm{r}(G) \geq 6$*, and the action of* G *on* X *is not free in codimension* 1*. Let* $\delta \in G$ *be an element such that* $\dim \mathrm{Fix}(\delta) = 2$ *and let* $D \subset \mathrm{Fix}(\delta)$ *be the union of all two-dimensional components. Then* $\mathrm{r}(G) = 6$ *and* D *is a Du Val member of* $|-K_X|$*. Moreover,* $\iota(X) = 1$ *except, possibly, for the case where* $\iota(X) = 2$ *and* $-\frac{1}{2}K_X$ *is not very ample.*

Proof. Since G is abelian, $\mathrm{Fix}(\delta)$ and D are G-invariant and so $-K_X \sim_{\mathbb{Q}} \lambda D$ for some $\lambda \in \mathbb{Q}$. In particular, D is a $\mathbb{Q}$-Cartier divisor. Since X has only terminal Gorenstein singularities, D must be Cartier. Clearly, D is smooth outside of $\mathrm{Sing}(X)$. Further, D is ample and so it must be connected. Since D is a reduced Cohen–Macaulay scheme with $\dim \mathrm{Sing}(D) \leq 0$, it is irreducible and normal.

Let $X \hookrightarrow \mathbb{P}^{g+1}$ be the anticanonical embedding. The action of δ on X is induced by an action of a linear involution of $\mathbb{P}^{g+1}$. There are two disjointed linear subspaces $V_+, V_- \subset \mathbb{P}^{g+1}$ of δ-fixed points and the divisor D is contained in one of them. This means that D is a component of a hyperplane section $S \in |-K_X|$ and so $\lambda \geq 1$.

Since $\mathrm{r}(G) \geq 6$, by Corollary 4.3 we have $\lambda = 1$ and $-K_X \sim D$ (because $\mathrm{Pic}(X)$ is a torsion free group). Since D is irreducible, the case (i) of Proposition 4.6 holds.

Finally, if $\iota(X) > 1$, then by Remark 6.7 we have $\iota(X) = 2$. If furthermore the divisor A is very ample, then it defines an embedding $X \hookrightarrow \mathbb{P}^N$ so that D spans $\mathbb{P}^N$. In this case the action of δ must be trivial, a contradiction. □

Lemma 6.9. *If* $\rho(X) > 1$*, then* $\mathrm{r}(G) \leq 5$.

Proof. We use the classification of G-Fano threefolds with $\rho(X) > 1$ [17]. By this classification $\rho(X) \leq 4$. Let G_0 be the kernel of the action of G on $\mathrm{Pic}(X)$.

Consider the case $\rho(X) = 2$. Then $[G : G_0] = 2$. In the cases (1.2.1) and (1.2.4) of [17] the variety X has a structure of G_0-equivariant conic bundle over $\mathbb{P}^2$. As in Proposition 3.2 we have $\mathrm{r}(G_0) \leq 4$ and $\mathrm{r}(G) \leq 5$ in these cases. In the cases (1.2.2) and (1.2.3) of [17] the variety X has two birational contractions to $\mathbb{P}^3$ and a quadric $Q \subset \mathbb{P}^4$, respectively. As above we get $\mathrm{r}(G) \leq 5$ by Lemma 6.4 and Corollary 6.3.

Consider the case $\rho(X) = 3$. We show that in this case $\mathrm{Pic}(X)^G \not\simeq \mathbb{Z}$ (and so this case does not occur). Since G is a 2-elementary abelian group, its action on $\mathrm{Pic}(X) \otimes \mathbb{Q}$ is diagonalizable. Since, $\mathrm{Pic}(X)^G = \mathbb{Z} \cdot K_X$, the group G contains an element τ that acts on $\mathrm{Pic}(X) \simeq \mathbb{Z}^3$ as the reflection with respect to the orthogonal complement to K_X. Since the group G preserves the natural bilinear form $\langle \mathbf{x}_1, \mathbf{x}_2 \rangle := \mathbf{x}_1 \cdot \mathbf{x}_2 \cdot K_X$, the action must be as follows:

$$\tau : \mathbf{x} \longmapsto \mathbf{x} - \lambda K_X, \qquad \lambda = \frac{2\mathbf{x} \cdot K_X^2}{K_X^3}.$$

Hence λK_X is an integral element for any $\mathbf{x} \in \mathrm{Pic}(X)$. This gives a contradiction in all cases (1.2.5)–(1.2.7) of [17, Th. 1.2]. For example, in the case (1.2.5) of [17, Th. 1.2] our variety X has a structure (non-minimal) del Pezzo fibration of degree 4 and $-K_X^3 = 12$. For the fiber F we have $F \cdot K_X^2 = K_F^2 = 4$ and λK_X is not integral, a contradiction.

Finally, consider the case $\rho(X) = 4$. Then according to [17] X is a divisor of multidegree $(1, 1, 1, 1)$ in $(\mathbb{P}^1)^4$. All the projections $\varphi_i : X \to \mathbb{P}^1$, $i = 1, \dots, 4$ are G_0-equivariant. We claim that natural maps $\varphi_{i*} : G_0 \to \mathrm{Aut}(\mathbb{P}^1)$ are injective. Indeed, assume that $\varphi_{1*}(\vartheta)$ is the identity map in $\mathrm{Aut}(\mathbb{P}^1)$ for some $\vartheta \in G$. This means that $\vartheta \circ \varphi_1 = \varphi_1$. Since $\mathrm{Pic}(X)^G = \mathbb{Z}$, the group G permutes the classes $\varphi_i^* \mathscr{O}_{\mathbb{P}^1}(1) \in \mathrm{Pic}(X)$. Hence, for any $i = 1, \dots, 4$, there exists $\sigma_i \in G$ such that $\varphi_i = \varphi_1 \circ \sigma_i$. Then

$$\vartheta \circ \varphi_i = \vartheta \circ \varphi_1 \circ \sigma_i = \varphi_1 \circ \sigma_i = \varphi_i.$$

Hence, $\varphi_{i*}(\vartheta)$ is the identity for any i. Since $\varphi_1 \times \cdots \times \varphi_4$ is an embedding, ϑ must be the identity as well. This proves our clam. Therefore, $\mathrm{r}(G_0) \leq 2$. The group G/G_0 acts on $\mathrm{Pic}(X)$ faithfully. By the same reason as above, an element of G/G_0 cannot act as the reflection with respect to K_X. Therefore, $\mathrm{r}(G/G_0) \leq 2$ and $\mathrm{r}(G) \leq 4$. □

Now we consider the case of del Pezzo threefolds.

Lemma 6.10. *If* $\iota(X) = 2$, *then* $\mathrm{r}(G) \le 5$.

Proof. By Lemma 6.9 we may assume that $\rho(X) = 1$. Let $A := -\frac{1}{2}K_X$ and let $d := A^3$ be the degree of X. Since $\rho(X) = 1$, we have $d \le 5$ (see e.g. [16]). Consider the possibilities for d case by case. We use the classification (see [21] and [16]).

If $d = 1$, then the linear system $|A|$ has a unique base point. This point is smooth and must be G-invariant. By Lemma 2.1 $\mathrm{r}(G) \le 3$. If $d = 2$, then the linear system $|A|$ defines a double cover $\varphi : X \to \mathbb{P}^3$. Then the image of G in $\mathrm{Aut}(\mathbb{P}^3)$ is a 2-elementary group $\bar{G}$ with $\mathrm{r}(\bar{G}) \ge \mathrm{r}(G) - 1$, where $\mathrm{r}(\bar{G}) \le 4$ by Lemma 6.4. If $d = 3$, then $X = X_3 \subset \mathbb{P}^4$ is a cubic hypersurface. By Lemma 6.2 $\mathrm{r}(G) \le 4$. If $d = 5$, then X is smooth, unique up to isomorphism, and $\mathrm{Aut}(X) \simeq PGL_2(\Bbbk)$ (see [7]).

Finally, consider the case $d = 4$. Then $X = Q_1 \cap Q_2 \subset \mathbb{P}^5$ is an intersection of two quadrics (see e.g. [21]). Let $\mathscr{Q}$ be the pencil generated by Q_1 and Q_2. Since X has a isolated singularities and it is not a cone, a general member of $\mathscr{Q}$ is smooth by Bertini's theorem and for any member $Q \in \mathscr{Q}$ we have $\dim \mathrm{Sing}(Q) \le 1$. Let D be the divisor of degree 6 on $\mathscr{Q} \simeq \mathbb{P}^1$ given by the vanishing of the determinant. The elements of $\mathrm{Supp}(D)$ are exactly degenerate quadrics. Clearly, for any point $P \in \mathrm{Sing}(X)$ there exists a unique quadric $Q \in \mathscr{Q}$ which is singular at P. This defines a map $\pi : \mathrm{Sing}(X) \to \mathrm{Supp}(D)$. Let $Q \in \mathrm{Supp}(D)$. Then $\pi^{-1}(Q) = \mathrm{Sing}(Q) \cap X = \mathrm{Sing}(Q) \cap Q'$, where $Q' \in \mathscr{Q}$, $Q' \ne Q$. In particular, $\pi^{-1}(Q)$ consists of at most two points. Hence the cardinality of $\mathrm{Sing}(X)$ is at most 12.

Assume that $\mathrm{r}(G) \ge 6$. Let $S \in |-K_X|$ be an invariant member. We claim that $S \supset \mathrm{Sing}(X)$ and $\mathrm{Sing}(X) \ne \emptyset$. Indeed, otherwise $S \cap \mathrm{Sing}(X) = \emptyset$. By Proposition 4.6 S is reducible: $S = S_1 + \cdots + S_N$, $N \ge 2$. Since $\iota(X) = 2$, we get $N = 2$ and $S_1 \sim S_2$, i.e., S_i is a hyperplane section of $X \subset \mathbb{P}^5$. As in the proof of Corollary 4.3 we see that S_i is rational. This contradicts Proposition 4.6 (ii). Thus $\emptyset \ne \mathrm{Sing}(X) \subset S$. By Lemma 6.8 the action of G on X is free in codimension 1. By Remark 2.2 for the stabilizer G_P of a point $P \in \mathrm{Sing}(X)$ we have $\mathrm{r}(G_P) \le 3$. Then by the above estimate the variety X has exactly 8 singular points and G acts on $\mathrm{Sing}(X)$ transitively.

Note that our choice of S is not unique: there is a basis $s^{(1)}, \dots, s^{(g+2)} \in H^0(X, -K_X)$ consisting of eigensections. This basis gives us G-invariant divisors $S^{(1)}, \dots, S^{(g+2)}$ generating $|-K_X|$. By the above $\mathrm{Sing}(X) \subset S^{(i)}$ for all i. Thus $\mathrm{Sing}(X) \subset \cap S^{(i)} = \mathrm{Bs}|-K_X|$. This contradicts the fact that $-K_X$ is very ample.

□

The following two examples show that the inequality $\mathrm{r}(G) \le 5$ in the above lemma is sharp.

Example 6.11. Let $X = X_{2\cdot 2} \subset \mathbb{P}^5$ be the variety given by $\sum x_i^2 = \sum \lambda_i x_i^2 = 0$ with $\lambda_i \ne \lambda_j$ for $i \ne j$ and let $G \subset \mathrm{Aut}(X)$ be the 2-elementary abelian subgroup generated by involutions $x_i \mapsto -x_i$. Then X is a *rational* del Pezzo threefold of degree 4 and $\mathrm{r}(G) = 5$.

Example 6.12 (suggested by the referee). Let A be the Jacobian of a curve of genus 2 and let Θ be its theta-divisor. The linear system $|2\Theta|$ defines a finite morphism $\alpha : A \to B \subset \mathbb{P}^3$ of degree 2 whose image $B = \alpha(A)$ is a quartic with 16 nodes [2, Chap. VIII, Exercises]. Let $\varphi : X \to \mathbb{P}^3$ be the double cover branched along B. Then X is a del Pezzo threefold of degree 2 whose singular locus consists of 16 nodes. In this situation, the rank of the Weil divisor class group $\mathrm{Cl}(X)$ equals to 7 (see [16, Th. 7.1]) and X has a small resolution which can be obtained by blowing up of six points in general position on $\mathbb{P}^3$ (see e.g. [4, 23, Chap. 3] or [16, Th. 7.1]). In particular, X is rational. The translation by a two-torsion point $a \in A$ induces a projective involution τ_a of $B \subset \mathbb{P}^3$. These involutions lift to X and generate a 2-elementary subgroup $H \subset \mathrm{Aut}(X)$ with $\mathrm{r}(H) = 4$. The Galois involution γ of the double cover φ is contained in the center of $\mathrm{Aut}(X)$. Hence γ and H generate a 2-elementary subgroup $G \subset \mathrm{Aut}(X)$ of rank 5.

Note that the fixed point locus of γ on X is a Kummer surface isomorphic to B. On the other hand, the fixed point loci of involutions acting on $X_{2\cdot 2}$ are either rational surfaces or subvarieties of dimension ≤ 1. Hence the groups constructed in Examples 6.11 and 6.12 are not conjugate to each other in the Cremona group.

From now on we assume that $\mathrm{Pic}(X) = \mathbb{Z} \cdot K_X$. Let $g := \mathrm{g}(X)$.

Lemma 6.13. *If* $g \leq 4$, *then* $\mathrm{r}(G) \leq 5$. *If* $g = 5$, *then* $\mathrm{r}(G) \leq 6$.

Proof. We may assume that $-K_X$ is very ample. Automorphisms of X are induced by projective transformations of $\mathbb{P}^{g+1}$ that preserve $X \subset \mathbb{P}^{g+1}$. On the other hand, there is a natural representation of G on $H^0(X, -K_X)$ which is faithful. Thus the composition

$$\mathrm{Aut}(X) \hookrightarrow GL(H^0(X, -K_X)) = GL_{g+2}(\Bbbk) \to PGL_{g+2}(\Bbbk)$$

is injective. Since G is abelian, its image $\bar{G} \subset GL_{g+2}(\Bbbk)$ is contained in a maximal torus and by the above $\bar{G}$ contains no scalar matrices. Hence, $\mathrm{r}(G) \leq g + 1$. □

Example 6.14. Let G be the two-torsion subgroup of the diagonal torus of $PGL_7(\Bbbk)$. Then X faithfully acts on the Fano threefold in $\mathbb{P}^6$ given by the equations $\sum x_i^2 = \sum \lambda_i x_i^2 = \sum \mu_i x_i^2 = 0$. This shows that the bound $\mathrm{r}(G) \leq 6$ in the above lemma is sharp. Note however that X is not rational if it is smooth [1]. Hence in this case our construction does not give any embedding of G to $\mathrm{Cr}_3(\Bbbk)$.

Lemma 6.15. *If in the above assumptions* $\mathrm{g}(X) \geq 6$, *then* X *has at most* 29 *singular points.*

Proof. According to [12] the variety X has a *smoothing*. This means that there exists a flat family $\mathfrak{X} \to \mathfrak{T}$ over a smooth one-dimensional base $\mathfrak{T}$ with special fiber $X = \mathfrak{X}_0$ and smooth general fiber $X_t = \mathfrak{X}_t$. Using the classification of Fano threefolds [6] (see also [7]) we obtain $h^{1,2}(X_t) \leq 10$. Then by Namikawa [12] we have

$$\#\mathrm{Sing}(X) \leq 21 - \frac{1}{2}\mathrm{Eu}(X_t) = 20 - \rho(X_t) + h^{1,2}(X_t) \leq 29.$$

Proof of Proposition 6.1. Assume that $\mathrm{r}(G) \geq 7$. Let $S \in |-K_X|$ be an invariant member. By Corollary 4.5 the singularities of S are worse than Du Val. So S satisfies the conditions (ii) of Proposition 4.6. Write $S = \sum_{i=1}^{N} S_i$. By Proposition 4.6 the group $G_\bullet$ acts on S_i faithfully and

$$N = 2^{\mathrm{r}(G)-\mathrm{r}(G_\bullet)} \geq 4.$$

First we consider the case where X is smooth near S. Since $\rho(X) = 1$, the divisors S_i's are linear equivalent to each other and so $\iota(X) \geq 4$. This contradicts Lemma 6.10.

Therefore, $S \cap \mathrm{Sing}(X) \neq \emptyset$. By Lemma 6.8 the action of G on X is free in codimension 1 and by Remark 2.2 we see that $\mathrm{r}(G_P) \leq 3$, where G_P is the stabilizer of a point $P \in \mathrm{Sing}(X)$. Then by Lemma 6.15 the variety X has exactly 16 singular points and G acts on $\mathrm{Sing}(X)$ transitively. Since $S \cap \mathrm{Sing}(X) \neq \emptyset$, we have $\mathrm{Sing}(X) \subset S$. On the other hand, our choice of S is not unique: there is a basis $s^{(1)}, \dots, s^{(g+2)} \in H^0(X, -K_X)$ consisting of eigensections. This basis gives us G-invariant divisors $S^{(1)}, \dots, S^{(g+2)}$ generating $|-K_X|$. By the above $\mathrm{Sing}(X) \subset S^{(i)}$ for all i. Thus $\mathrm{Sing}(X) \subset \cap S^{(i)} = \mathrm{Bs}|-K_X|$. This contradicts Lemma 6.6. $\square$

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