p-elementary subgroups of the Cremona group of rank 3

Yuri Prokhorov*

Abstract. For the subgroups of the Cremona group $\mathrm{Cr}_3(\mathbb{C})$ having the form $(\boldsymbol{\mu}_p)^s$, where p is prime, we obtain an upper bound for s. Our bound is sharp if $p \ge 17$.

2010 Mathematics Subject Classification. 14E07

1. Introduction

Let $\Bbbk$ be an algebraically closed field. The *Cremona group* $\mathrm{Cr}_n(\Bbbk)$ is the group of birational transformations of $\mathbb{P}^n_{\Bbbk}$, or equivalently the group of $\Bbbk$-automorphisms of the field $\Bbbk(x_1, \dots, x_n)$. Finite subgroups of $\mathrm{Cr}_2(\mathbb{C})$ are completely classified (see [DI09] and references therein). In contrast, subgroups of $\mathrm{Cr}_n(\Bbbk)$ for $n \ge 3$ are not studied well (cf. [Pro09]).

In the present paper we study a certain kind of abelian subgroups of $\mathrm{Cr}_3(\mathbb{C})$. Let p be a prime number. We say that a group G is *p-elementary* if $G \simeq (\boldsymbol{\mu}_p)^s$ for some positive integer s. In this case s is called the *rank* of G and denoted by $\operatorname{rk} G$.

Theorem 1.1 ([Bea07]). *Let p be a prime $\neq \operatorname{char}(\Bbbk)$ and let $G \subset \mathrm{Cr}_2(\Bbbk)$ be a p-elementary subgroup. Then:*

$$\operatorname{rk} G \le 2 + \delta_{p,3} + 2\delta_{p,2}$$

where $\delta_{i,j}$ is Kronecker's delta. Moreover, for any such p this bound is attained for some G. These "maximal" groups G are classified up to conjugacy in $\mathrm{Cr}_2(\Bbbk)$.

More generally, instead of $\mathrm{Cr}_n(\Bbbk)$ we also can consider the group $\mathrm{Bir}(X)$ of birational automorphisms of an arbitrary rationally connected variety X. Our main result is the following

Theorem 1.2. *Let X be a rationally connected threefold defined over a field of characteristic 0, let p be a prime, and let $G \subset \mathrm{Bir}(X)$ be a p-elementary subgroup. Then*

$$\operatorname{rk} G \le \begin{cases} 7 & \text{if } p = 2, \\ 5 & \text{if } p = 3, \\ 4 & \text{if } p = 5,\ 7,\ 11, \text{ or } 13, \\ 3 & \text{if } p \ge 17. \end{cases} \tag{1.3}$$

*The author was partially supported by RFBR, grant No. 08-01-00395-a and Leading Scientific Schools (grants No. 1983.2008.1, 1987.2008.1).

For any prime $p \geq 17$ *this bound is attained for some subgroup* $G \subset \mathrm{Cr}_3(\mathbb{C})$. (*However we do not assert that the bound* (1.3) *is sharp for* $p \leq 13$).

Remark 1.4. (i) Note that $\mathrm{Cr}_1(\Bbbk) \simeq \mathrm{PGL}_2(\Bbbk)$. Hence for any prime $p \neq \mathrm{char}(\Bbbk)$ and any p-elementary subgroup $G \subset \mathrm{Cr}_1(\Bbbk)$, we have $\mathrm{rk}\, G \leq 1 + \delta_{p,2}$ (see, e.g., [Bea07, Lemma 2.1]).

(ii) Since $\mathrm{Cr}_1(\Bbbk) \times \mathrm{Cr}_2(\Bbbk)$ admits (a lot of) embeddings into $\mathrm{Cr}_3(\Bbbk)$, the group $\mathrm{Cr}_3(\Bbbk)$ contains a p-elementary subgroup G of rank $3 + \delta_{p,3} + 3\delta_{p,2}$. This shows the last assertion of our theorem.

The following consequence of Theorem 1.2 was proposed by A. Beauville.

Corollary 1.5. *The group* $\mathrm{Cr}_3(\mathbb{C})$ *is not isomorphic to* $\mathrm{Cr}_n(\mathbb{C})$ *for* $n \neq 3$ *as an abstract group.*

Proof. Denote by $\xi(n, p)$ the maximal rank of a p-elementary group contained in $\mathrm{Cr}_n(\mathbb{C})$. Then $\xi(2, 17) = 2 < \xi(3, 17) = 3$ and $\xi(n, 17) \geq n$ by Theorems 1.1 and 1.2. □

Our method is a generalization of the method used for the study of finite subgroups of $\mathrm{Cr}_2(\Bbbk)$ [Bea07], [DI09]. Similarly to [Pro09] we use the equivariant three-dimensional minimal model program. This easily allows us to reduce the problem to the study of automorphism groups of some (not necessarily smooth) Fano threefolds.

Acknowledgments. I would like to thank J.-P. Serre for asking me the question considered here and for useful comments. I am also grateful to A. Beauville for proposing me Corollary 1.5 and for his interest in my paper. This paper was written at the IHES (Bures-sur-Yvette) during my visit in 2009. I am grateful to IHES for the support and hospitality. Finally I would like to thank the referee for several suggestions that make the paper more readable and clear.

2. Preliminaries

Clearly, we may assume that $\Bbbk = \mathbb{C}$. All the groups in this paper are multiplicative. In particular, we denote a cyclic group of order n by $\boldsymbol{\mu}_n$.

2.1. Terminal singularities. We need a few facts on the classification of three-dimensional terminal singularities (see [Mor85], [Rei87]). Let $(X \ni P)$ be a germ of a three-dimensional terminal singularity. Then $(X \ni P)$ is isolated, i.e., $\mathrm{Sing}(X) = \{P\}$. The *index* of $(X \ni P)$ is the minimal positive integer r such that rK_X is Cartier. If $r = 1$, then $(X \ni P)$ is Gorenstein. In this case $(X \ni P)$ is analytically isomorphic to a hypersurface singularity in $\mathbb{C}^4$ of multiplicity 2. Moreover, any Weil $\mathbb{Q}$-Cartier divisor D on $(X \ni P)$ is Cartier. If $r > 1$, then there is a cyclic, étale outside of P cover $\pi : (X^\sharp \ni P^\sharp) \to (X \ni P)$ of degree r such that $(X^\sharp \ni P^\sharp)$ is a Gorenstein terminal singularity (or a smooth point). This π is called the *index-one cover* of $(X \ni P)$.

Theorem 2.2 ([Mor85], [Rei87]). *In the above notation* $(X^\sharp \ni P^\sharp)$ *is analytically* $\boldsymbol{\mu}_r$*-isomorphic to a hypersurface in* $\mathbb{C}^4$ *with* $\boldsymbol{\mu}_r$*-semi-invariant* [1] *coordinates* $x_1, \dots, x_4$*, and the action is given by*

$$(x_1, \dots, x_4) \longmapsto (\varepsilon^{a_1} x_1, \dots, \varepsilon^{a_4} x_4)$$

for some primitive r*-th root of unity* ε*, where one of the following holds:*

(i) $(a_1, \dots, a_4) \equiv (1, -1, a_2, 0) \mod r, \quad \gcd(a_2, r) = 1,$

(ii) $r = 4$ *and* $(a_1, \dots, a_4) \equiv (1, -1, 1, 2) \mod 4.$

Definition 2.3. A *G-variety* is a variety X provided with a biregular faithful action of a finite group G. We say that a normal G-variety X is *$G\mathbb{Q}$-factorial* if any G-invariant Weil divisor on X is $\mathbb{Q}$-Cartier. A projective normal G-variety X is called *$G\mathbb{Q}$-Fano* if it is $G\mathbb{Q}$-factorial, has at worst terminal singularities, $-K_X$ is ample, and $\operatorname{rk}\operatorname{Pic}(X)^G = 1$.

Lemma 2.4. *Let* $(X \ni P)$ *be a germ of a threefold terminal singularity and let* $G \subset \operatorname{Aut}(X \ni P)$ *be a* p*-elementary subgroup. Then* $\operatorname{rk} G \le 3 + \delta_{2,p}$.

Proof. Assume that $\operatorname{rk} G \ge 4 + \delta_{2,p}$. First we consider the case where $(X \ni P)$ is Gorenstein. The group G acts faithfully on the Zariski tangent space $T_{P,X}$, so $G \subset \operatorname{GL}(T_{P,X})$, where $\dim T_{P,X} = 3$ or 4. If $\dim T_{P,X} = 3$, then G is contained in a maximal torus of $\operatorname{GL}_3(\mathbb{C})$, so $\operatorname{rk} G \le 3$ and we are done. Thus we may assume that $\dim T_{P,X} = 4$. Take semi-invariant coordinates $x_1, \dots, x_4$ in $T_{P,X}$. There is a G-equivariant analytic embedding $(X \ni P) \subset \mathbb{C}^4_{x_1,\dots,x_4}$. As above, $\operatorname{rk} G \le 4$. Thus we may assume that $\operatorname{rk} G \le 4$ and $p > 2$. Let $\phi(x_1, \dots, x_4) = 0$ be an equation of X, where ϕ is a G-semi-invariant function. Regard ϕ as a power series and write $\phi = \sum_d \phi_d$, where ϕ_d is the sum of all monomials of degree d. Since the action of G on $x_1, \dots, x_4$ is linear, all the ϕ_d's are semi-invariants of the same G-weight $w = \operatorname{wt}\phi_d$. Hence, for any ϕ_d, $\phi_{d'} \ne 0$ we have $d - d' \equiv 0 \mod p$. Since $(X \ni P)$ is a terminal singularity, $\phi_2 \ne 0$ and so $\phi_3 = 0$. Recall that $G \simeq (\boldsymbol{\mu}_p)^4$, $p \ge 3$. In this case, ϕ_2 must be a monomial. Thus up to permutations of coordinates and scalar multiplication we get either $\phi_2 = x_1^2$ or $\phi_2 = x_1x_2$. In particular, we have $\operatorname{rk}\phi_2 \le 2$ and $\phi_3 = 0$. This contradicts the classification of terminal singularities [Mor85], [Rei87].

Now assume that $(X \ni P)$ is non-Gorenstein of index $r > 1$. Consider the index-one cover $\pi\colon (X^\sharp \ni P^\sharp) \to (X \ni P)$ (see 2.1). Here $(X^\sharp \ni P^\sharp)$ is a Gorenstein terminal point and the map $X^\sharp \backslash \{P^\sharp\} \to X \backslash \{P\}$ can be regarded as the topological universal cover. Hence there exists a natural lifting $G^\sharp \subset \operatorname{Aut}(X^\sharp \ni P^\sharp)$ fitting in the following exact sequence

$$1 \longrightarrow \boldsymbol{\mu}_r \longrightarrow G^\sharp \longrightarrow G \longrightarrow 1. \tag{2.5}$$

It is sufficient to show that there exists a subgroup $G^\bullet \subset G^\sharp$ isomorphic to G (but we do not assert that the sequence splits). Indeed, in this case $G^\bullet \simeq G$

[1] In invariant theory people often say "relative invariant" rather than "semi-invariant". We prefer to use the terminology of [Mor85].

acts faithfully on the terminal Gorenstein singularity $(X^\sharp \ni P^\sharp)$ and we can apply the case considered above. We may assume that $G^\sharp$ is not abelian (otherwise a subgroup $G^\bullet \simeq G$ obviously exists). The group $G^\sharp$ permutes eigenspaces of $\boldsymbol{\mu}_r$. By Theorem 2.2 the subspace $T := \{x_4 = 0\} \subset \mathbb{C}^4_{x_1,\dots,x_4}$ is $G^\sharp$-invariant and $\boldsymbol{\mu}_r$ acts on any eigenspace $T_1 \subset T$ faithfully. On the other hand, by (2.5) we see that the derived subgroup $[G^\sharp, G^\sharp]$ is contained in $\boldsymbol{\mu}_r$. In particular, $[G^\sharp, G^\sharp]$ is abelian and also acts on any eigenspace $T_1 \subset T$ faithfully. Since $\dim T = 3$, this implies that the representation of $G^\sharp$ on T is irreducible (otherwise T has a one-dimensional subrepresentation, say T_1, and the kernel of the map $G \to \mathrm{GL}(T_1) \simeq \mathbb{C}^*$ must contain $[G^\sharp, G^\sharp]$). Hence eigenspaces of $\boldsymbol{\mu}_r$ have the same dimension and so $\boldsymbol{\mu}_r$ acts on T by scalar multiplication. By Theorem 2.2 this is possible only if $r = 2$.

Let $G^\sharp_p \subset G^\sharp$ be a Sylow p-subgroup. If $\boldsymbol{\mu}_r \cap G^\sharp_p = \{1\}$, then $G^\sharp_p \simeq G$ and we are done. Thus we assume that $\boldsymbol{\mu}_r \subset G^\sharp_p$, so $p = r = 2$ and $G^\sharp_p = G^\sharp$. But then $G^\sharp$ is a 2-group, so the dimension of its irreducible representation must be a power of 2. Hence T is reducible, a contradiction. □

Lemma 2.6. *Let X be a G-threefold with isolated singularities.*

(i) *If there is a curve $C \subset X$ of G-fixed points, then* $\operatorname{rk} G \le 2$.

(ii) *If there is a surface $S \subset X$ of G-fixed points, then* $\operatorname{rk} G \le 1$. *If moreover S is singular along a curve, then $G = \{1\}$.*

Sketch of the Proof. Consider the action of G on the tangent space to X at a general point of C (resp. S). □

G-equivariant minimal model program. Let X be a rationally connected three-dimensional algebraic variety and let $G \subset \mathrm{Bir}(X)$ be a finite subgroup. By shrinking X we may assume that G acts on X biregularly. The quotient $Y = X/G$ is quasiprojective, so there exists a projective completion $\hat{Y} \supset Y$. Let $\hat{X}$ be the normalization of $\hat{Y}$ in the function field $\mathbb{C}(X)$. Then $\hat{X}$ is a projective variety birational to X admitting a biregular action of G. There is an equivariant resolution of singularities $\tilde{X} \to \hat{X}$, see [AW97]. Run the G-equivariant minimal model program: $\tilde{X} \to \bar{X}$, see [Mor88, 0.3.14]. Running this program we stay in the category of projective normal varieties with at worst terminal $G\mathbb{Q}$-factorial singularities. Since X is rationally connected, on the final step we get a Fano-Mori fibration $f : \bar{X} \to Z$. Here $\dim Z < \dim X$, Z is normal, f has connected fibers, the anticanonical Weil divisor $-K_{\bar{X}}$ is ample over Z, and the relative G-invariant Picard number $\rho(\bar{X})^G$ is one. Obviously, we have the following possibilities:

(i) Z is a rational surface and a general fiber $F = f^{-1}(y)$ is a conic;

(ii) $Z \simeq \mathbb{P}^1$ and a general fiber $F = f^{-1}(y)$ is a smooth del Pezzo surface;

(iii) Z is a point and $\bar{X}$ is a $G\mathbb{Q}$-Fano threefold.

Proposition 2.7. *In the above notation assume that Z is not a point. Then* $\operatorname{rk} G \le 3 + \delta_{p,3} + 3\delta_{p,2}$. *In particular,* (1.3) *holds.*

Proof. Let $G_0 \subset G$ be the kernel of the homomorphism $G \to \mathrm{Aut}(Z)$. The group $G_1 := G/G_0$ acts effectively on Z and G_0 acts effectively on a general fiber $F \subset X$ of f. Hence, $G_1 \subset \mathrm{Aut}(Z)$ and $G_0 \subset \mathrm{Aut}(F)$. Clearly, G_0 and G_1 are p-elementary groups with $\mathrm{rk}\, G_0 + \mathrm{rk}\, G_1 = \mathrm{rk}\, G$. Assume that $Z \simeq \mathbb{P}^1$. Then $\mathrm{rk}\, G_1 \le 1 + \delta_{p,2}$. By Theorem 1.1 we obtain $\mathrm{rk}\, G_0 \le 2 + \delta_{p,3} + 2\delta_{p,2}$. This proves our assertion in the case $Z \simeq \mathbb{P}^1$. The case $\dim Z = 2$ is treated similarly. □

2.8. Main assumption. Thus from now on we assume that we are in the case (iii). Replacing X with $\bar{X}$ we may assume that our original X is a $G\mathbb{Q}$-Fano threefold.

The group G acts naturally on $H^0(X, -K_X)$. If $H^0(X, -K_X) \neq 0$, then there exists a G-semi-invariant section $s \in H^0(X, -K_X)$ (because G is an abelian group). This section gives us an invariant member $S \in |-K_X|$.

Lemma 2.9. *Let X be a $G\mathbb{Q}$-Fano threefold, where G is a p-elementary group with $\mathrm{rk}\, G \ge \delta_{p,2} + 4$. Let S be an invariant Weil divisor such that $-(K_X + S)$ is nef. Then the pair (X, S) is log canonical (LC).*

Proof. Assume that the pair (X, S) is not LC. Since S is G-invariant and $\rho(X)^G = 1$, we see that S is numerically proportional to K_X. Since $-(K_X + S)$ is nef, S is ample. We apply quite standard connectedness arguments of Shokurov [Sho93] (see, e.g., [MP09, Prop. 2.6]): for a suitable G-invariant boundary D, the pair (X, D) is LC, the divisor $-(K_X + D)$ is ample, and the minimal locus V of log canonical singularities is also G-invariant. Moreover, V is either a point or a smooth rational curve. By Lemma 2.4 we may assume that G has no fixed points. Hence, $V \simeq \mathbb{P}^1$ and we have a map $\varsigma : G \to \mathrm{Aut}(\mathbb{P}^1)$. If $p > 2$, then $\varsigma(G)$ is a cyclic group, so G has a fixed point, a contradiction. Let $p = 2$ and let $G_0 = \ker \varsigma$. By Lemma 2.6 $\mathrm{rk}\, G_0 \le 2$. Therefore $\mathrm{rk}\, \varsigma(G_0) \ge 3$. Again we get a contradiction. □

Lemma 2.10. *Let X be a $G\mathbb{Q}$-Fano threefold, where G is a p-elementary group with*

$$\mathrm{rk}\, G \ge \begin{cases} 7 & \text{if } p = 2, \\ 5 & \text{if } p = 3, \\ 4 & \text{if } p \ge 5. \end{cases} \tag{2.11}$$

Let $S \in |-K_X|$ be a G-invariant member. Then we have

(i) *Any component $S_i \subset S$ is either rational or birationally ruled over an elliptic curve.*

(ii) *The group G acts transitively on the components of S.*

(iii) *For the stabilizer G_{S_i} we have $\mathrm{rk}\, G_{S_i} \le \delta_{p,2} + 4$.*

(iv) *The surface S is reducible (and reduced).*

Proof. By Lemma 2.9 the pair (X, S) is LC. Assume that S is normal (and irreducible). By the adjunction formula $K_S \sim 0$. We claim that S has at worst Du Val singularities. Indeed, otherwise by the Connectedness Principle [Sho93, Th.

6.9] S has at most two non-Du Val points. If $p > 2$, these points must be G-fixed. This contradicts Lemma 2.4. Otherwise $p = 2$ and these points are fixed for an index two subgroup $G^{\bullet} \subset G$. Again we get a contradiction by Lemma 2.4. Thus we may assume that S has at worst Du Val singularities. Let Γ be the image of G in $\mathrm{Aut}(S)$. By Lemma 2.6 $\operatorname{rk} G \le \operatorname{rk}\Gamma + 1$. Let $\tilde{S} \to S$ be the minimal resolution. Here $\tilde{S}$ is a smooth K3 surface. The natural representation of Γ on $H^{2,0}(\tilde{S})$ induces the exact sequence (see [Nik80])

$$1 \longrightarrow \Gamma_0 \longrightarrow \Gamma \longrightarrow \Gamma_1 \longrightarrow 1,$$

where Γ_0 (resp. Γ_1) is the kernel (resp. image) of the representation of Γ on $H^{2,0}(\tilde{S})$. The group Γ_1 is cyclic. Hence either $\Gamma_1 = \{1\}$ or $\Gamma_1 \simeq \boldsymbol{\mu}_p$. In the second case by [Nik80, Cor. 3.2] $p \le 19$. Further, according to [Nik80, Th. 4.5] we have

$$\operatorname{rk}\Gamma_0 \le \begin{cases} 4 & \text{if } p = 2 \\ 2 & \text{if } p = 3 \\ 1 & \text{if } p = 5 \text{ or } 7 \\ 0 & \text{if } p > 7. \end{cases}$$

Combining this we obtain a contradiction with (2.11).

Now assume that S is not normal. Let $S_i \subset S$ be an irreducible component (the case $S_i = S$ is not excluded). Let $\nu\colon S' \to S_i$ be the normalization. Write $0 \sim \nu^*(K_X + S_i) = K_{S'} + D'$, where D' is the different, see [Sho93, §3]. Here D' is an effective reduced divisor and the pair is LC [Sho93, 3.2]. Since S is not normal, $D' \ne 0$. Consider the minimal resolution $\mu\colon \tilde{S} \to S'$ and let $\tilde{D}$ be the crepant pull-back of D', that is, $\mu_*\tilde{D} = D'$ and

$$K_{\tilde{S}} + \tilde{D} = \mu^*(K_{S'} + D') \sim 0.$$

Here $\tilde{D}$ is again an effective reduced divisor. Hence $\tilde{S}$ is a ruled surface. If it is not rational, consider the Albanese map $\alpha : \tilde{S} \to C$. Clearly α is Γ-equivariant and the action of Γ on C is not trivial. Let $\tilde{D}_1 \subset \tilde{D}$ be an α-horizontal component. By adjunction $\tilde{D}_1$ is an elliptic curve. So is C. This proves (i).

If the action on components $S_i \subset S$ is not transitive, we have an invariant divisor $S' < S$. Since X is $G\mathbb{Q}$-factorial and $\rho(X)^G = 1$, we can take S' so that $-(K_X + 2S')$ is nef. This contradicts Lemma 2.9. So, (ii) is proved.

Now we prove (iii). Let Γ be the image of G_{S_i} in $\mathrm{Aut}(S_i)$. By Lemma 2.6 $\operatorname{rk} G_{S_i} \le \operatorname{rk}\Gamma + 1$. If S_i is rational, then we get the assertion by Theorem 1.1. Assume that S_i is a birationally ruled surface over an elliptic curve. As above, let $\tilde{S}_i \to S_i$ be the composition of the normalization and the minimal resolution, and let $\alpha : \tilde{S}_i \to C$ be the Albanese map. Then Γ acts faithfully on $\tilde{S}_i$ and α is Γ-equivariant. Thus we have a homomorphism $\alpha_* : \Gamma \to \mathrm{Aut}(C)$. Here $\operatorname{rk}\Gamma \le \operatorname{rk}\alpha_*(\Gamma) + 1 + \delta_{p,2}$. Note that $\alpha_*(\Gamma)$ is a p-elementary subgroup of the automorphism group of an elliptic curve. Hence, $\operatorname{rk}\alpha_*(\Gamma) \le 2$. This implies (iii).

It remains to prove (iv). Assume that S is irreducible. By (iii) the surface S is not rational. So, S is birational to a ruled surface over an elliptic curve. By Lemma 2.6 the group G acts faithfully on S. Hence, in the above notation, $\operatorname{rk} G = \operatorname{rk}\Gamma \le \operatorname{rk}\alpha_*(\Gamma) + 1 + \delta_{p,2} \le 3 + \delta_{p,2}$, a contradiction. □

3. Proof of Theorem 1.2

3.1. In this section we prove Theorem 1.2. As in 2.8 we assume that X is a $G\mathbb{Q}$-Fano threefold, where G is a p-elementary subgroup of $\mathrm{Aut}(X)$.

First we consider the case where X is non-Gorenstein, i.e., it has at least one point of index > 1.

Proposition 3.2. *Let G be a p-elementary group and let X be a non-Gorenstein $G\mathbb{Q}$-Fano threefold. Then*

$$\operatorname{rk} G \leq \begin{cases} 7 & \text{if } p = 2, \\ 5 & \text{if } p = 3, \\ 4 & \text{if } p = 5,\ 7,\ 11,\ 13, \\ 3 & \text{if } p \geq 17. \end{cases}$$

Proof. Let P_1 be a point of index $r > 1$ and let $P_1, \dots, P_l$ be its G-orbit. Here $l = p^t$ for some t with $t \geq s - \delta_{2,p} - 3$, where $s = \operatorname{rk} G$ (see Lemma 2.4). By the orbifold Riemann-Roch formula [Rei87] and a form of the Bogomolov-Miyaoka inequality [Kaw92], [KMMT00] we have

$$\sum \left(r_{P_i} - \frac{1}{r_{P_i}} \right) < 24.$$

Since $r_i - 1/r_i \geq 3/2$, we have $3l/2 < 24$ and so

$$p^{s - \delta_{2,p} - 3} \leq l < 16.$$

This gives us the desired inequality. □

From now on we assume that our $G\mathbb{Q}$-Fano threefold X is Gorenstein, i.e., K_X is a Cartier divisor. Recall (see, e.g., [IP99]) that the Picard group of a Fano variety X with at worst (log) terminal singularities is a torsion free finitely generated abelian group ($\simeq H^2(X, \mathbb{Z})$). Then we can define the *Fano index* of X as the maximal positive integer that divides $-K_X$ in $\mathrm{Pic}(X)$.

Proposition-Definition 3.3 (see, e.g., [IP99]). *Let X be a Fano threefold with at worst terminal Gorenstein singularities. The positive integer $-K_X^3$ is called the* degree *of X. We can write $-K_X^3 = 2g - 2$, where g is an integer ≥ 2 called the* genus *of X. Then* $\dim |-K_X| = g + 1 \geq 3$.

Corollary-Notation 3.4. *In notation* 3.1 *the linear system $|-K_X|$ is not empty, so there exists a G-invariant member $S \in |-K_X|$. Write $S = \sum_{i=1}^{N} S_i$, where S_i are irreducible components.*

Theorem 3.5 ([Nam97]). *Let X be a Fano threefold with terminal Gorenstein singularities. Then X is smoothable, that is, there is a flat family X_t such that $X_0 \simeq X$ and a general member X_t is a smooth Fano threefold of the same degree,* *Fano index and Picard number. Furthermore, the number of singular points is bounded as follows:*

$$|\operatorname{Sing}(X)| \le 20 - \rho(X_t) + h^{1,2}(X_t), \tag{3.6}$$

where $h^{1,2}(X_t)$ is the Hodge number.

Combining the above theorem with the classification of smooth Fano threefolds [Isk80], [MM82] (see also [IP99]) we get the following

Theorem 3.7. *Let X be a Fano threefold with at worst terminal Gorenstein singularities and let X_t be its smoothing. Let g and q be the genus and Fano index of X, respectively.*

(i) $q \le 4$.

(ii) *If $q = 4$, then $X \simeq \mathbb{P}^3$.*

(iii) *If $q = 3$, then X is a quadric in $\mathbb{P}^4$ (with $\dim \operatorname{Sing}(X) \le 0$).*

(iv) *If $q = 2$, then $\rho(X) \le 3$ and $-K_X^3 = 8d$, where $1 \le d \le 7$. Moreover $\rho(X) = 1$ if and only if $d \le 5$.*

(v) *If $q = 1$ and $\rho(X) = 1$, then there are the following possibilities:*

g	2	3	4	5	6	7	8	9	10	12
$h^{1,2}(X_t)$	52	30	20	14	10	7	5	3	2	0

Lemma 3.8. *Let G be a p-elementary group and let X be a Gorenstein $G\mathbb{Q}$-Fano threefold. If the linear system $|-K_X|$ is not base point free, then $\operatorname{rk} G \le 3 + \delta_{p,2}$.*

Proof. Assume that $\operatorname{Bs}|-K_X| \ne \emptyset$. Clearly, $\operatorname{Bs}|-K_X|$ is G-invariant. By [Isk80], [Shi89] $\operatorname{Bs}|-K_X|$ is either a single point or a smooth rational curve. In the first case the assertion immediately follows by Lemma 2.4. In the second case G acts on the curve $C = \operatorname{Bs}|-K_X|$. Since $C \simeq \mathbb{P}^1$, the assertion follows by Lemma 2.6. □

Proposition 3.9. *Let G be a p-elementary group, where $p \ge 5$, and let X be a Gorenstein $G\mathbb{Q}$-Fano threefold. Then*

$$\operatorname{rk} G \le \begin{cases} 4 & \text{if } p = 5,\ 7,\ 11,\ 13, \\ 3 & \text{if } p \ge 17. \end{cases}$$

Proof. Assume that the above inequality does not hold. We use the notation of 3.4. In particular, N denotes the number of components of $S = \sum S_i \in |-K_X|$. By Lemma 2.10 $N = p^l$, where $l \ge 1$. Hence p divides $-K_X^3 = 2g - 2 = \big((-K_X)^2 \cdot S_i\big)N$. First we claim that $\rho(X) = 1$. Indeed, if $\rho(X) > 1$, then the natural representation of G on $\operatorname{Pic}_{\mathbb{Q}}(X) := \operatorname{Pic}(X) \otimes \mathbb{Q}$ is decomposed as $\operatorname{Pic}_{\mathbb{Q}}(X) =$

$V_1 \oplus V$, where V_1 is a trivial subrepresentation generated by the class of $-K_X$ and V is a subrepresentation such that $V^G = 0$. Since G is a p-elementary group, $\dim V \geq p-1$. Hence, $\rho(X) \geq p \geq 5$ and by the classification [MM82] we have two possibilities:

- $-K_X^3 = 6(11 - \rho(X)), \quad 5 \leq \rho(X) \leq 10$, or
- $-K_X^3 = 28, \quad \rho(X) = 5$.

In the last case $p = 5$, so $-K_X^3 \not\equiv 0 \mod p$, a contradiction. In the first case p divides $-K_X^3$ only if $p = 5$. Then $\rho(X) = 6$. So, $\dim V = 5$ and $V^G \neq 0$. Again we get a contradiction.

Therefore, $\rho(X) = 1$. Let q be the Fano index of X. We claim that X is singular. Indeed, otherwise all the S_i are Cartier divisors. Then $-K_X = NS_1$, where $N \geq p$, and so $q \geq 5$. This contradicts (i) of Theorem 3.7. Hence X is singular. By Lemma 2.4 and our assumption we have $|\operatorname{Sing}(X)| \geq p$. In particular, $q \leq 2$ (see Theorem 3.7). If $q = 1$, then by Theorem 3.7 either $2 \leq g \leq 10$ or $g = 12$. Thus $N = p$ and we get the following possibilities: $(p, g) = (5, 6)$, $(7, 8)$, or $(11, 12)$. Moreover, $(-K_X)^2 \cdot S_i = (2g-2)/N = 2$. Therefore, the restriction $|-K_X||_{S_i}$ of the (base point free) anticanonical linear system defines either an isomorphism to a quadric $S_i \to Q \subset \mathbb{P}^3$ or a double cover $S_i \to \mathbb{P}^2$. In both cases the image is rational, so we get a map $G_i \to \mathrm{Cr}_2(\mathbb{C})$ whose kernel is of rank ≤ 1 by Lemma 2.6 and because $p > 2$. Then by Theorem 1.1 $\operatorname{rk} G_{S_i} \leq 3$. Hence, $\operatorname{rk} G \leq 4$ which contradicts our assumption.

Finally, consider the case $q = 2$. Then $-K_X = 2H$ for some ample Cartier divisor H and $d := H^3 \leq 7$. Therefore, $NS_i \cdot H^2 = S \cdot H^2 = 2d$. Since $\rho(X) = 1$, by Theorem 3.7 we get $p = d = 5$. Then we apply (3.6). In this case, $h^{1,2}(X_t) = 0$ (see [IP99]). So, $|\operatorname{Sing}(X)| \leq 19$. On the other hand, $|\operatorname{Sing}(X)| \geq 25$ by Lemma 2.4 and our assumption. The contradiction proves the proposition. □

We need the following result which is a very weak form of Shokurov's much more general toric conjecture [McK01], [Pro03].

Lemma 3.10. *Let V be a smooth Fano threefold and let $D \in |-K_V|$ be a divisor such that the pair (V, D) is LC. Then D has at most $3 + \rho(V)$ irreducible components.*

Proof. Write $D = \sum_{i=1}^n D_i$. If $\rho(V) = 1$, then all the D_i are linearly proportional: $D_i \sim n_i H$, where H is an ample generator of $\operatorname{Pic}(V)$. Then $-K_V \sim \sum n_i H$ and by Theorem 3.7 we have $\sum n_i = q \leq 4$.

If V is a blowup of a curve on another smooth Fano threefold W, then we can proceed by induction replacing V with W. Thus we assume that V cannot be obtained by blowing up of a curve on another smooth Fano threefold. In this situation V is called *primitive* ([MM83]). According to [MM83, Th. 1.6] we have $\rho(V) \leq 3$ and V has a conic bundle structure $f : V \to Z$, where $Z \simeq \mathbb{P}^2$ (resp. $Z \simeq \mathbb{P}^1 \times \mathbb{P}^1$) if $\rho(V) = 2$ (resp. $\rho(V) = 3$). Let ℓ be a general fiber. Then $2 = -K_V \cdot \ell = \sum D_i \cdot \ell$. Hence D has at most two f-horizontal components and at least $n-2$ vertical ones. Now let $h : V \to W$ be an extremal contraction other than

f and let ℓ' be any curve in a non-trivial fiber of h. For any f-vertical component $D_i \subset D$ we have $D_i = f^{-1}(\Gamma_i)$, where $\Gamma_i \subset Z$ is a curve, so $D_i \cdot \ell' = \Gamma_i \cdot f_* \ell' \geq 0$. If $\rho(V) = 2$, then $D_i \cdot l' \geq 1$. Hence, $-K_V \cdot \ell' \geq n-2$. On the other hand, $-K_V \cdot \ell' \leq 3$ (see [MM83, §3]). This immediately gives us $n \leq 5$ as claimed. Finally consider the case $\rho(V) = 3$. Assume that $n \geq 7$. Then we can take h so that ℓ' meets at least three f-vertical components, say D_1, D_2, D_3. As above, $-K_V \cdot \ell' \geq 3$ and by the classification of extremal rays (see [MM83, §3]) h is a del Pezzo fibration. This contradicts our assumption $\rho(V) = 3$. □

Proposition 3.11. *Let G be a 2-elementary group and let X be a Gorenstein $G\mathbb{Q}$-Fano threefold. Then* $\operatorname{rk} G \leq 7$.

Proof. Assume that $\operatorname{rk} G \geq 8$. By Lemma 2.10 we have $\operatorname{rk} G_{S_i} \leq 5$. Hence, $N \geq 8$. If X is smooth, then by Lemma 3.10 we have $\rho(X) \geq 5$. If furthermore $X \simeq Y \times \mathbb{P}^1$, where Y is a del Pezzo surface, then the projection $X \to Y$ must be G-equivariant. This contradicts $\rho(X)^G = 1$. Therefore, $\rho(X) = 5$ and $-K_X^3 = 28$ or 36 (see [MM82]). On the other hand, $-K_X^3$ is divisible by N, a contradiction.

Thus X is singular. Assume that $|\operatorname{Sing}(X)| \geq 32$. Then for a smoothing X_t of X by (3.6) we have $h^{1,2}(X_t) \geq 13$. Since N divides $-K_X^3 = -K_{X_t}^3$, using the classification of Fano threefolds [Isk80], [MM82] (see also [IP99]) we get:

$$\rho(X) = 1, \quad -K_X^3 = 8, \quad N = 8, \quad |\operatorname{Sing}(X)| = 32.$$

Consider the representation of G on $H^0(X, -K_X)$. Since

$$7 = \dim H^0(X, -K_X) < \operatorname{rk} G,$$

this representation is not faithful (otherwise G is contained in a maximal torus of $\mathrm{GL}(H^0(X, -K_X)) = \mathrm{GL}_7(\mathbb{C})$). Therefore, the linear system $|-K_X|$ is not very ample. On the other hand, $|-K_X|$ is base point free (see Lemma 3.8). Hence $|-K_X|$ defines a double cover $X \to Y \subset \mathbb{P}^6$ [Isk80]. Here Y is a variety of degree 4 in $\mathbb{P}^6$, a variety of minimal degree. If Y is smooth, then according to the Enriques theorem (see, e.g., [Isk80, Th. 3.11]) Y is a rational scroll $\mathbb{P}_{\mathbb{P}^1}(\mathscr{E})$, where $\mathscr{E}$ is a rank 3 vector bundle on $\mathbb{P}^1$. Then X has a G-equivariant projection to a curve. This contradicts $\rho(X)^G = 1$. Hence Y is singular. In this case, Y is a cone (again by the Enriques theorem [Isk80, Th. 3.11]). If its vertex $O \in Y$ is zero-dimensional, then $\dim T_{O,Y} = 6$. On the other hand, X has only hypersurface singularities (see 2.1). Therefore the double cover $X \to Y$ is not étale over O and so G has a fixed point on X. This contradicts Lemma 2.4. Thus Y is a cone over a rational normal curve of degree 4 with vertex along a line. Then X cannot have isolated singularities, a contradiction.

Therefore, $|\operatorname{Sing}(X)| < 32$. Then for any point $P \in \operatorname{Sing}(X)$ by Lemma 2.4 we have $\operatorname{rk} G_P \geq 4$. Hence the orbit of P contains 16 elements and coincides with $\operatorname{Sing}(X)$, i.e., the action of G on $\operatorname{Sing}(X)$ is transitive. Since $S \cap \operatorname{Sing}(X) \neq \emptyset$, we have $\operatorname{Sing}(X) \subset S$. On the other hand, our choice of S in 2.8 is not unique: there is a basis $s^{(1)}, \ldots, s^{(g+2)} \in H^0(X, -K_X)$ consisting of eigensections. This basis gives us G-invariant divisors $S^{(1)}, \ldots, S^{(g+2)}$ generating $|-K_X|$. By the above,

$\mathrm{Sing}(X) \subset S^{(i)}$ for all i. Thus $\mathrm{Sing}(X) \subset \cap S^{(i)} = \mathrm{Bs}\,|-K_X|$. This contradicts Lemma 3.8. Proposition 3.11 is proved. □

Proposition 3.12. *Let G be an 3-elementary group and let X be a Gorenstein $G\mathbb{Q}$-Fano threefold. Then* $\mathrm{rk}\,G \le 5$.

Proof. Assume that $\mathrm{rk}\,G \ge 6$. By Lemma 2.10 we have $\mathrm{rk}\,G_{S_i} \le 5$. Hence, $N \ge 9$. If X is smooth, then by Lemma 3.10 we have $\rho(X) \ge 6$ and so $X \simeq Y \times \mathbb{P}^1$, where Y is a del Pezzo surface [MM82]. Then the projection $X \to Y$ must be G-equivariant. This contradicts $\rho(X)^G = 1$. Therefore, X is singular. By Lemma 2.4 $|\mathrm{Sing}(X)| \ge 3^{6-3} = 27$. Hence, for a smoothing X_t of X by (3.6) we have $h^{1,2}(X_t) \ge 7 + \rho(X)$. Recall that N divides $-K_X^3 = -K_{X_t}^3$. Then we use the classification of smooth Fano threefolds [Isk80], [MM82] and get a contradiction. □

Now Theorem 1.2 is a consequence of Propositions 3.2, 3.9, 3.11, and 3.12.

References

[AW97] D. Abramovich and J. Wang. Equivariant resolution of singularities in characteristic 0. *Math. Res. Lett.*, 4(2-3):427–433, 1997.

[Bea07] A. Beauville. p-elementary subgroups of the Cremona group. *J. Algebra*, 314(2):553–564, 2007.

[DI09] I. V. Dolgachev and V. A. Iskovskikh. Finite subgroups of the plane Cremona group. To appear in Algebra, Arithmetic, and Geometry in Honor of Yu. I. Manin, Progr. Math., 269, 2009.

[IP99] V. A. Iskovskikh and Y. Prokhorov. *Fano varieties. Algebraic geometry. V. Encyclopaedia Math. Sci.*, **47**, Springer, Berlin, 1999.

[Isk80] V. A. Iskovskikh. Anticanonical models of three-dimensional algebraic varieties. *J. Sov. Math.*, 13:745–814, 1980.

[Kaw92] Y. Kawamata. Boundedness of **Q**-Fano threefolds. In *Proceedings of the International Conference on Algebra, Part* 3 (*Novosibirsk, 1989*), *Contemp. Math.* **131**, 439–445, Providence, RI, 1992. Amer. Math. Soc.

[KMMT00] J. Kollár, Y. Miyaoka, S. Mori, and H. Takagi. Boundedness of canonical **Q**-Fano 3-folds. *Proc. Japan Acad. Ser. A Math. Sci.*, 76(5):73–77, 2000.

[McK01] J. McKernan. A simple characterization of toric varieties. In *Proc. Algebraic Geom. Symp. Kinosaki*, 59–72, 2001.

[MM83] S. Mori and S. Mukai. On Fano 3-folds with $B_2 \ge 2$. In *Algebraic varieties and analytic varieties (Tokyo, 1981)*, *Adv. Stud. Pure Math.*, **1**, 101–129. North-Holland, Amsterdam, 1983.

[MM82] S. Mori and S. Mukai. Classification of Fano 3-folds with $B_2 \ge 2$. *Manuscripta Math.*, 36(2):147–162, 1981/82. Erratum: Manuscripta Math. 110 (2003), 407.

[Mor85] S. Mori. On 3-dimensional terminal singularities. *Nagoya Math. J.*, 98:43–66, 1985.

[Mor88] S. Mori. Flip theorem and the existence of minimal models for 3-folds. *J. Amer. Math. Soc.*, 1(1):117–253, 1988.

[MP09] S. Mori and Y. Prokhorov. Multiple fibers of del Pezzo fibrations. *Proc. Steklov Inst. Math.*, 264(1):131–145, 2009.

[Nam97] Y. Namikawa. Smoothing Fano 3-folds. *J. Algebraic Geom.*, 6(2):307–324, 1997.

[Nik80] V. V. Nikulin. Finite automorphism groups of Kähler K3 surfaces. *Trans. Mosc. Math. Soc.*, 2:71–135, 1980.

[Pro03] Y. Prokhorov. An application of the canonical bundle formula. *Proc. Steklov Inst. Math.*, 241:195–201, 2003.

[Pro09] Y. Prokhorov. Simple finite subgroups of the Cremona group of rank 3. arXiv: 0908.0678, 2009.

[Rei87] M. Reid. Young person's guide to canonical singularities. In *Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, *Proc. Sympos. Pure Math.*, **46**, 345–414. Amer. Math. Soc., Providence, RI, 1987.

[Shi89] K.-H. Shin. 3-dimensional Fano varieties with canonical singularities. *Tokyo J. Math.*, 12(2):375–385, 1989.

[Sho93] V. V. Shokurov. 3-fold log flips. *Russ. Acad. Sci., Izv. Math.*, 40(1):95–202, 1993.

Yuri Prokhorov: Department of Algebra, Faculty of Mathematics, Moscow State University, Moscow 117234, Russia

E-mail: prokhoro@gmail.com