RELATIONS IN THE CREMONA GROUP OVER PERFECT FIELDS

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Abstract. For perfect fields $\mathbf{k}$ satisfying $[\bar{\mathbf{k}}:\mathbf{k}]>2$, we construct new normal subgroups of the plane Cremona group and provide an elementary proof of its non-simplicity, following the melody of the recent proof by Blanc, Lamy and Zimmermann that the Cremona group of rank n over (subfields of) the complex numbers is not simple for $n \geqslant 3$.

Contents

1. Introduction

The Cremona group $\mathrm{Cr}_n(\mathbf{k}) = \mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ is the group of birational transformations of the projective n-space over a field $\mathbf{k}$. In dimension $n=2$ it is known [CL13, She13, Lon16] that the Cremona group over any field is not simple. For algebraically closed fields, the plane Cremona group $\mathrm{Cr}_2(\mathbf{k})$ is a perfect group [CD13, Corollaire 5.15], meaning that all group homomorphisms from the Cremona group to an abelian group are trivial. For many perfect fields, however, Lamy and Zimmermann constructed a surjective group homomorphism from the plane Cremona group to a free product of $\mathbb{Z}/2\mathbb{Z}$ [LZ19, Theorem C], implying non-perfectness and thus reproving

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non-simplicity of the Cremona group in these cases. Recently, Blanc, Lamy and Zimmermann managed to construct a surjective group homomorphism from the high-dimensional Cremona group $\mathrm{Cr}_n(\mathbf{k})$ to a free product of direct sums of $\mathbb{Z}/2\mathbb{Z}$, where $n \geqslant 3$ and $\mathbf{k} \subset \mathbb{C}$ is a subfield [BLZ19]. For the high-dimensional case, it turned out that it is more suitable not to use the high-dimensional analogy of [LZ19] but to take a different construction. The goal of this article is to adapt the strategy of [BLZ19] back to dimension two over perfect fields and find new normal subgroups of $\mathrm{Cr}_2(\mathbf{k})$. No knowledge of [BLZ19] is required to read our paper but we will highlight the connections to their proof. In fact, only classical algebraic geometry is used, and the well-established decomposition of birational maps into Sarkisov links (proven in [Isk96, Theorem 2.5] and [Cor95, Appendix], see Theorem 4 below). Therefore, the following result can be seen as an elementary proof of non-simplicity of the Cremona group over perfect fields whose extension degree of the algebraic closure is at least 2 (and thus infinite by Artin-Schreier):

Theorem 1. *For each perfect field* $\mathbf{k}$ *such that* $[\bar{\mathbf{k}} : \mathbf{k}] > 2$*, there exists a group homomorphism*

$$\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2) \twoheadrightarrow \bigoplus_I \mathbb{Z}/2\mathbb{Z}$$

where the indexing set I *is infinite and whose kernel contains* $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2) = \mathrm{PGL}_3(\mathbf{k})$ *such that the restriction to the subgroup that is locally given by*

$$\{(x,y) \mapsto (xp(y), y) \mid p \in \mathbf{k}(x)\backslash\{0\}\}$$

is surjective. In particular, the Cremona group $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ *is not perfect and thus not simple.*

The result is thus a 2-dimensional analogue of [BLZ19, Theorem A]. Note that over an algebraically closed field, the plane Cremona group is generated by $\mathrm{PGL}_3(\mathbf{k})$ and the standard quadratic transformation. Over non-closed fields, such a nice set of generators is not known (a set of generators can be found in [IKT93]). However, instead of looking at the group $\mathrm{Bir}(X)$ for a (smooth and projective) surface X we consider the groupoid $\mathrm{BirMori}(X)$, which is the set consisting of birational maps between surfaces that are birational to X. For a Mori fiber space (a simple fibration, see Definition 2.2), the groupoid $\mathrm{BirMori}(X)$ is generated by Sarkisov links of type I to IV (simple birational maps, see Definition 2.8) [Isk96, Theorem 2.5]. Whereas over an algebraically closed field the Sarkisov links are just the blow-up of one point (type III), or its inverse (type I), or the blow-up of one point followed by the contraction of one curve (type II), or an exchange of the fibration of $\mathbb{P}^1 \times \mathbb{P}^1$ (type IV), over a perfect field one has to consider orbits of the Galois action of $\mathrm{Gal}(\bar{\mathbf{k}}/\mathbf{k})$ on X. In this paper, the size of the orbits that lie in the base locus of a birational map is going to be important. We say that the *cardinality* of a birational map φ is the maximal size of an orbit that lies in the base locus of φ or φ^{-1}.

So we do not directly construct a group homomorphism from $\mathrm{Bir}(X)$, but we first construct a groupoid homomorphism from $\mathrm{BirMori}(X)$ and then take the restriction to $\mathrm{Bir}(X)$. For this one has to study relations of the groupoid. Note that in [Isk96] there is a long and complicated list of generating relations. In [LZ19] the focus lies on Bertini involutions (the blow-up of an orbit of size 8 in $\mathbb{P}^2$, followed by the contraction of an orbit of curves of size 8). For higher dimensions, the focus lies on links of type II between conic bundles that have a large covering gonality (see [BLZ19] for definitions). Translating this back to the 2-dimensional case, we focus on links of type II between conic bundles (that is, a Mori fiber space $X \to B$ where B is a curve) that have a large cardinality and find the following generating relations:

Theorem 2. *Let X be a Mori fiber space. Relations of the groupoid* $\mathrm{BirMori}(X)$ *are generated by the trivial relations and relations of the following form:*

(a) $\varphi_n \cdots \varphi_1 = \mathrm{id}$*, where the cardinality of all* φ_i *is* $\leqslant 15$*, and*
(b) $\chi_4\chi_3\chi_2\chi_1 = \mathrm{id}$ *where* $\chi_i \colon X_{i-1} \dashrightarrow X_i$ *are links of type II between conic bundles such that the links* χ_1 *and* χ_3 *are equivalent and* χ_2 *and* χ_4 *are equivalent.*

(For the notion of equivalent links see Remark 3.4 Definition 5.2.) This can be compared with [BLZ19, Proposition 5.5]. It would be nice to have the number 8 instead of 15, but some technicalities in Lemma 3.10 deny us this pleasure. One may however observe that in dimension $n \geqslant 3$ the bound on the covering gonality given in [BLZ19], the analogue of the cardinality, is not explicit. Using these generating relations, we are finally able to construct a groupoid homomorphism. (For the notation: $\mathrm{CB}(X)$ denotes the set of equivalence classes of conic bundles, and $\mathrm{M}(C)$ denotes the set of equivalence classes of Sarkisov links between conic bundles equivalent to C; see Definitions 5.1 and 5.2.)

Theorem 3. *Let X be a Mori fiber space. There exists a groupoid homomorphism*

$$\mathrm{BirMori}(X) \to \mathop{\ast}_{C \in \mathrm{CB}(X)} \bigoplus_{\chi \in \mathrm{M}(C)} \mathbb{Z}/2\mathbb{Z}$$

that sends each Sarkisov link χ of type II between conic bundles that is of cardinality $\geqslant 16$ *onto the generator indexed by its equivalence class, and all other Sarkisov links and all automorphisms of Mori fiber spaces birational to X onto zero.*

Moreover it restricts to group homomorphisms

$$\mathrm{Bir}(X) \to \mathop{\ast}_{C \in \mathrm{CB}(X)} \bigoplus_{\chi \in \mathrm{M}(C)} \mathbb{Z}/2\mathbb{Z}, \qquad \mathrm{Bir}(X/W) \to \bigoplus_{\chi \in \mathrm{M}(X/W)} \mathbb{Z}/2\mathbb{Z}.$$

This is analogue to [BLZ19, Theorem D]. Note that the group homomorphism of Theorem 3 is trivial if the field does not admit large orbits. If the group homomorphism is not trivial, the kernel is a non-trivial normal subgroup of $\mathrm{Bir}(X)$. For perfect fields $\mathbf{k}$ that admit a large orbit and $X = \mathbb{P}^2_{\mathbf{k}}$, we provide an example that shows that the group homomorphism is not trivial and therefore obtain Theorem 1.

The paper is structured as follows: In Section 2 we introduce the notion of Mori fiber space and Sarkisov link, and state some basic but important remarks about the cardinality of birational maps. In Section 3 we study relations of $\mathrm{BirMori}(X)$ and prove Theorem 2. Then, we make a detour to Galois theory in Section 4 to establish that a perfect field $\mathbf{k}$ with $[\bar{\mathbf{k}} : \mathbf{k}] > 2$ has arbitrarily large orbits (Lemma 4.2), and that there are such fields that do not have an orbit of size exactly 8 (Lemma 4.3). The latter is to contrast our result with [LZ19]. We end with Section 5, where we prove Theorem 3 and finally Theorem 1.

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2. Preliminaries

Consider a perfect field $\mathbf{k}$ and the Galois group $\Gamma = \mathrm{Gal}(\bar{\mathbf{k}}/\mathbf{k}) = \mathrm{Aut}(\bar{\mathbf{k}}/\mathbf{k})$. We will work over the algebraic closure $\bar{\mathbf{k}}$, equipped with the Galois action of Γ. A perfect field is a field such that every algebraic extension is separable. We will use the following property of perfect fields: A field $\mathbf{k}$ is perfect if and only if the extension $\bar{\mathbf{k}}/\mathbf{k}$ is normal and separable, which means that $\bar{\mathbf{k}}/\mathbf{k}$ is Galois. In particular, the field fixed by the action of Γ is exactly $\mathbf{k}$ [Lan05, Theorem 1.2, Chapter VI]. So a point is fixed by the Galois action if and only if it is defined over $\mathbf{k}$.

We are interested in surfaces. **In the sequel, we assume all surfaces to be smooth and projective.** A (rational) map $\varphi\colon X \dashrightarrow Y$ is always supposed to be defined over $\mathbf{k}$ (and thus X and Y are defined over $\mathbf{k}$, too). However, we will look at $\bar{\mathbf{k}}$-points and $\bar{\mathbf{k}}$-curves on our surfaces.

2.1. Spaces of interest: Mori fiber spaces.

Definition 2.1. Let X be a surface and $\pi\colon X \to B$ a surjective morphism to a smooth variety B. The *relative Picard group* is the quotient $\mathrm{Pic}_{\mathbf{k}}(X/B) := \mathrm{Pic}_{\mathbf{k}}(X)/\pi^* \mathrm{Pic}_{\mathbf{k}}(B)$.

We study Mori fiber spaces and only consider surfaces X over $\mathbf{k}$. So the definition is as follows:

Definition 2.2. A surjective morphism $X \to B$, where X is a smooth surface and B is smooth, is called a *Mori fiber space* if the following conditions are satisfied:

(1) $\dim(B) < \dim(X)$,
(2) $\mathrm{rk}\,\mathrm{Pic}_{\mathbf{k}}(X/B) = 1$ (relative Picard rank),
(3) $-K_X \cdot D > 0$ for all $D \in \mathrm{Pic}_{\mathbf{k}}(X/B)$.

If B is 1-dimensional, we say that $X \to B$ is a *conic bundle*.

Remark 2.3. Any Mori fiber space $X \to B$ is of one of the two following forms, depending on the dimension of the base B:

(1) If $B = \{*\}$ then $\operatorname{rk}\operatorname{Pic}_{\mathbf{k}}(X) = 1$ and so X is a del Pezzo surface.
(2) If B is a curve, then $X \to B$ is a conic bundle, and the fiber of each $\bar{\mathbf{k}}$-point of B is isomorphic to a reduced conic in $\mathbb{P}^2$ (irreducible or reducible). So a general fiber is isomorphic to $\mathbb{P}^1$ and any singular fiber is the union of two (-1)-curves intersecting at one point. Moreover, the two irreducible components of any singular fiber lie in the same Galois orbit. (Otherwise, the orbit of one component consists of disjoint (-1)-curves. Contracting them yields a surface $S \to B$ with $\operatorname{rk}(\operatorname{Pic}(S/B)) \geqslant 1$. Then $\operatorname{rk}(\operatorname{Pic}_{\mathbf{k}}(X/B)) = \operatorname{rk}(\operatorname{Pic}_{\mathbf{k}}(S/B)) + 1 \geqslant 2$, a contradiction.)

If B is 1-dimensional and X is geometrically rational with a $\mathbf{k}$-point, the following lemma implies that $B = \mathbb{P}^1$ and so $\operatorname{rk}\operatorname{Pic}_{\mathbf{k}}(X) = 2$.

Lemma 2.4. *Let $\pi\colon X \to B$ be a conic bundle. Then X is geometrically rational if and only if the genus of B is 0. In this case, either $B = \mathbb{P}^1$ or B is a smooth projective curve without $\mathbf{k}$-point.*

Proof. Assume that X is geometrically rational, so there is a birational map $\varphi\colon X_{\bar{\mathbf{k}}} \dashrightarrow \mathbb{P}^2$. Hence, $\pi\varphi^{-1}\colon \mathbb{P}^2 \dashrightarrow B_{\bar{\mathbf{k}}}$ is a dominant rational map and so B is geometrically unirational. As B is a curve, the solution of Lüroth's problem implies that B is geometrically rational. So the genus of B is zero.

The converse direction is a corollary of Tsen's theorem [Kol99, Corollary 6.6.2, p. 232], which states that $X_{\bar{\mathbf{k}}}$ is birational to $B_{\bar{\mathbf{k}}} \times \mathbb{P}^1$ and hence $X_{\bar{\mathbf{k}}}$ is rational. □

Definition 2.5. Let $X_1 \to B_1$ and $X_2 \to B_2$ be two Mori fiber spaces. We say that a birational map $\varphi\colon X_1 \dashrightarrow X_2$ *preserves the fibration* if the diagram

$$\begin{tikzcd} X_1 \arrow[r, dashed, "\varphi"] \arrow[d] & X_2 \arrow[d] \\ B_1 \arrow[r, phantom, "\simeq"] & B_2 \end{tikzcd}$$

commutes. Moreover, if φ is also an isomorphism we say that $\varphi\colon X_1 \xrightarrow{\sim} X_2$ is an *isomorphism of Mori fiber spaces*.

2.2. **Maps of interest: Sarkisov links.**

Definition 2.6. Let $X \to B$ be a Mori fibration. We denote by $\operatorname{BirMori}(X)$ the groupoid consisting of all birational maps $\varphi\colon X_1 \dashrightarrow X_2$ where $X_i \to B_i$ are Mori fiber spaces for $i = 1, 2$ such that X_1 and X_2 are birational to X.

Definition 2.7. For a Mori fiber space $X \xrightarrow{\pi} W$ we denote by $\operatorname{Bir}(X/W) \subset \operatorname{Bir}(X)$ the subgroup of birational maps $f \in \operatorname{Bir}(X)$ that preserve the fibration.

Definition 2.8. A *Sarkisov link* (or simply *link*) is a birational map $\varphi\colon X_1 \dashrightarrow X_2$ between two Mori fibrations $\pi_i\colon X_i \to B_i$, $i = 1, 2$, that is of one of the following four types:

Type I
$$\begin{tikzcd} X_1 \arrow[r, dashed, "\varphi"] \arrow[d] & X_2 \arrow[d] \\ \{*\} = B_1 & B_2 \arrow[l] \end{tikzcd}$$
where $\varphi^{-1}\colon X_2 \to X_1$ is the blow up of one orbit of $\mathrm{Gal}(\bar{\mathbf{k}}/\mathbf{k})$.

Type II
$$\begin{tikzcd} & Z \arrow[dl, "\sigma_1"'] \arrow[dr, "\sigma_2"] & \\ X_1 \arrow[rr, dashed, "\varphi"] \arrow[d] & & X_2 \arrow[d] \\ B_1 \arrow[rr, "\simeq"] & & B_2 \end{tikzcd}$$
where $\sigma_i\colon Z \to X_i$ is a blow-up of one orbit.

Type III
$$\begin{tikzcd} X_1 \arrow[r, "\varphi"] \arrow[d] & X_2 \arrow[d] \\ B_1 \arrow[r] & \{*\} = B_2 \end{tikzcd}$$
where $\varphi\colon X_1 \to X_2$ is the blow-up of one orbit.

Type IV
$$\begin{tikzcd} X_1 \arrow[r, "\varphi", "\simeq"'] \arrow[d] & X_2 \arrow[d] \\ B_1 & B_2 \end{tikzcd}$$
where φ is an isomorphism that does not preserve the fibration and B_1, B_2 are curves.

Theorem 4 ([Isk96, Theorem 2.5] and [Cor95, Appendix])**.** *Let X be a geometrically rational Mori fiber space. The groupoid* $\mathrm{BirMori}(X)$ *is generated by Sarkisov links and isomorphisms of Mori fiber spaces.*

The following two lemmas follow from the classification of Sarkisov links [Isk96, Theorem 2.6]. In order to keep the promise of providing elementary proofs, we reprove the statement for completeness.

Lemma 2.9. *Let $\varphi\colon X_1 \to X_2$ be a link of type III.*

(1) *The cone of effective curves $\mathrm{NE}_{\mathbb{Q}}(X_1)$ equals $\mathbb{Q}_{\geqslant 0}f + \mathbb{Q}_{\geqslant 0}E$, where f is a fiber of the Mori fiber space $X_1 \to B_1$ and E is the exceptional locus of φ.*
(2) *X_1 is a del Pezzo surface.*
(3) *The cardinality r of the orbit blown-up by φ is less or equal than* 8.

Proof. As E is an orbit of r disjoint (-1)-curves, none of these is contained in a fiber of $X_1 \to B_1$ (see Remark 2.3). Hence, $f \cdot E > 0$ and so f and E are not linearly equivalent because $f^2 = 0$ and $E^2 = -r < 0$. As the rank of the Picard group $\mathrm{Pic}_{\mathbf{k}}(X_1)$ is 2 (since the rank of the Picard group of X_2 is 1), any curve C in $\mathrm{NE}_{\mathbb{Q}}(X_1)$ is linearly equivalent to $\alpha f + \beta E$ for some $\alpha, \beta \in \mathbb{Q}$. We want to prove that $\alpha, \beta \geqslant 0$. If $C = E$, then $\alpha = 0$ and $\beta = 1$, so assume that $C \neq E$. We compute

$$0 \leqslant f \cdot C = \beta E \cdot f,$$

hence $\beta \geqslant 0$. We also find

$$0 \leqslant E \cdot C = \alpha E \cdot f + \beta E^2,$$

which implies that $\alpha \geqslant 0$, since $\beta E^2 \leqslant 0$ and $E \cdot f > 0$. This proves (1).

For (2), we prove ampleness of $-K_{X_1}$ using Kleiman's Criterion. Note that the expression of $\mathrm{NE}_{\mathbb{Q}}(X_1)$ in (1) is closed, hence $\overline{\mathrm{NE}_{\mathbb{Q}}(X_1)}\backslash\{0\} = (\mathbb{Q}_{\geqslant 0}f + \mathbb{Q}_{\geqslant 0}E)\backslash\{0\}$. We compute for $(\alpha, \beta) \in \mathbb{Q}^2_{\geqslant 0}\backslash\{(0,0)\}$

$$-K_{X_1}(\alpha f + \beta E) = \alpha(-K_{X_1}f) + \beta(-K_{X_1}E) = 2\alpha + \beta r > 0,$$

where we used the adjunction formula to compute $-K_{X_1}f$ and $-K_{X_1}E$. Therefore, Kleiman's Criterion implies that $-K_{X_1}$ is ample, and so (2) holds.

For (3), note that (2) implies in particular that $0 < (-K_{X_1})^2 \leqslant 9$. Since $\varphi\colon X_1 \to X_2$ is a blow-up of r points (over $\bar{\mathbf{k}}$), we also have $(-K_{X_1})^2 = (-K_{X_2})^2 - r$. This gives $r < (-K_{X_1})^2 \leqslant 9$, hence $r \leqslant 8$. □

Lemma 2.10. *For $i = 1, 2$, let $X_i \to \{*\}$ be two Mori fiber spaces and let $\varphi\colon X_1 \dashrightarrow X_2$ be a link of type II that has a resolution $\sigma_i\colon Z \to X_i$, where σ_i is the blow-up of an orbit of size r_i with exceptional divisor E_i.*

(1) *The cone of effective curves $\mathrm{NE}_{\mathbb{Q}}(Z)$ equals $\mathbb{Q}_{\geqslant 0}E_1 + \mathbb{Q}_{\geqslant 0}E_2$.*
(2) *Z is a del Pezzo surface. In particular, $r_i \leqslant 8$ for $i = 1, 2$.*

Proof. For (1) it is enough to show that E_1 and E_2 are (different) extremal rays in the cone of effective 1-cycles $\mathrm{NE}(Z)$ since $\mathrm{rk}\,\mathrm{Pic}_{\mathbf{k}}(Z) = 2$. First, we remark that $\mathrm{NE}_{\mathbb{Q}}(X_1) = -K_{X_1} \cdot \mathbb{Q}_{\geqslant 0}$: Having $\mathrm{rk}\,\mathrm{Pic}(X_1) = 1$, there is a curve C on X_1 such that $\mathrm{NE}_{\mathbb{Q}}(X_1) = \mathbb{Q}_{\geqslant 0}C$. As $-K_{X_1}$ is ample (since X_1 is del Pezzo), $-K_{X_1}$ is effective and non-zero, hence $-K_{X_1} = \lambda C$ for some $\lambda > 0$. Therefore, $\mathrm{NE}_{\mathbb{Q}}(X_1) = -K_{X_1}\mathbb{Q}_{\geqslant 0}$. Now, let $D \in \mathrm{NE}_{\mathbb{Q}}(Z)$ be such that D is no multiple of E_1. Hence, $\pi_*(D)$ is effective. As $D \in \mathrm{Pic}(Z) = \mathbb{Q}E_1 + \mathbb{Q}\pi^*(-K_{X_1})$, we can write $D \sim a\pi^*(-K_{X_1}) + bE_1$ with $a \geqslant 0$. Therefore, $\mathrm{NE}_{\mathbb{Q}}(Z)$ lies in $\mathbb{Q}_{\geqslant 0}\pi^*(-K_{X_1}) + \mathbb{Q}E_1$. Hence, E_1 is extremal. The same argument works for E_2. The two extremal rays E_1 and E_2 are different because E_1 is effective with $E_1^2 < 0$ but $E_1E_2 \geqslant 0$ (φ is not an isomorphism, hence E_1 and E_2 are distinct).

For (2), compute with the adjunction formula $-K_ZE_i = r_i$. For $\alpha, \beta \in \mathbb{Q}_{\geqslant 0}$, not both zero, this gives

$$-K_Z(\alpha E_1 + \beta E_2) = \alpha r_1 + \beta r_2 > 0$$

and Kleiman criterion implies that $-K_Z$ is ample, hence Z is del Pezzo. Note that $r_i \leqslant 8$ follows in the same way as the proof of (3) of Lemma 2.9. □

This leads us to one of the main points of this article: The *cardinality* of birational maps, which plays in our article the role of the covering gonality in [BLZ19, Theorem D].

2.3. **Cardinality of birational maps.**

Definition 2.11. Let $\varphi\colon X \dashrightarrow X'$ be a birational map between surfaces. The *cardinality* of φ is the maximal size of all orbits contained in the base loci of φ and φ^{-1}.

Remark 2.12. A Sarkisov link $\varphi\colon X_1 \dashrightarrow X_2$ has cardinality $\leqslant 8$ – except if it is a link of type II between conic bundles. Indeed, the statement for links of type I and III is implied by Lemma 2.9, for type II it is Lemma 2.10, and links of type IV do not have base points so they have cardinality 0.

Remark 2.13. Any $f \in \mathrm{BirMori}(X)$ can be decomposed as

$$\begin{tikzcd}
Y_0 \arrow[r, dashed, "\Psi_1"] & X_1 \arrow[r, dashed, "\Phi_1"] \arrow[d] & Y_1 \arrow[r, dashed, "\Psi_2"] \arrow[d] & X_2 \arrow[r, dashed] \arrow[d] & Y_{N-1} \arrow[r, dashed, "\Psi_N"] \arrow[d] & X_N \arrow[r, dashed, "\Phi_N"] \arrow[d] & Y_N \arrow[r, dashed, "\Psi_{N+1}"] \arrow[d] & X_{N+1}, \\
 & V_1 \arrow[r, phantom, "\simeq"] & W_1 & V_2 & W_{N-1} & V_N \arrow[r, phantom, "\simeq"] & W_N &
\end{tikzcd}$$

where $\Psi_i\colon Y_{i-1} \dashrightarrow X_i$ are compositions of links of cardinality $\leqslant 8$, and $\Phi_i\colon X_i \dashrightarrow Y_i$ are a composition of links of type II. The $X_i \to V_i, Y_i \to W_i$ are conic bundles for $i = 1, \ldots, N$.

3. Relations

Let X be a Mori fiber space (of dimension 2). By Theorem 4, the groupoid $\mathrm{BirMori}(X)$ is generated by Sarkisov links and isomorphisms of Mori fiber spaces. We study the set of relations of these generators. The following two relations will be called *trivial*:

- $\alpha\beta = \gamma$, where α, β, γ are isomorphisms of Mori fiber spaces,
- $\alpha\psi^{-1}\varphi = \mathrm{id}_X$, where $\varphi\colon X \dashrightarrow Y$ and $\psi\colon Z \dashrightarrow Y$ are Sarkisov links and $\alpha\colon Z \xrightarrow{\sim} X$ is an isomorphism of Mori fiber spaces.

Any element of $\mathrm{BirMori}(X)$ that is not an isomorphism of Mori fiber spaces is a product of Sarkisov links.

3.1. **Relations between conic bundles.**

Lemma 3.1. *Let $X \to V$ and $Y \to W$ be two conic bundles and let $\varphi\colon X \dashrightarrow Y$ be a birational map such that every curve contracted by φ is contained in a fiber of $X \to V$. Then there is a composition $\varphi = \alpha\varphi_n \circ \cdots \circ \varphi_1$ of Sarkisov links φ_i of type II between conic bundles such that each φ_i is the blow-up of one orbit of $r_i \geqslant 1$ distinct points on r_i distinct smooth fibers, followed by the contraction of the strict transforms of these fibers, where α is an isomorphism (not necessarily of Mori fiber spaces).*

Proof. Consider the minimal resolution

$$\begin{tikzcd}
 & S \arrow[dl, "\sigma"'] \arrow[dr, "\tau"] & \\
X \arrow[rr, dashed, "\varphi"'] & & Y,
\end{tikzcd}$$

where σ and τ are compositions of blow-ups in points. Let $C \subset S$ be a (-1)-curve contracted by τ. So $\sigma(C)$ is either a smooth fiber or a component of a singular fiber.

Let us show that it is not possible that $\sigma(C) = F$ is a component of a singular fiber. In this case, the self-intersection of F is -1. Hence, no point on F is a base point of σ. (Otherwise, $C^2 \leqslant -2$.) Let E be the other irreducible component of the fiber containing F, so $E^2 = -1$. By Remark 2.3, E and F lie in one orbit. As τ contracts $C = \tilde{E}$, it also contracts $\tilde{F}$ (since they are in the same orbit). This is not possible, since after the contraction of C onto a point, the push-forward of $\tilde{F}$ is a curve of self-intersection 0 and can therefore not be contracted.

Hence, $\sigma(C) = F$ is a smooth fiber and so its self-intersection is 0. As σ is a composition of blow-ups, there is one base point $p \in F$ of σ, and it is the only base point of σ on the fiber $\sigma(C)$. (Otherwise, the self-intersection of C would be $\leqslant -2$.) So all points of the orbit of p are base points of σ, and no two lie on the same fiber. (Otherwise, there would also be a second base point p' on $\sigma(C)$.) Therefore, τ contracts all fibers through the orbit. Let φ_1 be the blow-up of the orbit of p followed by the contraction of the strict transforms of the fibers through the orbit. This is a link of type II between conic bundles, and φ factors through φ_1. Moreover, $\varphi \circ \varphi^{-1}$ has fewer base points.

Repeating this process for all (-1)-curves that are contracted by τ gives an isomorphism $\alpha = \varphi \circ \varphi_1^{-1} \circ \cdots \circ \varphi_n^{-1}$, that is, $\varphi = \alpha\varphi_n \circ \cdots \circ \varphi_1$ where all φ_i are blow-ups of an orbit followed by the contraction of the strict transforms of the fibers through it, as in the statement of the lemma. □

Corollary 3.2. *Let $X \to V$ and $Y \to W$ be two Mori fibers and let $\varphi\colon X \dashrightarrow Y$ be a birational map that preserves the fibration. If φ is not an isomorphism, then $\varphi = \varphi_n \circ \cdots \circ \varphi_1$ for Sarkisov links φ_i of type II between conic bundles such that each φ_i is the blow-up of one orbit of $r_i \geqslant 1$ distinct points on r_i distinct smooth fibers, followed by the contraction of the strict transforms of these fibers.*

Proof. As in the proof of Lemma 3.1, consider the minimal resolution $\sigma\colon S \to X$ and $\tau\colon S \to Y$ of φ. Let $C \subset S$ be a (-1)-curve contracted by τ. Since φ preserves the fibration, $\sigma(C)$ lies in a fiber. By the minimality of the resolution, $\sigma(C)$ is not a point. So $\sigma(C)$ is a curve contained in a fiber of $X \to V$ and Lemma 3.1 can be applied. Hence, $\varphi = \alpha\varphi_n \circ \cdots \circ \varphi_1$ where the φ_i are links of type II as desired and α is an isomorphim. As φ and each of the φ_i preserve the fibration, also α preserves the fibration and is therefore an isomorphism of Mori fiber spaces. Hence $\alpha\varphi_n$ is also a link of type II between conic bundles and the corollary follows. □

Lemma 3.3. *Let X be a Mori fiber space that is not geometrically rational. Then,* $\mathrm{BirMori}(X)$ *is generated by links of type II and isomorphisms of Mori fiber spaces.*

Proof. Let $\varphi\colon X_1 \dashrightarrow X_2$ be a birational map in $\operatorname{BirMori}(X)$, where $X_i \to B_i$ are Mori fiber spaces where B_i are curves of genus at least 1. Any curve C in X_1 contracted by φ is rational, and so every morphism $C \to B_1$ is constant (by Riemann-Hurwitz). Therefore, C is contained in a fiber. Lemma 3.1 can be applied and therefore $\varphi = \alpha\varphi_n \cdots \varphi_1$, where α is an isomorphism and φ_i are links of type II between conic bundles. If α preserves the fibration, then we are in the case of Corollary 3.2 and so φ is a product of links of type II. If α does not preserve the fibration, $\alpha\colon Y_1 \xrightarrow{\sim} Y_2$ is a link of type IV, where $Y_i \to C_i$ are Mori fiber spaces. Such a link does not exist for geometrically non-rational surfaces: Consider $F_2 \subset Y_2$ a general fiber (hence isomorphic to $\mathbb{P}^1$) of $Y_2 \to C_2$ and consider its image $F_1 = \alpha^{-1}(F_2)$ in Y_1, which is also isomorphic to $\mathbb{P}^1$. As F_2 was chosen to be a general fiber and α does not preserve the fibration, the restriction of the fibration $Y_1 \to C_1$ to F_1 is surjective. This gives a contradiction, since every map from $\mathbb{P}^1 \to C_1$ is constant as before. $\square$

Remark 3.4. Let $\chi_1\colon X_0 \dashrightarrow X_1$ and $\chi_2\colon X_1 \dashrightarrow X_2$ be two links of type II between conic bundles $X_i \to B_i$ such that no fiber of $X_1 \to B_1$ is contracted by both χ_1^{-1} and χ_2. Then, the composition $\chi_2\chi_1$ can be written as $\chi_3^{-1}\chi_4^{-1}$, where $\chi_3\colon X_2 \dashrightarrow X_3$ is the blow-up of $\chi_2(\operatorname{Bas}(\chi_1^{-1}))$ followed by the contraction of the strict transform of the corresponding fiber, and $\chi_4\colon X_3 \dashrightarrow X_4$ is the blow-up of $\chi_3(\operatorname{Bas}(\chi_2^{-1}))$ followed by the contraction of the strict transform of the corresponding fiber. So $\chi_4\chi_3\chi_2\chi_1$ is an isomorphism of Mori fiber spaces and χ_4 can be chosen such that $\chi_4\chi_3\chi_2\chi_1 = \mathrm{id}$.

Note that χ_1 and χ_3 have the same cardinality, as well as χ_2 and χ_4.

Lemma 3.5. *Let X be a conic bundle and let $\varphi_n \cdots \varphi_1 = \mathrm{id}_X$ be a relation in* $\operatorname{BirMori}(X)$ *such that $\varphi_i\colon X_{i-1} \dashrightarrow X_i$ is a link of type II between conic bundles, where $X_0 = X_n = X$. This relation is generated in* $\operatorname{BirMori}(X)$ *by the trivial relations and those of the form* $\chi_4\chi_3\chi_2\chi_1 = \mathrm{id}$ *as in Remark 3.4, where $\chi_1, \ldots, \chi_4$ are links of type II between conic bundles.*

Proof. For each link φ_i of type II we call $p_{i-1} \subset X_{i-1}$ the base orbit of φ_i, and $q_i \subset X_i$ the base orbit of φ_i^{-1}.

In the following, we show that using relations of the form $\chi_4\chi_3\chi_2\chi_1 = \mathrm{id}$ and the trivial relations, we can reduce the word $\varphi = \varphi_n \cdots \varphi_1$ to the empty word.

Starting from a fiber $F = F_0 \subset X_0$ and the value $\mathcal{N}(F, 0) = 0$, we define a sequence of subsets $F_i = (\varphi_i \cdots \varphi_1)(F) \subset X_i$, and a sequence of values $\mathcal{N}(F, i) \in \mathbb{N}$ for $i = 1, \ldots, n$ that "keep track of what happens to F" in each step of our fixed decomposition $\varphi_n \circ \cdots \circ \varphi_1$.

We inductively define $\mathcal{N}(F, i) \geqslant 0$ in the following way:

(1) If φ_i is a local isomorphism on the fiber containing F_{i-1}, then

$$\mathcal{N}(F, i) = \mathcal{N}(F, i-1).$$

(2) Otherwise, F_{i-1} lies on the same fiber as a point of p_{i-1}. We define

$$\mathcal{N}(F,i) = \begin{cases} 1 & \text{if } F_{i-1} \text{ is a fiber (so } F_i \text{ is a point in } q_i), \\ \mathcal{N}(F,i-1)-1 & \text{if } F_{i-1} \text{ is a point in } p_{i-1}, \\ \mathcal{N}(F,i-1)+1 & \text{if } F_{i-1} \text{ is a point but not in } p_{i-1} \\ & \text{(again, } F_i \text{ is a point in } q_i). \end{cases}$$

Observe that $\mathcal{N}(F,i)$ is the number of base points of $\varphi_i \circ \cdots \circ \varphi_1$ that are equal or infinitely near to a base point on F. Note that the sequence

$$\Sigma_F = \big(\mathcal{N}(F,0), \mathcal{N}(F,1), \ldots, \mathcal{N}(F,n)\big)$$

is the same for each fiber in the same orbit as F. We consider connected subsequences of Σ_F and note that the last value $\mathcal{N}(F,n)$ is zero as the product of all the φ_i is the identity.

(1) First, we look at subsequences of the form $(m-1, m, m-1)$ for $m \geqslant 1$ with corresponding links φ_i and φ_{i+1}. This occurs only if φ_i^{-1} and φ_{i+1} have common base points, so $\varphi_{i+1}\varphi_i$ equals an isomorphism. Hence, this part of the sequence is equivalent to the empty word modulo the trivial relations.

(2) Now, we consider subsequences of the form $(m-1, m, m)$ with corresponding links φ_i, which is not an isomorphism on F, and φ_{i+1}, which is a local isomorphism on F. In this case, the fibers of $X_i \to B_i$ that are contracted by φ_i^{-1} are not contracted by φ_{i+1}. By Remark 3.4, there exist links χ_i and χ_{i+1} of type II between conic bundles such that χ_i has $\varphi_{i+1}(\mathrm{Bas}(\varphi_i^{-1}))$ as base points, and χ_{i+1} has base points $\chi_i(\mathrm{Bas}(\varphi_{i+1}))$, and $\varphi_{i+1}\varphi_i = \chi_i^{-1}\chi_{i+1}^{-1}$ is satisfied. So by replacing $\varphi_{i+1}\varphi_i$ with $\chi_i^{-1}\chi_{i+1}^{-1}$ in the factorisation of φ, we can change this part of the sequence to $(m-1, m-1, m)$, leaving the rest of it invariant.

Using these two kinds of reduction modulo the said relations, we proceed by induction over

$$m = m_F = \max\{\mathcal{N}(F,i) \mid i = 0, \ldots, n\}.$$

There exists at least one subsequence $\Sigma' = (m-1, m, \ldots, m, m-1)$ of size $k+2$ for some $k \geqslant 1$. Using (2) multiple times we can change this part of the sequence to $(m-1, \ldots, m-1, m, m-1)$. This can then be reduced to $(m-1, \ldots, m-1)$ of size $k+1$ with (1). Doing this for any such Σ', we get a new sequence (corresponding to the new factorisation of φ) whose maximum is $m-1$. By induction, we find the sequence $(0, \ldots, 0)$. Hence, φ is a local isomorphism on F.

Note that $m_F \geqslant 1$ only for finitely many fibers F. We repeat the described process for each fiber F with $m_F \geqslant 1$ and can therefore reduce φ to an isomorphism using the trivial relations and compositions of four links of type II between conic bundles. $\square$

Corollary 3.6. *Let $\Delta \geqslant 1$. Let $X_1 \to B_1$ and $X_2 \to B_2$ be two conic bundles. Let $\varphi = \varphi_n \cdots \varphi_1 \colon X_1 \dashrightarrow X_2$ be a composition of links of type II. Then we can write φ modulo the trivial relations and those of the form $\chi_4\chi_3\chi_2\chi_1$, where χ_i are links of type II between conic bundles, as $\psi_n \cdots \psi_1$, where $\psi_1, \dots, \psi_r$ are of cardinality $\geqslant \Delta$ and $\psi_{r+1}, \dots, \psi_n$ are of cardinality $< \Delta$.*

Proof. We use relations of Lemma 3.5. If φ_i and φ_{i+1} are centered at the same fiber (that is, $\mathrm{Bas}(\varphi_i^{-1})$ lies on the same orbit of fibers as $\mathrm{Bas}(\varphi_{i+1})$), then they have the same cardinality. If φ_i and φ_{i+1} are centered at different fibers and the cardinality of φ_i is $< \Delta$ and the one of φ_{i+1} is $\geqslant \Delta$, let $\chi_1 = \varphi_i$ and $\chi_2 = \varphi_{i+1}$. As in Remark 3.4 there are links χ_3 (of the same cardinality as χ_1, hence $< \Delta$) and χ_4 (of the same cardinality as χ_2, hence $\geqslant \Delta$) such that $\chi_4\chi_3\chi_2\chi_1 = \mathrm{id}$. Therefore, by replacing $\chi_2\chi_1$ with $\chi_3^{-1}\chi_4^{-1}$ we have the desired order of orbit sizes on these two elements. In this manner we can move all small orbits to the end of the composition. □

Lemma 3.7. *Let $X \to B$ be a conic bundle that is not geometrically rational. Relations of the groupoid $\mathrm{BirMori}(X)$ are generated by the following relations:*

- *(a) $\varphi_n \circ \cdots \circ \varphi_1 = \mathrm{id}$, where the cardinality of all φ_i is $\leqslant 15$, and*
- *(b) $\chi_4\chi_3\chi_2\chi_1 = \mathrm{id}$ where $\chi_i \colon X_{i-1} \dashrightarrow X_i$ are links of type II between conic bundles.*

Proof. By Lemma 2.4, the genus of B is $\geqslant 1$. Links of type I or III do not occur as they are between del Pezzo surfaces (see Lemma 2.9), which are geometrically rational. Also no links of type IV are possible by Lemma 3.3. Therefore, only links of type II between conic bundles occur. By Lemma 3.5, they are generated by links as in (b). □

We want to prove the same theorem for the more interesting case of geometrically rational Mori fiber spaces.

3.2. **Geometrically rational conic bundles.**

Lemma 3.8. *Let $X \to B$ be a geometrically rational conic bundle. Then, each effective divisor D on X is linearly equivalent to $-aK_X + bf$ for $a, b \in \frac{1}{2}\mathbb{Z}$ and $a \geqslant 0$. Moreover, if the support of D is not contained in fibers, then $a \geqslant \frac{1}{2}$.*

Proof. By the adjunction formula, we have $-K_X f = 2$. Hence, $-K_X$ and f are linearly independent as $f^2 = 0$. By the geometrical rationality of X, we have that $B_{\overline{\mathbf{k}}} = \mathbb{P}^1_{\overline{\mathbf{k}}}$, hence the Picard rank of X is 2. Hence, there are $a, b \in \mathbb{Q}$ with $D \sim -aK_X + bf$. As D is effective, we have $0 \leqslant Df = 2a \in \mathbb{N}$. Moreover, if the support of D is not contained in fibers, $a > 0$. So $a \geqslant 0$ and $a \in \mathbb{Z}\frac{1}{2}$. Since X is geometrically rational, there exists a section s on X defined over $\overline{\mathbf{k}}$ (by a corollary to Tsen's theorem [Kol99, Corollary 6.6.2, p.232]). As $-K_X s$ is an integer and $Ds = a(-K_X s) + b$, we also find that $b \in \frac{1}{2}\mathbb{Z}$. □

Lemma 3.9. *Let $X_1 \to W_1$ and $X_2 \to W_2$ be two conic bundles that are geometrically rational and let $\varphi\colon X_1 \dashrightarrow X_2$ be a birational map that preserves the fibration, that is the diagram*

$$\begin{tikzcd} X_1 \arrow[r, dashed, "\varphi"] \arrow[d] & X_2 \arrow[d] \\ W_1 \arrow[r, phantom, "\simeq"] & W_2 \end{tikzcd}$$

commutes. Let $H_1 \sim -\lambda_1 K_{X_1} + \nu_1 f$ be a linear system without fixed component on X, and let $H_2 \sim -\lambda_2 K_{X_2} + \nu_2 f$ be the strict transform of H_1 on X_2. Then,

1. $\lambda_1 = \lambda_2 =: \lambda$,
2. $\nu_2 = \nu_1 + \sum|\omega_1|\left(\lambda - m_{\omega_1}\right)$, *where the sum runs over all orbits ω_1 in* $\mathrm{Bas}(\varphi)$,
3. $m_{\omega_2}(H_2) = 2\lambda - m_{\omega_1}$ *for orbits ω_2 in* $\mathrm{Bas}(\varphi^{-1})$ *(and ω_1 the corresponding orbit in* $\mathrm{Bas}(\varphi)$*),*

where $|\omega_i|$ denotes the size of the orbit ω_i, and $m_{\omega_i} = m_{\omega_i}(H_i)$ denotes the multiplicity of H_i at the points in ω_i, for $i = 1, 2$.

Proof. Consider the minimal resolution

$$\begin{tikzcd} & S \arrow[dl, "\sigma_1"'] \arrow[dr, "\sigma_2"] & \\ X_1 \arrow[rr, dashed, "\varphi"'] & & X_2. \end{tikzcd}$$

We consider the case where φ is one link of type II, that is, σ_i is the blow-up of one orbit ω_i. Let $E_i \subset S$ be the exceptional divisor of the blow-up σ_i for $i = 1, 2$, and let $\hat{f}$ be a general fiber on S. So $E_1 + E_2 = |\omega_1|\,\hat{f}$. We compute

$$\begin{aligned} \tilde{H}_i &= \sigma_i^*(H_i) - m_{\omega_i} E_i \\ &= -\lambda_i \sigma_i^*(K_{X_i}) + \nu_i \sigma_i^*(f) - m_{\omega_i} E_i \\ &= -\lambda_i (K_S - E_i) + \nu_i \hat{f} - m_{\omega_i} E_i \\ &= -\lambda_i K_S + \nu_i \hat{f} + (\lambda_i - m_{\omega_i}) E_i \end{aligned}$$

Replacing $E_1 = |\omega_1|\,\hat{f} - E_2$ in $\tilde{H}$ we get

$$\tilde{H}_1 = -\lambda_1 K_S + \left(\nu_1 + |\omega_1|\,(\lambda_1 - m_{\omega_1})\right)\hat{f} + (m_{\omega_1} - \lambda_1) E_2.$$

Comparing $\tilde{H}_1 = \tilde{H}_2$ we find $\lambda_2 = \lambda_1$, $\nu_2 = \nu_1 + |\omega_1|\,(\lambda - m_{\omega_1})$, and $m_{\omega_2} = 2\lambda - m_{\omega_1}$. Repeating this for every orbit $\omega_1 \in \mathrm{Bas}(\varphi)$, we find $\lambda_2 = \lambda_1$ and $\nu_2 = \nu + \sum|\omega_1|\,(\lambda - m_{\omega_1})$. □

Lemma 3.10. *Let $\Delta = 16$ and $\delta = \frac{1}{2}$. Let $X \to V$, $Y \to W$ and $X' \to V'$ be three minimal conic bundles and let $\Phi\colon X \dashrightarrow Y$ be a birational map of cardinality $\geqslant \Delta$ and $\Psi\colon Y \dashrightarrow X'$ a birational map of cardinality $< \Delta$ such that*

$$\begin{array}{ccccc} X & \overset{\Phi}{\dashrightarrow} & Y & \overset{\Psi}{\dashrightarrow} & X' \\ \downarrow & & \downarrow & & \downarrow \\ V & \simeq & W & \not\rightarrow & V'. \end{array}$$

where we mean by "$W \not\rightarrow V'$" that there is no such morphism making the diagram commute.

Let H be a linear system on X without fixed component, so

$$H \sim -\lambda K_X + \nu f,$$

where f is a general fiber of $X \to V$. Let $H_Y \sim -\lambda_Y K_Y + \nu_Y f_Y$ be the strict transform of H on Y, and let $H' \sim -\lambda' K_{X'} + \nu' f'$ be the one on X'.

Let μ (respectively μ') be the maximum of the multiplicities $m_\omega(H)$ (respectively $m_\omega(H')$) for all orbits ω with $|\omega| \geqslant \Delta$.

Assume $\mu < \delta\lambda$. Then $\lambda' > \lambda$ and $\mu' < \delta\lambda'$.

Proof. By Lemma 3.2, we can write Φ as a composition of links of type II between conic bundles and, by Corollary 3.6, move the links of cardinality $< \Delta$ to the end. So we can assume that each base orbit of Φ has cardinality $\geqslant \Delta$.

Since Ψ is an isomorphism on the points lying in an orbit of size $\geqslant \Delta$, the maximal multiplicity μ' on X' of H' equals the multiplicity of H_Y at a point in Y. Hence, $\mu' \leqslant H_Y \cdot f_Y = 2\lambda_Y$ (when taking f_Y to be the fiber through a point of maximal multiplicity). As $\lambda_Y = \lambda$ by Lemma 3.9, we get $\mu' \leqslant 2\lambda$. So if we show that $\lambda' > \frac{2}{\delta}\lambda = 4\lambda$, then $\delta\lambda' > 2\lambda \geqslant \mu'$ and $\lambda' > \lambda$ are implied.

Let $g \subset Y$ be the pull back of a general fiber f' of $X' \to V'$ under Ψ. We write $g \sim -aK_Y + bf$. As g is not a fiber, $a \geqslant \frac{1}{2}$ by Lemma 3.8. We will use $H' \cdot f' = 2\lambda'$ to find a lower bound for λ'. Consider a minimal resolution

$$\begin{array}{ccc} & T & \\ \swarrow & & \searrow \\ Y & \overset{\Psi}{\dashrightarrow} & X'. \end{array}$$

As f' is a general fiber, we have $\tilde{H}' \cdot \tilde{f}' = H' \cdot f'$ for the strict transforms of H' respectively f' in T. So we find

$$\begin{aligned} 2\lambda' &= H' \cdot f' \\ &= \tilde{H}_Y \cdot \tilde{g} \\ &= H_Y \cdot g - \sum_j m_{p_j}(H_Y) m_{p_j}(g), \end{aligned}$$

where the points $p_j \in Y$ are the points blown up in $T \to Y$ and infinitely near ones. Since Ψ is of cardinality $< \Delta$, the p_j form orbits of size $< \Delta$. As we have remarked in the beginning of the proof, the base points of Φ consist of orbits of size $\geqslant \Delta$, hence the map Φ is a local isomorphism onto the points p_j. Hence, $M_j := m_{p_j}(H_Y) = m_{\Phi^{-1}(p_j)}(H)$. We have $H^2 - \sum m_q(H)^2 \geqslant 0$, where the sum goes over all points (including infinitely near ones) $q \in X$ (since H is a linear system, hence nef), so also $H^2 - \sum M_j^2 \geqslant 0$. This gives an upper bound $\sum M_j^2 \leqslant H^2$.

Let $N_j = m_{p_j}(g)$. As $\tilde{g}^2 = \tilde{f}'^2 = 0$, we find $g^2 = \tilde{g}^2 + \sum N_j^2 = \sum N_j^2$. By Cauchy-Schwarz, we have $\left(\sum M_j N_j\right)^2 \leqslant (\sum M_j^2)(\sum N_j^2)$ and, with the above discussion, get the inequality

$$\begin{aligned} 2\lambda' &\geqslant H_Y \cdot g - \sqrt{\left(\sum M_j^2\right)\left(\sum N_j^2\right)} \\ &\geqslant H_Y \cdot g - \sqrt{H^2 g^2}. \end{aligned} \tag{1}$$

Let β be such that $\nu_Y = \nu + \beta\lambda$, namely

$$\beta = \sum |\omega| \, (1 - \frac{m_\omega}{\lambda}) > \Delta(1-\delta) = 8,$$

where the notation is from Lemma 3.9 and the inequalities come from our assumptions that $|\omega| \geqslant \Delta = 16$ and $m_\omega \leqslant \mu < \delta\lambda = \frac{1}{2}\lambda$.

To compare $H_Y \cdot g$ with the square root of $H^2 g^2$, let $d = K_Y^2$ and denote by e_1 the expression $\frac{1}{\lambda^2} H^2 = \lambda(\lambda d + 4\nu) = d + 4\frac{\nu}{\lambda}$, and similary $e_2 = \frac{1}{a^2} g^2 = d + 4\frac{b}{a}$. We compute

$$\begin{aligned} H_Y \cdot g &= (-\lambda_Y K_Y + \nu_Y f) \cdot (-a K_Y + bf) \\ &= a\lambda d + 2b\lambda + 2a\nu_Y \\ &= \lambda(ad + 2b) + 2a(\nu + \beta\lambda) \end{aligned}$$

and so

$$\frac{1}{a\lambda} H_Y \cdot g = d + 2\frac{b}{a} + 2\frac{\nu}{\lambda} + 2\beta = \frac{1}{2}(e_1 + e_2) + 2\beta.$$

Therefore,

$$\begin{aligned} \frac{2\lambda'}{a\lambda} &\geqslant \frac{1}{a\lambda}\left(H_Y \cdot g - \sqrt{H^2 g^2}\right) \\ &= \frac{e_1}{2} + \frac{e_2}{2} + 2\beta - \sqrt{e_1 e_2} \\ &\geqslant 2\beta, \end{aligned}$$

where the last inequality holds because of the inequality of the arithmetic and the geometric mean. Hence, using $a \geqslant \frac{1}{2}$, we conclude the proof with

$$\lambda' \geqslant a\beta\lambda > 8a\lambda \geqslant 4\lambda.$$

So we have $\lambda' > \lambda$, and $\delta\lambda' = \frac{1}{2}\lambda' > \frac{1}{2}4\lambda = 2\lambda \geqslant \mu'$. □

Corollary 3.11. *Let $\Delta = 16$. Assume $N \geqslant 1$. For $i = 0, \ldots, N$ let $X_i \to V_i$ and $Y_i \to W_i$ be conic bundles that are geometrically rational with birational maps $\Phi_i \colon X_i \dashrightarrow Y_i$ of cardinality $\geqslant \Delta$ and birational maps (for $i \neq 0$) $\Psi_i \colon Y_{i-1} \dashrightarrow X_i$ of cardinality $< \Delta$ such that the diagram*

$$\begin{tikzcd}
X_0 \arrow[r, dashed, "\Phi_0"] \arrow[d] & Y_0 \arrow[r, dashed, "\Psi_1"] \arrow[d] & X_1 \arrow[r, dashed] \arrow[d] & Y_{N-1} \arrow[r, dashed, "\Psi_N"] \arrow[d] & X_N \arrow[r, dashed, "\Phi_N"] \arrow[d] & Y_N \arrow[d] \\
V_0 \arrow[r, phantom, "\simeq"] & W_0 \arrow[r, "|" marking] & V_1 & W_{N-1} \arrow[r, "|" marking] & V_N \arrow[r, phantom, "\simeq"] & W_N
\end{tikzcd}$$

commutes, where we mean by "$W_{i-1} \nrightarrow V_i$" that there is no such morphism making the diagram commute.

Let $\varphi = \Phi_N\Psi_N\Phi_{N-1}\cdots\Phi_1\Psi_1\Phi_0$, and let $H \sim -\lambda K_X + \nu f$ be a linear system without fixed comoponent on $X = X_0$ and let $H' \sim -\lambda' K_{X'} + \nu' f$ be its strict transform in $X' = Y_N$ under φ. Then $\lambda' > \lambda$. In particular, there is no morphism $V_0 \to W_N$ making the diagram

$$\begin{tikzcd}
X_0 \arrow[r, dashed, "\varphi"] \arrow[d] & Y_N \arrow[d] \\
V_0 \arrow[r, "|" marking] & W_N
\end{tikzcd}$$

commute, hence φ is not an isomorphism of conic bundles.

Proof. This is a direct corollary from Lemma 3.10: We can assume that H is smooth, hence $\mu = 0$, and we can apply the lemma.

For the last part: If φ would preserve the fibration, then it would be of type II, hence we would have $\lambda' = \lambda$, a contradiction to Lemma 3.9. $\square$

3.3. **Generating relations.**

Proof of Theorem 2. The statement was already proven in Lemma 3.7 if X is not geometrically rational. So we assume now that X is geometrically rational.

Let $\varphi_n \cdots \varphi_1 = \mathrm{id}$ be a relation in $\mathrm{BirMori}(X)$, where $\varphi_i \colon Z_{i-1} \dashrightarrow Z_i$ is a Sarkisov link of cardinality d_i. If all $d_i \leqslant 15$, we are in situation (a).

Otherwise, the base points of at least one of the φ_i contains an orbit of size $\geqslant 16$. In particular, $\varphi_i \colon Z_{i-1} \dashrightarrow Z_i$ is a link of type II between conic bundles (since these are the only links of big cardinality, see Remark 2.12). We will prove that we are always in the situation of Lemma 3.5 using Corollary 3.11. By replacing the relation with

$$\varphi_{i-1} \cdots \varphi_1 \varphi_n \cdots \varphi_{i+1} \varphi_i,$$

we can assume that Z_0 is a conic bundle. We consider the relator $\varphi = \varphi_n \cdots \varphi_1$ and write it – as in Remark 2.13 – as

$$\varphi = \Phi_N \Psi_N \cdots \Phi_1 \Psi_1 \Phi_0,$$

where for $i = 0, \dots, N$ the $X_i \to V_i$ and $Y_i \to W_i$ are conic bundles with birational maps $\Phi_i \colon X_i \dashrightarrow Y_i$ that are a composition of links of type II between conic bundles, and birational maps (for $i \neq 0$) $\Psi_i \colon Y_{i-1} \dashrightarrow X_i$ of cardinality $\leqslant 15$.

If $N = 0$ then $\varphi = \Phi_0$ is a composition of links of type II between conic bundles. The result follows with Lemma 3.5.

If $N \geqslant 1$, we can assume with Corollary 3.6 that each Φ_i is either a product of links of cardinality $\geqslant 16$ or an isomorphism. Now, we change our decomposition of the relator φ such that it is of the form of Corollary 3.11. If one of the Φ_i is an isomorphism, we look at the birational map $\Psi_i' = \Psi_{i+1}\Phi_i\Psi_i : Y_{i-1} \dashrightarrow X_{i+1}$. There are two possibilities: Either Ψ_i' preserves the fibration, or it does not. If it does not, we replace $\Psi_{i+1}\Phi_i\Psi_i$ with Ψ_i' in the decomposition of φ. Note that Ψ_i' is of cardinality $\leqslant 15$. If Ψ_i' preserves the fibration, we replace Φ_{i-1} with $\Phi_{i-1}' = \Phi_{i+1}\Psi_i'\Phi_{i-1} \colon X_{i-1} \dashrightarrow Y_{i+1}$. Applying Corollary 3.6 once more, we can assume that Φ_{i-1}' is a product of links of cardinality $\geqslant 16$ or an isomorphism. In the latter case we repeat the process.

In this way, we arrive either at the case $N = 0$ and we are done, or we are in the situation of Corollary 3.11, which implies that the relator φ is not an isomorphism, a contradiction. □

4. Detour to Galois theory for non-experts

In this section we will prove that a perfect field $\mathbf{k}$ with $[\overline{\mathbf{k}} : \mathbf{k}] > 2$ contains an arbitrarily large Galois orbit. We recall first the statements from Galois theory that we need.

Recall that a field $\mathbf{k}$ is called *perfect* if every algebraic extension is separable. In particular, any finite extension of a perfect field is again perfect. A finite field extension L/K is called *Galois* if it is normal and separable. In this case, the extension degree $[L : K]$ equals the number of elements in the Galois group $\mathrm{Gal}(L/K)$.

Moreover, for any splitting field L of an irreducible polynomial $f \in \mathbf{k}[x]$, the extension $L/\mathbf{k}$ is normal. So if $\mathbf{k}$ is perfect, then $L/\mathbf{k}$ is Galois. We will use the Artin-Schreier Theorem [Lan05, Corollary 9.3, Chapter IV] and the Primitive Element Theorem [Lan05, Theorem 4.6, Chapter V].

Lemma 4.1. *Let $L/\mathbf{k}$ be a finite Galois extension. For $\gamma \in L$ the degree of $[\mathbf{k}(\gamma) : \mathbf{k}]$ equals the length of the orbit of γ under the action of* $\mathrm{Gal}(L/\mathbf{k})$.

Proof. As $L/\mathbf{k}$ is normal and separable by assumption, also $L/\mathbf{k}(\gamma)$ is normal [Lan05, Chapter V, Theorem 3.4] and separable, hence it is Galois. Note that the stabilizer of γ is $\mathrm{Gal}(L/\mathbf{k}(\gamma))$. By the orbit formula, the length of the orbit of γ under the action of $\mathrm{Gal}(L/\mathbf{k})$ equals $[\mathrm{Gal}(L/\mathbf{k}) : \mathrm{Gal}(L/\mathbf{k}(\gamma))]$, which is equal to

$$\frac{|\mathrm{Gal}(L/\mathbf{k})|}{|\mathrm{Gal}(L/\mathbf{k}(\gamma))|} = \frac{[L : \mathbf{k}]}{[L : \mathbf{k}(\gamma)]} = [\mathbf{k}(\gamma) : \mathbf{k}].$$

□

Lemma 4.2. *Let $\mathbf{k}$ be a perfect field with $[\overline{\mathbf{k}} : \mathbf{k}] > 2$ and let $\Delta \geqslant 1$. Then $\overline{\mathbf{k}}$ contains an orbit of length $\geqslant \Delta$ under the action of* $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$.

Proof. The Artin-Schreier Theorem directly implies that the degree $[\bar{\mathbf{k}} : \mathbf{k}]$ is infinite, and hence for a finite field extension $L/\mathbf{k}$ with $L \subset \bar{\mathbf{k}}$ we have that L is not equal to the algebraic closure $\bar{\mathbf{k}}$. We inductively construct a series of finite field extensions $L_n/\mathbf{k}$ such that $[L_n : \mathbf{k}] \geqslant 2^n$. For the base case, set $L_0 = \mathbf{k}$ so $[L_0 : \mathbf{k}] = 1 = 2^0$ is finite. For the induction step $n-1 \to n$, assume that there is a finite field extension $L_{n-1}/\mathbf{k}$ with $[L_{n-1} : \mathbf{k}] \geqslant 2^{n-1}$. Hence, $L \neq \bar{\mathbf{k}}$ and so there exists $\alpha_n \in \bar{\mathbf{k}} \backslash L_{n-1}$. Set $L_n = L_{n-1}(\alpha_n)$, so L_n/L_{n-1} is an algebraic and even finite extension. As $L_n \neq L_{n-1}$ we have $[L_n : L_{n-1}] \geqslant 2$. The induction hypothesis implies with $[L_n : \mathbf{k}] = [L_n : L_{n-1}][L_{n-1} : \mathbf{k}]$ that

$$\infty > [L_n : \mathbf{k}] \geqslant 2 \cdot 2^{n-1} = 2^n,$$

which means that $L_n/\mathbf{k}$ is a finite extension of degree $\geqslant 2^n$.

Now, choose n such that $2^n \geqslant \Delta$. As $L_n/\mathbf{k}$ is an algebraic extension of the perfect field $\mathbf{k}$, it is a separable extension. The Primitive Element Theorem can be applied and provides the existence of $\gamma \in L_n$ such that $L_n = \mathbf{k}(\gamma)$. Take a finite Galois extension $L/\mathbf{k}$ with $\gamma \in L \subset \bar{\mathbf{k}}$. (For example take L to be the splitting field in $\bar{\mathbf{k}}$ of the minimal polynomial of γ over $\mathbf{k}$.) By Lemma 4.1, the orbit of γ is of length $[\mathbf{k}(\gamma) : \mathbf{k}] = [L_n : \mathbf{k}] \geqslant \Delta$. □

Note that we do not claim that an orbit of exact size Δ exists. In fact, for any $\Delta \geqslant 2$ there exists a perfect field with no Galois orbit of exact size Δ, namely the following example that was provided to me by Lars Kuehne.

Lemma 4.3. *Let $\Delta \geqslant 2$. There exists a perfect field $\mathbf{k}$ with $[\bar{\mathbf{k}} : \mathbf{k}] = \infty$ such that no element in $\bar{\mathbf{k}}$ has an orbit of length Δ under the action of $\mathrm{Gal}(\bar{\mathbf{k}}/\mathbf{k})$.*

Proof. Consider the field extension $\mathbb{Q} \subset \mathbf{k} \subset \overline{\mathbb{Q}}$, where $\mathbf{k}$ is the set consisting of elements $a \in \overline{\mathbb{Q}}$ such that there exists a tower of fields $\mathbb{Q} = L_0 \subset L_1 \subset \cdots \subset L_n \ni a$ such that L_i/L_{i-1} is the splitting field of a polynomial of degree Δ with coefficients in L_{i-1}.

Indeed, $\mathbf{k}$ is a perfect field: For $a, b \in \mathbf{k}$ let $L_0 \subset L_1 \subset \cdots \subset L_n \ni a$ be the tower of fields corresponding to a, and let g_i be the polynomials of degree Δ corresponding to the splitting fields corresponding to b for $i = 1, \ldots, m$. There exists a tower of fields $L_0 \subset \cdots \subset L_n \subset L_{n+1} \subset \cdots L_{n+m}$ with $a + b, ab \in L_{n+m}$, where the L_{n+i}/L_{n+i-1} are the splitting fields of g_i. Therefore, $\mathbf{k}$ is a field. It is perfect because its characteristic is zero.

To prove that $[\overline{\mathbb{Q}} : \mathbf{k}] = \infty$, we assume that $[\overline{\mathbb{Q}} : \mathbf{k}] = N$ for some $N \in \mathbb{N}$. Let $p > \max\{N, \Delta!\}$ be a prime number. First, we prove that there exists a Galois extension $F/\mathbb{Q}$ of degree p. By Dirichlet's Theorem, one can choose a prime $q \equiv 1 \mod p$. Let $\mathbb{Q}(\mu_q)$ be the cyclotomic extension of $\mathbb{Q}$, where μ_q is a q^{th} root of unity. The Galois group of $\mathbb{Q}(\mu_q)/\mathbb{Q}$ is the multiplicative group $(\mathbb{Z}/q\mathbb{Z})^\times$, which is cyclic of order $q-1$. As p divides $q-1$, there exists a (normal) subgroup $H \subset \mathrm{Gal}(\mathbb{Q}(\mu_q)/\mathbb{Q})$ of order $\frac{q-1}{p}$. Let $F \subset \mathbb{Q}(\mu_q)$ be the field that is fixed by H. By Galois Theory, the extension

$F/\mathbb{Q}$ is Galois and of degree p (using that the extension degree of $\mathbb{Q}(\mu_q)/\mathbb{Q}$ is $q-1$) [Lan05, Chapter VI, Theorem 1.1 and 1.8].

By the Primitive Element Theorem, we can choose $\alpha \in \overline{\mathbb{Q}}$ such that $F = \mathbb{Q}(\alpha)$. Hence $\mathbb{Q}(\alpha)/\mathbb{Q}$ is a Galois extension with $[\mathbb{Q}(\alpha) : \mathbb{Q}] = p$. We prove that $\alpha \in \mathbf{k}$. As $\mathbf{k}/\mathbb{Q}$ is an (arbitrary) extension, the degree $[\mathbf{k}(\alpha) : \mathbf{k}]$ divides $[\mathbb{Q}(\alpha) : \mathbb{Q}] = p$ [Lan05, Chapter VI, Corollary 1.13] (using that the compositum $\mathbb{Q}(\alpha)\mathbf{k}$ of the two fields $\mathbb{Q}(\alpha)$ and $\mathbf{k}$ equals $\mathbf{k}(\alpha)$). By the transitivity of the degree, we also find that

$$N = [\overline{\mathbb{Q}} : \mathbf{k}] = [\overline{\mathbb{Q}} : \mathbf{k}(\alpha)][\mathbf{k}(\alpha) : \mathbf{k}],$$

so $[\mathbf{k}(\alpha) : \mathbf{k}]$ divides N. Since it also divides the prime number $p > N$, the only possibility is $[\mathbf{k}(\alpha) : \mathbf{k}] = 1$ and so $\alpha \in \mathbf{k}$.

Now, we find a contradiction to $p > \Delta!$. As α lies in $\mathbf{k}$, there exists a tower of fields $L_0 \subset \cdots \subset L_n \ni \alpha$ such that L_i/L_{i-1} is the splitting field of a polynomial of degree Δ. Hence, $[L_i : L_{i-1}] \leqslant \Delta!$. So $[L_n : \mathbb{Q}] = [L_n : L_{n-1}] \cdots [L_1 : \mathbb{Q}]$ is a product of numbers smaller or equal to $\Delta!$. Note that $\mathbb{Q}(\alpha) \subset L_n$, hence $[L_n : \mathbb{Q}] = [L_n : \mathbb{Q}(\alpha)][\mathbb{Q}(\alpha) : \mathbb{Q}]$ and so $p = [\mathbb{Q}(\alpha) : \mathbb{Q}]$ divides $[L_n : \mathbb{Q}]$. As p is a prime, it implies that $p \leqslant \Delta!$, which is a contradiction to $p > \Delta!$.

Finally, we prove that $\mathbf{k}$ has no Galois orbit of size Δ. Assume that there exists $\beta \in \overline{\mathbb{Q}}$ such that $\mathbf{k}(\beta)/\mathbf{k}$ is finite and such that the length of the Galois orbit of β is Δ. In particular, $\beta \in \overline{\mathbb{Q}} \setminus \mathbf{k}$. Consider the minimal polynomial $\min_{\mathbf{k}}(\beta)$ of β over $\mathbf{k}$ and its splitting field L. So $\beta \in L$ and $L/\mathbf{k}$ is finite and Galois. Hence with Theorem 4.1 we have that the size of the Galois orbit of β, which is Δ, equals $[\mathbf{k}(\beta) : \mathbf{k}]$, which in turn is the degree of the minimal polynomial $\min_{\mathbf{k}}(\beta)$. By the construction of our field $\mathbf{k}$, this implies that β already lies in $\mathbf{k}$, a contradiction. □

5. Group homomorphism

Definition 5.1. We say that two conic bundles X/W and X'/W' are *equivalent* if there exists a birational map $X \dashrightarrow X'$ that preserves the fibration (see Definition 2.5) and maps singular fibers onto singular fibers.

We denote the set of equivalence classes of conic bundles birational to X by $\mathrm{CB}(X)$.

Definition 5.2. We say that two Sarkisov links χ and χ' of conic bundles of type II are *equivalent*, if

(1) the conic bundles are equivalent,
(2) the Sarkisov links have the same cardinality.

For an equivalence class $C \in \mathrm{CB}(X)$ of conic bundles, we denote by $\mathrm{M}(C)$ the set of equivalence classes of Sarkisov links of type II (between conic bundles in the equivalence class of C). That is, an element of $\mathrm{M}(C)$ is the class of Sarkisov links of type II between equivalent conic bundles of the same cardinality.

Proof of Theorem 3. One has to show that the homomorphism is well defined, that is, to show that every relator is mapped onto the identity.

By construction, relators that consist of Sarkisov links of cardinality $\leqslant 15$ are mapped on the identity.

The trivial relation $\alpha\beta = \gamma$, where α, β, γ are isomorphisms of Mori fiber spaces, is mapped onto the identity by construction. Trivial relations of the form $\alpha\psi^{-1}\phi = \mathrm{id}$ satisfy $\mathrm{Bas}(\phi) = \mathrm{Bas}(\psi)$, hence they have the same cardinality and are therefore in the same equivalence class of Sarkisov links. Hence ψ and ϕ have the same image and so the relator is mapped onto the identity.

Relations of the form $\chi_4\chi_3\chi_2\chi_1 = \mathrm{id}$, where χ_i are Sarkisov links of type II between conic bundles, are such that χ_4 and χ_2, as well as χ_3 and χ_1 have the same cardinality as in Remark 3.4. As they are all links between conic bundles of the same equivalence class of conic bundles, χ_1 and χ_3, as well as χ_2 and χ_4 have the same image. Therefore, the relator is mapped onto the identity. This proves the existence of the groupoid homomorphism. The fact that it restricts to a group homomorphism from $\mathrm{Bir}(X)$ is immediate, and the fact that it restricts to a group homomorphism from $\mathrm{Bir}(X/W)$ is a consequence of Lemma 3.2. □

Example 5.3. Consider the birational map

$$(x, y) \mapsto (xp(y), y),$$

and its extension to a birational map $\varphi\colon \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ that is given by

$$[x_0 : x_1; y_0 : y_1] \mapsto [x_0 y_1^d : x_1 p(y_0, y_1); y_0 : y_1],$$

where $p \in \mathbf{k}[y_0, y_1]$ is an irreducible polynomial of degree $d \geqslant 16$. Since $\mathbf{k}$ is perfect, $p(t, 1)$ has d different zeroes $t_1, \ldots, t_d \in \bar{\mathbf{k}}$. So φ is not defined on $[1 : 0; 1 : 0]$ and on the points $p_i = [0 : 1; t_i : 1]$ for $i = 1, \ldots, d$. One can check that φ is the composition of a link $\varphi_0\colon \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{F}_d$ of type II centered at the orbit $\{p_1, \ldots, p_d\}$, followed by d links $\varphi_n\colon \mathbb{F}_n \dashrightarrow \mathbb{F}_{n-1}$ of type II of cardinality 1 for $n = d, \ldots, 1$. Note that φ_0 is not mapped onto the identity (its image is $1 \in \mathbb{Z}/2$ corresponding to the equivalence class of φ_0 in $\mathrm{M}(\mathbb{P}^1 \times \mathbb{P}^1)$), whereas all φ_n for $n = 1, \ldots, d$ are mapped onto the identity. Therefore, the image of $\varphi \in \mathrm{Bir}(\mathbb{P}^1 \times \mathbb{P}^1)$ under the group homomorphism is non-trivial.

Proof of Theorem 1. We take the group homomorphism from Theorem 3. For a constant polynomial $p \in \mathbf{k}$, the local map $(x, y) \mapsto (px, y)$ is an automorphism and therefore it is mapped onto the identity. For the surjectivity, using Lemma 4.2 we can construct an infinite and countable indexing set I such that for each $d \in I$ there exists an irreducible polynomial $p \in \mathbf{k}[y]$ of degree d, and each $d \in I$ is at least 16. For each such polynomial we consider $\varphi\colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ as in Example 5.3. Let $\alpha\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ be the blow-up of two points in $\mathbb{P}^2$ that are defined over $\mathbf{k}$, followed by the contraction of the line connecting the two points. Then $\alpha^{-1}\varphi\alpha$ lies in $\mathrm{Bir}(\mathbb{P}^2)$ and, since

α and α^{-1} have cardinality 1, its image under the group homomorphism of Theorem 3 is non-trivial on the index of I corresponding to the degree of p. $\square$

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